mathematics

Review

Partial Inverse Sturm-Liouville Problems

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Abstract: This paper presents a review of both classical and modern results pertaining to partial inverse spectral problems for differential operators. Such problems consist in the recovery of differential expression coefficients in some part of the domain (a finite interval or a geometric graph) from spectral characteristics, while the coefficients in the remaining part of the domain are known a priori. Usually, partial inverse problems require less spectral data than complete inverse problems. In this review, we pay considerable attention to partial inverse problems on graphs and to the unified approach based on the reduction of partial inverse problems to Sturm-Liouville problems with entire analytic functions in a boundary condition. We not only describe the results of selected studies but also compare them with each other and establish interconnections.

Keywords: inverse spectral problems; Sturm-Liouville operator; differential operators on graphs; Hochstadt-Lieberman problem; half-inverse problem

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1. Introduction

This paper contains an overview of results pertaining to partial inverse spectral problems for ordinary differential operators. Such problems consist in the recovery of differential expression coefficients on some part of the domain (a finite interval or a geometric graph) from spectral characteristics, while the coefficients on the remaining part of the domain are known a priori. Usually, partial inverse problems require fewer spectral data than complete inverse problems. In the literature, partial inverse problems are also called half-inverse problems, Hochstadt–Lieberman-type problems, inverse problems with mixed data, and incomplete inverse problems.

We begin with some classical results regarding complete inverse spectral problems. The greatest success in inverse spectral theory has been achieved for the second-order Sturm-Liouville (one-dimensional Schrödinger) equation (see the monographs [1–5] and the references therein):

$$-y'' + q(x)y = \lambda y, \quad x \in (0,1), \tag{1}$$

where the function $q(x)$ is usually called the potential, and λ is the spectral parameter. In 1946, Borg [6] proved that the potential $q(x)$ is uniquely specified by the two spectra $\{\lambda_{n,j}\}_{n \ge 1}$ and $j = 0, 1$ of the boundary value problems for Equation (1) subject to the boundary conditions

$$y(0) = y^{(j)}(1) = 0, \quad j = 0, 1.$$

In their seminal paper [7], Gelfand and Levitan developed a constructive method for solving the inverse Sturm-Liouville problem. This method allowed the authors to obtain the necessary and sufficient conditions of the inverse problem's solvability. Since then, inverse spectral theory has been developing all over the world for various classes of differential operators with applications in classical and quantum mechanics, geophysics, nanotechnology, acoustics, electronics, and other fields of science and engineering.

In 1978, Hochstadt and Lieberman [8] proved that, if the potential $q(x)$ of the Sturm-Liouville (Schrödinger) Equation (1) is known a priori on the half-interval $\left(\frac{1}{2}, 1\right)$, then, in contrast to the Borg problem, the spectrum $\{\lambda_{n,0}\}_{n\geq 1}$ alone is sufficient for the unique specification of $q(x)$ on $\left(0, \frac{1}{2}\right)$. Thus, knowledge of the potential on part of the interval reduces the amount of spectral data needed for the operator reconstruction. The Hochstadt–Lieberman problem was the first partial inverse problem. Later on, various generalizations of this problem were considered by Hald [9], Gestezy and Simon [10], Horváth [11,12], and other scholars. Constructive methods and solvability conditions for the Hochstadt–Lieberman problem have been obtained by Sakhnovich [13], Hryniv and Mykytyuk [14], Buterin [15,16], and Martinyuk and Pivovarchik [17,18].

In recent years, considerable attention has been paid by mathematicians and physicists to the inverse transmission eigenvalue problem, which has applications in acoustics. In [19], McLaughlin and Polyakov presented an inverse transmission problem statement, which generalized the Hochstadt–Lieberman problem. The investigation of the McLaughlin–Polyakov problem continued in [20–23] and other studies, offering a series of new results in the theory of partial inverse problems.

A variety of partial inverse problems arise for differential operators on geometrical graphs, also called quantum graphs. Such operators are used to model various processes in graph-like structures and networks in organic chemistry, mesoscopic physics, nanotechnology, hydrodynamics, waveguide theory, and other applications (see, e.g., the monographs [24,25] and the references therein). A basic introduction to quantum graphs can be found in [26]. There is an extensive literature on inverse spectral problems for differential operators on graphs (see the survey [27] on this topic). In this review, we focus on partial inverse problems. Such problems on graphs arise when differential operator coefficients (for example, Sturm-Liouville potentials) are known a priori for part of the graph. These coefficients can be obtained by either measurements or a reconstruction method. In the second case, the solution of partial inverse problems can be used as an auxiliary step in solving complete inverse problems on graphs.

The first results on partial inverse problems on graphs were obtained by Pivovarchik [28], Yurko [29], and Yang et al. [30–32]. However, the results of these papers were limited to uniqueness theorems for the Sturm-Liouville (Schrödinger) operators on graphs of an elementary structure (star-shaped graphs and simple graphs with loops). Later on, Bondarenko developed a constructive method to solve partial inverse problems on graphs of various types. Using this method, a number of new results have been obtained for differential operators and pencils on star-shaped graphs [33–36], simple graphs with cycles [37,38], tree graphs (graphs without cycles) [39], and even graphs of an arbitrary geometrical structure [40,41]. These results included not only uniqueness theorems but also constructive algorithms for the solution, solvability, and stability of partial inverse problems.

Following the investigation of partial inverse problems on graphs, a unified approach to various classes of partial inverse problems arose [41–44]. This approach was based on the reduction of a partial inverse problem on either an interval or a graph to an inverse problem for a differential operator on an "unknown" interval with entire analytic functions in one of the boundary conditions. In [41–44], an inverse problem theory was created for Sturm-Liouville operators with entire functions in a boundary condition. This theory included the necessary and sufficient conditions of uniqueness, constructive methods for a solution, global solvability, local solvability, and stability. These results have been applied to the Hochstadt–Lieberman problem, the inverse transmission eigenvalue problem, and partial inverse problems on graphs. Later on, this approach was developed in [45] for differential pencils and in [46] for Sturm-Liouville operators with polynomial boundary conditions.

In addition, it is worth mentioning that partial inverse problems have been considered for other types of operators, in particular, for integro-differential operators [47–49], functional-differential operators with a constant delay [50–52], higher-order differential operators [53,54], and matrix Sturm-Liouville operators [55,56].

The goal of this review was to summarize classical and recent work on partial inverse problems. Below, we describe some features of this review. Since the amount of literature on partial inverse problems is enormous, this review includes only the results of selected papers, which, in the author's opinion, could help the reader to form a general picture. Most attention was paid to partial inverse problems on geometrical graphs and the unified approach, which has been investigated by the author in recent years. However, we also paid attention to classical results and different modern directions of research. In view of the huge amount of information available, we focused on describing *the results* of the selected papers. Unfortunately, we could not provide a full description of *the methods* by which these results were obtained. Nevertheless, the reader can find more details in the referenced literature. In this review, we compare the results for different problems and establish connections between them.

The paper is organized as follows. In Section 2, we consider the Hochstadt–Lieberman problem and its generalizations on intervals. Section 2.1 is devoted to the uniqueness theorems, and Section 2.2 focuses on constructive methods and solvability conditions. Section 2.3 is concerned with the inverse transmission eigenvalue problem (mostly the McLaughlin–Polyakov problem). In Section 3, we describe the known results on partial inverse problems for differential operators on graphs. Star-shaped graphs are considered in Section 3.1, simple graphs with loops in Section 3.2, and graphs of a general structure in Section 3.3. Section 4 is concerned with the unified approach to various classes of partial inverse problems. In Section 4.1, the inverse spectral theory of the Sturm-Liouville problem with entire functions in a boundary condition is presented. In Section 4.2, this theory is applied to partial inverse problems. In Section 5, we consider partial inverse problems for classes of operators other than Sturm-Liouville operators and pencils. Section 6 contains the conclusions.

Here, we present a few remarks about notations. When describing the results, we mostly preserve the notations of the original papers. Therefore, the notations included throughout the review can have different meanings. The symbol λ usually denotes the spectral parameter, unless stated otherwise. In the formulations of the uniqueness theorems, along with one problem (e.g., problem L), we often consider another problem (e.g., $\tilde{L}$) of the same form but with different coefficients. If a symbol α denotes an object related to the problem without a tilde, then the symbol $\tilde{\alpha}$ denotes the analogous object related to the problem with a tilde. In addition, all the boundary value problems in this review were considered on finite intervals or compact graphs, so their spectra are countable sets of eigenvalues.

2. Hochstadt–Lieberman Problem and Generalizations

2.1. Uniqueness Theorems

Let us begin with the famous result of Hochstadt and Lieberman [8]. They considered the Sturm-Liouville problem

$$\left.\begin{array}{c} -y'' + q(x)y = \lambda y, \quad x \in (0,1), \\ y(0)\cos\alpha + y'(0)\sin\alpha = 0, \quad y(1)\cos\beta + y'(1)\cos\beta = 0, \end{array}\right\} \tag{2}$$

where $q \in L_1(0,1)$ and $\alpha, \beta \in [0,\pi)$. The Hochstadt–Lieberman problem is formulated as follows:

Problem 1 ([8])**.** *Suppose that the potential $q(x)$ on $\left(\frac{1}{2}, 1\right)$ and the constants α and β are known a priori. Given the spectrum $\{\lambda_n\}_{n\geq 1}$ of the problem (2), find $q(x)$ on $\left(0, \frac{1}{2}\right)$.*

Hochstadt and Lieberman proved the following uniqueness theorem for Problem 1:

Theorem 1 ([8]). *Let $\{\lambda_n\}_{n\ge 1}$ be the spectrum of the problem (2), and let $\{\tilde\lambda_n\}_{n\ge 1}$ be the spectrum of a similar problem with an integrable potential $\tilde q(x)$. Suppose that $q(x) = \tilde q(x)$ on $\left(\frac{1}{2}, 1\right)$, and $\lambda_n = \tilde\lambda_n$, $n \ge 1$. Then, $q(x) = \tilde q(x)$ a.e. on $(0,1)$.*

Hald [9] generalized Hochstadt and Lieberman's findings to the Sturm-Liouville problem with discontinuity. In addition, Hald showed that the coefficient in the left boundary condition is also uniquely specified by the spectrum.

Theorem 2 ([9]). *Consider the eigenvalue problem*

$$-y'' + q(x)y = \lambda y, \quad x \in (0,\pi),$$

with the boundary conditions

$$y'(0) - hy(0) = y'(\pi) + Hy(\pi) = 0$$

and the jump conditions

$$y(d+) = ay(d-), \quad y'(d+) = a^{-1}y(d-) + by(d-),$$

where q is an integrable function; $0 < d < \frac{\pi}{2}$; $a > 0$; and $|a-1| + |b| > 0$. Let $\{\lambda_n\}_{n\ge 0}$ be the eigenvalues. Consider the eigenvalue problem with a, b, d, h, H, λ, and q replaced by $\tilde a, \tilde b, \tilde d, \tilde h, \tilde H, \tilde\lambda$, and $\tilde q$, respectively. If $\lambda_n = \tilde\lambda_n$, $H = \tilde H$, and $q = \tilde q$ a.e. on $\left(\frac{\pi}{2}, \pi\right)$, then $a = \tilde a$, $b = \tilde b$, $d = \tilde d$, $h = \tilde h$, and $q = \tilde q$ a.e. on $(0,\pi)$.

Gesztesy and Simon [10] investigated a case where the potential $q(x)$ on an interval $(0,a)$, $a > \frac{1}{2}$ is known. Then, the potential is uniquely determined by a fractional part of the spectrum.

Theorem 3 ([10]). *Let $\sigma(H)$ denote the spectrum of the operator $H = -\frac{d^2}{dx^2} + q$ in $L_2(0,1)$ with the boundary conditions*

$$y'(0) + h_0 y(0) = 0, \quad y'(1) + h_1 y(1) = 0, \quad h_0, h_1 \in \mathbb{R}.$$

Then, q on $\left[0, \frac{\pi}{2} + \frac{\alpha}{2}\right]$ for some $\alpha \in (0,1)$; h_0; and a subset $S \subseteq \sigma(H)$ satisfying

$$\#\{\lambda \in S \colon \lambda \le \lambda_0\} \ge (1-\alpha)\#\{\lambda \in \sigma(H) \colon \lambda \le \lambda_0\} + \frac{\alpha}{2} \tag{3}$$

for all sufficiently large $\lambda_0 \in \mathbb{R}$ uniquely determine h_1 and q on $[0,1]$.

The uniqueness of recovering the Sturm-Liouville potential from parts of spectra described by conditions analogous to (3) was also investigated in [57–59].

Horváth [11] noticed that, to recover the potential $q(x)$, one can use eigenvalues of several spectra $\sigma_j = \sigma(q, \alpha_j, \beta)$, $j = 1, \dots, N$ of the Sturm-Liouville problems

$$\left.\begin{array}{c} -y'' + q(x)y = \lambda y, \quad x \in (0,\pi), \\ y(0)\cos\alpha_j + y'(0)\sin\alpha_j = 0, \quad y'(\pi)\cos\beta + y(\pi)\sin\beta = 0. \end{array}\right\} \tag{4}$$

The following main result of [11] generalized Theorem 3 of Gestezy and Simon.

Theorem 4 ([11]). *Suppose that $\lambda_n^{(j)} \in \sigma_j$ is known for $n \in S_j$ and let*

$$n_j(t) = \#\{n \in S_j \colon \lambda_n^{(j)} < t^2\}, \quad t \ge 0.$$

Let $0 \le a < \pi$, $0 \le \gamma \le 1$, *and suppose that there exist* $t_0 > 0$ *and* $\delta > 0$ *such that, for* $t \ge t_0$,

$$\sum_{j=1}^{N} n_j(t) \ge \begin{cases} 2(1-\frac{a}{\pi})\{\gamma[t+\frac{1}{2}]+(1-\gamma)([t]+\frac{1}{2})\}+O(t^{-\delta}), & \text{if } \sin\beta \ne 0, \\ 2(1-\frac{a}{\pi})\{\gamma[t+\frac{1}{2}]+(1-\gamma)([t]+\frac{1}{2})\}-1+O(t^{-\delta}), & \text{if } \sin\beta = 0. \end{cases}$$

Then, q *on* $(0,a)$ *and the eigenvalues* $\left\{\lambda_n^{(j)} : n \in S_j\right\}$, $j = 1,\dots,N$, *determine* q *a.e. on* $(0,\pi)$.

An analogous result was obtained in [11] for Dirac operators.

The disadvantage of Theorems 3 and 4 is that their conditions are sufficient but not necessary for the unique specification of the potential by part of the spectrum. In [12], Horváth obtained the necessary and sufficient conditions for the uniqueness of a solution for the following inverse problem in terms of closed exponential systems.

Problem 2 ([12])**.** *Given the eigenvalues* $\{\lambda_n\}_{n\ge1}$, *where each* λ_n *belongs to the spectrum* $\sigma(q,\alpha_n,\beta)$ *of the Sturm-Liouville problem* (4)*, find the potential* q.

For definiteness, we provide the results of [12] for $\sin\beta = 0$.

Theorem 5 ([12])**.** *Let* $1 \le p \le \infty$, $q \in L_p(0,\pi)$, *and* $0 \le a < \pi$, *and let* $\lambda_n \in \sigma(q,\alpha_n,0)$ *be real numbers with* $\lambda_n \not\to -\infty$. *Then,* $\beta = 0$, q *on* $(0,a)$ *and the eigenvalues* λ_n *determine* q *in* $L_p(0,\pi)$, *and*

$$e(\Lambda) = \left\{e^{\pm 2i\mu x}, e^{\pm 2i\sqrt{\lambda_n}x} : n \ge 1\right\} \tag{5}$$

is closed in $L_p(a-\pi,\pi-a)$ *for* $\mu \ne \pm\sqrt{\lambda_n}$. *Note that, if the sequence* $e(\Lambda)$ *is closed for at least one* $\mu \ne \pm\sqrt{\lambda_n}$, *then it is closed for any such value of* μ.

