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Global Existence and Uniform Blow-Up to a Nonlocal Parabolic System with Nonlinear Boundary Conditions Arising in a Thermal Explosion Theory

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Abstract: This paper deals with a nonlinear nonlocal parabolic system with nonlinear heat-loss boundary conditions, which arise in the thermal explosion model. Firstly, we prove a comparison principle for some kinds of parabolic systems under nonlinear boundary conditions. Using this, we improve a new theorem of the sub-and-super solution. Secondly, based on the new sub-and-super solution theorem, the sufficient conditions that the solution exists and blows up uniformly in finite time are presented. Then, we generalize some of the lemmas related to uniform blow-up solutions, which are used to introduce the uniform blow-up profiles of solutions. Finally, we give several numerical simulations to illustrate the existence and uniform blow-up of solutions.

Keywords: nonlocal sources; nonlinear boundary conditions; global existence; blow-up profile

MSC: 35K57; 35K61



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1. Introduction

In this paper, we discuss the following semi-linear parabolic system with nonlinear nonlocal sources and nonlinear boundary conditions:

$$\begin{cases} u_t = \Delta u + k_1 \int_{\Omega} u^m(x, t) v^n(x, t) dx, & (x, t) \in Q_T, \\ v_t = \Delta v + k_2 \int_{\Omega} u^p(x, t) v^q(x, t) dx, & (x, t) \in Q_T, \\ \mathbf{n} \cdot \nabla u + g_1(u)u = \mathbf{n} \cdot \nabla v + g_2(v)v = 0, & (x, t) \in S_T, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), & x \in \Omega, \end{cases} \quad (1)$$

where k_1, k_2, m, n, p, q are positive constants, $Q_T = \Omega \times (0, T]$, $S_T = \partial\Omega \times (0, T]$, $T > 0$ and Ω is a bounded region in R^N ($N > 1$ is an integer) with a smooth boundary $\partial\Omega$, $\mathbf{n}$ is an outward unit normal vector of $\partial\Omega$, g_1, g_2, u_0, v_0 are nonnegative continuous functions satisfying some assumptions, and $|\Omega|$ denotes the Lebesgue measure of Ω .

Our study is motivated by [1,2], in which the existence and blow-up of partial differential equations under nonlinear boundary conditions were researched. As stated in [1], in certain thermal explosion problems that require long induction times (times prior to ignition), such as the safe storage of energetic materials or nuclear waste, the Dirichlet boundary condition $u = v = 0$ is no longer applicable because during this time the boundary of the reaction material will be preheated to a temperature significantly higher than the surrounding temperature. As a result, we need the heat-loss boundary conditions in (1) to describe the temperature distribution of the boundary. In recent years, many authors have investigated the reaction-diffusion system with Dirichlet boundary conditions as follows:

$$\begin{cases} u_t - d_1(x, t)\Delta u = f(u, v), & (x, t) \in \Omega \times (0, T), \\ v_t - d_2(x, t)\Delta v = g(u, v), & (x, t) \in \Omega \times (0, T), \\ u(x, t) = v(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), & x \in \Omega, \end{cases} \quad (2)$$

For details, see [3–7] and their references. In [3], the blow-up and global existence criteria of the nonlocal parabolic system

$$u_t = f(u) \left(\Delta u + a \int_{\Omega} v dx \right), \quad v_t = g(v) \left(\Delta v + b \int_{\Omega} u dx \right) \quad (3)$$

were obtained. In [4,7], the authors studied nonlocal nonlinear parabolic systems of the form

$$u_t - u^{s_1} \Delta u = a u^m \int_{\Omega} v^n dx, \quad v_t - v^{s_2} \Delta v = b v^p \int_{\Omega} u^q dx \quad (4)$$

where $m, n, p, q, s_1, s_2 > 0$ are constants. They all gave some results on the existence of solutions, the blow-up solutions, and the blow-up rate or profile. In [5], Li et al. suggested that if one of $\alpha > 1, \beta > 1$ or $pq > (1 - \alpha)(1 - \beta)$ holds, the solution of (2) blows up in finite time for sufficiently large initial data, when

$$f(u, v) = \int_{\Omega} u^{\alpha} v^{\beta} dx, \quad g(u, v) = \int_{\Omega} u^q v^{\beta} dx \quad (5)$$

and $d_i = 1, i = 1, 2$. Furthermore, they also obtained the blow-up profile of the solution

$$\begin{aligned} \lim_{t \rightarrow T^*} u(x, t) (T^* - t)^{\frac{p-\beta+1}{pq-(1-\alpha)(1-\beta)}} &= \left(\frac{q-\alpha+1}{p-\beta+1} \right)^{\frac{p}{pq-(1-\alpha)(1-\beta)}} \\ &\times \left(\frac{p-\beta+1}{|\Omega|(pq-(1-\alpha)(1-\beta))} \right)^{\frac{p-\beta+1}{pq-(1-\alpha)(1-\beta)}}, \\ \lim_{t \rightarrow T^*} v(x, t) (T^* - t)^{\frac{q-\alpha+1}{pq-(1-\alpha)(1-\beta)}} &= \left(\frac{p-\beta+1}{q-\alpha+1} \right)^{\frac{q}{pq-(1-\alpha)(1-\beta)}} \\ &\times \left(\frac{q-\alpha+1}{|\Omega|(pq-(1-\alpha)(1-\beta))} \right)^{\frac{q-\alpha+1}{pq-(1-\alpha)(1-\beta)}}, \end{aligned}$$

uniformly on any compact subset of Ω , if $q > \alpha - 1, p > \beta - 1$, and $pq > (1 - \alpha)(1 - \beta)$. Thereafter, in [8], Kong and Wang extended these results to parabolic systems with nonlocal boundaries

$$u = \int_{\Omega} \varphi(x, y) u(y, t) dy, \quad v = \int_{\Omega} \psi(x, y) v(y, t) dy \quad (6)$$

by improving the methods used in [9,10].

In addition, there are some interesting results on the global well-posed theory related to our work that can be found in [11–14].

In this paper, we present the global existence, the uniform blow-up criteria, and the uniform blow-up profiles of system (1) with nonlinear heat-loss boundary conditions. To this end, we first give the following assumptions.

- (H1) $u_0, v_0 \in C^{2+\alpha}(\Omega) \cap C^1(\overline{\Omega})$ ($0 < \alpha < 1$) are nonnegative, bounded, and $\frac{\partial u_0}{\partial n}, \frac{\partial v_0}{\partial n} < 0$;
- (H2) $g_i > 0, i = 1, 2$ satisfy the local Lipschitz condition;
- (H3) $g_i, i = 1, 2$ are increasing functions.

This paper is divided into five sections. In Section 2, we prove some new comparison principles for a modification of our system and using the obtained principles, we can

obtain the local existence of (1) if our system has sub- and super-solutions as we define. In Section 3, we give some new definitions of uniform blow-up solutions and global solutions and present the global existence and uniform blow-up criteria of solutions to system (1), respectively, with the help of the methods used in [8,15]. In Section 4, we generalize several relevant lemmas from [9,10] and based on them give the blow-up profile of (1), which describes the asymptotic behavior of the solution near the blow-up time. In Section 5, we simulate numerically the global existence and uniform blow-up of solutions.

2. Local Existence

In this section, we intend to prove the local existence of solutions to (1) using the monotone iterative technique (see [16–18]) and the fixed-point theorem. To do this, we need to define the sub- and super-solutions, as well as to generalize the comparison principle from [2,8,15]. Here we define the sub- and super-solutions as follows.

Definition 1. A pair of nonnegative functions $(\bar{u}(x, t), \bar{v}(x, t))$ is called a super-solution of (1) if $(\bar{u}, \bar{v}) \in (C^{2,1}(Q_T) \cap C^1(\bar{Q}_T)) \times (C^{2,1}(Q_T) \cap C^1(\bar{Q}_T))$, $0 < T \leq +\infty$ and satisfies

$$\begin{cases} \bar{u}_t \geq \Delta \bar{u} + k_1 \int_{\Omega} \bar{u}^m(x, t) \bar{v}^n(x, t) dx, & (x, t) \in Q_T, \\ \bar{v}_t \geq \Delta \bar{v} + k_2 \int_{\Omega} \bar{u}^p(x, t) \bar{v}^q(x, t) dx, & (x, t) \in Q_T, \\ \mathbf{n} \cdot \nabla \bar{u} + g_1(\bar{u}) \bar{u} \geq 0, \mathbf{n} \cdot \nabla \bar{v} + g_1(\bar{v}) \bar{v} \geq 0, & (x, t) \in S_T, \\ \bar{u}(x, 0) \geq u_0(x), \bar{v}(x, 0) \geq v_0(x), & x \in \Omega. \end{cases} \quad (7)$$

Similarly, a sub-solution can be defined with reversed inequalities in (7).

Then, we prove the following comparison principle.

Lemma 1. Suppose that a_i and b_i are the continuous and bounded functions on $\bar{Q}_T$, respectively; c_i and d_i are the nonnegative and bounded functions in Q_T ; and f_i and h_i are the nonnegative functions and the positive increasing functions on $\bar{\Omega}$, respectively. If $u, v \in C^{2,1}(Q_T) \cap C^1(\bar{Q}_T)$ satisfy

$$\begin{cases} u_t - a_1 \Delta u \geq b_1 u + f_1(x) \int_{\Omega} (c_1 u + d_1 v) dx, & (x, t) \in Q_T, \\ v_t - a_2 \Delta v \geq b_2 v + f_2(x) \int_{\Omega} (c_2 u + d_2 v) dx, & (x, t) \in Q_T, \\ \mathbf{n} \cdot \nabla u + h_1(u) u \geq 0, \mathbf{n} \cdot \nabla v + h_2(v) v \geq 0, & (x, t) \in S_T, \\ u(x, 0) = u_0(x) \geq 0, v(x, 0) = v_0(x) \geq 0, & x \in \Omega \end{cases} \quad (8)$$

then $u, v \geq 0$ in Q_T .

Proof. Firstly, we prove that if u, v satisfy

$$\begin{cases} \mathbf{n} \cdot \nabla u + h_1(u) u > 0, \mathbf{n} \cdot \nabla v + h_2(v) v > 0, & (x, t) \in S_T, \\ u(x, 0) = u_0(x) > 0, v(x, 0) = v_0(x) > 0, & x \in \Omega \end{cases} \quad (9)$$

then $u, v > 0$ in Q_T .

Let $\bar{b}_i = \max_{\bar{Q}_T} b_i$, $i = 1, 2$ and $\gamma_1 > \max\{\bar{b}_1, \bar{b}_2\}$. Set $w = e^{-\gamma_1 t} u$, $z = e^{-\gamma_1 t} v$, then we have

$$\begin{cases} w_t - a_1 \Delta w + (\gamma_1 - b_1)w \geq f_1(x) \int_{\Omega} (c_1 w + d_1 z) dx, & (x, t) \in Q_T, \\ z_t - a_2 \Delta z + (\gamma_1 - b_2)z \geq f_2(x) \int_{\Omega} (c_2 w + d_2 z) dx, & (x, t) \in Q_T, \\ \mathbf{n} \cdot \nabla w + h_1(u)w > 0, \mathbf{n} \cdot \nabla z + h_2(v)z > 0, & (x, t) \in S_T, \\ w(x, 0) = u_0(x) > 0, z(x, 0) = v_0(x) > 0, & x \in \Omega. \end{cases} \quad (10)$$

By boundary conditions and Assumption (H1), $w, z > 0$ on S_T can be obtained. Considering initial conditions $w(x, 0) > 0, z(x, 0) > 0$, by continuity, there exists a $t > 0$ such that $w(x, t) > 0, z(x, t) > 0$ for all $(x, t) \in \Omega \times [0, t]$. Let $A = \{t < T : w, z > 0 \text{ in } \Omega \times [0, t]\}$ and take $\bar{t} = \sup A$, then we have $0 < \bar{t} \leq T$. If $\bar{t} = T$, the conclusion is proven, and if $\bar{t} < T$, with no loss of generality we can suppose that there are some $\bar{x} \in \bar{\Omega}$ such that $w(\bar{x}, \bar{t}) = 0$. Therefore, w takes the minimum on $\bar{\Omega} \times (0, \bar{t}]$. According to the first inequality of (10), we obtain

$$w_t - a_1 \Delta w \geq (b_1 - \gamma_1)w, \quad (x, t) \in \Omega \times (0, \bar{t}]. \quad (11)$$

By the strong maximum principle (Theorem 1 in [19] Chap. 2), $w \equiv 0$ can be obtained, which is a contradiction. Hence, $w, z > 0$; that is $u, v > 0$.

Then, we prove that if u, v satisfy

$$\begin{cases} \mathbf{n} \cdot \nabla u + h_1(u)u \geq 0, \mathbf{n} \cdot \nabla v + h_2(v)v \geq 0, & (x, t) \in S_T, \\ u(x, 0) = u_0(x) \geq 0, v(x, 0) = v_0(x) \geq 0, & x \in \Omega \end{cases} \quad (12)$$

then $u, v \geq 0$.

Let $\bar{b}_i = \max_{Q_T} |b_i|, \bar{c}_i = \max_{Q_T} (c_i + d_i), \bar{f}_i = \max_{\Omega} f_i, i = 1, 2$ and

$$w(x, t) = u(x, t) + \eta e^{\gamma_2 t}, \quad z(x, t) = v(x, t) + \eta e^{\gamma_2 t}, \quad (13)$$

where $\gamma_2 \geq \max_{i=1,2} (\bar{b}_i + \bar{f}_i \bar{c}_i |\Omega|)$, η is an arbitrary positive real number. Then, we know $w(x, 0) = u_0(x) + \eta > 0, z(x, 0) = v_0(x) + \eta > 0$. Due to the boundary conditions, we have

$$\begin{cases} \mathbf{n} \cdot \nabla w + h_1(w)w \geq \mathbf{n} \cdot \nabla u + h_1(u)u + \eta h_1(w)e^{\gamma_2 t} \geq \eta h_1(w)e^{\gamma_2 t} > 0, \\ \mathbf{n} \cdot \nabla z + h_2(z)z \geq \mathbf{n} \cdot \nabla v + h_2(v)v + \eta h_2(z)e^{\gamma_2 t} \geq \eta h_2(z)e^{\gamma_2 t} > 0, \end{cases} \quad (14)$$

which holds in S_T . Further, in Q_T , we have

$$\begin{cases} w_t - a_1 \Delta w - b_1 w - f_1 \int_{\Omega} (c_1 w + d_1 z) dx \geq (\gamma_2 - b_1 - f_1(c_1 + d_1)|\Omega|)\eta e^{\gamma_2 t} \geq 0, \\ z_t - a_2 \Delta z - b_2 z - f_2 \int_{\Omega} (c_2 w + d_2 z) dx \geq (\gamma_2 - b_2 - f_2(c_2 + d_2)|\Omega|)\eta e^{\gamma_2 t} \geq 0. \end{cases} \quad (15)$$

Therefore, we know that $w, z > 0$, then, letting $\eta \rightarrow 0^+, u, v \geq 0$ is obtained. The proof of Lemma 1 is completed. $\square$

We give the following conclusion about the properties of the sub-solutions and super-solutions.

Lemma 2. Suppose Assumptions (H1)–(H3) hold, g_1, g_2 are derivable, and the nonnegative $(\underline{u}, \underline{v}), (\bar{u}, \bar{v}) \in (C^{2,1}(Q_T) \cap C^1(\bar{Q}_T)) \times (C^{2,1}(Q_T) \cap C^1(\bar{Q}_T)), 0 < T \leq +\infty$ are the sub- and super-solutions to system (1), respectively. $(u, v) \in (C^{2,1}(Q_T) \cap C^1(\bar{Q}_T)) \times (C^{2,1}(Q_T) \cap C^1(\bar{Q}_T))$ is a solution to problem (1). If Case (i)

$$\begin{cases} \mathbf{n} \cdot \nabla \bar{u} + g_1(\bar{u})\bar{u} > 0, \mathbf{n} \cdot \nabla \bar{v} + g_1(\bar{v})\bar{v} > 0, & (x, t) \in S_T, \\ \bar{u}(x, 0) > u_0(x), \bar{v}(x, 0) > v_0(x) & x \in \Omega \end{cases} \quad (16)$$

is satisfied, we have $\bar{u} > u$, $\bar{v} > v$; otherwise, if Case (ii)

$$\begin{cases} \mathbf{n} \cdot \nabla \underline{u} + g_1(\underline{u})\underline{u} < 0, \mathbf{n} \cdot \nabla \underline{v} + g_1(\underline{v})\underline{v} < 0, & (x, t) \in S_T, \\ \underline{u}(x, 0) < u_0(x), \underline{v}(x, 0) < v_0(x) & x \in \Omega \end{cases} \quad (17)$$

is satisfied, we have $u > \underline{u}$, $v > \underline{v}$.

Proof. Case (i). We assume that (16) holds. Let $w = \bar{u} - u$, $z = \bar{v} - v$, and $\psi_1(s) = g'_1(s)s + g_1(s)$, $\psi_2(s) = g'_2(s)s + g_2(s)$; obviously, $\psi_1, \psi_2 > 0$. According to (1), and using the mean value theorem, we have

$$\begin{cases} w_t - \Delta w \geq k_1 \int_{\Omega} (\bar{u}^m \bar{v}^n - u^m v^n) dx \\ \quad = k_1 \int_{\Omega} (m \bar{\xi}_1^{m-1} \bar{\xi}_2^n w + n \bar{\xi}_1^m \bar{\xi}_2^{n-1} z) dx, & (x, t) \in Q_T, \\ z_t - \Delta z \geq k_2 \int_{\Omega} (\bar{u}^p \bar{v}^q - u^p v^q) dx \\ \quad = k_2 \int_{\Omega} (p \bar{\xi}_1^{p-1} \bar{\xi}_2^q w + q \bar{\xi}_1^p \bar{\xi}_2^{q-1} z) dx, & (x, t) \in Q_T, \\ \mathbf{n} \cdot \nabla w + \psi_1(\bar{\xi}_1)w > 0, \mathbf{n} \cdot \nabla z + \psi_2(\bar{\xi}_2)z > 0, & (x, t) \in S_T, \\ w(x, 0) > 0, z(x, 0) > 0, & x \in \Omega. \end{cases} \quad (18)$$

where $\bar{\xi}_1, \bar{\xi}_2$ are between u and $\bar{u}$; $\bar{\xi}_2, \bar{\xi}_2$ are between v and $\bar{v}$. Similar to the proof of Lemma 1, let $\tilde{w} = e^{-t}w$, $\tilde{z} = e^{-t}z$, then, (18) can be transformed into

$$\begin{cases} \tilde{w}_t - \Delta \tilde{w} + \tilde{w} \geq k_1 \int_{\Omega} (m \bar{\xi}_1^{m-1} \bar{\xi}_2^n w + n \bar{\xi}_1^m \bar{\xi}_2^{n-1} z) dx, & (x, t) \in Q_T, \\ \tilde{z}_t - \Delta \tilde{z} + \tilde{z} \geq k_2 \int_{\Omega} (p \bar{\xi}_1^{p-1} \bar{\xi}_2^q w + q \bar{\xi}_1^p \bar{\xi}_2^{q-1} z) dx, & (x, t) \in Q_T, \\ \mathbf{n} \cdot \nabla \tilde{w} + \psi_1(\bar{\xi}_1)\tilde{w} > 0, \mathbf{n} \cdot \nabla \tilde{z} + \psi_2(\bar{\xi}_2)\tilde{z} > 0, & (x, t) \in S_T, \\ \tilde{w}(x, 0) > 0, \tilde{z}(x, 0) > 0, & x \in \Omega. \end{cases} \quad (19)$$

By boundary conditions and $\psi_1, \psi_2 > 0$, we have $\tilde{w}, \tilde{z} > 0$ on S_T . Since $\tilde{w}(x, 0), \tilde{z}(x, 0) > 0$, by continuity, there exists a $t > 0$ such that $\tilde{w}, \tilde{z} > 0$ for all $(x, t) \in \Omega \times [0, t]$. Let $A = \{t < T : \tilde{w}, \tilde{z} > 0 \text{ in } \Omega \times [0, t]\}$ and $\bar{t} = \sup A$, then we have $0 < t \leq T$. When $\bar{t} = T$, the conclusion is obvious and we discuss the case $\bar{t} < T$. Supposing that there are some $\bar{x} \in \bar{\Omega}$ such that $\tilde{w}(\bar{x}, \bar{t}) = 0$, $\tilde{w}$ takes the minimum on $\bar{\Omega} \times (0, \bar{t}]$. According to the first inequality of (19), we obtain

$$\tilde{w}_t - \Delta \tilde{w} \geq -\tilde{w}, \quad (x, t) \in \Omega \times (0, \bar{t}].$$

By the strong maximum principle (Theorem 1 in [19] Chap. 2), $\tilde{w} \equiv 0$ can be obtained, which is a contradiction. We have $\tilde{w}, \tilde{z} > 0$; that is, $w, z > 0$. $\bar{u} > u$ and $\bar{v} > v$ are proved.

Case (ii). The proof of Case (ii) is similar to that of Case (i), and we omit it. $\square$

Based on Definition 1 and Lemma 1, we give the theorem describing the local existence of (1).

Theorem 1. Suppose Assumptions (H1)–(H3) hold, the nonnegative $(\underline{u}, \underline{v}), (\bar{u}, \bar{v}) \in (C^{2,1}(Q_T) \cap C^1(\bar{Q}_T)) \times (C^{2,1}(Q_T) \cap C^1(\bar{Q}_T))$, $0 < T \leq +\infty$ are the sub- and super-solutions to system (1),

respectively, and $\underline{u} \leq \bar{u}$, $\underline{v} \leq \bar{v}$, then there exists a pair of functions (u, v) satisfying $\underline{u} \leq u \leq \bar{u}$, $\underline{v} \leq v \leq \bar{v}$, which is a solution to system (1).