Furthermore, in [12], Horváth noticed that Problem 2 was closely related to the reconstruction of the potential $q(x)$ from the values of the Weyl function at a countable set of points. Let $v(x,\lambda)$ denote the solution to the following initial value problem:

$$-v'' + q(x)v = \lambda v, \quad x \in (0,\pi), \quad v(\pi,\lambda) = \sin\beta, \quad v'(\pi,\lambda) = -\cos\beta.$$

Then, the Weyl function is defined as follows:

$$m_\beta(\lambda) = \frac{v'(0,\lambda)}{v(0,\lambda)}.$$

According to the classical results [1,6], the Weyl function $m_\beta(\lambda)$ uniquely specifies the potential $q(x)$. Horváth [12] obtained the following necessary and sufficient conditions for the uniqueness of the potential reconstruction using the values $\{m_\beta(\lambda_n)\}_{n\ge1}$.

Problem 3 ([12])**.** *Given the values* $\{m_\beta(\lambda_n)\}_{n\ge1}$, *find* q.

Theorem 6 ([12])**.** *Let* $1 \le p \le \infty$ *and* λ_n, $n \ge 1$ *be different arbitrary real numbers with* $\lambda_n \not\to -\infty$. *Let* $\beta = 0$, $q, \tilde{q} \in L_p(0,\pi)$ *and consider the Weyl functions* $m_0(\lambda)$, *and* $\tilde{m}_0(\lambda)$, *defined by* q *and* $\tilde{q}$, *respectively. Then, the relation*

$$m_0(\lambda_n) = \tilde{m}_0(\lambda_n), \quad n \ge 1 \tag{6}$$

implies that $m_0(\lambda) \equiv \tilde{m}_0(\lambda)$ *if and only if the system* $e(\Lambda)$ *defined by* (5) *is closed in* $L_p(-\pi,\pi)$.

Note that both sides of (6) are allowed to be infinite. Results analogous to Theorems 5 and 6 for the case $\sin\beta \ne 0$ can also be found in [12].

The results of Horváth [11,12] motivated the further study of Problem 2 and its analogs. In particular, Horváth and Kiss [60,61] investigated the stability of the problem. Horváth and Sáfár [62] obtained the necessary and sufficient conditions for the uniqueness of the potential reconstruction on a subinterval by some of the eigenvalues and norming constants.

Note that the Hochstadt–Lieberman problem and the abovementioned generalizations deal with cases in which the potential is known on the right-hand (left-hand) subinterval, as in Figure 1. Naturally, the following question arises: if the potential is known on either the middle subinterval, as in Figure 2, or the boundary subintervals, as in Figure 3, then is the potential on the remaining part of the interval uniquely specified by the spectrum or any other spectral data?

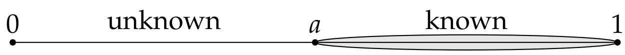

Figure 1. Hochstadt–Lieberman-type problems.

Figure 2. When $q(x)$ is known on the middle subinterval.

Figure 3. When $q(x)$ is unknown on the middle subinterval.

The question regarding the case in Figure 2 was answered by Guo and Wei [63]. Let us formulate their result. Let L denote the Sturm-Liouville operator $-y'' + q(x)y$ subject to the boundary conditions

$$y'(0) - hy(0) = 0, \quad y'(1) + Hy(1) = 0,$$

where $q \in L_1(0,1)$ is a real-valued function, and $h, H \in \mathbb{R}$. Let $\sigma(L) = \{\lambda_n\}_{n \ge 0}$ be the spectrum of L, and let $\psi(x,\lambda)$ be the solution of the Sturm-Liouville equation under the initial conditions $\psi(1,\lambda) = 1$, $\psi'(1,\lambda) = -H$. For a set $A = \{x_n\}_{n \ge 0}$ of positive reals, define

$$N_A(t) := \{n \in \mathbb{N} \cup \{0\} \colon x_n \le t\}.$$

Theorem 7 ([63])**.** *Let $[a_1, a_2] \subset [0,1]$ with $a_1 \le \frac{1}{2}$ and $a_1 + a_2 \ge 1$, where the two equalities do not occur simultaneously. Then, q on $[a_1, a_2]$ together with the subset S of $\sigma(L)$ and the interior spectral data $\dfrac{\psi(a_2,\lambda_n)}{\psi'(a_2,\lambda_n)}$ for $\lambda_n \in S'$, $S' \subset S$, where the subsets S and S' satisfy*

$$N_S(t) \ge 2a_1 N_\sigma(t) - a_1, \quad N_{S'}(t) \ge 2(1-a_2)N_\sigma(t) + a_2 - 1$$

for all sufficiently large values of t, uniquely determine h, H, and q a.e. on $[0,1]$.

Thus, given the potential $q(x)$ on an interior subinterval $[a_1, a_2]$, some part of the spectrum together with additional spectral data related to the point a_2 uniquely specify the operator. It is interesting that, in Theorem 7, $[a_1, a_2]$ can be an arbitrarily small interval containing $\frac{1}{2}$.

The case in Figure 3, to the best of the author's knowledge, remains an open problem.

2.2. Solvability Conditions and Constructive Solution

The results of the previous subsection were concerned only with the uniqueness theorems. In this subsection, we consider constructive methods for solving the Hochstadt–Lieberman problem and the existence of its solution.

The first results in this direction were obtained by Sakhnovich [13]. He considered the Sturm-Liouville problem

$$-y'' + q(x)y = \lambda y, \quad x \in [0,1], \tag{7}$$
$$y(0) = y(1) = 0, \tag{8}$$

where the potential $q(x)$ is real-valued and continuous. Let $y(x,\lambda)$ denote the solution of Equation (7) satisfying the initial conditions $y(0,\lambda) = 0$, $y'(0,\lambda) = 1$. The main result of [13] was the following theorem, which provided sufficient conditions for solvability of the Hochstadt–Lieberman problem.

Theorem 8 ([13])**.** *Let the given functions $h(t)$, $t \in [0,1]$, and $p(x)$, $x \in [0,\frac{1}{2}]$ satisfy the following conditions:*

1. *The function $h(t)$ has a bounded derivative, and $h(0) = 0$.*
2. *The function $p(x)$ is bounded on the segment $[0,\frac{1}{2}]$.*
3. *The following inequality holds:*

$$\sup_{0 \le t \le 1} |h'(t)| + \frac{1}{4} \sup_{0 \le x \le \frac{1}{2}} |p(x)| < \frac{1}{2}. \tag{9}$$

Then, there exists a bounded function $q(x)$, $x \in [0,1]$, such that

$$q(x) = p(x), \quad x \in [0,\tfrac{1}{2}], \tag{10}$$

and the corresponding function $y(1,\lambda)$ has the form

$$y(1,\lambda) = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} + \int_0^1 \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}} h(t)\, dt. \tag{11}$$

It is important to note that $y(1,\lambda)$ is the characteristic function of the problem (7)–(8), that is, the zeros of $y(1,\lambda)$ coincide with the eigenvalues $\{\lambda_n\}_{n\ge1}$ of (7)–(8). Using the eigenvalues $\{\lambda_n\}_{n\ge1}$, one can construct the function $y(1,\lambda)$ using Hadamard's factorization theorem and then find the function $h(t)$ satisfying (11) using the Fourier transform. Thus, Theorem 8 provides the sufficient conditions for the existence of the potential $q(x)$ that satisfies (10) and has the given spectrum $\{\lambda_n\}_{n\ge1}$.

Theorem 8 is proved by a constructive method that finds the potential $q(x)$ from $p(x)$ and $h(x)$ via approximations. For the convergence of these approximations, the inequality (9) is crucial. Thus, the result of [13] has a local nature.

The necessary and sufficient conditions for the Hochstadt–Lieberman problem's solvability, to the best of the author's knowledge, were obtained for the first time by Hryniv and Mykytyuk [14]. They considered the Sturm-Liouville equation (Equation (1)) with the potential q of class $W_2^{-1}(0,1)$. In this case, it is convenient to write the Sturm-Liouville differential expression $-y'' + q(x)y$ as $\ell_\sigma(y) = -(y^{[1]})' - \sigma y'$, where $q = \sigma'$, $\sigma \in L_2(0,1)$, and $y^{[1]} := y' - \sigma y$ is the quasi-derivative. Let us use the notation $\operatorname{Re} L_2(0,a)$ for the class of real-valued functions of $L_2(0,a)$, $a > 0$. For $\sigma \in \operatorname{Re} L_2(0,1)$ and $h \in \mathbb{R}$, let $T_{\sigma,h}$ denote the operator in $L_2(0,1)$ that acts as $T_{\sigma,h} y = \ell_\sigma(y)$ on the domain

$$\operatorname{dom} T_{\sigma,h} = \{y \in W_1^1(0,1) \colon y^{[1]} \in W_1^1(0,1),\, \ell_\sigma(y) \in L_2(0,1),\, y^{[1]}(0) = 0,\, y^{[1]}(1) = hy(1)\}.$$

The operator $T_{\sigma,h}$ is self-adjoint, and its spectrum is a countable set of real simple eigenvalues $\{\lambda_n^2\}_{n\ge0}$ satisfying $\{(\lambda_n - n)\}_{n\ge0} \in l_2$. Using the shift $\sigma(x) := \sigma(x) + cx$, $h := h - c$, one can achieve the positivity $\lambda_n > 0$, $n \ge 0$. In [14], the following analog of the Hochstadt–Lieberman problem was considered:

Problem 4 ([14]). *Given a function $\sigma_0(x)$, $x \in (0, \frac{1}{2})$ and reals $\{\lambda_n\}_{n\ge 0}$, find a function $\sigma \in \operatorname{Re} L_2(0,1)$ and $h \in \mathbb{R}$ such that $\sigma(x) = \sigma_0(x)$ a.e. on $(0, \frac{1}{2})$ and the spectrum of $T_{\sigma,h}$ coincides with $\{\lambda_n^2\}_{n\ge 0}$.*

In order to formulate the results of [14], one needs some additional definitions. Let $\mathfrak{L}$ denote the set of all strictly increasing sequences $\Lambda = \{\lambda_n\}_{n\ge 0}$, in which $\lambda_n > 0$ and $\{(\lambda_n - \pi n)\}_{n\ge 0} \in l_2$. Let us fix an arbitrary $\Lambda = \{\lambda_n\} \in \mathfrak{L}$ and denote by Π_Λ the set of all real-valued functions $\psi \in L_2(0,1)$ of the form

$$\psi(x) = \sum_{n=0}^{\infty} (\alpha_n \cos(\lambda_n x) - \cos(\pi n x)) + \frac{1}{2}, \tag{12}$$

where $\{\alpha_n\}_{n\ge 0}$ is a sequence of positive numbers such that $\{(\alpha_n - 1)\}_{n\ge 0} \in l_2$.

For $\sigma_0 \in \operatorname{Re} L_2(0, \frac{1}{2})$, let $y_0(x,\lambda)$ denote the solution of the initial value problem

$$\ell_{\sigma_0}(y_0) = \lambda^2 y_0, \quad x \in (0, \tfrac{1}{2}), \quad y_0(0,\lambda) = 1, \quad y_0^{[1]}(0,\lambda) = 0.$$

Let $K(x,t)$ be the transformation operator kernel (see the details in [14]):

$$\cos \lambda x = y_0(x,\lambda) + \int_0^x K(x,t) y_0(t,\lambda)\, dt.$$

The necessary and sufficient conditions for the solvability of Problem 4 are provided by the following theorem:

Theorem 9 ([14]). *Assume that $\Lambda = \{\lambda_n\}_{n\ge 0} \in \mathfrak{L}$, $\sigma_0 \in \operatorname{Re} L_2(0, \frac{1}{2})$ and*

$$\phi_0(2x) := -\frac{1}{2}\sigma_0(x) + \int_0^x K^2(x,t)\, dt, \quad x \in (0, \tfrac{1}{2}).$$

1. *Problem 4 is solvable for the mixed spectral data (σ_0, Λ) if and only if $\phi_0 \in \Pi_\Lambda$.*
2. *If $\phi_0 \in \Pi_\Lambda$, then the solution of Problem 4 is unique, that is, there exists a unique $\sigma \in \operatorname{Re} L_2(0,1)$ and a unique $h \in \mathbb{R}$ such that σ is an extension of σ_0 and the spectrum of $T_{\sigma,h}$ coincides with $\Lambda^2 = \{\lambda_n^2\}_{n\ge 0}$.*

As corollaries of Theorem 9, Hryniv and Mykytuyk [14] also obtained some results for the case of the regular potential $q \in L_2(0,1)$. The proof of Theorem 9 is based on the transformation operator method (see [1,2]). Note that the numbers α_n in Expansion (12) for $\phi_0(x)$ equal the weight numbers $\|y_n\|^{-2}_{L_2(0,1)}$, where $\{y_n(x)\}_{n\ge 0}$ are the eigenfunctions of the operator $T_{\sigma,h}$ corresponding to the eigenvalues $\{\lambda_n^2\}_{n\ge 0}$. Hence, the requirement $\phi_0 \in \Pi_\Lambda$ means that the weight numbers are positive and have the asymptotics $\{(\alpha_n - 1)\}_{n\ge 0} \in l_2$, which is valid by necessity. Roughly speaking, the method of [14] reduced Problem 4 to the classical inverse problem using the spectral data $\{\lambda_n^2, \alpha_n\}_{n\ge 0}$ and required the necessary and sufficient conditions for the solvability of the latter problem. Such conditions in the case of a singular potential of class $W_2^{-1}(0,1)$ were obtained in [64].

Theorem 10 ([64]). *For the numbers $\{\lambda_n^2, \alpha_n\}_{n\ge 0}$ to be the spectral data of a positive Sturm-Liouville operator $T_{\sigma,h}$ with $\sigma \in \operatorname{Re} L_2(0,1)$ and $h \in \mathbb{R}$, it is necessary and sufficient that $\Lambda = \{\lambda_n\}_{n\ge 0} \in \mathfrak{L}$, $\alpha_n > 0$ for $n \ge 0$ and $\{(\alpha_n - 1)\}_{n\ge 0} \in l_2$.*

An alternative approach to the solution of the Hochstadt–Lieberman problem was developed in parallel by Buterin [15,16] and by Martinyuk and Pivovarchik [17]. In [15], Buterin considered the Sturm-Liouville problem

$$-y'' + q(x)y = \lambda y, \quad x \in (0,\pi), \tag{13}$$

$$y'(0) - hy(0) = 0, \quad y'(\pi) + Hy(\pi) = 0, \tag{14}$$

where $q \in L_1(0, \pi)$.

Let $S(x, \lambda)$, $\varphi(x, \lambda)$, and $\psi(x, \lambda)$ denote, respectively, the solutions of Equation (13) satisfying the initial conditions

$$S(0, \lambda) = 0, \quad S'(0, \lambda) = 1, \quad \varphi(0, \lambda) = 1, \quad \varphi'(0, \lambda) = h, \quad \psi(\pi, \lambda) = 1, \quad \psi'(\pi, \lambda) = -H.$$

Then, the eigenvalues $\{\lambda_n\}_{n \ge 0}$ of the boundary value problem (13)–(14) coincide with the zeros of the characteristic function $\Delta(\lambda) = \varphi'(\pi, \lambda) + H\varphi(\pi, \lambda)$. Thus,

$$\begin{gathered} \Delta^0(\lambda) = \psi(0, \lambda), \quad \Delta_1(\lambda) = -\psi'(\pi/2, \lambda), \quad \Delta_1^0(\lambda) = \psi(\pi/2, \lambda), \\ \Theta(\lambda) = \varphi(\pi/2, \lambda), \quad \Xi(\lambda) = \varphi'(\pi/2, \lambda). \end{gathered} \tag{15}$$

The main idea of [15] consists in the fact that, if the potential $q(x)$ on $(0, \pi/2)$ is known, then the functions $\Theta(\lambda)$ and $\Xi(\lambda)$ can be found. Thus, using the following relations between the characteristic functions,

$$\begin{aligned} \Delta_1^0(\lambda) &= \Delta^0(\lambda)\Theta(\lambda) - \Delta(\lambda)S(\pi/2, \lambda), \\ -\Delta_1(\lambda) &= \Delta^0(\lambda)\Xi(\lambda) - \Delta(\lambda)S'(\pi, \lambda), \end{aligned}$$

one can find $\Delta_1^0(\lambda)$ and $\Delta_1(\lambda)$ by the interpolation of entire functions and recover the potential $q(x)$ and the coefficient H from the Weyl function $M(\lambda) = -\dfrac{\Delta_1(\lambda)}{\Delta_1^0(\lambda)}$ for the interval $(\pi/2, \pi)$. Indeed, the zeros of the functions $\Delta_1(\lambda)$ and $\Delta_1^0(\lambda)$ are the two spectra of the Borg problem on this interval.

Let $\{\xi_n\}_{n \ge 0}$ and $\{\theta_n\}_{n \ge 0}$ denote the zeros of the entire functions $\Xi(\lambda)$ and $\Theta(\lambda)$, respectively. If these zeros are simple, the Hochstadt–Lieberman problem can be solved by the following constructive algorithm:

Method 1 ([15]). *Suppose that the spectrum $\{\lambda_n\}_{n \ge 0}$, the coefficient h, and the potential $q(x)$ on the interval $(0, \pi/2)$ are given. We must find $q(x)$ on $(\pi/2, \pi)$ and H.*

1. *Find $\Delta(\lambda)$ by the formula*

$$\Delta(\lambda) = \pi(\lambda_0 - \lambda) \prod_{n=1}^{\infty} \frac{\lambda_n - \lambda}{n^2}.$$

2. *Construct the functions $\Theta(\lambda)$ and $\Xi(\lambda)$ using (15) and find their zeros θ_n, ξ_n, $n \ge 0$.*
3. *Calculate the numbers*

$$\begin{aligned} d(\xi_n) &= \Delta(\xi_n)S'(\pi/2, \xi_n) + \sqrt{\xi_n}\sin(\sqrt{\xi_n}\pi/2), \\ d_0(\theta_n) &= -\Delta(\theta_n)S(\pi/2, \theta_n) - \cos(\sqrt{\theta}_n\pi/2). \end{aligned}$$

4. *By interpolation, find the functions*

$$d(\lambda) = \sum_{n=0}^{\infty} d(\xi_n) \frac{\Xi(\lambda)}{(\lambda - \xi_n)\Xi'(\xi_n)}, \quad d_0(\lambda) = \sum_{n=0}^{\infty} d_0(\theta_n) \frac{\Theta(\lambda)}{(\lambda - \theta_n)\Theta'(\theta_n)}.$$

5. *Let $\Delta_1(\lambda) = -\sqrt{\lambda}\sin(\sqrt{\lambda}\pi/2) + d(\lambda)$, $\Delta_1^0(\lambda) = \cos(\sqrt{\lambda}\pi/2) + d_1(\lambda)$.*
6. *Recover $q(x)$ on $(\pi/2, \pi)$ and H from the Weyl function $M(\lambda) = -\dfrac{\Delta_1^0(\lambda)}{\Delta_1(\lambda)}$.*

Note that Method 1 does not require the self-adjointness of the problem (13)–(14) and so works for complex-valued potentials $q(x)$, h, and H. The last step of Method 1 in the non-self-adjoint case can be implemented by an algorithm presented in [65]. Method 1 is also valid for multiple eigenvalues $\{\xi_n\}$ and $\{\theta_n\}$ after minor modifications.

In [16], Buterin generalized Method 1 to quadratic differential pencils of the form

$$y'' + (\rho^2 - 2\rho q_1(x) - q_0(x))y = 0, \quad x \in (0,\pi),$$
$$y'(0) - (h_1\rho + h_0)y(0) = 0, \quad y'(\pi) + (H_1\rho + H_0)y(\pi) = 0,$$

where ρ is the spectral parameter; $q_j(x) \in W_1^j[0,1]$ are complex-valued functions; $h_j, H_j \in \mathbb{C}$; $j = 0,1$; $h_1 \neq \pm i$; and $H_1 \neq \pm i$. The half-inverse problem of [16] consists in the recovery of the coefficients q_0, q_1, H_0, and H_1 from the spectrum $\{\rho_n\}$, while q_0 and q_1 on $(0,\pi/2)$, h_0, and h_1 are known a priori. A similar problem for the quadratic differential pencil with another type of boundary conditions was investigated in [66].

An analogous approach was used by Martinyuk and Pivovarchik [17] to obtain the necessary and sufficient conditions of the Hochstadt–Lieberman problem's solvability. They considered the Sturm-Liouville problem

$$-y'' + q(x)y = \lambda y, \quad x \in (0,a), \quad y(0) = y(a) = 0$$

in the following equivalent form:

$$-y_j'' + q_j(x)y_j = \lambda^2 y_j, \quad x \in [0,a/2], \quad j = 1,2, \tag{16}$$
$$y_j(0) = 0, \quad j = 1,2, \quad y_1(a/2) = y_2(a/2), \quad y_1'(a/2) + y_2'(a/2) = 0. \tag{17}$$

The boundary value problem (16)–(17) can be treated as the Sturm-Liouville problem on a two-edge star-shaped graph with the standard matching conditions in the interior vertex (see Figure 4). In the Hochstadt–Lieberman problem, the potential q_1 on the first edge is known, and the potential q_2 on the second edge must be recovered from the spectrum $\{\lambda_k\}_{k\in\mathbb{Z}_0}$ of the boundary value problem (16)–(17), $\mathbb{Z}_0 := \mathbb{Z} \setminus \{0\}$.

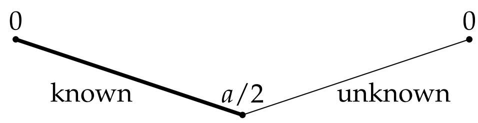

Figure 4. Two-edge graph.

The authors of [17] assumed that $q_j \in L_2(0,a/2)$ and denoted by $s_j(\lambda,x)$, $j = 1,2$ the solutions of the corresponding Equation (16) satisfying the initial conditions $s_j(\lambda,0) = 0$, $s_j'(\lambda,0) = 1$.

Theorem 11 ([17])**.** *Let a real-valued function $q_1 \in L_2(0,a/2)$ be given, together with a set $\{\lambda_k\}_{k\in\mathbb{Z}_0}$ of numbers that satisfy the conditions:*

1. $\lambda_k = -\lambda_k$,
2. $-\infty < \lambda_1^2 < \lambda_2^2 < \cdots < \lambda_k^2 < \dots,$
3. $\lambda_k = \dfrac{\pi k}{a} + \dfrac{K}{\pi k} + \dfrac{\beta_k}{k}$.

   Here, $K \in \mathbb{R}$, and $\{\beta_k\}_{k\in\mathbb{Z}_0} \in l_2$.

   If the function $\dfrac{s_2(\sqrt{\lambda},a/2)}{s_2'(\sqrt{\lambda},a/2)}$ belongs to the Nevanlinna class, then there exists a real-valued function $q_2(x) \in L_2[0,a/2]$ such that the spectrum of the problems (16) and (17) generated by q_1 and q_2 coincides with $\{\lambda_k\}_{k\in\mathbb{Z}_0}$.

It is supposed in Theorem 11 that the functions $s_2(\lambda,a/2)$ and $s_2'(\lambda,a/2)$ are recovered from q_1 and $\{\lambda_k\}_{k\in\mathbb{Z}_0}$ by a procedure analogous to Method 1. Note that the conditions of Theorem 11 are not only sufficient but also necessary. Indeed, the conditions 1–3 are the standard properties of Sturm-Liouville eigenvalues, and the function $\dfrac{s_2(\sqrt{\lambda},a/2)}{s_2'(\sqrt{\lambda},a/2)}$ is the Weyl function of the Sturm-Liouville problem on the second edge, which belongs to the Nevanlinna class by necessity. In fact, Martinyuk and Pivovarchik [17] reduced the Hochstadt–Lieberman problem to the classical inverse problem using the Weyl function on a subinterval corresponding to the second edge and then imposed an additional requirement of belonging to the Nevanlinna class. Analogous results for the Robin boundary conditions were obtained in [18].

Thus, both Theorems 9 and 11 pertaining to the necessary and sufficient conditions proposed by Hryniv and Mykytyuk [14] and by Martinyuk and Pivovarchik [17], respectively, contain some a posteriori conditions, which have to be checked after the implementation of several steps of a constructive procedure for solution. It seems that such conditions are unavoidable for Hochstadt–Lieberman-type problems.

Additionally, numerical techniques for solving the Hochstadt–Lieberman problem were developed by Rundell and Sacks [67]. An overview of some other work on the Hochstadt-Leiberman problems on an interval can be found in [68].

2.3. McLaughlin–Polyakov Problem

In this section, we consider the so-called transmission eigenvalue problem

$$-y'' + q(x)y = \lambda y, \quad x \in (0,1), \tag{18}$$

$$y(0) = 0, \quad y(1)\cos\rho a - y'(1)\frac{\sin\rho a}{\rho} = 0, \quad \rho = \sqrt{\lambda}, \tag{19}$$

where q is a real-valued potential of $L_2(0,1)$ and $a \ge 0$. The boundary condition at $x = 1$ has a non-linear and even non-polynomial dependence on the spectral parameter λ.

The problem (18)–(19) arise in connection with the investigation of the acoustic inverse scattering problem in a non-homogeneous medium (see [19]). *Transmission eigenvalues* are the eigenvalues k^2 of the boundary value problem

$$\begin{array}{ll} \Delta u + k^2 n(x) u = 0, & x \in B_1, \\ \Delta v + k^2 v = 0, & x \in B_1, \\ u(x) = v(x), & x \in \partial B_1, \\ \frac{\partial}{\partial r}(u(x) - v(x)) = 0, & x \in \partial B_1, \end{array} \tag{20}$$

where B_1 is the ball in $\mathbb{R}^3$ of radius 1 centered at the origin, ∂B_1 is its boundary, $n(x) > 0$ is the refractive index, and $\frac{\partial}{\partial r}$ is the normal derivative. The inverse transmission eigenvalue problems consist in the recovery of the function $n(x)$ (related to the speed of sound in acoustics) from the transmission eigenvalues. In spherically symmetric cases, the problem (20) can be reduced to the one-dimensional form (18)–(19) using the separation of variables and subsequent transforms (see [19]).

Difficulties in the investigation of the problem (18)–(19) are caused by the non-regularity of its boundary conditions in the Birkhoff and Stone senses (see [69]). Therefore, the transmission problem involves more complex spectral behavior than the classical Sturm-Liouville problems.

McLaughlin and Polyakov [19] showed that, for $a \ne 1$, the transmission eigenvalue problem has a subspectrum $\{\lambda_n\}_{n\ge1}$ with the asymptotics

$$\sqrt{\lambda_n} = \frac{\pi n}{1-a} + \frac{\omega_0}{\pi n} + \frac{\varkappa_n}{n}, \quad \omega_0 := \frac{1}{2}\int_0^1 q(x)dx, \quad n \ge 1, \tag{21}$$

Furthermore, $\lambda_n \in \mathbb{R}$ for a sufficiently large n. Buterin and Yang [70] suggested that an eigenvalue sequence $\{\lambda_n\}_{n\ge1}$ possessing these properties should be called *an almost real subspectrum*. Note that a finite number of the first eigenvalues in an almost real subspectrum may be complex and/or multiple. Since the potential $q(x)$ is real-valued, then, without a loss of generality, we can assume that an almost real subspectrum is symmetrical with respect to the real axis, that is, values λ and $\overline{\lambda}$ are only contained in the sequence

$\{\lambda_n\}_{n\geq 1}$ simultaneously and have the same multiplicity. An almost real subspectrum can be non-unique. The results of this section are valid for any almost real subspectrum.

In [19], an investigation of the following inverse transmission eigenvalue problem was initiated.

Problem 5 ([19])**.** *Given an almost real subspectrum $\{\lambda_n\}_{n\geq 1}$ and the potential $q(x)$ on the interval $(\alpha, 1)$, where $\alpha := \min\{|a-1|/2, 1\}$, find $q(x)$ on $(0, \alpha)$.*

We call Problem 5 *the McLaughlin–Polyakov problem*. Obviously, if $a = 0$, then the McLauglin-Polyakov problem coincides with the Hochstadt–Lieberman problem, and an almost real subspectrum coincides with the whole spectrum. McLaughin and Polyakov [19] proved the uniqueness theorem for the solution of Problem 5.

Theorem 12 ([19])**.** *Suppose that $a \geq 0$, $a \neq 1$. If $\lambda_n = \tilde{\lambda}_n$ for $n \geq 1$ and $q(x) = \tilde{q}(x)$ a.e. on $(\alpha, 1)$, then $q(x) = \tilde{q}(x)$ a.e. on $(0, \alpha)$.*

Note that, for $a \geq 3$, an almost real subspectrum uniquely specifies the potential on the whole interval $(0, 1)$, and some part of an almost real subspectrum is sufficient for $a > 3$. The investigation of Problem 5 was continued by McLauglin et al. in [20] for $a \geq 3$ and in [21] for $a \in (0, 1) \cup (1, 3)$. The authors of [20,21] developed numerical methods for reconstructing the potential based on the ideas of Rundell and Sacks [67]. However, they did not study the existence and stability of the solution.

In [22], Bondarenko and Buterin proved the following theorem on the local solvability and stability of the McLaughlin–Polyakov problem:

Theorem 13 ([22])**.** *Fix $a \in [0, 1) \cup (1, 3]$. For any real-valued potential $q \in L_2(0, 1)$, there exists $\delta > 0$ such that, for any sequence $\{\tilde{\lambda}_n\}_{n\geq 1}$ symmetric with respect to the real axis and an arbitrary real-valued function $q_1 \in L_2(\alpha, 1)$ satisfying*

$$\int_\alpha^1 q_1(x)\,dx = \int_\alpha^1 q(x)\,dx,$$

the closeness

$$\Lambda := \sqrt{\sum_{n=1}^{\infty} |\lambda_n - \tilde{\lambda}_n|^2} \leq \delta, \quad Q := \|q - q_1\|_{L_2(\alpha,1)} \leq \delta$$

implies the existence of a unique function $\tilde{q}(x) \in L_2(0, 1)$ such that $\tilde{q}(x) = q_1(x)$ a.e. on $(\alpha, 1)$ and $\{\tilde{\lambda}_n\}_{n\geq 1}$ is an almost real subspectrum of the boundary value problems (18) and (19) with $q(x)$ replaced by $\tilde{q}(x)$. Moreover, the following estimate holds:

$$\|q - \tilde{q}\|_{L_2(0,\alpha)} \leq C(\Lambda + Q),$$

where C does not depend on $\{\tilde{\lambda}_n\}_{n\geq 1}$ and $q_1(x)$.

Theorem 13 represents the first existence result for the inverse transmission eigenvalue problem. Furthermore, for $a = 0$, it provides the first full stability result for the Hochstadt–Lieberman problem, in which perturbations of both the spectrum and the potential on $(1/2, 1)$ are taken into account. In addition, Theorem 13 implies the minimality of the McLaughlin–Polyakov data in the case $a \in [0, 1) \cup (1, 3]$. For $a > 3$, the stability does not hold, since Problem 5 is overdetermined. The proof of Theorem 13 is constructive. Later on, by relying on the ideas of [22], a unified approach to partial inverse problems was developed (see Section 4).

In [23], the methods of Bondarenko and Buterin [22] were used to obtain further stability results for Problem 5. It is worth mentioning that inverse transmission eigenvalue problems were studied using statements other than the McLaughlin–Polyakov statement in [69–74] and other papers.

3. Partial Inverse Problems on Graphs

In this section, we consider generalizations of the Hochstadt–Lieberman problem on metric graphs. We treat the boundary value problems on graphs as differential systems. The geometrical graph structure is used only for defining the matching conditions. In interior graph vertices, the problems in this section mostly feature the standard matching conditions, which, from a physical viewpoint, express Kirchoff's law in electrical circuits, the balance of tension in elastic string networks, etc.

3.1. Star-Shaped Graphs

The majority of results on partial inverse problems for metric graphs have been obtained for star-shaped graphs. We start with the complete inverse problem statement for the Sturm-Liouville equations on such a graph.

Let G be a star-shaped graph containing $m \geq 3$ edges $\{e_j\}_{j=1}^m$ of equal lengths π. Each edge e_j joins the internal vertex v_0 with the boundary vertex v_j. For each edge e_j, we introduce the parameter $x_j \in [0, \pi]$. The value $x_j = 0$ corresponds to the boundary vertex v_j, and the value $x_j = \pi$ corresponds to the internal vertex v_0 (see Figure 5).