Proof. According to Assumptions (H1)(H2), we know that for any $u_1 \geq u_2$ given, since the function $-g_1(u)u$ is Lipschitz continuous, there exists a fixed real number L_1 such that

$$|-g_1(u_1)u_1 - (-g_1(u_2)u_2)| \leq L_1|u_1 - u_2|, \quad (20)$$

thus

$$-g_1(u_1)u_1 - (-g_1(u_2)u_2) \geq -L_1(u_1 - u_2). \quad (21)$$

Let $G_{l_1} = -g_1(u)u + L_1u$ and similarly $G_{l_2}(v) = -g_2(v)v + L_2v$, which are increasing. Then, we consider the following auxiliary problem

$$\begin{cases} w_t - \Delta w + w = u + k_1 \int_{\Omega} u^m v^n dx, & (x, t) \in Q_T, \\ z_t - \Delta z + z = v + k_2 \int_{\Omega} u^p v^q dx, & (x, t) \in Q_T, \\ \mathbf{n} \cdot \nabla w + L_1 w = G_{l_1}(u), \mathbf{n} \cdot \nabla z + L_2 z = G_{l_2}(v), & (x, t) \in S_T, \\ w(x, 0) = u_0(x), z(x, 0) = v_0(x), & x \in \Omega, \end{cases} \quad (22)$$

where $u, v \in C^{2,1}(Q_T) \cap C^1(\bar{Q}_T)$, $0 < T \leq +\infty$ are two nonnegative functions. The auxiliary problem (22) has Robin boundary conditions, which means that there exists a unique solution (w, z) to it. Define nonlinear operator $T_1 : [\underline{u}, \bar{u}] \rightarrow [\underline{u}, \bar{u}]$ and $T_2 : [\underline{v}, \bar{v}] \rightarrow [\underline{v}, \bar{v}]$ such that $w = T_1 u$ and $z = T_2 v$. Then, construct the following sequences

$$\begin{aligned} \tilde{w}_1 &= T_1 \bar{u}, & \tilde{w}_2 &= T_1 \tilde{w}_1, & \dots, & \tilde{w}_n &= T_1 \tilde{w}_{n-1}, \\ \hat{w}_1 &= T_1 \underline{u}, & \hat{w}_2 &= T_1 \hat{w}_1, & \dots, & \hat{w}_n &= T_1 \hat{w}_{n-1}; \\ \tilde{z}_1 &= T_2 \bar{v}, & \tilde{z}_2 &= T_2 \tilde{z}_1, & \dots, & \tilde{z}_n &= T_2 \tilde{z}_{n-1}, \\ \hat{z}_1 &= T_2 \underline{v}, & \hat{z}_2 &= T_2 \hat{z}_1, & \dots, & \hat{z}_n &= T_2 \hat{z}_{n-1}. \end{aligned} \quad (23)$$

Next, we will show that the operators T_1 and T_2 are increasing. For any $\underline{u} \leq u_1 \leq u_2 \leq \bar{u}$ and $\underline{v} \leq v_1 \leq v_2 \leq \bar{v}$, let $w_1 = T_1 u_1$, $w_2 = T_1 u_2$, and $z_1 = T_2 v_1$, $z_2 = T_2 v_2$, and take $W = w_2 - w_1$, $Z = z_2 - z_1$. Then, we have

$$\begin{cases} W_t - \Delta W + W = k_1 \int_{\Omega} (u_2^m v_2^n - u_1^m v_1^n) dx + (u_2 - u_1) \geq 0, & (x, t) \in Q_T, \\ Z_t - \Delta Z + Z = k_2 \int_{\Omega} (u_2^p v_2^q - u_1^p v_1^q) dx + (v_2 - v_1) \geq 0, & (x, t) \in Q_T, \\ \mathbf{n} \cdot \nabla W + L_1 W = G_{l_1}(u_2) - G_{l_1}(u_1) \geq 0, & (x, t) \in S_T, \\ \mathbf{n} \cdot \nabla Z + L_2 Z = G_{l_2}(v_2) - G_{l_2}(v_1) \geq 0, & (x, t) \in S_T, \\ W(x, 0) = Z(x, 0) = 0, & x \in \Omega. \end{cases} \quad (24)$$

Applying Lemma 1 with $a_i = 1$, $b_i = -1$, $c_i = d_i = 0$, $h_i = L_i$, $i = 1, 2$, we know $W, Z \geq 0$; that is, $w_2 \geq w_1$, $z_2 \geq z_1$. Letting $W = \bar{u} - \tilde{w}_1$ and $Z = \bar{v} - \tilde{z}_1$, we can obtain

$$\begin{cases} W_t - \Delta W + W = k_1 \int_{\Omega} (\bar{u}^m \bar{v}^n - \bar{u}^m \bar{v}^n) dx + (\bar{u} - \bar{u}) = 0, & (x, t) \in Q_T, \\ Z_t - \Delta Z + Z = 0, & (x, t) \in Q_T, \\ \mathbf{n} \cdot \nabla W + L_1 W \geq G_{l_1}(\bar{u}) - G_{l_1}(\bar{u}) = 0, & (x, t) \in S_T, \\ \mathbf{n} \cdot \nabla Z + L_2 Z \geq 0, & (x, t) \in S_T, \\ W(x, 0) = Z(x, 0) = 0, & x \in \Omega, \end{cases} \quad (25)$$

thus, $\bar{u} \geq \tilde{w}_1$ and $\bar{v} \geq \tilde{z}_1$ can be deduced. Similarly, $\hat{w}_1 \geq \underline{u}$ and $\hat{z}_1 \geq \underline{v}$ can be deduced, too. By mathematical induction on n , we have

$$\begin{aligned} \underline{u} &\leq \hat{w}_1 \leq \hat{w}_2 \leq \cdots \leq \hat{w}_n \leq \cdots \leq \tilde{w}_n \leq \tilde{w}_{n-1} \leq \cdots \leq \tilde{w}_1 \leq \bar{u}, \\ \underline{v} &\leq \hat{z}_1 \leq \hat{z}_2 \leq \cdots \leq \hat{z}_n \leq \cdots \leq \tilde{z}_n \leq \tilde{z}_{n-1} \leq \cdots \leq \tilde{z}_1 \leq \bar{v}, \end{aligned} \quad (26)$$

which shows that the sequences $\{\tilde{w}_n\}$, $\{\hat{w}_n\}$, $\{\tilde{z}_n\}$, $\{\hat{z}_n\}$ are increasing and bounded. Therefore, limits

$$\begin{aligned} \tilde{w}_0 &= \lim_{n \rightarrow \infty} \tilde{w}_n, & \hat{w}_0 &= \lim_{n \rightarrow \infty} \hat{w}_n, \\ \tilde{z}_0 &= \lim_{n \rightarrow \infty} \tilde{z}_n, & \hat{z}_0 &= \lim_{n \rightarrow \infty} \hat{z}_n \end{aligned} \quad (27)$$

satisfying $\tilde{w}_0 = T_1 \tilde{w}_0$, $\tilde{z}_0 = T_2 \tilde{z}_0$ and $\hat{w}_0 = T_1 \hat{w}_0$, $\hat{z}_0 = T_2 \hat{z}_0$, exist. Considering the compactness of the nonlinear operators T_1 , T_2 and $|\Omega| < \infty$, we know that $(\tilde{w}_0, \tilde{z}_0)$ and $(\hat{w}_0, \hat{z}_0)$ are the solutions to problem (1). The local existence of the solutions to problem (1) is proved. $\square$

3. Global Existence and Uniform Blow-Up Results

In this section, we will use the sub-and-super solution theorem and combine it with Lemma 2 to give the global existence and uniform blow-up conditions of system (1). Firstly, we define the uniform blow-up solution.

Definition 2. The nonnegative solution (u, v) to problem (1) blows up uniformly in finite time if there exists a positive real number $T < \infty$ such that

$$\limsup_{t \rightarrow T} \min_{x \in \bar{\Omega}} (u(x, t) + v(x, t)) = +\infty. \quad (28)$$

Further, the solution (u, v) of problem (1) exists globally if for all $t \in (0, +\infty)$,

$$\max_{x \in \bar{\Omega}} (u(x, t) + v(x, t)) < +\infty. \quad (29)$$

Then, we refer to [8,15] to give Theorems 2 and 3, which are about global existence and blow-up, respectively. Further, the proofs are given using the sub-and-super solution theorem.

Theorem 2. Suppose Assumptions (H1)–(H3) hold, g_1, g_2 are derivable and

$$m < 1, \quad p < 1, \quad nq \leq (1 - m)(1 - p). \quad (30)$$

Then, the solution (u, v) of this problem exists globally of system (1).

Proof. First, according to condition (30), we have

$$\frac{1 - m}{n} \cdot \frac{1 - p}{q} \geq 1, \quad (31)$$

so there exist two positive constants $\alpha_1, \beta_1 > 1$ such that

$$\frac{1 - m}{n} \geq \frac{\beta_1}{\alpha_1}, \quad \frac{1 - p}{q} \geq \frac{\alpha_1}{\beta_1}. \quad (32)$$

Due to the continuity of functions g_1 and g_2 , let $\delta_0 = \min\{\min g_1, \min g_2\}$ and $c_0 > \frac{B}{\delta_0} \max\{\alpha_1, \beta_1\}$. We consider the following eigenvalue problem

$$\begin{cases} \Delta \varphi + \lambda \varphi = 0, & x \in \Omega, \\ \varphi = c_0, & x \in \partial\Omega. \end{cases} \quad (33)$$

The first eigenvalue of problem (33) is denoted by λ . Take $B = \max_{\Omega} |\nabla \varphi|$, $\gamma_0 = \max_{\Omega} \varphi$ and $\bar{k}_1 = \gamma_0^{m+n} k_1 |\Omega|$, $\bar{k}_2 = \gamma_0^{p+q} k_2 |\Omega|$. Then, let $s(t)$ be the unique solution to the Cauchy problem

$$\begin{cases} \frac{ds}{dt} = -\lambda s(t) + (\alpha_1 + \beta_1 - 2)c_0^{-2} B^2 s(t) + \bar{k}_1 \alpha_1^{-1} c_0^{-\alpha_1} s^{m\alpha_1+n\beta_1-\alpha_1+1}(t) \\ \quad + \bar{k}_2 \beta_1^{-1} c_0^{-\beta_1} s^{p\alpha_1+q\beta_1-\beta_1+1}(t), & t > 0, \\ s(0) = \frac{1}{c_0} \max \left\{ \left(\max_{\Omega} u_0(x) \right)^{\frac{1}{\alpha_1}}, \left(\max_{\Omega} v_0(x) \right)^{\frac{1}{\beta_1}} \right\} + 1. \end{cases}$$

where $m\alpha_1 + n\beta_1 - \alpha_1 + 1$, $p\alpha_1 + q\beta_1 - \beta_1 + 1 < 1$ such that $s(t)$ exists globally. Set $\bar{u} = s^{\alpha_1}(t)\varphi^{\alpha_1}(x)$, $\bar{v} = s^{\beta_1}(t)\varphi^{\beta_1}(x)$, $(x, t) \in \Omega \times (0, +\infty)$. Then, we assert that $(\bar{u}, \bar{v})$ is a super-solution to problem (1). In $\Omega \times (0, +\infty)$, we have