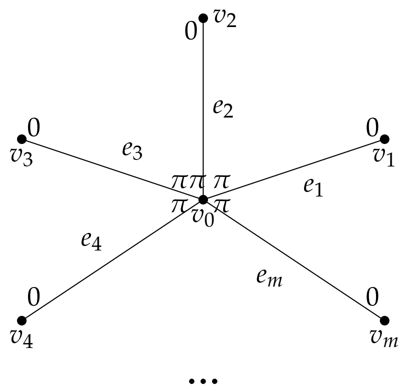

Figure 5. Star-shaped graph.

On the graph G, consider the Sturm-Liouville equations

$$-y_j''(x_j) + q_j(x_j)y_j(x_j) = \lambda y_j(x_j), \quad x_j \in (0, \pi), \quad j = \overline{1, m}, \tag{22}$$

with real-valued potentials $q_j \in L_2(0, \pi)$, $j = \overline{1, m}$, and the standard matching conditions at the internal vertex:

$$y_1(\pi) = y_j(\pi), \quad j = \overline{2, m}, \qquad \sum_{j=1}^{m} y_j'(\pi) = 0. \tag{23}$$

Let Λ and Λ_k, $k = \overline{1, m}$, denote the spectra of the boundary value problems L and L_k, $k = \overline{1, m}$, respectively, for Equation (22), subject to the matching conditions (23) and the following boundary conditions:

$$\begin{aligned} L: &\quad y_j(0) = 0, \quad j = \overline{1, m}, \\ L_k: &\quad y_k'(0) = 0, \quad y_j(0) = 0, \quad j = \overline{1, m} \setminus k. \end{aligned} \tag{24}$$

The spectra Λ and Λ_k, $k = \overline{1, m}$, are countable sets of real eigenvalues.

The complete inverse Sturm-Liouville problem on graph G is formulated as follows:

Problem 6 ([75]). *Given the spectra Λ and Λ_k, $k = \overline{1, m-1}$, find the potentials $\{q_j\}_{j=1}^m$.*

Problem 6 is a special case of the well-studied inverse spectral problems for Sturm-Liouville operators on trees (see [27,75]). In [75], the uniqueness of the inverse problem solution was proved, and a constructive algorithm for its solution based on the method of spectral mappings [4] was developed. Note that, for the recovery of the potentials on the whole graph, a sufficiently large amount of spectral data must be used (m spectra). To the best of the author's knowledge, the minimality of these data is an open question. In addition, the following question arises:

Can the amount of spectral data for reconstruction be reduced if the potentials on some edges are given a priori?

The first partial inverse problems on graphs were considered by Pivovarchik [28]. He studied the Sturm-Liouville problem (22)–(24) on a three-edge star graph ($m = 3$) with real-valued non-negative potentials $q_j \in L_2(0, \pi)$, $j = 1, 2, 3$. In addition, for $j = 1, 2, 3$, the spectrum of the Sturm-Liouville Equation (22) on the edge e_j subject to the boundary conditions $y_j(0) = y_j(\pi) = 0$ is denoted by $\mathfrak{S}_j$. The main results of [28] were concerned with the following inverse problem:

Problem 7 ([28])**.** *Given the spectra Λ and $\mathfrak{S}_j$, $j = 1, 2, 3$, find q_j for $j = 1, 2, 3$.*

A disadvantage of Problem 7 is that the spectra $\mathfrak{S}_j$, which carry information not from the whole graph but from the separate edges, are used for reconstruction. Nevertheless, as a corollary of the main results, the uniqueness of the solution was proved in [28] for the following partial inverse problem:

Problem 8 ([28])**.** *Given the potentials q_1 and q_2 and the spectrum Λ, find q_3.*

In fact, Problem 8 is overdetermined. Yang [30] showed that, for the unique recovery of the potential on one edge, the fractional part $\frac{2}{m}$ of the spectrum is sufficient if the potentials on all the other edges are supposed to be known. In [30], a Sturm-Liouville problem on a star-shaped graph G with general boundary conditions was considered:

$$y_j(0, \lambda) \cos \alpha_j + y_j'(0, \lambda) \sin \alpha_j = 0, \quad \alpha_j \in [0, \pi), \quad j = \overline{1, m}.$$

For simplicity, we formulate the results of Yang [30] for the Dirichlet boundary conditions (24).

Consider the Sturm-Liouville problem L presented by (22)–(24) with real-valued potentials of class $L_1(0, \pi)$. For each $j = \overline{1, m}$, let $S_j(x_j, \lambda)$ denote the solution of Equation (22) satisfying the initial conditions $S_j(0, \lambda) = 0$, $S_j'(0, \lambda) = 1$. The eigenvalues of L coincide with the zeros of the characteristic function

$$\Delta(\lambda) := \sum_{j=1}^{m} S_j'(\pi, \lambda) \prod_{\substack{k=1 \\ k \neq j}}^{m} S_k(\pi, \lambda) \tag{25}$$

and can be denoted as $\{\lambda_{nk}\}_{n \geq 1, k = \overline{1, m}}$ (counting with multiplicities), so that the following asymptotic relations hold:

$$\sqrt{\lambda_{n1}} = n - \frac{1}{2} + O\left(n^{-1}\right), \tag{26}$$

$$\sqrt{\lambda_{nk}} = n + O\left(n^{-1}\right), \quad k = \overline{2, m}. \tag{27}$$

The partial inverse problem of [30] is formulated as follows:

Problem 9 ([30])**.** *Suppose that the potentials $\{q_j\}_{j=2}^{m}$ are known a priori. Given a subspectrum $\Omega := \{\lambda_{nk}\}_{n \geq 1, k = 1, 2}$, find q_1.*

In view of symmetry, one can replace the potential q_1 with an arbitrary q_j, $j = \overline{2, m}$, and the eigenvalues $\{\lambda_{n2}\}_{n \ge 1}$ with a sequence $\{\lambda_{nk}\}_{n \ge 1}$ containing an arbitrary fixed $k = \overline{3, m}$ in the problem statement. Note that the subspectrum Ω can contain a finite number of multiple eigenvalues. Furthermore, Ω is not uniquely determined by the asymptotics (26) and (27), so any suitable subspectrum can be considered. Obviously, in the case $m = 2$, Ω is the whole spectrum, and Problem 9 turns into the Hochstadt–Lieberman problem.

In [30], the following uniqueness theorem for Problem 9 was proved:

Theorem 14 ([30])**.** *Let $\Omega = \{\lambda_{nk}\}_{n \ge 1, k = 1,2}$ be a subspectrum of problem L satisfying the asymptotics* (26)–(27) *and the condition $\Omega \cap \mathfrak{S}_j = \varnothing$ for $j = \overline{2, m}$. If $q_j(x) = \tilde{q}_j(x)$ a.e. on $(0, \pi)$ for $j = \overline{2, m}$ and $\Omega = \tilde{\Omega}$, then $q_1(x) = \tilde{q}_1(x)$ a.e. on $(0, \pi)$.*

The condition $\Omega \cap \mathfrak{S}_j = \varnothing$ is crucial for the uniqueness. Suppose that this condition is violated, that is, there exist $j \in \{2, \dots, m\}$ and λ_0 such that $\lambda_0 \in \Omega \cap \mathfrak{S}_j$. Obviously, $\lambda_0 \in \mathfrak{S}_j$ implies $S_j(\pi, \lambda_0) = 0$. Taking (25) into account, we conclude that $S_i(\pi, \lambda_0) = 0$, and so $\lambda_0 \in \mathfrak{S}_i$ for some $i \ne j$. Thus, λ_0 is the eigenvalue of the two Dirichlet problems for separate edges e_i and e_j. If $i \ne 1$, this eigenvalue carries no information about the potential q_1. In [42], the validity of Theorem 14 was proved for complex-valued potentials $\{q_j\}_{j=1}^m$ and the condition $\Omega \cap \mathfrak{S}_j = \varnothing$, $j = \overline{2, m}$, replaced by the following less restrictive condition:

Condition 1. *For every $\lambda_{nk} \in \Omega$, there do not exist indices i and j such that $2 \le i, j \le m$, $i \ne j$ and $S_i(\pi, \lambda_{nk}) = S_j(\pi, \lambda_{nk}) = 0$.*

It is worth mentioning the paper by Yurko [29] in which uniqueness was studied for the following partial inverse problem.

Problem 10 ([29])**.** *Suppose that $\{q_j\}_{j=2}^m$ are known a priori and q_1 is known on the subinterval (b, π), where $b < \pi$. Given part of the spectrum Λ of the problem* (22)–(24)*, find q_1 on $(0, b)$.*

The investigation of Problem 9 was continued by Bondarenko [33]. In [33], a constructive algorithm for solution was developed, and the local solvability and stability were proved. In order to formulate the results of [33], we needed the following precise eigenvalue asymptotics:

Theorem 15 ([76])**.** *The eigenvalues $\{\lambda_{nk}\}_{n \ge 1, k = \overline{1,m}}$ (counting with multiplicities) of the boundary value problem L with real-valued potentials $q_j \in L_2(0, \pi)$, $j = \overline{1, m}$, can be numbered so that*

$$\sqrt{\lambda_{n1}} = n - \frac{1}{2} + \frac{\hat{\omega}}{\pi n} + \frac{\varkappa_{n1}}{n}, \tag{28}$$

$$\sqrt{\lambda_{nk}} = n + \frac{z_{k-1}}{\pi n} + \frac{\varkappa_{nk}}{n}, \quad k = \overline{2, m}, \tag{29}$$

where $\{\varkappa_{nk}\}_{n \in \mathbb{N}} \in l_2$, $k = \overline{1, m}$, $\hat{\omega} = \frac{1}{m} \sum\limits_{j=1}^m \omega_j$, $\omega_j = \frac{1}{2} \int\limits_0^\pi q_j(x)\, dx$ and $\{z_k\}_{k=1}^{m-1}$ are the roots of the characteristic polynomial

$$P(z) = \frac{d}{dz} \prod_{k=1}^m (z - \omega_k).$$

In [33], the following theorem regarding the local solvability and stability of Problem 9 was proved.

Theorem 16 ([33])**.** *Suppose that the boundary value problem L of the forms* (22) *and* (24) *with potentials $q_j \in L_2(0, \pi)$, $j = \overline{1, m}$ and its subspectrum $\{\lambda_{nk}\}_{n \ge 1, k = \overline{1, m}}$ satisfy the following assumptions:*

1. *All the eigenvalues* $\{\lambda_{nk}\}_{n\in\mathbb{N}, k=1,2}$ *are distinct;*
2. $\lambda_{nk} > 0$, $n \in \mathbb{N}$, $k = 1, 2$;
3. $S_j(\pi, \lambda_{nk}) \neq 0$, $j = \overline{1, m}$, $n \in \mathbb{N}$, $k = 1, 2$;
4. $z_1 \neq \omega_j$, $j = \overline{1, m}$;
5. $S_1(\pi, 0) \neq 0$, $S_1'(\pi, 0) \neq 0$.

Then, there exists $\varepsilon_0 > 0$ *such that, for arbitrary real numbers* $\{\tilde{\lambda}_{nk}\}_{n\in\mathbb{N}, k=1,2}$ *satisfying*

$$\left(\sum_{n=1}^{\infty} \sum_{k=1,2} \left(n \left(\lambda_{nk}^{1/2} - \tilde{\lambda}_{nk}^{1/2} \right) \right)^2 \right)^{1/2} < \varepsilon, \quad \varepsilon \leq \varepsilon_0,$$

there exists a unique real function $q_1 \in L_2(0, \pi)$, *which is the solution of Problem 9 for* $\{\tilde{\lambda}_{nk}\}_{n\in\mathbb{N}, k=1,2}$ *and* q_j, $j = \overline{2, m}$. *Moreover, the following estimate holds:*

$$\|q_1 - \tilde{q}_1\|_{L_2(0,\pi)} < C\varepsilon,$$

where the constant C *depends only on* L *and* ε_0.

Let us show that Theorem 16 implies the minimality of the spectral data of Problem 9. Suppose that problem L and the subspectrum $\Omega = \{\lambda_{nk}\}_{n\geq 1, k=1,2}$ satisfy the hypothesis of Theorem 16 and exclude one eigenvalue: $\Omega^- := \Omega \setminus \{\lambda_{11}\}$. Then, the subspectrum Ω^- does not uniquely specify q_1 if $\{q_j\}_{j=2}^m$ are fixed. Indeed, for any real number $\tilde{\lambda}_{11}$ sufficiently close to λ_{11}, Problem 9 with the data $\Omega^- \cup \{\tilde{\lambda}_{11}\}$ has a solution $\tilde{q}_1 \neq q_1$. Thus, there are two potentials, q_1 and $\tilde{q}_1$, corresponding to the subspectrum Ω^-.

In [34], the boundary value problem (22)–(23) on a star-shaped graph G was considered with complex-valued potentials $q_j \in L_2(0, \pi)$ and conditions of different types (Robin and Dirichlet) in the boundary vertices:

$$y_j'(0) - h_j y_j(0) = 0, \; j = \overline{1, p}, \quad y_j(0) = 0, \; j = \overline{p+1, m}, \tag{30}$$

where $1 \leq p < m$ and $\{h_j\}_{j=1}^p$ are complex constants.

In [34], the following asymptotic formulas were obtained for the eigenvalues $\{\lambda_{nk}\}_{n\in\mathbb{N}, k=\overline{1,m}}$ of the boundary value problem (22), (23), and (30):

$$\begin{aligned}
\sqrt{\lambda_{n1}} &= n - 1 + \frac{\alpha}{\pi} + \frac{\sigma}{\pi n} + \frac{\varkappa_{n1}}{n}, \quad \{\varkappa_{n1}\} \in l_2, \\
\sqrt{\lambda_{n2}} &= n - \frac{\alpha}{\pi} + \frac{\sigma}{\pi n} + \frac{\varkappa_{n2}}{n}, \quad \{\varkappa_{n2}\} \in l_2, \\
\sqrt{\lambda_{nk}} &= n - \frac{1}{2} + \frac{z_k}{\pi n} + \frac{\varkappa_{nk}}{n}, \quad k \in \mathcal{I}_3, \quad \varkappa_{n3} = o(1), \\
\sqrt{\lambda_{nk}} &= n + \frac{t_k}{\pi n} + \frac{\varkappa_{nk}}{n}, \quad k \in \mathcal{I}_4, \quad \varkappa_{n4} = o(1),
\end{aligned}$$

where α, σ, z_k, and t_k are certain constants, and $\mathcal{I}_3$ and $\mathcal{I}_4$ are fixed sets of indices such that $\mathcal{I}_3 \cup \mathcal{I}_4 = \overline{3, m}$, $\mathcal{I}_3 \cap \mathcal{I}_4 = \varnothing$, $|\mathcal{I}_3| = p - 1$, and $|\mathcal{I}_4| = m - p - 1$. To be precise, we assumed that $3 \in \mathcal{I}_3$ and $4 \in \mathcal{I}_4$ if these sets are nonempty.

The author of [34] was concerned with the following partial inverse problem for all possible cases depending on p and $1 \leq k_1 < k_2 \leq 4$:

Problem 11 ([34]). *Let the potentials* q_j, $j = \overline{2, m}$, *the coefficients* h_j, $j = \overline{2, p}$, *and the sequence* $\{\lambda_{nk}\}_{n\in\mathbb{N}, k\in\{k_1,k_2\}}$ *of the eigenvalues of* L *be given. Find the potential* q_1 *and the coefficient* h_1.

The results of [34] included:

- Eigenvalue asymptotics;
- Uniqueness;
- A constructive solution.

The proof technique of [34] was derived from the study of the Riesz basis property for the sequences $\{\sin(n+\beta)t\}_{n\ge1}$ and $\{1\}\cup\{\cos(n+\beta)t\}_{n\ge1}$ [77].

In [35], the Sturm-Liouville problem $\mathcal{L}$ with singular potentials was considered on a star-shaped graph with different edge lengths $\{d_j\}_{j=1}^m$:

$$\ell_j y_j = -(y_j^{[1]})' - \sigma_j(x_j)y_j^{[1]} - \sigma_j^2(x_j)y_j, \quad x_j \in (0, d_j), \quad \sigma_j \in L_2(0, d_j), \quad j = \overline{1, m}, \tag{31}$$

with standard matching conditions

$$y_1(d_1) = y_j(d_j), \quad j = \overline{2, m}, \quad \sum_{j=1}^m y_j^{[1]}(d_j) = 0,$$

and Dirichlet boundary conditions

$$y_j(0) = 0, \quad j = \overline{1, m}.$$

Fix an integer p, $1 \le p < m$. Let $\{\lambda_n\}_{n\in T}$, $T \subseteq \mathbb{N}$ be some subset of the spectrum.

Problem 12 ([35]). *Given the potentials $\{\sigma_j\}_{j=p+1}^m$, the subspectrum $\{\lambda_n\}_{n\in T}$, and the sequence $\{\omega_k\}_{k\ge1}$, find $\{\sigma_j\}_{j=1}^p$.*

The numbers $\{\omega_k\}_{k\ge1}$ are defined as follows. For $j = \overline{1, m}$, let $S_j(x_j, \lambda)$ be the solution of Equation (31) on the edge e_j satisfying the initial conditions $S_j(0, \lambda) = 0$, $S_j^{[1]}(0, \lambda) = 1$, and let $\{\lambda_{nj}\}_{n\ge1}$ be the zeros of $S_j(d_j, \lambda)$. Since the function σ_j is real-valued, then the zeros of $\{\lambda_{nj}\}_{n\ge1}$ are real and distinct as the eigenvalues of the corresponding operator.

Assume that the functions $S_j(d_j, \lambda)$ and $j = \overline{1, p}$ do not have any common zeros. Let $\{\mu_k\}_{k\ge1}$ denote the union $\bigcup_{j=1}^p \{\lambda_{nj}\}_{n\ge1}$ by arranging the numbers in increasing order: $\mu_k < \mu_{k+1}$, $k \in \mathbb{N}$. In view of our assumption, for every $k \in \mathbb{N}$, there exists exactly one index $j =: \omega_k \in \{1, \dots, p\}$, such that $\mu_k \in \{\lambda_{nj}\}_{n\ge1}$. The sequence $\{\omega_k\}_{k\ge1}$ is used as additional data for the partial inverse problem.

Using a subspectrum $\{\lambda_n\}_{n\in T}$, it is possible to recover only the sum of the Weyl functions $\sum_{j=1}^p M_j(\lambda)$ for separate edges $\{e_j\}_{j=1}^p$. In order to find $M_j(\lambda)$ separately, one also needs $\{\omega_k\}_{k\ge1}$.

Impose the assumptions:

(A_1) $S_j(d_j, \lambda_n) \ne 0$, $j = \overline{1, m}$, $n \in T$.

(A_2) The functions $S_j(d_j, \lambda)$ and $j = \overline{1, p}$ do not have any common zeros.

(A_3) $\lambda_n \ne \lambda_k$, $n \ne k$, $n, k \in T$.

(A_4) $\lambda_n > 0$, $n \in T$.

(A_5) $\lambda_{nj} > 0$, $n \in \mathbb{N}$, $j = \overline{1, p}$.

In [35], three approaches to uniqueness for the solution of Problem 12 were suggested. The first approach was based on the estimate of the infinite product

$$\Delta_T(\lambda) := \prod_{n\in T}\left(1 - \frac{\lambda}{\lambda_n}\right).$$

Theorem 17 ([35]). *Suppose that $\sigma_j = \tilde{\sigma}_j$ in $L_2(0, d_j)$ for $j = \overline{p+1, m}$, $\{\lambda_n\}_{n\in T} = \{\tilde{\lambda}_n\}_{n\in\tilde{T}}$, and $\omega_k = \tilde{\omega}_k$, $k \ge 1$. Moreover, let the assumptions (A_1)–(A_5) hold for both problems $\mathcal{L}$ and $\tilde{\mathcal{L}}$ and the corresponding subspectra, and let the estimate*

$$|\Delta_T(\lambda)| \ge C|\lambda|^{(1-2p)/2}\exp(2l|Im\sqrt{\lambda}|), \quad |\lambda| \ge \lambda^*, \quad \arg\lambda = \varphi, \tag{32}$$

be valid, where $\varphi \in (0, 2\pi)$ *and* $\lambda^* > 0$ *are fixed numbers,* $l := \sum_{j=1}^{p} d_j$. *Then,* $\sigma_j = \tilde{\sigma}_j$ *in* $L_2(0, d_j)$ *for* $j = \overline{1, p}$.

The second approach relied on the ideas of Gesztesy and Simon [10] and generalized Theorem 3.

Theorem 18 ([35])**.** *Suppose that* $\sigma_j = \tilde{\sigma}_j$ *in* $L_2(0, d_j)$ *for* $j = \overline{p+1, m}$, $\{\lambda_n\}_{n \in T} = \{\tilde{\lambda}_n\}_{n \in \tilde{T}}$, *and* $\omega_k = \tilde{\omega}_k$, $k \in \mathbb{N}$. *Moreover, let the assumptions* (A_1)–(A_5) *hold for both problems* $\mathcal{L}$ *and* $\tilde{\mathcal{L}}$ *and the corresponding subspectra, and for all sufficiently large* $t > 0$, *we have*

$$\#\{n \in T \colon \lambda_n < t\} \geq \alpha \#\{n \in \mathbb{N} \colon \lambda_n < t\} + \beta,$$

where

$$L = \sum_{j=1}^{m} d_j, \quad \alpha = \frac{2l}{L}, \quad \beta = \frac{1}{2}(\alpha(m-1) - 2p + 1).$$

Then, $\sigma_j = \tilde{\sigma}_j$ *in* $L_2(0, d_j)$ *for* $j = \overline{1, p}$.

The third approach of [35] was based on the construction of a special sequence of vector functions in the Hilbert space $L_2(0, l) \oplus L_2(0, l)$. The completeness of this sequence implies the uniqueness of the partial inverse problem solution.

3.2. Simple Graphs with Loops

The study of partial inverse problems on graphs with loops began with [31,32] for graph G presented in Figure 6. Graph G has the vertices $\{v_j\}_{j=0}^{r}$ and the edges $\{e_j\}_{j=1}^{r+r_1}$, where $e_j = [v_j, v_0]$ for $j = \overline{1, r}$ are boundary edges and e_j for $j = \overline{r+1, r_1}$ are loops.

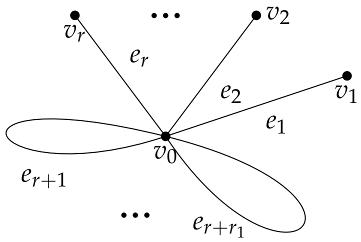

Figure 6. Graph with loops.

The Sturm-Liouville problem on G is given by the equations

$$-y_j'' + q_j(x) y_j = \lambda y_j, \quad x \in (0, 1), \quad j = \overline{1, r + r_1},$$

subject to the matching conditions in the internal vertex v_0:

$$y_j(1) = y_i(0), \quad j = \overline{1, r + r_1}, \quad i = \overline{r+1, r_1 + 1},$$
$$\sum_{j=1}^{r+r_1} y_j'(1) = \sum_{i=r+1}^{r+r_1} y_i'(0),$$

and the boundary conditions in the vertices v_j, $j = \overline{1, r}$. In [31], the Robin boundary conditions $y_j'(0) - h_j y_j(0) = 0$, $j = \overline{1, r}$ were considered and, in [32], the Dirichlet boundary conditions $y_j(0) = 0$, $j = \overline{1, r}$. The potentials $\{q_j\}_{j=1}^{r+r_1}$ were assumed to be real-valued and integrable.

In [31,32], the uniqueness theorems for the solution of the following partial inverse problem were proved:

Problem 13 ([31,32])**.** *Suppose that the potentials* $\{q_j\}_{j=2}^{r+r_1}$ *on* $(0,1)$ *and the potential* q_1 *on the subinterval* $(b,1)$*,* $b<1$ *are known a priori. Given a subspectrum, find* q_1 *on* $(0,b)$*.*

In [31], the constants $\{h_j\}_{j=2}^{r}$ of the boundary conditions were assumed to be known, and h_1 had to be recovered together with q_1 on $(0,b)$. Problem 13 was studied under a separation condition and the completeness condition of a cosine system. These conditions guaranteed the uniqueness of the solution.

Yang and Bondarenko [37] investigated a partial inverse problem on a lasso graph (see Figure 7).

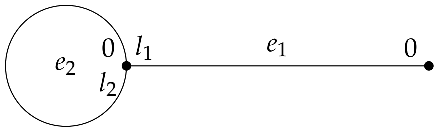

Figure 7. Lasso graph.

In [37], the following Sturm-Liouville problem on a lasso graph with singular potentials $q_j = W_2^{-1}(0,l_j)$, $q_j=\sigma_j'$, $j=1,2$ was considered:

$$\ell_j y_j = -(y_j^{[1]})' - \sigma_j(x_j)y_j^{[1]} - \sigma_j^2(x_j)y_j = \lambda y_j, \quad \sigma_j \in L_2(0,l_j), \quad j=1,2,$$
$$y_1(0)=0, \quad y_1(l_1)=y_2(0)=y_2(l_2), \quad y_1^{[1]}(l_1) - y_2^{[1]}(0) + y_2^{[1]}(l_1) = 0,$$

where $y_j^{[1]} = y_j' - \sigma_j y_j$, $j=1,2$, $l_1 = m \in \mathbb{N}$, $l_2 = 1$.

Problem 14 ([37])**.** *Given the function* σ_1*, the subspectrum* Λ*, and the signs* Ω*, find the function* σ_2*.*

The signs Ω in the problem statement are defined as follows. Let $S_2(x,\lambda)$ and $C_2(x,\lambda)$ be the solutions of equation $\ell_2 y_2 = \lambda y_2$ under the initial conditions $S_2(0,\lambda) = C_2^{[1]}(0,\lambda) = 0$, $S_2^{[1]}(0,\lambda) = C_2(0,\lambda) = 1$. Define

$$h(\lambda) := S_2(1,\lambda), \quad H(\lambda) := C_2(1,\lambda) - S_2^{[1]}(1,\lambda), \quad d(\lambda) := C_2(1,\lambda) + S_2^{[1]}(1,\lambda) - 2.$$

The zeros $\{\nu_n\}_{n\ge1}$ of $h(\lambda)$ are the eigenvalues of the Dirichlet boundary value problem:

$$\ell_2 y_2 = \lambda y_2, \quad y_2(0) = y_2(1) = 0.$$

The zeros $\{\mu_n\}_{n\in\mathbb{Z}}$ of $d(\lambda)$ are the eigenvalues of the periodic problem:

$$\ell_2 y_2 = \lambda y_2, \quad y_2(0) = y_2(1), \quad y_2^{[1]}(0) = y_2^{[1]}(1).$$

Let $\omega_n := \operatorname{sign} H(\nu_n)$ and $\Omega := \{\omega_n\}_{n\ge1}$. The partial inverse problem on the lasso graph (Problem 14) is reduced to the following periodic inverse problem on a finite interval:

Problem 15 ([37])**.** *Given the sequences* $\{\nu_n\}_{n\ge1}$ *and* $\{\mu_n\}_{n\in\mathbb{Z}}$ *and the sequence of signs* Ω*, construct the function* σ_2*.*

In [37], the solution of Problem 15 was obtained for the case of singular potentials $q \in W_2^{-1}(0,1)$. Thus, [37] contained the following results for the partial inverse problem:

- Eigenvalue asymptotics;
- Uniqueness;
- Algorithm;
- The solution of the inverse periodic problem with a singular potential.

Bondarenko and Shieh [38] studied partial inverse problems for a quadratic differential pencil on the graph with a cycle presented in Figure 8.

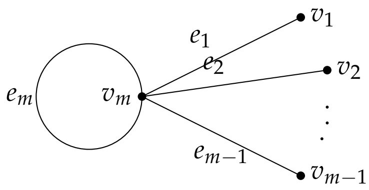

Figure 8. Graph with a cycle.

In [38], the following boundary value problem with nonlinear dependence on the spectral parameter λ was considered:

$$-y_j'' + q_j(x_j)y_j + 2\lambda p_j(x_j)y_j = \lambda^2 y_j, \quad x_j \in (0,\pi), \quad j = \overline{1,m},$$
$$y_j(0) = 0, \quad j = \overline{1,m-1},$$
$$y_m(0) = y_j(\pi), \quad j = \overline{1,m}, \qquad y_m'(0) = \sum_{j=1}^{m} y_j'(\pi),$$

where $p_j \in AC[0,\pi]$ and $q_j \in L_1(0,\pi)$, $j = \overline{1,m}$, are complex-valued functions. The following two partial inverse problems were studied.

Problem 16 ([38])**.** *Given the functions $\{p_j\}_{j=2}^m$ and $\{q_j\}_{j=2}^m$ and a subspectrum Λ, find p_1 and q_1.*

Problem 17 ([38])**.** *Let the functions $\{p_j\}_{j=1}^{m-1}$ and $\{q_j\}_{j=1}^{m-1}$, a subspectrum Λ, and the sequence of signs Ω be given. Find p_m and q_m.*

In Problem 17, Ω is the sequence of signs for the auxiliary periodic problem (see [38] for details). The results of [38] included:

- Eigenvalue asymptotics;
- Uniqueness;
- A constructive solution.

In [38], methods of working with multiple eigenvalues and vector-functional sequences containing exponents were developed. An important role in the proofs was played by the Riesz basicity of exponential systems $\{\exp(i\lambda_k t)\}$ in $L_2(-\pi,\pi)$. Later on, these methods were generalized to graphs of an arbitrary structure (see [40]).

3.3. Graphs of a General Structure

The analysis of partial inverse problems on star-shaped graphs and simple graphs with loops showed that such problems present specific features for each case. Therefore, it is difficult to obtain results for graphs of a general structure. Until now, only the following cases have been studied:

- The reconstruction of the potentials on an arbitrary tree graph (graph without cycles) by several spectra, while the potential on one edge is known a priori (see [78]).
- The reconstruction of the potential on one boundary edge of an arbitrary graph using part of the spectrum, while the potentials on all other edges are known a priori (see [40,41]).
- For a tree graph, the reconstruction of the potentials on a connected subtree from parts of several spectra, while the potentials on the remaining edges are known a priori (see [39]).

In this section, we briefly describe these results.

Proceeding to the statement of the Sturm-Liouville problem on a graph of a general structure, let $\mathcal{G}$ be a graph with a set of vertices $\mathcal{V}$ and edges $\{e_j\}_{j=1}^m$ with the corresponding lengths $\{T_j\}_{j=1}^m$. The graph may contain cycles, loops, and multiple edges. For each edge e_j, $j = \overline{1,m}$, introduce the parameter $x_j \in [0, T_j]$. Let us denote the ends of e_j as w_{2j-1} and w_{2j} so that $x_j = 0$ corresponds to w_{2j-1} and $x_j = T_j$ to w_{2j}. Every vertex v of the graph $\mathcal{G}$ can be considered as the equivalence class of all the ends w_j incident to this vertex: $v = \{w_{j_1}, w_{j_2}, \dots, w_{j_r}\}$. The number of elements in this class is called *the degree* of the vertex. We assume that the graph $\mathcal{G}$ does not have vertices of degree 2. Otherwise, the two edges incident to such vertices could be merged into one edge. The vertices of degree 1 are called *the boundary vertices*, and the others are called *the internal vertices*. An edge incident to a boundary vertex is called *a boundary edge*. Let $\partial\mathcal{G}$ and $\operatorname{int}\mathcal{G}$ denote the sets of the boundary vertices and the internal vertices, respectively, $\mathcal{V} = \partial\mathcal{G} \cup \operatorname{int}\mathcal{G}$.

A *function* on the graph $\mathcal{G}$ is a vector function $y = [y_j]_{j=1}^m$ with components $y_j = y_j(x_j)$, $x_j \in [0, T_j]$. A function y belongs to a class $\mathscr{A}(\mathcal{G})$ if $y_j \in \mathscr{A}[0, T_j]$ for $j = \overline{1,m}$, where $\mathscr{A} = L_1$, AC, etc. In order to define matching and boundary conditions, one needs the following notations:

$$\begin{array}{ll} y_{|w_{2j-1}} = y_j(0), & y_{|w_{2j}} = y_j(T_j), \\ y'_{|w_{2j-1}} = -y'_j(0), & y'_{|w_{2j}} = y'_j(T_j), \end{array} \qquad j = \overline{1,m}.$$

For $v \in \partial\mathcal{G}$, we write $y(v)$ and $y'(v)$ for $y_{|w_k}$ and $y'_{|w_k}$, respectively, where $w_k \in v$.