$$\begin{aligned} \Delta \bar{u} + k_1 \int_{\Omega} \bar{u}^m \bar{v}^n dx &= s^{\alpha_1}(t) \Delta \varphi^{\alpha_1} + k_1 s^{m\alpha_1+n\beta_1}(t) \int_{\Omega} \varphi^{m\alpha_1+n\beta_1} dx \\ &\leq s^{\alpha_1}(t) \left(\alpha_1 \varphi^{\alpha_1-1} \Delta \varphi + \alpha_1 (\alpha_1 - 1) \varphi^{\alpha_1-2} (\nabla \varphi)^2 \right) + \bar{k}_1 s^{m\alpha_1+n\beta_1}(t) \\ &= s^{\alpha_1}(t) \left(-\lambda \alpha_1 \varphi^{\alpha_1} + \alpha_1 (\alpha_1 - 1) \varphi^{\alpha_1-2} (\nabla \varphi)^2 \right) + \bar{k}_1 s^{m\alpha_1+n\beta_1}(t) \\ &\leq \alpha_1 s^{\alpha_1-1}(t) \varphi^{\alpha_1} \left(-\lambda + (\alpha_1 - 1) c_0^{-2} B^2 + \bar{k}_1 \alpha_1^{-1} c_0^{-\alpha_1} s^{m\alpha_1+n\beta_1-\alpha_1}(t) \right) s(t) \\ &\leq \alpha_1 s^{\alpha_1-1}(t) s'(t) \varphi^{\alpha_1} = \bar{u}_t, \end{aligned} \quad (34)$$

similarly,

$$\begin{aligned} \Delta \bar{v} + k_2 \int_{\Omega} \bar{u}^p \bar{v}^q dx &= s^{\beta_1}(t) \Delta \varphi^{\beta_1} + k_2 s^{p\alpha_1+q\beta_1}(t) \int_{\Omega} \varphi^{p\alpha_1+q\beta_1} dx \\ &\leq s^{\beta_1}(t) \left(\beta_1 \varphi^{\beta_1-1} \Delta \varphi + \beta_1 (\beta_1 - 1) \varphi^{\beta_1-2} (\nabla \varphi)^2 \right) + \bar{k}_2 s^{p\alpha_1+q\beta_1}(t) \\ &= s^{\beta_1}(t) \left(-\lambda \beta_1 \varphi^{\beta_1} + \beta_1 (\beta_1 - 1) \varphi^{\beta_1-2} (\nabla \varphi)^2 \right) + \bar{k}_2 s^{p\alpha_1+q\beta_1}(t) \\ &\leq \beta_1 s^{\beta_1-1}(t) \varphi^{\beta_1} \left(-\lambda + (\beta_1 - 1) c_0^{-2} B^2 + \bar{k}_2 \beta_1^{-1} c_0^{-\beta_1} s^{p\alpha_1+q\beta_1-\beta_1}(t) \right) s(t) \\ &\leq \beta_1 s^{\beta_1-1}(t) s'(t) \varphi^{\beta_1} = \bar{v}_t. \end{aligned} \quad (35)$$

Since $\varphi = c_0$ on $\partial\Omega$, we have that

$$\mathbf{n} \cdot \nabla \bar{u} + g_1(\bar{u}) \bar{u} = \varphi^{\alpha_1-1} s^{\alpha_1} (\alpha_1 \mathbf{n} \cdot \nabla \varphi + c_0 g(\bar{u})) > 0, \quad (36)$$

and

$$\mathbf{n} \cdot \nabla \bar{v} + g_2(\bar{v}) \bar{v} = \varphi^{\beta_1-1} s^{\beta_1} (\beta_1 \mathbf{n} \cdot \nabla \varphi + c_0 g(\bar{v})) > 0 \quad (37)$$

hold in $\partial\Omega \times (0, +\infty)$. Considering the initial values, we have

$$\bar{u}(x, 0) = s^{\alpha_1}(0) \varphi^{\alpha_1} > \frac{\varphi^{\alpha_1}}{c_0^{\alpha_1}} \max_{\Omega} u_0 \geq u_0, \quad (38)$$

and

$$\bar{v}(x, 0) = s^{\beta_1}(0) \varphi^{\beta_1} > \frac{\varphi^{\beta_1}}{c_0^{\beta_1}} \max_{\Omega} v_0 \geq v_0. \quad (39)$$

Hence, $(\bar{u}, \bar{v})$ is a super-solution to problem (1), and satisfies (16). Applying Lemma 2, the solution (u, v) to problem (1) satisfies $u < \bar{u}$, $v < \bar{v}$. Since $\bar{u}, \bar{v}$ exist globally, Theorem 2 is proven. $\square$

Theorem 3. Suppose Assumptions (H1)–(H3) hold and g_1, g_2 are derivable.

Case (i). If one of the following conditions holds

$$(a) m > 1; \quad (b) q > 1; \quad (c) np > (1 - m)(1 - q), \quad (40)$$

and $u_0(x), v_0(x)$ are sufficiently large, then the solution (u, v) of problem (1) blows up uniformly in a finite time.

Case (ii). If (40) is satisfied and $u_0(x), v_0(x)$ are sufficiently small, then the solution (u, v) exists globally.

Proof. Case (i) (a). Assuming that $m > 1$, we consider the eigenvalue problem (33) with $c_0 = 0$, and for convenience, let $\max_{\Omega} \varphi = \frac{1}{2}$. Take $\alpha_2, \beta_2 > 1$ such that $p\alpha_2 + q\beta_2 - \beta_2 > 0$ and set

$$\gamma = \min\{m\alpha_2 + n\beta_2 - \alpha_2 + 1, p\alpha_2 + q\beta_2 - \beta_2 + 1\}. \quad (41)$$

Obviously, $\gamma > 1$. Let $h(t)$ be the unique solution to the following Cauchy problem

$$\begin{cases} \frac{dh}{dt} = -\lambda h(t) + \min\left\{\frac{k_1}{\alpha_2} \int_{\Omega} \varphi^{m\alpha_2+n\beta_2} dx, \frac{k_2}{\beta_2} \int_{\Omega} \varphi^{p\alpha_2+q\beta_2} dx\right\} h^{\gamma}(t), & t > 0, \\ h(0) = h_0, \end{cases} \quad (42)$$

where $h_0 > 0$ is sufficiently large such that the solution to (42) blows up in a finite time.

Let $u_0(x), v_0(x)$ be sufficiently large such that $\min\{(\min_{\Omega} u_0)^{\frac{1}{\alpha_2}}, (\min_{\Omega} v_0)^{\frac{1}{\beta_2}}\} > h_0$ and $\underline{u} = h^{\alpha_2}(t)(\varphi(x) + \delta)^{\alpha_2}$, $\underline{v} = h^{\beta_2}(t)(\varphi(x) + \delta)^{\beta_2}$, $(x, t) \in \Omega \times (0, T^* - \varepsilon]$, where $\varepsilon \in (0, T^*)$ and $\delta \in (0, \frac{1}{2})$ are arbitrarily real numbers. Then, we will show that $(\underline{u}, \underline{v})$ is a sub-solution to problem (1). In $\Omega \times (0, T^* - \varepsilon]$, we yield

$$\begin{aligned} \Delta \underline{u} + k_1 \int_{\Omega} \underline{u}^m \underline{v}^n dx &= h^{\alpha_2}(t) \Delta(\varphi + \delta)^{\alpha_2} + k_1 h^{m\alpha_2+n\beta_2}(t) \int_{\Omega} (\varphi + \delta)^{m\alpha_2+n\beta_2} dx \\ &\geq h^{\alpha_2}(t) \left(-\lambda \alpha_2 (\varphi + \delta)^{\alpha_2} + \alpha_2 (\alpha_2 - 1) (\varphi + \delta)^{\alpha_2-2} (\nabla \varphi)^2 \right) \\ &\quad + k_1 h^{m\alpha_2+n\beta_2}(t) \int_{\Omega} \varphi^{m\alpha_2+n\beta_2} dx \\ &\geq \alpha_2 h^{\alpha_2-1}(t) (\varphi + \delta)^{\alpha_2} \left(-\lambda + \frac{k_1}{\alpha_2} h^{m\alpha_2+n\beta_2-\alpha_2}(t) \int_{\Omega} \varphi^{m\alpha_2+n\beta_2} dx \right) h(t) \\ &\geq \alpha_2 h^{\alpha_2-1}(t) h'(t) (\varphi + \delta)^{\alpha_2} = \underline{u}_t, \end{aligned} \quad (43)$$

and

$$\begin{aligned} \Delta \underline{v} + k_2 \int_{\Omega} \underline{u}^p \underline{v}^q dx &= h^{\beta_2}(t) \Delta(\varphi + \delta)^{\beta_2} + k_2 h^{p\alpha_2+q\beta_2}(t) \int_{\Omega} (\varphi + \delta)^{p\alpha_2+q\beta_2} dx \\ &\geq h^{\beta_2}(t) \left(-\lambda \beta_2 (\varphi + \delta)^{\beta_2} + \beta_2 (\beta_2 - 1) (\varphi + \delta)^{\beta_2-2} (\nabla \varphi)^2 \right) \\ &\quad + k_2 h^{p\alpha_2+q\beta_2}(t) \int_{\Omega} \varphi^{p\alpha_2+q\beta_2} dx \\ &\geq \beta_2 h^{\beta_2-1}(t) (\varphi + \delta)^{\beta_2} \left(-\lambda + \frac{k_2}{\beta_2} h^{p\alpha_2+q\beta_2-\beta_2}(t) \int_{\Omega} \varphi^{p\alpha_2+q\beta_2} dx \right) h(t) \\ &\geq \beta_2 h^{\beta_2-1}(t) h'(t) (\varphi + \delta)^{\beta_2} = \underline{v}_t. \end{aligned} \quad (44)$$

On $\partial\Omega \times (0, T^* - \varepsilon]$, we yield

$$\begin{cases} \mathbf{n} \cdot \nabla \underline{u} + g_1(\underline{u}) \underline{u} = \delta^{\alpha_2-1} h^{\alpha_2}(t) (\alpha_2 \mathbf{n} \cdot \nabla \varphi + \delta g_1(\underline{u})) < 0, \\ \mathbf{n} \cdot \nabla \underline{v} + g_2(\underline{v}) \underline{v} = \delta^{\beta_2-1} h^{\beta_2}(t) (\beta_2 \mathbf{n} \cdot \nabla \varphi + \delta g_2(\underline{v})) < 0, \end{cases} \quad (45)$$

when δ is sufficiently close to 0. Since

$$\begin{cases} \underline{u}(x, 0) = h_0^{\alpha_2}(\varphi + \delta)^{\alpha_2} < u_0(x), & x \in \Omega, \\ \underline{v}(x, 0) = h_0^{\beta_2}(\varphi + \delta)^{\beta_2} < v_0(x), & x \in \Omega, \end{cases} \quad (46)$$

we can obtain that $(\underline{u}, \underline{v})$ is a sub-solution to problem (1) and satisfies (17). Applying Lemma 2, we know that $u > \underline{u}$, $v > \underline{v}$ in $\Omega \times (0, T^* - \varepsilon]$. Due to the arbitrary nature of ε , letting $\varepsilon \rightarrow 0^+$, $u \geq \underline{u}$, $v \geq \underline{v}$ in $\Omega \times (0, T^*)$ can be obtained.

Specifically, we have $u > \underline{u} = \delta^{\alpha_2} h^{\alpha_2}(t)$ and $v > \underline{v} = \delta^{\beta_2} h^{\beta_2}(t)$ on $\partial\Omega \times (0, T^*)$. Hence, the solution (u, v) blows up globally when $m > 1$ and $u_0(x)$, $v_0(x)$ are sufficiently large.

Case (i) (b). We omit the proof for the comparable case in which $q > 1$.