Bondarenko and Shieh [78] considered the Sturm-Liouville equations

$$-y_j'' + q_j(x_j) y_j = \lambda y_j, \quad x_j \in (0, T_j), \quad j = \overline{1,m}, \tag{33}$$

on a tree graph $\mathcal{G}$ (a graph without cycles) with the potential $q = [q_j]_{j=1}^m \in L_1(\mathcal{G})$ and the standard matching conditions

$$\left.\begin{array}{lll} y_{|w_j} = y_{|w_k}, & w_j, w_k \in v & \text{(continuity conditions)} \\ \sum\limits_{w_j \in v} y'_{|w_j} = 0 & & \text{(Kirchhoff's condition)} \end{array}\right\} \quad v \in \operatorname{int}\mathcal{G}. \tag{34}$$

Let L_0 and L_k, $v_k \in \partial\mathcal{G}$, be the boundary value problems for the system (33) with the matching conditions (34) and the following conditions in the boundary vertices:

$$\begin{array}{l} L_0\colon y(v_i) = 0, \quad v_i \in \partial\mathcal{G}, \\ L_k\colon y'(v_k) = 0, \quad y(v_i) = 0, \quad v_i \in \partial\mathcal{G} \backslash \{v_k\}. \end{array}$$

The problems L_k have discrete spectra, which are the countable sets of eigenvalues $\Lambda_k = \{\lambda_{ks}\}_{s \ge 1}$, $k = 0$ or $v_k \in \partial\mathcal{G}$.

Fix a vertex $v_r \in \partial\mathcal{G}$ as *a root* of the tree $\mathcal{G}$. Let e_r be the edge incident to v_r. Then, the uniqueness theorem for the *complete* inverse problem on the tree is formulated as follows:

Theorem 19 ([75])**.** *The spectra Λ_0 and Λ_k, $k \in \partial\mathcal{G} \setminus \{v_r\}$, uniquely determine the potential q on the whole tree $\mathcal{G}$.*

Thus, if the number of boundary vertices is b, then b spectra are required for the recovery of the potentials. Bondarenko and Shieh [78] assumed that the potential q_f is known a priori on one edge e_f and proved that the remaining potentials can be uniquely recovered from $(b-1)$ spectra. If an internal edge e_f is removed, then the tree $\mathcal{G}$ splits into two parts. Let us denote them by P_1 and P_2 and their sets of boundary vertices by ∂P_1 and ∂P_2, respectively.

Theorem 20 ([78])**.** *Let the potential q_f on an edge e_f ($f \neq r$) be known.*

1. *If e_f is a boundary edge, the spectra Λ_0 and Λ_k, $v_k \in \partial\mathcal{G} \setminus \{v_f, v_r\}$, uniquely determine the potential q on the whole graph $\mathcal{G}$.*
2. *If e_f is an internal edge, the spectra Λ_0 and Λ_k $v_k \in \partial\mathcal{G} \setminus \{v_{r1}, v_{r2}\}$, where $v_{r1} \in \partial P_1$ and $v_{r2} \in \partial P_2$, uniquely determine the potential q on the whole graph $\mathcal{G}$.*

Theorem 20 was proved by a constructive method, developing from the ideas of [75]. The case of the internal edge e_f is the most difficult. It is crucial that the two end vertices of the internal edge e_f have degrees of at least 3. Consequently, Theorem 20 cannot be applied to an interval with the potential given on a middle subinterval.

Bondarenko [39] investigated another type of partial inverse problem on a tree $\mathcal{G}$. The edge lengths T_j in [39] were assumed to be equal π. The Sturm-Liouville equation presented in Equation (33) was considered with the singular potentials $q_j \in W_2^{-1}(0, \pi)$, $j = \overline{1, m}$. Therefore, Equation (33) was represented in the form

$$-(y_j^{[1]})' - \sigma_j(x_j)y_j^{[1]} - \sigma_j^2(x_j)y_j = \lambda y_j, \quad x \in (0, T_j), \quad j = \overline{1, m}, \tag{35}$$

where $q_j = \sigma_j'$, $\sigma_j \in L_2(0, T_j)$, $y_j^{[1]} = y_j' - \sigma_j y_j$, $j = \overline{1, m}$.

Let $\{\gamma_j\}_{j=1}^m$ be some real constants. In order to define the matching conditions, one uses the following notations:

$$\begin{array}{ll} y_{|w_{2j-1}} = y_j(0), & y_{|w_{2j}} = y_j(T_j), \\ y^{[1]}_{|w_{2j-1}} = -y_j^{[1]}(0), & y^{[1]}_{|w_{2j}} = y_j^{[1]}(T_j) + \gamma_j y_j(T_j), \end{array} \quad j = \overline{1, m}. \tag{36}$$

For $v \in \partial\mathcal{G}$, $y(v)$ and $y^{[1]}(v)$ are written for $y_{|w_k}$ and $y^{[1]}_{|w_k}$, respectively, where $w_k \in v$. Let us divide the set of the boundary vertices into two disjoint subsets:

$$\partial\mathcal{G} = \mathcal{V}^D \cup \mathcal{V}^N, \quad \mathcal{V}^D \cap \mathcal{V}^N = \varnothing.$$

Thus, in [39], the boundary value problem L for the Sturm-Liouville equation presented in Equation (35) was considered subject to the matching conditions

$$\left.\begin{array}{r} y_{|w_j} = y_{|w_k}, \quad w_j, w_k \in v \\ \sum\limits_{w_j \in v} y^{[1]}_{|w_j} = 0 \end{array}\right\} \quad v \in \operatorname{int}\mathcal{G} \tag{37}$$

and the boundary conditions

$$y(v) = 0, \quad v \in \mathcal{V}^D, \qquad y^{[1]}(v) = 0, \quad v \in \mathcal{V}^N. \tag{38}$$

Let the tree $\mathcal{G}$ be divided into two subtrees $\mathcal{G}_{known}$ and $\mathcal{G}_{unknown}$ with a common vertex $w \in \operatorname{int}\mathcal{G}$ (see Figure 9). Let E_{known} and $E_{unknown}$ denote the edge sets of $\mathcal{G}_{known}$ and $\mathcal{G}_{unknown}$, respectively. Let $\{v_k\}_{k=1}^b$ denote the boundary vertices $\partial G_{unknown} \setminus \{w\}$. For each $k = \overline{1, b}$, let L_k denote the boundary value problem (35), (37) with the boundary conditions (38) for $v \in \partial\mathcal{G} \setminus \{v_k\}$ and $y(v_k) = 0$ if $v_k \in \mathcal{V}^N$, or $y^{[1]}(v_k) = 0$ if $v_k \in \mathcal{V}^D$. In other words, if the problem L has the Dirichlet boundary condition in v_k, then L_k has the Neumann boundary condition, and vice versa.

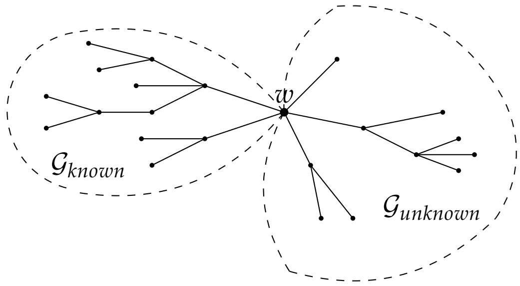

Figure 9. Tree graph.

Problem 18 ([39]). *Suppose that the functions σ_j on the edges $e_j \in E_{known}$ and the constants $\{\gamma_j\}_{j=1}^m$ are known a priori. Given some subspectra of the problems L and L_k for $k = \overline{1, b-1}$, find σ_j for all $e_j \in E_{unknown}$.*

Note that, in view of Theorem 19, the full spectra of L and L_k for $k = \overline{1, b-1}$ determine the potentials on the whole tree $\mathcal{G}$. Strictly speaking, Yurko [75] proved Theorem 19 for regular potentials $q_j \in L_2(0, T_j)$. For singular potentials $q_j \in W_2^{-1}(0, T_j)$, similar results were obtained by Vasiliev [79].

In [39], it was shown that, if the potentials σ_j are known on E_{known}, then only part of the spectra can be used for reconstruction. Sufficient conditions for the uniqueness were formulated in terms of completeness for some special vector functional sequences, which were constructed using the known functions σ_j and the given subspectra. Furthermore, the uniqueness conditions in terms of the eigenvalue asymptotics were obtained. In addition, in [39], a constructive algorithm for solving Problem 18 was developed. This algorithm allows one to reduce the partial inverse problem to a complete inverse problem for the "unknown" subtree.

Proceeding to general graphs containing cycles, for such graphs, partial inverse problems were investigated only for case in which a potential is unknown on one edge. Let $\mathcal{G}$ be a graph of an arbitrary structure with arbitrary edge lengths $\{T_j\}_{j=1}^m$. In [40], Sturm-Liouville differential equations with quadratic dependence on the spectral parameter λ were considered on the graph $\mathcal{G}$:

$$-y_j''(x_j) + (q_j(x_j) + 2\lambda p_j(x_j) - \lambda^2) y_j(x_j) = \lambda y_j(x_j), \quad x_j \in (0, T_j), \quad j = \overline{1, m}, \tag{39}$$

where $y = [y_j]_{j=1}^m$, $p = [p_j]_{j=1}^m$, and $q = [q_j]_{j=1}^m$ are complex-valued functions on $\mathcal{G}$, $y \in W_2^2(\mathcal{G})$, $p \in AC(\mathcal{G})$, $q \in L_1(\mathcal{G})$.

Let γ_{jk} be some complex numbers, defined for the ends $w_j \in v$, $v \in \operatorname{int} \mathcal{G}$, $k = \overline{1, 4}$, $\gamma_{jk} \neq 0$ for $k = 1, 2$. Define the linear forms

$$U_j(y) := y'_{|w_j} + (\lambda \gamma_{j3} + \gamma_{j4}) y_{|w_j}.$$

Thus, in [40], the differential pencil L given by Equation (39) subject to the following conditions was considered:

$$\gamma_{j1} y_{|w_j} = \gamma_{k1} y_{|w_k}, \quad w_j, w_k \in v, \quad v \in \operatorname{int} \mathcal{G},$$
$$\sum_{w_j \in v} \gamma_{j2} U_j(y) = 0, \quad v \in \operatorname{int} \mathcal{G},$$
$$y_{|w_j} = 0, \quad w_j \in v, \quad v \in \partial \mathcal{G}.$$

For certainty, we assume that e_1 is a boundary edge.

Problem 19 ([40]). *Suppose that $\{p_j\}_{j=2}^m$, $\{q_j\}_{j=2}^m$, and $\{\gamma_{jk}\}$ are known a priori. Given a subspectrum Λ' of the pencil L, find p_1 and q_1 (see Figure 10).*

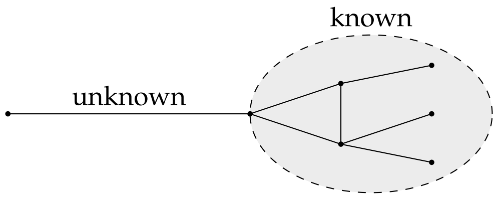

Figure 10. Graph of an arbitrary structure.

In [40], Problem 19 was studied under a separation condition, which generalized Condition 1 for a star-shaped graph. For a general graph, the separation condition had a complicated technical formulation, so we omit it here. The results of [40] for Problem 19 included:

- Uniqueness in the general case;
- A constructive solution for rationally dependent edge lengths.

In particular, the recovery of the coefficients p_1 and q_1 from the whole spectrum Λ of the pencil L was investigated. The characteristic function of L satisfies the following asymptotic relation:

$$\Delta(\lambda) = \lambda^r(\Delta_0(\lambda) + O\Big(|\lambda|^{-1}\exp(M|\mathrm{Im}\,\lambda|)\Big)), \quad |\lambda| \to \infty,$$

where $r \in \mathbb{Z}$, $M = \sum\limits_{j=1}^{m} T_j$, and $\Delta_0(\lambda)$ is a polynomial of $\cos(\lambda T_j)$ and $\sin(\lambda T_j)$, $j = \overline{1,m}$, $\Delta_0(\lambda) = O(\exp(M|\mathrm{Im}\,\lambda|))$. In [40], the following regularity condition was imposed:

$$|\Delta_0(i\tau)| \ge C\exp(M|\tau|), \quad \tau \in \mathbb{R}, \quad |\tau| \ge \tau^*, \tag{40}$$

for some $\tau^* > 0$. Under the conditions in (40) and the separation condition, the spectrum Λ uniquely specifies p_1 and q_1 if $T_1 < M/2$ or $T_1 = M/2, r = -2$ (see Theorem 2 in [40] for details). In other words, the length of the "unknown" edge has to be less than or equal to the total length of the graph for the unique determination of the pencil coefficients on this edge by the spectrum.

In the case of rationally dependent edge lengths $T_j = \pi n_j$, $n_j \in \mathbb{N}$, $j = \overline{1,m}$, the spectrum Λ of the regular pencil L contains subsequences of eigenvalues $\{\lambda_{nk}\}_{n\in\mathbb{Z}}$, $k = \overline{1,s}$, satisfying the asymptotic relation

$$\lambda_{nk} = 2n + \beta_k + o(1), \quad |n| \to \infty.$$

It was shown in [40] that one can choose a certain number of such subsequences to uniquely recover p_1 and q_1. The constructive method of [40] developed the ideas of [33,38] and other papers. This method was based on the reduction of the partial inverse problem (Problem 19) to a complete inverse problem for the Sturm-Liouville quadratic pencil on the interval $(0, T_1)$.

The most complete results for a partial inverse problem on an arbitrary graph were obtained in [41] for the boundary value problem given by (35), (37), and (38):

$$\left.\begin{array}{c} -(y_j^{[1]})' - \sigma_j(x_j)y_j^{[1]} - \sigma_j^2(x_j)y_j = \lambda y_j, \quad x \in (0, T_j), \quad j = \overline{1,m}, \\ y_{|w_j} = y_{|w_k}, \quad w_j, w_k \in v, \quad \sum\limits_{w_j \in v} y^{[1]}_{|w_j} = 0, \quad v \in \mathrm{int}\,\mathcal{G}, \\ y(v) = 0, \quad v \in \mathcal{V}^D, \qquad y^{[1]}(v) = 0, \quad v \in \mathcal{V}^N, \end{array}\right\} \tag{41}$$

where $y_{|w_k}$ are defined by (36), with $\gamma_j = 0$. The edge lengths were assumed to be rationally dependent: $T_j = 2\pi n_j$, $n_j \in \mathbb{N}$, $j = \overline{1,m}$.

For certainty, assume that v_1 is a boundary vertex corresponding to the end $w_1 \sim x_1 = 0$ of the edge e_1 and $v_1 \in \mathcal{V}^D$.

Problem 20 ([41]). *Suppose that the functions $\{\sigma_j\}_{j=2}^m$ are known a priori. Given a subspectrum Λ, find σ_1 (see Figure 10).*

In [41], the following results were obtained for Problem 20:

- A uniqueness theorem;
- A constructive solution;
- Sufficient conditions for global solvability;
- Local solvability and stability.

Let us formulate a uniqueness theorem for Problem 20. For this purpose, one first needs to construct the characteristic function of the Sturm-Liouville problem on an arbitrary graph. For each fixed $j = \overline{1,m}$, introduce the solutions $C_j(x_j,\lambda)$ and $S_j(x_j,\lambda)$ of Equation (35) satisfying the initial conditions

$$C_j(0,\lambda) = S_j^{[1]}(0,\lambda) = 1, \quad C_j^{[1]}(0,\lambda) = S_j(0,\lambda) = 0.$$

Every solution $[y_j]_{j=1}^m$ of system (35) can be represented in the form

$$y_j(x_j,\lambda) = a_j(\lambda)C_j(x_j,\lambda) + b_j(\lambda)S_j(x_j,\lambda), \quad j = \overline{1,m}, \tag{42}$$

with some coefficients $a_j(\lambda)$ and $b_j(\lambda)$ independent of x. Substituting (42) into (37) and (38), one obtains the system of linear equations $\mathscr{S}$ with respect to $a_j(\lambda)$ and $b_j(\lambda)$, $j = \overline{1,m}$. The determinant $\Delta(\lambda)$ of this system is the characteristic function of L, that is, the spectrum of the problem (41) coincides with the zeros of $\Delta(\lambda)$.

The characteristic function can be represented in the form

$$\Delta(\lambda) = S_1(T_1,\lambda)\Delta^K(\lambda) + S_1^{[1]}(T_1,\lambda)\Delta^\Pi(\lambda), \tag{43}$$

where $\Delta^K(\lambda)$ and $\Delta^\Pi(\lambda)$ are the determinants of the linear systems obtained from $\mathscr{S}$ by the replacements $S_1(T_1,\lambda) \mapsto 1$, $S_1^{[1]}(T_1,\lambda) \mapsto 0$ and $S_1(T_1,\lambda) \mapsto 0$, $S_1^{[1]}(T_1,\lambda) \mapsto 1$, respectively. Note that our construction defines the functions $\Delta(\lambda)$, $\Delta^\Pi(\lambda)$, and $\Delta^K(\lambda)$ uniquely up to the sign, which depends on the order of equations and variables in the system $\mathscr{S}$. However, it is possible to fix such signs so that Formula (43) is valid. Clearly, the functions $\Delta(\lambda)$, $\Delta^K(\lambda)$, and $\Delta^\Pi(\lambda)$ are entire, and $\Delta^K(\lambda)$ and $\Delta^\Pi(\lambda)$ do not depend on σ_1.

The separation condition for Problem 20 reads as follows:

Condition 2. *For each $\lambda \in \Lambda$, $\Delta^\Pi(\lambda) \neq 0$ or $\Delta^K(\lambda) \neq 0$.*

Condition 2 is essential, since otherwise, if $\Delta^\Pi(\lambda) = \Delta^K(\lambda) = 0$, then (43) readily implies $\Delta(\lambda) = 0$, but such an eigenvalue λ is related to the "known" part of the graph (see Figure 10) and carries no information on σ_1.

The spectrum of the problem (41) consists of eigenvalue subsequences with the asymptotics

$$\sqrt{\lambda_{nk}} = n + r_k + \varkappa_{nk}, \quad \{\varkappa_{nk}\} \in l_2, \tag{44}$$

where $k = \overline{1,N}$, $N := 2\sum\limits_{j=1}^m n_j$, $n_j = \frac{T_j}{2\pi}$, $n \in \mathbb{N}$ or $n \in N \cup \{0\}$ depending on k, and $\{r_k\}_{k=1}^N \subseteq [0,1)$. Furthermore, for each $r_k \neq 0$, there exists $r_s = 1 - r_k$. The numbers $\{r_k\}_{k=1}^N$ depend on the graph structure and not on $\{\sigma_j\}_{j=1}^m$. The asymptotics (44) are obtained by the reduction of the Sturm-Liouville problem to the matrix form.

Let us impose the following condition on the subspectrum Λ in Problem 20:

Condition 3. $\Lambda = \{\lambda_{nk}\}_{n\geq 0, k\in\mathcal{K}}$, *where λ_{nk} satisfies the asymptotics in (44) and a subset $\mathcal{K} \subseteq \{1,\ldots,N\}$ fulfills the following conditions:*

1. *All the values* $\{r_k\}_{k \in \mathcal{K}}$ *from* (44) *are distinct.*
2. $r_k \notin \{0, \frac{1}{2}\}$, $k \in \mathcal{K}$.
3. *For each* $k \in \mathcal{K}$*, there exists* $s \in \mathcal{K}$ *such that* $r_k + r_s = 1$.
4. $|\mathcal{K}| = 4n_1$.

Note that Λ is non-uniquely determined by $\mathcal{K}$, and any finite number of eigenvalues in Λ can be chosen arbitrarily. In particular, Λ can contain a finite number of multiple eigenvalues. The condition $|\mathcal{K}| = 4n_1$ connects the length of the "unknown" edge $T_1 = 2\pi n_1$ with the "size" of the subspectrum, which is used for the reconstruction. The following theorem asserts the uniqueness of the solution to Problem 20.

Theorem 21 ([41])**.** *Let* Λ *be a subspectrum of the problem in* (41) *satisfying Conditions 2 and 3. If* $\sigma_j = \tilde{\sigma}_j$ *in* $L_2(0, T_j)$ *for* $j = \overline{2, m}$ *and* $\Lambda = \tilde{\Lambda}$ *(with respect to multiplicities), then* $\sigma_1 = \tilde{\sigma}_1$ *in* $L_2(0, T_1)$*.*

The proof of Theorem 21, the constructive solution, and the study of the solvability and stability of the partial inverse problem in [41] were based on the unified approach, which is described in the next section.

4. Unified Approach

In this section, we describe a unified approach to partial inverse problems on intervals and graphs that was developed in [41–44] and subsequent studies. This approach allows one to reduce a partial inverse problem to a complete inverse problem on an "unknown" part of an interval or graph. The central role in the reduction technique is played by a special vector functional sequence $\{v_n\}_{n \ge 0}$ in the Hilbert space $\mathcal{H} = L_2(0, l) \oplus L_2(0, l)$, where l is the length of an "unknown" subinterval. The completeness and the Riesz basis property of this sequence imply uniqueness and a constructive solution to the corresponding partial inverse problem, respectively. The unified approach also allows one to obtain the solvability conditions and stability of partial inverse problems.

The initial ideas behind this approach appeared in [22,33], studies of the inverse transmission eigenvalue problem and a partial inverse Sturm-Liouville problem on a star-shaped graph, respectively. Later on, Bondarenko [42,43] noticed that partial inverse problems for various classes of differential operators can be represented as Sturm-Liouville problems with entire analytic functions in one of the boundary conditions, and an inverse spectral theory for such problems was created. As corollaries of this general theory, both well-known and novel results for the Hochstast-Lieberman problem and its generalizations have been deduced.

In Section 4.1, we provide the inverse spectral theory of the Sturm-Liouville equation with entire functions in a boundary condition, mostly based on the results of [42,43]. In Section 4.2, applications to partial inverse problems are discussed.

4.1. Sturm-Liouville Problem with Entire Functions in a Boundary Condition

Consider the following Sturm-Liouville problem $R(q, f_1, f_2)$:

$$-y''(x) + q(x)y(x) = \lambda y(x), \quad x \in (0, \pi), \tag{45}$$

$$y(0) = 0, \quad f_1(\lambda)y'(\pi) + f_2(\lambda)y(\pi) = 0, \tag{46}$$

where $q(x)$ is a complex-valued potential of $L_2(0, \pi)$, and $f_1(\lambda)$ and $f_2(\lambda)$ are entire analytic functions of the spectral parameter λ.

Let $S(x, \lambda)$ denote the solution of Equation (45) satisfying the initial conditions $S(0, \lambda) = 0$, $S'(0, \lambda) = 1$. The spectrum of $R(q, f_1, f_2)$ consists of the eigenvalues, which coincide with the zeros of the entire characteristic function

$$\Delta(\lambda) := f_1(\lambda)S'(\pi, \lambda) + f_2(\lambda)S(\pi, \lambda). \tag{47}$$

Depending on the functions $f_1(\lambda)$ and $f_2(\lambda)$, the spectrum can be at most countable or coincide with the whole complex plane. If there is no additional information on $f_1(\lambda)$ and $f_2(\lambda)$, then one cannot study specific properties of the spectrum (e.g., eigenvalue asymptotics). However, one can consider the following inverse problem:

Problem 21 ([42,43])**.** *Suppose that the functions $f_1(\lambda)$ and $f_2(\lambda)$ are known a priori. Given a subspectrum $\{\lambda_n\}_{n\ge 1}$ of the problem $R(q, f_1, f_2)$ and the number $\omega := \frac{1}{2}\int_0^\pi q(x)\,dx$, find the potential q.*

In [42,43], Problem 21 was studied under certain restrictions on $\{\lambda_n\}_{n\ge 1}$ that guaranteed the uniqueness and existence of the solution, etc. Note that, in applications to partial inverse problems, the number ω can usually be found from the eigenvalue asymptotics. However, in general cases, it has to be given.

Introduce the notations

$$s(x,\lambda) = \sqrt{\lambda}\sin(\sqrt{\lambda}x), \quad c(x,\lambda) = \cos(\sqrt{\lambda}x).$$

Then, the sine-type solution $S(x,\lambda)$ can be represented in terms of the transformation operator:

$$S(x,\lambda) = \frac{s(x,\lambda)}{\lambda} + \frac{1}{\lambda}\int_0^x \mathscr{K}(x,t)s(t,\lambda)\,dt. \tag{48}$$

Let $\eta_1(\lambda) := S(\pi,\lambda)$ and $\eta_2(\lambda) := S'(\pi,\lambda)$. Applying differentiation and integration by parts in (48), one can easily obtain the following standard relations:

$$\eta_1(\lambda) = \frac{s(\pi,\lambda)}{\lambda} - \frac{\omega c(\pi,\lambda)}{\lambda} + \frac{1}{\lambda}\int_0^\pi K(t)c(t,\lambda)\,dt, \tag{49}$$

$$\eta_2(\lambda) = c(\pi,\lambda) + \frac{\omega s(\pi,\lambda)}{\lambda} + \frac{1}{\lambda}\int_0^\pi N(t)s(t,\lambda)\,dt, \tag{50}$$

where

$$K(t) = \mathscr{K}_t(\pi,t), \quad N(t) = \mathscr{K}_x(\pi,t), \quad K, N \in L_2(0,\pi). \tag{51}$$

The pair of functions $\{K, N\}$ from (49) and (50) is called *the Cauchy data* of the potential q. This name was chosen because the eigenvalue problem for Equation (45) with the boundary conditions $y(0) = y(\pi) = 0$ is related to the initial value (Cauchy) problem

$$u_{tt} - u_{xx} + q(x)u = 0, \quad 0 \le |t| \le x \le \pi,$$
$$u(\pi,t) = \mathscr{K}(\pi,t), \quad u_x(\pi,t) = \mathscr{K}_x(\pi,t), \quad -\pi \le t \le \pi,$$

where $\mathscr{K}(x,t) = -\mathscr{K}(x,-t)$ for $t < 0$. This problem has the unique solution $u(x,t) \equiv \mathscr{K}(x,t)$. The initial data of the Cauchy problem are the functions $\{\mathscr{K}(\pi,t), \mathscr{K}_x(\pi,t)\}$, which are related to $K(t)$ and $N(t)$ by (51).

The method of [42,43] was based on the reduction of Problem 21 to the following auxiliary inverse problem.

Problem 22. *Given the Cauchy data $\{K, N\}$, find the potential q.*

Problem 22 is equivalent to classical inverse spectral problems. Indeed, using the Cauchy data, one can construct $S(\pi,\lambda)$ and $S'(\pi,\lambda)$ via (49) and (50) and the Weyl function $M(\lambda) := -\dfrac{S'(\pi,\lambda)}{S(\pi,\lambda)}$, which uniquely specifies q (see, e.g., [4]). Thus, the uniqueness of Problem 22's solution follows from the classical result obtained by Borg [6]. Its constructive solution can be obtained by the standard methods (see [4]). Some numerical techniques for the reconstruction of the potential using the Cauchy data are described in [67].

For simplicity, we assume that $\lambda_n \neq 0$ for $n \geq 1$ and the eigenvalues $\{\lambda_n\}_{n\geq 1}$ are simple, that is, $\lambda_n \neq \lambda_m$ for $n \neq m$. In [42,43], results were provided for the general case of multiple eigenvalues.

Introduce the complex Hilbert space of vector functions

$$\mathcal{H} := L_2(0,\pi) \oplus L_2(0,\pi) = \{h = [h_1, h_2] \colon h_j \in L_2(0,\pi),\ j = 1,2\}$$

with the following scalar product and norm:

$$(g,h)_{\mathcal{H}} := \int_0^\pi (\overline{g_1(t)}h_1(t) + \overline{g_2(t)}h_2(t))\,dt, \quad \|h\|_{\mathcal{H}} = \sqrt{(h,h)_{\mathcal{H}}},$$
$$g,h \in \mathcal{H}, \quad g = [g_1, g_2], \quad h = [h_1, h_2].$$

Substituting the representations (49) and (50) into (47) and letting $\lambda = \lambda_n$, one derives the relation

$$(u, v_n)_{\mathcal{H}} = w_n \tag{52}$$

for $n \geq 1$, where

$$u(t) := [\overline{N(t)}, \overline{K(t)}], \quad v_n(t) = v(t,\lambda_n), \quad w_n = w(\lambda_n), \quad n \geq 1, \tag{53}$$
$$v(t,\lambda) := [f_1(\lambda)s(t,\lambda), f_2(\lambda)c(t,\lambda)], \tag{54}$$
$$w(\lambda) := -f_1(\lambda)(\lambda c(\pi,\lambda) + \omega s(\pi,\lambda)) - f_2(\lambda)(s(\pi,\lambda) - \omega c(\pi,\lambda)). \tag{55}$$

Since the function $S(\pi,\lambda)$ is analytical at $\lambda = 0$, one obtains an additional relation of the form (52) for $n = 0$ from (49) with

$$v_0(t) := [0,1], \quad w_0 := \omega. \tag{56}$$

In [42], the following conditions were introduced:

(COMPLETE)—the sequence $\{v_n\}_{n\geq 0}$ is complete in $\mathcal{H}$.

(BASIS)—the sequence $\{v_n\}_{n\geq 0}$ is an unconditional basis in $\mathcal{H}$.

Clearly, the condition (BASIS) implies (COMPLETE). It was shown in [42] that the condition (COMPLETE) is necessary and sufficient for the uniqueness of Problem 21's solution.

Theorem 22 ([42])**.** *Let $\{\lambda_n\}_{n\geq 1}$ and $\{\tilde{\lambda}_n\}_{n\geq 1}$ be subspectra of the problems $R(q, f_1, f_2)$ and $R(\tilde{q}, f_1, f_2)$, respectively. Suppose that $R(q, f_1, f_2)$ and $\{\lambda_n\}_{n\geq 1}$ satisfy the condition* (COMPLETE)*, and let $\lambda_n = \tilde{\lambda}_n$, $n \geq 1$, $\omega = \tilde{\omega}$. Then, $q = \tilde{q}$ in $L_2(0,\pi)$.*

Theorem 23 ([42])**.** *Let $\{\lambda_n\}_{n\geq 1}$ be a subspectrum of the problem $R(q, f_1, f_2)$. Suppose that the sequence $\{v_n\}_{n\geq 0}$ is incomplete in $\mathcal{H}$. Then, there exists a complex-valued function $\tilde{q} \in L_2(0,\pi)$, $\tilde{q} \neq q$ such that $\omega = \tilde{\omega}$, and $\{\lambda_n\}_{n\geq 1}$ is a subspectrum of $R(\tilde{q}, f_1, f_2)$.*

Under the condition (BASIS), the following constructive algorithm for solving Problem 21 was obtained in [42]:

Method 2 ([42])**.** *Let the functions $f_j(\lambda)$, $j = 1,2$, the subspectrum $\{\lambda_n\}_{n\geq 1}$, and the number ω be given. One must construct the potential q.*

1. *Using $f_j(\lambda)$, $j = 1,2$, $\{\lambda_n\}_{n\geq 1}$, and ω, construct the vector functions $\{v_n\}_{n\geq 0}$ and the numbers $\{w_n\}_{n\geq 0}$ using Formulas (53)–(56).*
2. *For the basis $\{v_n\}_{n\geq 0}$, find the biorthonormal basis $\{v_n^*\}_{n\geq 0}$, that is, $(v_n, v_k^*)_{\mathcal{H}} = \delta_{nk}$, $n,k \geq 0$.*
3. *Construct the element $u \in \mathcal{H}$ satisfying (52) using the formula*

$$u = \sum_{n=0}^{\infty} \overline{w_n} v_n^*.$$

4. *Using the elements of* $u(t) = [\overline{N(t)}, \overline{K(t)}]$, *solve Problem 22 with the Cauchy data and find* q.