Case (i) (c). When $0 < m \leq 1$, $0 < q \leq 1$ and $np > (1 - m)(1 - q)$, we know that

$$\frac{n}{1 - m} - \frac{1 - q}{p} > 0. \quad (47)$$

Take $\varepsilon \in \left(0, \frac{n}{1 - m} - \frac{1 - q}{p}\right)$ and $\frac{\alpha_2}{\beta_2} = \frac{1 - q}{p} + \varepsilon$, then there exist two positive constants α_2 , $\beta_2 > 1$ such that

$$\frac{1 - q}{p} < \frac{\alpha_2}{\beta_2}, \quad \frac{1 - m}{n} < \frac{\beta_2}{\alpha_2}, \quad (48)$$

which means that $m\alpha_2 + n\beta_2 - \alpha_2 + 1 > 1$, $p\alpha_2 + q\beta_2 - \beta_2 + 1 > 1$. When $\gamma = \min\{m\alpha_2 + n\beta_2 - \alpha_2 + 1, p\alpha_2 + q\beta_2 - \beta_2 + 1\}$ is used, and the solution (u, v) also blows up, as shown in Case (i).

Case (ii). By the continuity of functions g_1 and g_2 , we have $\min g_1, \min g_2 > 0$. Let positive constant δ be sufficiently small such that $\delta < \min\{\min g_1, \min g_2\}$ and $e(x)$ be the unique solution to the elliptic problem

$$\begin{cases} -\Delta e(x) = 1, & x \in \Omega, \\ \mathbf{n} \cdot \nabla e(x) + \delta e(x) = 0, & x \in \partial\Omega \end{cases}$$

Then, there exists a positive constant M such that $0 \leq e(x) \leq M$. Set

$$\bar{u}(x, t) = \alpha_3(1 + e(x)), \quad \bar{v}(x, t) = \beta_3(1 + e(x)),$$

where

$$\alpha_3 \leq k_1 |\Omega| \alpha_3^m \beta_3^n (1 + M)^{m+n}, \quad \beta_3 \leq k_2 |\Omega| \alpha_3^p \beta_3^q (1 + M)^{p+q}.$$

When $x \in \partial\Omega$, we have

$$\begin{aligned} \mathbf{n} \cdot \nabla \bar{u} + g_1(\bar{u})\bar{u} &= \alpha_3 g_1(\bar{u}) + \alpha_3 e(x)(g_1(\bar{u}) - \delta) > 0, \\ \mathbf{n} \cdot \nabla \bar{v} + g_2(\bar{v})\bar{v} &= \beta_3 g_2(\bar{v}) + \beta_3 e(x)(g_2(\bar{v}) - \delta) > 0. \end{aligned}$$

Taking the initial values of problem (1) $u_0(x)$, $v_0(x)$ are sufficiently small such that $u_0(x) < \bar{u}(x, t)$, $v_0(x) < \bar{v}(x, t)$, we have that $(\bar{u}, \bar{v})$ is a super-solution to (1), and satisfies (16). Using Lemma 2, we obtain $\bar{u} > u$, $\bar{v} > v$. Thus, u and v exist globally. Theorem 3 is then fully proved. $\square$

4. Uniform Blow-Up Profile of the Solution

In this section, according to the ideas of [10], we discuss the uniform blow-up profile of the solution to (1).

We denote

$$\begin{aligned} F_1(t) &= k_1 \int_{\Omega} u^m v^n dx, & G_1(t) &= \int_0^t F_1(s) ds; \\ F_2(t) &= k_2 \int_{\Omega} u^p v^q dx, & G_2(t) &= \int_0^t F_2(s) ds. \end{aligned} \quad (49)$$

Lemma 3. Assume that (H1)-(H3) hold, $\Delta u_0, \Delta v_0 < 0$, and g_1, g_2 can be derived. Let $(u, v) \in (C^{2,1}(Q_T) \cap C^1(\overline{Q}_T)) \times (C^{2,1}(Q_T) \cap C^1(\overline{Q}_T))$ be a solution to problem (1). If u, v blow up at the finite time T^* simultaneously, then we have that for all $\varepsilon > 0$, there exists $C_1 > 0$ such that

$$\begin{aligned} C_1 \leq -\Delta u &\leq \max \left\{ \sup_{s \in [0, t]} F_1(s), C_1 \right\}, \\ C_1 \leq -\Delta v &\leq \max \left\{ \sup_{s \in [0, t]} F_2(s), C_1 \right\} \end{aligned} \quad (50)$$

in $\Omega \times [\frac{T^*}{2}, T^* - \varepsilon]$.

Proof. We take the first equation in (1) as an example. Let $w = -\Delta u$. Since $\Delta F_1(t) = 0$, we obtain

$$w_t - \Delta w = 0, \quad (x, t) \in \Omega \times [\frac{T^*}{2}, T^* - \varepsilon]. \quad (51)$$

It is easy to see that w can obtain its maximum and minimum in $\partial\Omega \times [\frac{T^*}{2}, T^* - \varepsilon]$.

On the one hand, if there is a point (x_0, t_0) , $x_0 \in \Omega$, $\frac{T^*}{2} \leq t_0 \leq T^* - \varepsilon$ such that $w(x_0, t_0) = \max_{\overline{\Omega} \times [\frac{T^*}{2}, T^* - \varepsilon]} w(x, t)$, then according to the strong maximum principle of heat equations, we know that w is a constant when $t \in [\frac{T^*}{2}, t_0]$.

On the other hand, when $t > t_0$, w cannot obtain its maximum and minimum in $\Omega \times (t_0, T^* - \varepsilon]$. For $(x, t) \in \partial\Omega \times (t_0, T^* - \varepsilon]$, set $\psi(u) = g'_1(u)u + g_1(u)$. Obviously, $\psi(u) > 0$. A straightforward computation yields

$$\frac{\partial}{\partial t} \frac{\partial u}{\partial n} = -\frac{\partial}{\partial t} (g_1(u)u) = -\psi(u)u_t. \quad (52)$$

Further, according to $w = -u_t + F_1(t)$, we deduce

$$\frac{\partial w}{\partial n} = -\frac{\partial u_t}{\partial n} = \psi(u)u_t. \quad (53)$$

Applying Theorem 6 in [20] Chap. 3, when w takes the minimum value, we have that

$$\frac{\partial w}{\partial n} = \psi(u)(F_1(t) - w) < 0. \quad (54)$$

Since $u, v, F_1(t) > 0$ and by continuity, there exists a constant $0 < C_1 \leq \max_{\overline{\Omega}} u(x, \frac{T^*}{2})$ such that

$$-\Delta u \geq C_1. \quad (55)$$

And, when w takes the maximum value, we have that

$$\frac{\partial w}{\partial n} = \psi(u)(F_1(t) - w) > 0. \quad (56)$$

So we can deduce that

$$-\Delta u \leq \sup_{s \in [0, t]} F_1(s) \quad (57)$$

for $t \in [\frac{T^*}{2}, T^* - \varepsilon]$. Combining (55) and (57), we have

$$C_1 \leq -\Delta u \leq \max \left\{ \sup_{s \in [0, t]} F_1(s), C_1 \right\}. \quad (58)$$

The proof of the inequality on $-\Delta v$ is similar to the one on $-\Delta u$, which is omitted. $\square$

According to (50), we have

$$\begin{aligned} u_t - C_1 &\leq u_t - \Delta u = F_1(t), \\ v_t - C_1 &\leq v_t - \Delta v = F_2(t). \end{aligned} \quad (59)$$

Integrating Equation (59) from $\frac{T^*}{2}$ to t yields the following lemma.

Lemma 4. Under the assumptions of Lemma 3, we have that for all $\varepsilon > 0$

$$\begin{aligned} C'_2 &\leq u(x, t) \leq C_2 + G_1(t), \\ C'_2 &\leq v(x, t) \leq C_2 + G_2(t) \end{aligned} \quad (60)$$

where $C'_2, C_2 > 0$ are constants and $t \in [\frac{T^*}{2}, T^* - \varepsilon]$.

Proof. After performing the integration, we have

$$\begin{aligned} u(x, t) - u(x, \frac{T^*}{2}) - C_1(t - \frac{T^*}{2}) &\leq G_1(t) - G_1(\frac{T^*}{2}) \leq G_1(t), \\ v(x, t) - v(x, \frac{T^*}{2}) - C_1(t - \frac{T^*}{2}) &\leq G_2(t) - G_2(\frac{T^*}{2}) \leq G_2(t), \end{aligned} \quad (61)$$

Taking $C_2 = \max \left\{ \max_{x \in \Omega} u(x, \frac{T^*}{2}) + \frac{T^*}{2} C_1, \max_{x \in \Omega} v(x, \frac{T^*}{2}) + \frac{T^*}{2} C_1 \right\}$,

$$\begin{aligned} u(x, t) &\leq C_2 + G_1(t), \\ v(x, t) &\leq C_2 + G_2(t) \end{aligned}$$

are obtained. Since $u, v > 0$ in $\bar{\Omega} \times (0, T^*)$ and due to their continuity, there exists a constant $C'_2 > 0$ such that $u, v \geq C'_2$. $\square$

Remark 1. An obvious conclusion deduced from Lemma 4 is that

$$\lim_{t \rightarrow T^*} G_1(t) = +\infty, \quad \lim_{t \rightarrow T^*} G_2(t) = +\infty, \quad (62)$$

when u, v blow up.

Set

$$H_1(t) = \int_0^t G_1(s) ds, \quad H_2(t) = \int_0^t G_2(s) ds, \quad (63)$$

where $t \in [0, T)$.

We consider the eigenvalue problem (33) with $c_0 \geq 0$. Set λ as the first eigenvalue of (33), and $\varphi > 0$ is normalized with $\int_{\Omega} \varphi dx = 1$. Define

$$K_1(t) = \int_{\partial\Omega} u \frac{\partial \varphi}{\partial \mathbf{n}} dS, \quad K_2(t) = \int_{\partial\Omega} v \frac{\partial \varphi}{\partial \mathbf{n}} dS, \quad (64)$$

where $t \in [0, T)$. We have $K_1(t), K_2(t) < 0$.

Lemma 5. Under the assumptions of Lemma 3, letting $x \in \overline{\Omega}$, and any $\rho > 0$ such that (i) $B(x, \rho) \subset \Omega^\circ$, if $x \in \Omega^\circ$; or (ii) $B(y, \rho) \cap \partial\Omega = \{x\}$, $B(y, \rho) \subset \Omega$ where $y \in \Omega^\circ$ is fixed, if $x \in \partial\Omega$; and $C_3 > 0$ is a constant, it holds that

$$\begin{aligned} \sup_{x \in K_\rho} [G_1(t) - u(x, t)] &\leq \frac{C_3}{\rho^N} (1 + H_1(t)), \\ \sup_{x \in K_\rho} [G_2(t) - v(x, t)] &\leq \frac{C_3}{\rho^N} (1 + H_2(t)), \end{aligned} \quad (65)$$

in $\Omega \times [\frac{T^*}{2}, T^* - \varepsilon]$, for all $\varepsilon > 0$, where $K_\rho = B(x, \rho)$ (or $B(y, \rho)$).

Proof. Case (i). Let $z(x, t) = G_1(t) - u(x, t)$, $c_0 = 0$ in (33) and $\beta(t) = \int_\Omega z(x, t) \varphi(x) dx$. By Green's formula, we have

$$\begin{aligned} \beta'(t) &= \int_\Omega (F_1(t) - u_t(x, t)) \varphi(x) dx = - \int_\Omega \varphi(x) \Delta u dx \\ &= - \int_\Omega u(x, t) \Delta \varphi dx + K_1(t) \\ &= \lambda \int_\Omega u(x, t) \varphi(x) dx + K_1(t) \\ &= -\lambda \beta + \lambda G_1(t) + K_1(t). \end{aligned} \quad (66)$$

Solving this ODE, we have

$$\begin{aligned} \beta(t) &= \beta(0) e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} (K_1(s) + \lambda G_1(s)) ds \\ &\leq \beta(0) e^{-\lambda t} + \lambda \int_0^t e^{-\lambda(t-s)} G_1(s) ds \\ &\leq C(1 + H_1(t)). \end{aligned} \quad (67)$$

On the other hand, Lemma 4 implies

$$\inf_\Omega z(x, t) \geq -C_2 \quad (68)$$

when $t \in [\frac{T^*}{2}, T^* - \varepsilon]$ for all $\varepsilon > 0$, which, combined with Equation (67), implies

$$\int_\Omega |z(x, t)| \varphi(x) dx \leq C'(1 + H_1(t)). \quad (69)$$

Further, by Lemma 3, we have

$$-\Delta z \leq -C_1 \leq 0. \quad (70)$$

Fixing K_ρ , whose center is noted as x , the function $z(y, t)$, $y \in K_\rho \subset \Omega^\circ$ is a subharmonic implied by (70). Applying the mean-value inequality for subharmonic functions, it follows that

$$z(x, t) \leq \frac{1}{|K_\rho|} \int_{K_\rho} z(y, t) dy \leq \frac{1}{|K_\rho|} \int_{K_\rho} |z(y, t)| dy. \quad (71)$$

We know that $\inf_{K_\rho} \varphi \geq c'_0$, where $c'_0 > 0$ is related to the distance from $\partial\Omega$, which, together with (69) and (71), can imply that

$$z(x, t) \leq \frac{C'}{\rho^N} \int_{K_\rho} |z(y, t)| \varphi(y) dy \leq \frac{C'}{\rho^N} (1 + H_1(t)). \quad (72)$$

Similarly, it follows that

$$G_2(t) - v(x, t) \leq \frac{C''}{\rho^N} (1 + H_2(t)). \quad (73)$$

Selecting $C_3 \geq \max\{C', C''\}$, Case (i) can be proved.

Case (ii). Letting $c_0 > 0$ in (33), considering the mean value theorem, (66) and (67) are transformed into

$$\beta'(t) = -\lambda\beta + \lambda G_1(t) + K_1(t) + c_0 \int_{\partial\Omega} \left(-\frac{\partial u}{\partial \mathbf{n}} \right) dS, \quad (74)$$

and

$$\begin{aligned} \beta(t) &= \beta(0)e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} \left(K_1(s) + \lambda G_1(s) + c_0 \int_{\partial\Omega} \left(-\frac{\partial u(x, s)}{\partial \mathbf{n}} \right) dS \right) ds \\ &\leq \beta(0)e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} \left(c_0 \int_{\Omega} (-\Delta u(x, s)) dx + \lambda G_1(s) \right) ds \\ &= \beta(0)e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} (-c_0 |\Omega| \Delta u(x_0, s) + \lambda G_1(s)) ds \\ &\leq C \left(1 + H_1(t) - \int_0^t \Delta u(x_0, s) ds \right). \end{aligned} \quad (75)$$

where $x_0 \in \Omega^\circ$, respectively. Similarly, since $\inf_{K_\rho} \geq c_0 > 0$ and $z(y, t)$, $y \in K_\rho \subset \overline{\Omega}$ is a subharmonic function. Lemma 3 deduces that $\int_0^t \Delta u ds$ is bounded on $\Omega \times (\frac{T^*}{2}, T^* - \varepsilon]$ for all $\varepsilon > 0$, so it follows that

$$z(x, t) \leq \frac{C'}{\rho^N} \int_{K_\rho} |z(y, t)| \varphi(y) dy \leq \frac{C'}{\rho^N} (1 + H_1(t)), \quad (76)$$

too. Choosing sufficiently large C_3 , Lemma 5 is proved completely. $\square$

Lemma 6. Under the assumptions of Lemma 3, we have that

$$\limsup_{t \rightarrow T^*} \min_{x \in \overline{\Omega}} u(x, t) = +\infty, \quad \limsup_{t \rightarrow T^*} \min_{x \in \overline{\Omega}} v(x, t) = +\infty \quad (77)$$

if and only if

$$\lim_{t \rightarrow T^*} G_1(t) = +\infty, \quad \lim_{t \rightarrow T^*} G_2(t) = +\infty. \quad (78)$$

Further, if (77) or (78) is fulfilled, then

$$\begin{aligned} \lim_{t \rightarrow T^*} \frac{u(x, t)}{G_1(t)} &= \lim_{t \rightarrow T^*} \frac{\|u(t)\|_\infty}{G_1(t)} = 1, \\ \lim_{t \rightarrow T^*} \frac{v(x, t)}{G_2(t)} &= \lim_{t \rightarrow T^*} \frac{\|v(t)\|_\infty}{G_2(t)} = 1, \end{aligned} \quad (79)$$

uniformly on $\overline{\Omega}$.

Proof. (77) $\Rightarrow$ (78) is a simple corollary of Lemma 4. Assuming (78) holds, we use the method in [10]. According to Lemma 5 Case (i), for any fixed $K_\rho \subset \Omega^\circ$, combining Lemma 4 and $G_1(t), G_2(t) > 0$ yields

$$\begin{aligned} -\frac{C_2}{G_1(t)} &\leq 1 - \frac{u(x, t)}{G_1(t)} \leq \frac{C_3}{\rho^N} \frac{1 + H_1(t)}{G_1(t)}, \\ -\frac{C_2}{G_2(t)} &\leq 1 - \frac{v(x, t)}{G_2(t)} \leq \frac{C_3}{\rho^N} \frac{1 + H_2(t)}{G_2(t)}. \end{aligned} \quad (80)$$

Since G_1, G_2 are increasing, it follows that for all $\varepsilon > 0$,

$$\begin{aligned} 0 &\leq \frac{1 + H_1(t)}{G_1(t)} \leq \frac{\int_0^{T^*-\varepsilon} G_1(s) ds}{G_1(t)} + \varepsilon, \\ 0 &\leq \frac{1 + H_2(t)}{G_2(t)} \leq \frac{\int_0^{T^*-\varepsilon} G_2(s) ds}{G_2(t)} + \varepsilon, \end{aligned} \quad (81)$$

which, together with $G_1(t), G_2(t) \rightarrow +\infty$ implies that $\lim_{t \rightarrow T^*} \frac{H_i(t)}{G_i(t)} = 0, i = 1, 2$.

$$\lim_{t \rightarrow T^*} \frac{u(x, t)}{G_1(t)} = \lim_{t \rightarrow T^*} \frac{v(x, t)}{G_2(t)} = 1 \quad (82)$$

on K_ρ is deduced.

Integrating the first two equations in (1) from 0 to t , we have

$$\begin{cases} u(x, t) - \int_0^t \Delta u(x, s) ds - u_0(x) = G_1(t), & (x, t) \in K_\rho \times [\frac{T^*}{2}, T^*) \\ v(x, t) - \int_0^t \Delta v(x, s) ds - v_0(x) = G_2(t), & (x, t) \in K_\rho \times [\frac{T^*}{2}, T^*). \end{cases} \quad (83)$$

Since (82), we know that

$$\lim_{t \rightarrow T^*} \frac{\int_0^t \Delta u ds}{G_1(t)} = \lim_{t \rightarrow T^*} \frac{\int_0^t \Delta v ds}{G_2(t)} = 0. \quad (84)$$

On the other hand, according to Lemma 4, it follows that $F_1(t), F_2(t) > 0$; that is,

$$u_t - \Delta u > 0, \quad v_t - \Delta v > 0.$$

Applying the maximum principle, we know that the solution (u, v) can take the minimum in $\partial\Omega$. For all $x \in \overline{\Omega}$, combining Lemmas 4 and 5 Case (ii) and $G_1(t), G_2(t) > 0$, there exists a fixed $K_\rho \subset \overline{\Omega}$ such that

$$\begin{aligned} -\frac{C_2}{G_1(t)} &\leq 1 - \frac{u(x, t)}{G_1(t)} \leq \frac{C_3}{\rho^N} \frac{1 + H_1(t) - \int_0^t \Delta u(x_{01}, s) ds}{G_1(t)}, \\ -\frac{C_2}{G_2(t)} &\leq 1 - \frac{v(x, t)}{G_2(t)} \leq \frac{C_3}{\rho^N} \frac{1 + H_2(t) - \int_0^t \Delta v(x_{02}, s) ds}{G_2(t)}. \end{aligned} \quad (85)$$

where $x_{01}, x_{02} \in \Omega^\circ$. Since $\lim_{t \rightarrow T^*} \frac{H_i(t)}{G_i(t)} = 0, i = 1, 2$ and (84), (85) can imply (79) uniformly on $\overline{\Omega}$. Further, we obtain

$$\begin{aligned} \limsup_{t \rightarrow T^*} \min_{x \in \overline{\Omega}} u(x, t) &= \limsup_{t \rightarrow T^*} \min_{x \in \partial\Omega} u(x, t) = +\infty, \\ \limsup_{t \rightarrow T^*} \min_{x \in \overline{\Omega}} v(x, t) &= \limsup_{t \rightarrow T^*} \min_{x \in \partial\Omega} v(x, t) = +\infty. \end{aligned}$$

The proof of Lemma 6 is completed. $\square$

Using the above lemmas and corollaries, we can obtain the following theorems describing blow-up solutions by the same method as in [8,21]. Theorems 4 and 5 give the necessary and sufficient conditions that u, v blow up simultaneously. Further, Theorem 6 describes the blow-up profile of (1) when u, v blow up simultaneously.

Theorem 4. Assuming (H1)–(H3) hold, $\Delta u_0, \Delta v_0 < 0$, g_1, g_2 are derivable. Let (u, v) be a classical solution to problem 1. If u and v blow up in a finite time T^* simultaneously, then (i) $p \geq m - 1$ and $n \geq q - 1$, or (ii) $p < m - 1$ and $n < q - 1$ must be satisfied.

Theorem 5. Under the assumption of Theorem 4. Let (u, v) be a classical blow-up solution to problem 1. If $p \geq m - 1 > 0$ and $n \geq q - 1 > 0$, then u and v blow up simultaneously and uniformly.

Theorem 6. Under the assumption of Theorem 4, the following results hold uniformly on $\overline{\Omega}$:

(i) if (a) $n > q - 1, p > m - 1$ or (b) $n < q - 1, p < m - 1$, then we have

$$\begin{cases} \lim_{t \rightarrow T^*} u(x, t)(T^* - t)^\theta = \lim_{t \rightarrow T^*} \|u(t)\|_\infty (T^* - t)^\theta = \left(\frac{k_1 |\Omega|}{\theta} \left(\frac{k_2}{k_1} \frac{\theta}{\sigma} \right)^{\frac{n}{n+1-q}} \right)^{-\theta}, \\ \lim_{t \rightarrow T^*} v(x, t)(T^* - t)^\sigma = \lim_{t \rightarrow T^*} \|v(t)\|_\infty (T^* - t)^\sigma = \left(\frac{k_2 |\Omega|}{\sigma} \left(\frac{k_1}{k_2} \frac{\sigma}{\theta} \right)^{\frac{p}{p+1-m}} \right)^{-\sigma}, \end{cases}$$

where

$$\theta = \frac{n+1-q}{np - (m-1)(q-1)}, \quad \sigma = \frac{p+1-m}{np - (m-1)(q-1)};$$

(ii) if $p = m - 1 > 0$ and $n > q - 1 > 0$, then we have

$$\begin{cases} \lim_{t \rightarrow T^*} u^p(x, t) [\ln u(x, t)]^{\frac{n}{n+1-q}} (T^* - t) = (pk_1 |\Omega|)^{-1} \left[\frac{k_2(n+1-q)}{k_1} \right]^{-\frac{n}{n+1-q}}, \\ \lim_{t \rightarrow T^*} v^{n+1-q} |\ln(T^* - t)|^{-1} = \frac{k_2}{k_1} \frac{n+1-q}{p}; \end{cases}$$

(iii) if $p > m - 1 > 0$ and $n = q - 1 > 0$, then we have

$$\begin{cases} \lim_{t \rightarrow T^*} u^{p+1-m}(x, t) |\ln(T^* - t)|^{-1} = \frac{k_1}{k_2} \frac{p+1-m}{n}, \\ \lim_{t \rightarrow T^*} v^n(x, t) [\ln v(x, t)]^{\frac{p}{p+1-m}} (T^* - t) = (nk_2 |\Omega|)^{-1} \left[\frac{k_1(p+1-m)}{k_2} \right]^{-\frac{p}{p+1-m}}; \end{cases}$$

(iv) if $p = m - 1 > 0$ and $n = q - 1 > 0$, then we have

$$\begin{cases} \lim_{t \rightarrow T^*} \frac{\ln u(x, t)}{|\ln(T^* - t)|} = \lim_{t \rightarrow T^*} \frac{\| \ln u(t) \|_\infty}{|\ln(T^* - t)|} = \frac{k_1}{pk_1 + nk_2}, \\ \lim_{t \rightarrow T^*} \frac{\ln v(x, t)}{|\ln(T^* - t)|} = \lim_{t \rightarrow T^*} \frac{\| \ln v(t) \|_\infty}{|\ln(T^* - t)|} = \frac{k_2}{pk_1 + nk_2}. \end{cases} \quad (86)$$

The proofs of Theorems 4–6 are very similar to [8]; we only need to pay attention to the constants k_1, k_2 , so we only give the proof by taking Theorem 6 Case (i) as an example, and omit the others.

Proof of Theorem 6 Case (i). We have

$$G_1'(t) = F_1(t) = k_1 \int_{\Omega} u^m(x, t) v^n(x, t) dx \sim k_1 |\Omega| G_1^m(t) G_2^n(t), \quad (87)$$

$$G_2'(t) = F_2(t) = k_2 \int_{\Omega} u^p(x, t) v^q(x, t) dx \sim k_2 |\Omega| G_1^p(t) G_2^q(t), \quad (88)$$

it follows that

$$k_2 G_1^{p-m}(t) dG_1 \sim k_1 G_2^{n-q}(t) dG_2. \quad (89)$$

When $n > q - 1$ and $p > m - 1$, integrating (89) from 0 to t , yields $\frac{k_2}{p+1-m} G_1^{p+1-m}(t) \sim \frac{k_1}{n+1-q} G_2^{n+1-q}(t)$. That is,

$$k_2(n+1-q) G_1^{p+1-m}(t) \sim k_1(p+1-m) G_2^{n+1-q}(t). \quad (90)$$

While $n < q - 1$ and $p < m - 1$, since $\lim_{t \rightarrow T^*} G_i = +\infty, i = 1, 2$, integrating (89) from t to T^* , we have

$$k_2(n+1-q) G_1^{p+1-m}(t) \sim k_1(p+1-m) G_2^{n+1-q}(t), \quad (91)$$

similarly. By (87) and (90), we find

$$\begin{aligned} G_1'(t) &\sim k_1 |\Omega| \left[\frac{k_2(n+1-q)}{k_1(p+1-m)} \right]^{\frac{n}{n+1-q}} G_1^{m+\frac{n(p+1-m)}{n+1-q}}(t) \\ &= k_1 |\Omega| \left(\frac{k_2 \theta}{k_1 \sigma} \right)^{\frac{n}{n+1-q}} G_1^{m+\frac{n\sigma}{\theta}}(t). \end{aligned} \quad (92)$$

Since $1 - m - \frac{n\sigma}{\theta} = -\frac{1}{\theta} < 0$ and $\lim_{t \rightarrow T^*} G_1(t) = +\infty$, integrating (92) from t to T^* , we obtain

$$G_1(t) \sim \left(\frac{k_1 |\Omega|}{\theta} \left(\frac{k_2 \theta}{k_1 \sigma} \right)^{\frac{n}{n+1-q}} (T^* - t) \right)^{-\theta}. \quad (93)$$

According to (93) and Lemma 6, it follows that uniformly on $\overline{\Omega}$,

$$u(x, t) \sim \|u(t)\|_\infty \sim \left(\frac{k_1 |\Omega|}{\theta} \left(\frac{k_2 \theta}{k_1 \sigma} \right)^{\frac{n}{n+1-q}} (T^* - t) \right)^{-\theta},$$

that is,

$$\lim_{t \rightarrow T^*} u(x, t)(T^* - t)^\theta = \lim_{t \rightarrow T^*} \|u(t)\|_\infty (T^* - t)^\theta = \left(\frac{k_1 |\Omega|}{\theta} \left(\frac{k_2 \theta}{k_1 \sigma} \right)^{\frac{n}{n+1-q}} \right)^{-\theta}$$

holds uniformly on $\overline{\Omega}$.

Combining (88) and (91), and applying the same proofs of G_2 and v , we obtain that

$$\lim_{t \rightarrow T^*} v(x, t)(T^* - t)^\sigma = \lim_{t \rightarrow T^*} \|v(t)\|_\infty (T^* - t)^\sigma = \left(\frac{k_2 |\Omega|}{\sigma} \left(\frac{k_1 \sigma}{k_2 \theta} \right)^{\frac{p}{p+1-m}} \right)^{-\sigma}$$

holds uniformly on $\overline{\Omega}$. $\square$

5. Numerical Simulations

In this section, we give numerical simulations for several specific cases to illustrate Theorems 2 and 3. All calculations are performed through Mathematica.

Firstly, letting $m = 0.2, n = 0.5, p = q = 0.3$ and $T = 10$, we obtain problem (1) of the following form

$$\begin{cases} u_t = \Delta u + \int_{\Omega} u^{0.2}(t, x, y) v^{0.5}(t, x, y) d\sigma, & (t, x, y) \in (0, 10) \times \Omega, \\ v_t = \Delta v + \int_{\Omega} u^{0.3}(t, x, y) v^{0.3}(t, x, y) d\sigma, & (t, x, y) \in (0, 10) \times \Omega, \\ \mathbf{n} \cdot \nabla u + u^2 = \mathbf{n} \cdot \nabla v + v^{1.5} = 0, & (t, x, y) \in (0, 10) \times \partial\Omega, \\ u(0, x, y) = v(0, x, y) = 1 - \frac{1}{2}(x^2 + y^2), & (x, y) \in \Omega, \end{cases} \quad (94)$$

where $\Omega = \{(x, y) : x^2 + y^2 \leq 1\}$. Because of the symmetry, we only need to simulate the values of its solution on $0 \leq x \leq 1$ as shown below.

In this case, $g_1(u) = u$, $g_2(v) = v^{\frac{1}{2}}$ satisfies (H2) and (H3); since $u_0(x, y) = v_0(x, y) = 1 - \frac{1}{2}(x^2 + y^2) > 0$,

$$\frac{\partial u_0}{\partial \mathbf{n}} = \frac{\partial v_0}{\partial \mathbf{n}} = (x, y) \cdot (-x, -y) = -(x^2 + y^2) = -1 < 0,$$

Further, (94) satisfies (H1) and $m = 0.2 < 1$, $p = 0.3 < 1$, $nq = 0.15 \leq 0.56 = (1 - m)(1 - p)$. Therefore, (94) satisfies the conditions of Theorem 2, so its solution should exist globally, which can be illustrated by Figure 1.

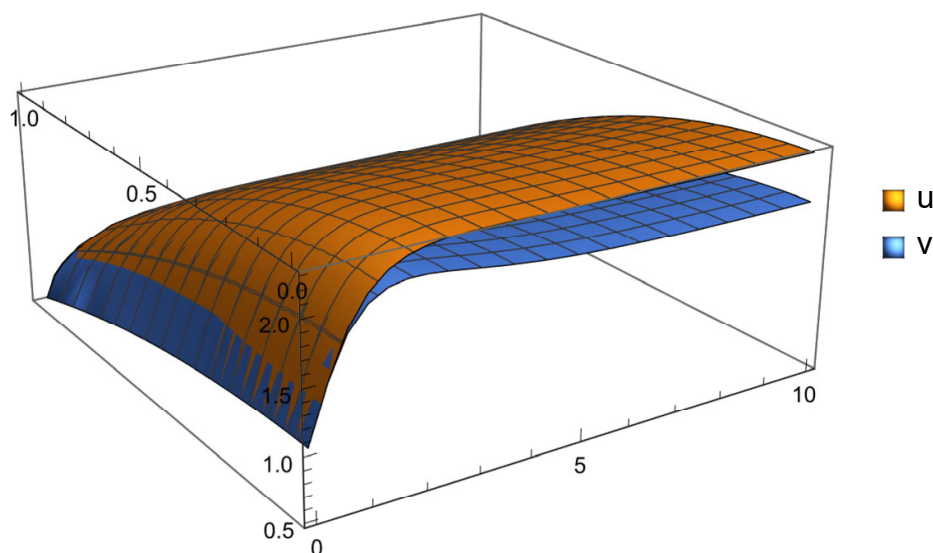


Figure 1. The solution (u, v) in $\{(t, x, y) : 0 < t < 10, 0 \leq x \leq 1, y = 0\}$.

Next, we present a numerical simulation to illustrate the blow-up profiles using Mathematica. As stated in Theorem 3, we consider problem (1) in the following form satisfying (40):

$$\begin{cases} u_t = \Delta u + \int_{\Omega} u^{1.2}(t, x, y) v^{0.4}(t, x, y) d\sigma, & (t, x, y) \in (0, T^*) \times \Omega, \\ v_t = \Delta v + \int_{\Omega} u^{0.3}(t, x, y) v^{1.3}(t, x, y) d\sigma, & (t, x, y) \in (0, T^*) \times \Omega, \\ \mathbf{n} \cdot \nabla u + u^2 = \mathbf{n} \cdot \nabla v + v^{1.5} = 0, & (t, x, y) \in (0, T^*) \times \partial\Omega, \\ u(0, x, y) = v(0, x, y) = 1 - \frac{1}{12}(x^4 + y^4), & (x, y) \in \Omega, \end{cases} \quad (95)$$

where $\Omega = \{(x, y) : x^2 + y^2 \leq 1\}$ and T^* is the blow-up time that will be given by the numerical simulation. In this case, the initial values satisfy

$$\frac{\partial u_0}{\partial \mathbf{n}} = \frac{\partial v_0}{\partial \mathbf{n}} = (x, y) \cdot \left(-\frac{1}{3}x^3, -\frac{1}{3}y^3\right) = -\frac{1}{3}(x^4 + y^4) < 0$$

and

$$\Delta u_0 = \Delta v_0 = -(x^2 + y^2) < 0.$$

According to Theorem 3, we know that the solution to (95) blows up in a finite time. Without loss of generality, we take as an example the value of function u when $t \in (0, 0.979)$, $x \in [0, 1]$, $y = 0$, as in Figure 2.

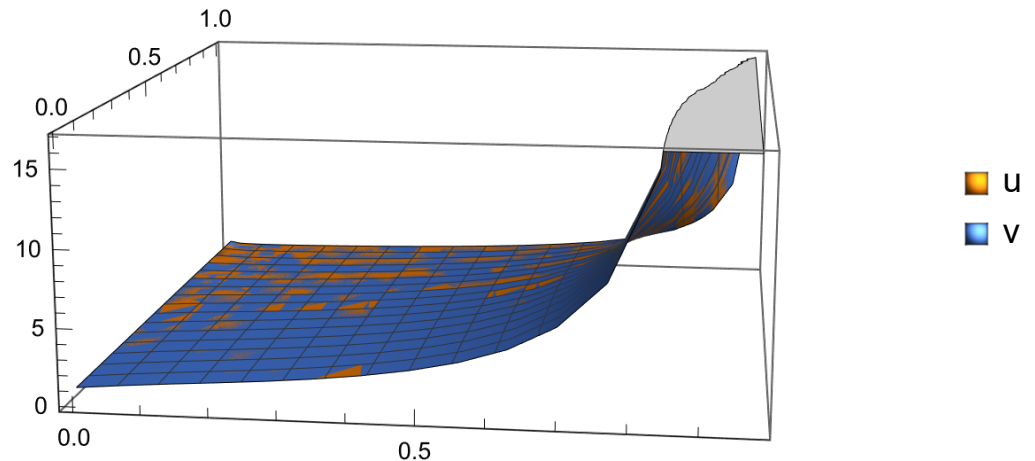


Figure 2. The solution (u, v) in $\{(t, x, y) : 0 < t < 0.979, 0 \leq x \leq 1, y = 0\}$.

As expected, Since (95) satisfies $p \geq m - 1 > 0$ and $n \geq q - 1 > 0$, the solution (u, v) blows up at approximately $T^* = 0.979$, simultaneously. We take $m = 1.2$, $n = 0.5$, $p = 1.3$, $q = 0.3$ and replace the initial value conditions with

$$u_0(x, y) = v_0(x, y) = \frac{1}{4} - \frac{1}{48}(x^4 + y^4),$$

which does not satisfy that $u_0(x), v_0(x)$ are sufficiently large in Theorem 3. We resolve the solution (u, v) of problem (95) (see Figure 3).

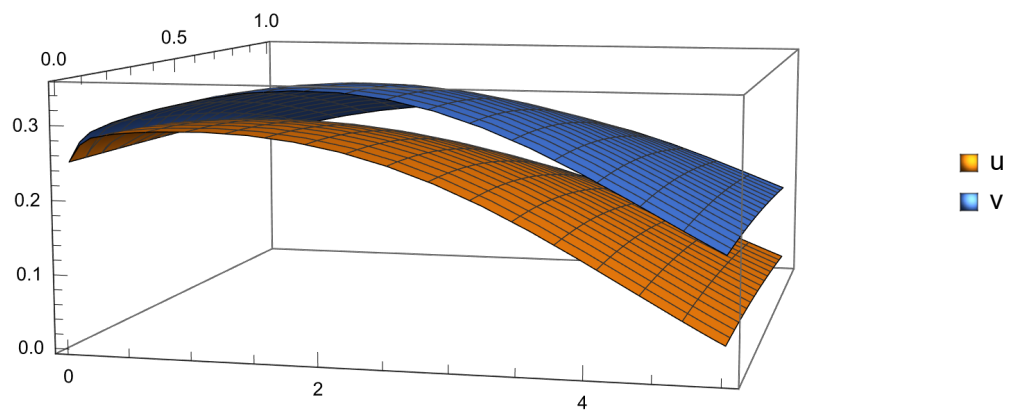


Figure 3. The solution (u, v) in $\{(t, x, y) : 0 < t < 5.02, 0 \leq x \leq 1, y = 0\}$.

Figure 3 shows that when the initial values are very small, the solution (u, v) does not blow up even if (40) is satisfied. Then, we change the values of m, n, p and q in (95) and perform numerical simulations for the rest of the cases mentioned in Theorem 6. This is shown in Figures 4–7.

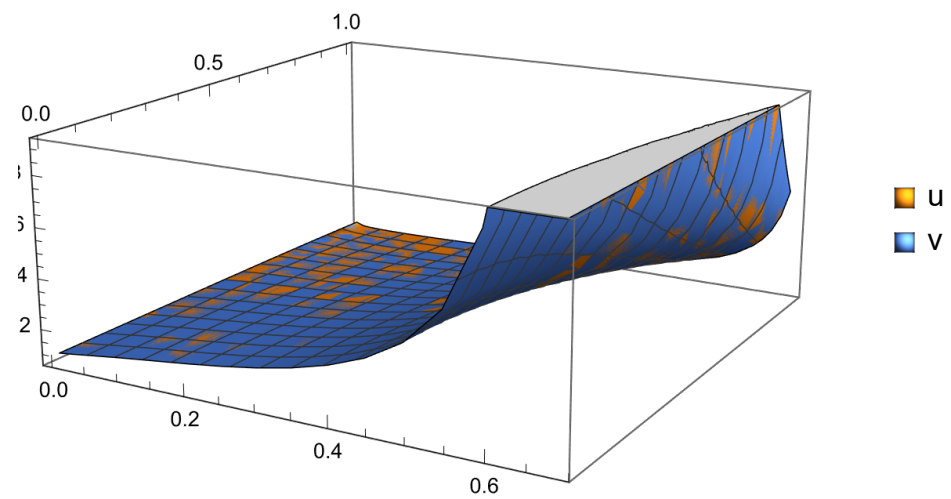


Figure 4. $m = 1.4, n = 0.4, p = 0.3, q = 1.5$.

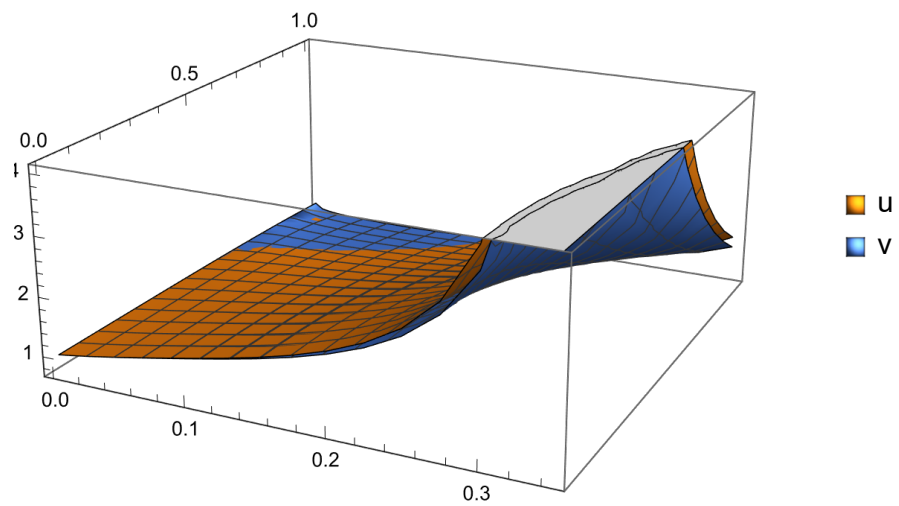


Figure 5. $m = 2, n = 0.4, p = 1, q = 1.3$.

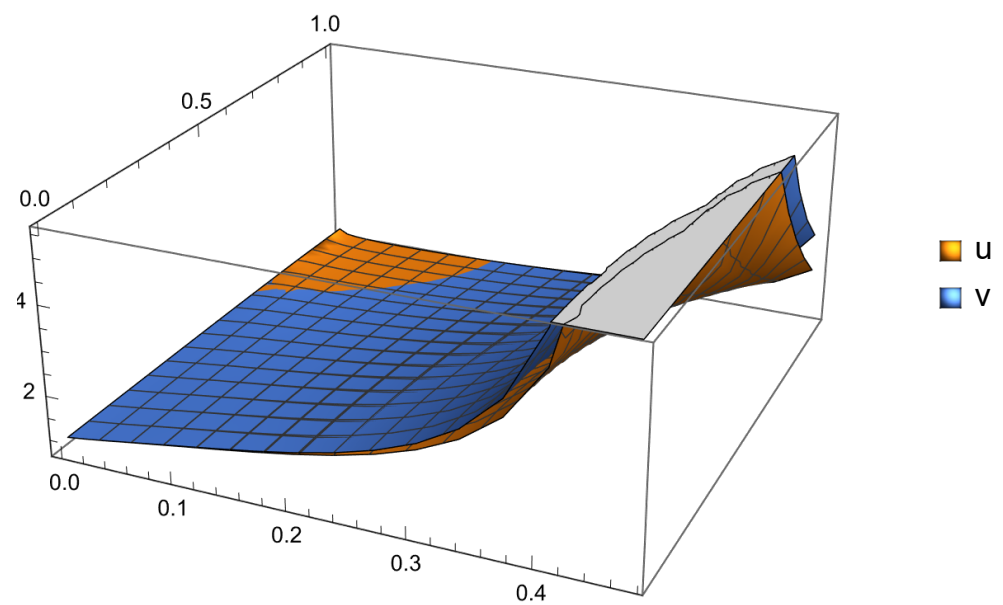


Figure 6. $m = 1.2, n = 0.8, p = 0.3, q = 1.8$.

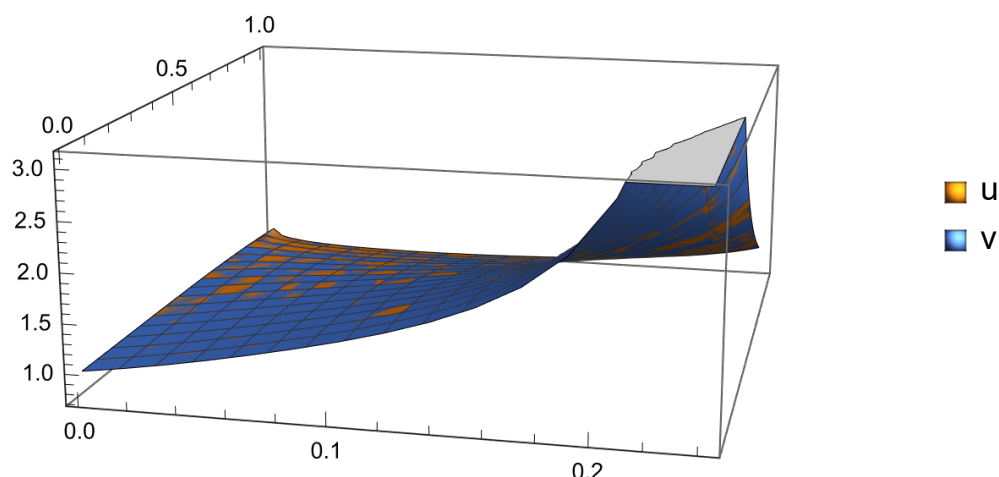


Figure 7. $m = 2, n = 0.8, p = 1, q = 1.8$.

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