It is worth noting that, in the case of simple eigenvalues $\{\lambda_n\}_{n\ge1}$, Problem 21 is a special case of Problem 2, which was studied by Horváth [12]. Indeed, the numbers $\{\lambda_n\}_{n\ge1}$ can be treated as the eigenvalues of different boundary value problems for Equation (45) subject to the boundary conditions

$$y(0) = 0, \quad y'(\pi)\cos\beta_n + y(\pi)\sin\beta_n = 0, \quad \beta_n := \arctan\frac{f_2(\lambda_n)}{f_1(\lambda_n)}.$$

On the other hand, using the given data of Problem 21, one can easily find the values of the Weyl function in the points $\{\lambda_n\}_{n\ge1}$:

$$M(\lambda_n) = -\frac{S'(\pi,\lambda_n)}{S(\pi,\lambda_n)} = \frac{f_2(\lambda_n)}{f_1(\lambda_n)}.$$

Thus, Problem 21 is closely related to the problem of potential reconstruction from the values $\{M(\lambda_n)\}_{n\ge1}$ (see Problem 3). The uniqueness of this problem solution was studied by Horváth [12]. A constructive solution is provided by Method 2 with necessary modifications. Namely, in the definitions of v_n and w_n, one should replace $f_1(\lambda_n)$ with 1 and $f_2(\lambda_n)$ with $M(\lambda_n)$. Thus, to the best of the author's knowledge, a constructive algorithm for the recovery of the potential q from the values $\{M(\lambda_n)\}_{n\ge1}$ of the Weyl function at a countable set of points was obtained for the first time in [42]. Effective numerical algorithms for this reconstruction were developed by Kravchenko and Torba [80]. The technique of [80] was based on the representations of the Sturm-Liouville equation solutions as Neumann series of Bessel functions. The methods of [80] can be applied to various classes of partial inverse problems.

Proceeding with the results of [42], in applications to partial inverse problems, it can be difficult to verify the conditions (COMPLETE) and (BASIS). Therefore, the following easily verified conditions are introduced (for convenience, let $\lambda_0 = 0$):

(COMPLETE C)—the sequence $\{\cos(\sqrt{\lambda_n}t)\}_{n\ge0}$ is complete in $L_2(0,2\pi)$.

(BASIS C)—the sequence $\{\cos(\sqrt{\lambda_n}t)\}_{n\ge0}$ is a Riesz in $L_2(0,2\pi)$.

(SEPARATION)—for every $n \ge 1$, there exists $f_1(\lambda_n) \ne 0$ or $f_2(\lambda_n) \ne 0$.

(ASYMPTOTICS)—$\operatorname{Im}\rho_n = O(1)$, $n \to \infty$, and $\{\rho_n^{-1}\}_{n\ge n_0} \in l_2$, where $\rho_n := \sqrt{\lambda_n}$, $\arg\rho_n \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Theorem 24 ([42])**.**

1. (SEPARATION) *and* (COMPLETE C) *together imply* (COMPLETE).
2. (SEPARATION), (ASYMPTOTICS), *and* (BASIS C) *together imply* (BASIS).

Thus, one can replace the condition (COMPLETE) in Theorem 22 with (SEPARATION) and (COMPLETE C) and the condition (BASIS) in Method 2 with (SEPARATION), (ASYMPTOTICS), and (BASIS C). The results remain valid.

The investigation of Problem 21 was continued in [43], which studied the solvability and stability of the inverse problem. In particular, the following sufficient conditions for the global solvability of Problem 21 were obtained:

Theorem 25 ([43])**.** *Let* $f_j(\lambda)$, $j = 1, 2$, *be entire functions, and let* $\{\lambda_n\}_{n\ge1}$ *and* ω *be complex numbers such that the sequence* $\{v_n\}_{n\ge0}$ *constructed by them satisfies the condition* (BASIS) *and* $\left\{\frac{w_n}{\|v_n\|_{\mathcal{H}}}\right\} \in l_2$. *Then, following Method 2, one can construct the functions* $K, N \in L_2(0,\pi)$. *If the zeros* $\{\theta_{nj}\}_{n\ge1}$, $j = 1, 2$, *of the corresponding functions* $\eta_j(\lambda)$, $j = 1, 2$, *defined by* (49) *and* (50) *are real and interlace in the sense*

$$\theta_{n2} < \theta_{n1} < \theta_{n+1,2}, \quad n \in \mathbb{N}, \tag{57}$$

then there exists a unique real-valued function $q \in L_2(0,\pi)$ such that the sequence $\{\lambda_n\}_{n\geq 1}$ is a subspectrum of $R(q, f_1, f_2)$ and $\frac{1}{2}\int_0^\pi q(x)\,dx = \omega$.

Note that the interlacing property (57) appears from the necessary and sufficient conditions for the solvability of the classical Borg problem:

Theorem 26 ([4])**.** *For sequences $\{\theta_{nj}\}_{n\geq 1}$, $j = 1,2$, of real numbers to be the spectra of the corresponding problems $L_j(q)$, $j = 1,2$, for Sturm-Liouville Equation (45) with a real-valued potential $q \in L_2(0,\pi)$ subject to the boundary conditions $y(0) = y^{(j-1)}(\pi) = 0$, it is necessary and sufficient to have the asymptotics*

$$\sqrt{\theta_{nj}} = n - \frac{j-1}{2} + \frac{\omega}{\pi n} + \frac{\varkappa_{nj}}{n}, \quad n \in \mathbb{N}, \quad j = 1,2, \quad \{\varkappa_{nj}\} \in l_2,$$

and the interlacing property (57).

In fact, Problem 21 is reduced to the Borg problem by Method 2, and then the a posteriori condition (57) is imposed. Analogous a posteriori conditions appeared in the papers of Hryniv and Mykytyuk [14] and Martinyuk and Pivovarchik [17] for the Hochstadt–Lieberman problem (see Theorems 9 and 11). As already mentioned in Section 2.2, such conditions seem to be unavoidable for the solvability of partial inverse problems.

Furthermore, in [43], the local solvability and stability of Problem 21 were obtained. In order to formulate these results, one needs the following additional condition:

(ESTIMATES)—there exist constants $a_j > 0$, $j = 1,2,3$, and $\{\alpha_n\}_{n\geq 1}$ such that

$$|f_j(\rho^2)| \leq a_1|\rho_n|^{\alpha_n+j-1}, \quad j = 1,2, \quad |\rho - \rho_n| \leq \frac{a_2}{|\rho_n|},$$
$$|w(\rho^2)| \leq a_1|\rho_n|^{\alpha_n+1}, \quad |\rho - \rho_n| \leq \frac{a_2}{|\rho_n|},$$
$$|f_1(\lambda_n)|^2 + |\lambda_n|^{-1}|f_2(\lambda_n)|^2 \geq a_3|\lambda_n|^{\alpha_n}, \quad n \geq 1.$$

Although these estimates look complicated, they naturally appear in applications involving partial inverse problems on graphs, the inverse transmission eigenvalue problem, etc.

Theorem 27 ([43])**.** *Let $R(q, f_1, f_2)$ be a fixed boundary value problem of the form (45) and (46), and let $\{\lambda_n\}_{n\geq 1}$ be a fixed subspectrum of $R(q, f_1, f_2)$. Suppose that the conditions* (BASIS), (ASYMPTOTICS)*, and* (ESTIMATES) *are fulfilled. Then, there exists $\varepsilon > 0$ (depending on $R(q, f_1, f_2)$ and $\{\lambda_n\}_{n\geq 1}$) such that, for every complex sequence $\{\tilde{\lambda}_n\}_{n\geq 1}$ satisfying the estimate*

$$\Xi := \left(\sum_{n=1}^{\infty} (|\rho_n| + 1)^{-2}|\rho_n - \tilde{\rho}_n|^2\right)^{1/2} \leq \varepsilon, \quad \tilde{\rho}_n := \sqrt{\tilde{\lambda}_n}, \tag{58}$$

there exists a complex-valued function $\tilde{q} \in L_2(0,\pi)$ such that $\omega = \tilde{\omega}$, and $\{\tilde{\lambda}_n\}_{n\geq 1}$ is a subspectrum of the corresponding problem $R(\tilde{q}, f_1, f_2)$. Moreover,

$$\|q - \tilde{q}\|_{L_2(0,\pi)} \leq C\Xi, \tag{59}$$

where the constant C depends only on $R(q, f_1, f_2)$, $\{\lambda_n\}_{n\geq 1}$ and not on $\{\tilde{\lambda}_n\}_{n\geq 1}$.

Note that here, Theorem 27 was formulated for simple eigenvalues $\{\lambda_n\}_{n\geq 1}$. However, in [43], it was proved for the general case of multiple eigenvalues. The multiplicities in the sequences $\{\lambda_n\}_{n\geq 1}$ and $\{\tilde{\lambda}_n\}_{n\geq 1}$ may be distinct, since the groups of multiple eigenvalues in $\{\lambda_n\}_{n\geq 1}$ may split into smaller groups under a small perturbation. This effect was taken into account in [43]. The proof of Theorem 27 relies on Method 2 and the local solvability and stability of Problem 22 using the Cauchy data, which was proved in [42]. In addition, note that Theorem 27 contains no a posteriori conditions of the type (57).

Thus, for Problem 21, the following results have been obtained:

- The necessary and sufficient conditions of uniqueness;
- A constructive solution;
- Simple sufficient conditions for uniqueness and the algorithm;
- Sufficient conditions for the global solvability;
- Local solvability and stability.

Below, we discuss the studies on inverse problems with entire functions in the boundary conditions for other types of operators. The Sturm-Liouville problem analogous to (45) and (46) with the Robin boundary condition $y'(0) - hy(0) = 0$ was considered in [44]. However, in [44], proofs were provided only for simple eigenvalues. Moreover, in the proof of the local solvability and stability theorem, reduction to the Borg problem by the two spectra was used. Unfortunately, the application of the Borg theorem in [44] allows us to obtain only the stability estimate $\|q - \tilde{q}\| \le C\Xi^{1/p}$, where p is the maximal eigenvalue multiplicity in the Borg problem (see [44] for details). The reduction to the inverse problem using the Cauchy data allows us to obtain a better estimate (59) without the power $1/p$.

In [41], the inverse problem analogous to Problem 21 was studied for a singular potential $q \in W_2^{-1}(0, \pi)$, and the results were applied to a partial inverse problem on an arbitrary graph (see Section 3.3 for details).

Kuznetsova [45] studied the inverse problem for the differential pencil

$$-y'' + q(x)y + 2\lambda p(x)y = \lambda^2 y, \quad x \in (0, \pi),$$

$$y(0) = 0, \quad f_1(\lambda)y^{[1]}(\pi) + f_2(\lambda)y(\pi) = 0,$$

where $q \in W_2^{-1}(0, \pi)$, $p \in L_2(0, \pi)$, $y^{[1]} = y' - \sigma y$, $q = \sigma'$, $\sigma \in L_2(0, \pi)$. The results of [45] included:

- Uniqueness;
- A constructive solution;
- Simple sufficient conditions for uniqueness and the algorithm;
- Application to Hochstadt–Lieberman-type problems.

Bondarenko and Chitorkin [46] investigated the inverse problem for the Sturm-Liouville equation (45) subject to the boundary conditions

$$p_1(\lambda)y'(0) + p_2(\lambda)y(0) = 0, \quad f_1(\lambda)y'(\pi) + f_2(\lambda)y(\pi) = 0,$$

where $p_1(\lambda)$ and $p_2(\lambda)$ are relative prime polynomials of the spectral parameter λ, and $f_1(\lambda)$ and $f_2(\lambda)$ are entire functions. In [46], the uniqueness of the inverse problem solution was studied, and the results were applied to Hochstadt–Lieberman-type problems with polynomial dependence on λ not only in the boundary conditions but also in the discontinuity conditions inside the interval.

4.2. Applications to Partial Inverse Problems

In this subsection, we show how partial inverse problems can be reduced to Problem 21 with entire functions in the boundary conditions. As examples, we consider the following partial inverse problems:

- The Hochstadt–Lieberman problem (Problem 1);
- The McLaughlin–Polyakov problem (Problem 5);
- A partial inverse problem on a star-shaped graph (Problem 9);
- A partial inverse problem on a graph of an arbitrary structure (Problem 20).

We start with the application to the Hochstadt–Lieberman problem, which is described in [42]. Consider the following eigenvalue problem:

$$-y''(x) + q(x)y(x) = \lambda y(x), \quad x \in (0, 2\pi), \tag{60}$$
$$y(0) = y(2\pi) = 0, \tag{61}$$

with a complex-valued potential $q \in L_2(0, 2\pi)$. Let $\{\lambda_n\}_{n\ge1}$ denote the eigenvalues of the problems presented in (60) and (61), counted with their multiplicities and numbered according to their asymptotics

$$\sqrt{\lambda_n} = \frac{n}{2} + \frac{\omega_{2\pi}}{\pi n} + o\left(n^{-1}\right), \quad n \to \infty, \tag{62}$$

where $\omega_{2\pi} := \frac{1}{2}\int_0^{2\pi} q(x)\,dx$. The Hochstadt–Lieberman problem in this case is formulated as follows:

Problem 23 ([42]). *Suppose that the potential $q(x)$ is known a priori for $x \in (\pi, 2\pi)$. Given the spectrum $\{\lambda_n\}_{n\ge1}$ (counting with multiplicities), find the potential $q(x)$ for $x \in (0, \pi)$.*

Let us show that Problem 23 can be reduced to Problem 21 with entire functions in the boundary condition. Let $S(x, \lambda)$ and $\psi(x, \lambda)$ denote the solution of Equation (60) satisfying the initial conditions

$$S(0, \lambda) = 0, \quad S'(0, \lambda) = 1, \quad \psi(2\pi, \lambda) = 0, \quad \psi'(2\pi, \lambda) = -1.$$

The eigenvalues of (60) and (61) coincide with the zeros of the characteristic function

$$\Delta(\lambda) = \psi(\pi, \lambda)S'(\pi, \lambda) - \psi'(\pi, \lambda)S(\pi, \lambda). \tag{63}$$

Comparing (63) with (47), one can conclude that the eigenvalue problems presented in (60) and (61) are equivalent to the problem $R(q, f_1, f_2)$ given by (45) and (46) with

$$f_1(\lambda) := \psi(\pi, \lambda), \quad f_2(\lambda) := -\psi'(\pi, \lambda). \tag{64}$$

Note that these functions $f_j(\lambda)$, $j = 1, 2$, are entire in the λ-plane and can be constructed by the known part of the potential $q(x)$, $x \in (\pi, 2\pi)$. The constant ω can also be found using the given data of Problem 23 by the formula

$$\omega = \omega_{2\pi} - \frac{1}{2}\int_\pi^{2\pi} q(x)\,dx,$$

where $\omega_{2\pi}$ can be determined from the asymptotics in (62). Thus, Problem 23 is reduced to Problem 21.

Suppose that the eigenvalues $\{\lambda_n\}_{n\ge1}$ of the problem (60)–(61) are simple. Then, one can easily show that the conditions (Basis C), (Separation), (Asymptotics), and (Estimates) of the previous subsection hold. Therefore, Theorems 22 and 24 imply the following corollary:

Corollary 1 ([42]). *Let $\{\lambda_n\}_{n\ge1}$ and $\{\tilde{\lambda}_n\}_{n\ge1}$ be the spectra of the boundary value problems of the form (60) and (61) with potentials q and $\tilde{q}$, respectively. Suppose that $q(x) = \tilde{q}(x)$ a.e. on $(\pi, 2\pi)$ and $\lambda_n = \tilde{\lambda}_n$ for all $n \ge 1$. Then, $q(x) = \tilde{q}(x)$ a.e. on $(0, \pi)$. In other words, the solution of Problem 23 is unique. This solution can be found using Method 2, taking (64) into account.*

Obviously, the uniqueness of Corollary 1 is similar to the Hochstadt–Lieberman theorem (Theorem 1) for complex-valued potentials. Method 2 generalizes the algorithms of Buterin [16,65] (see Method 1) and Martinyuk and Pivovarchik [17] for solving the Hochstadt–Lieberman problem.

Theorem 27 implies the following corollary on the local solvability and stability of the Hochstadt–Lieberman problem:

Corollary 2. *For any complex-valued function $q \in L_2(0, 2\pi)$, there exists $\varepsilon > 0$ such that, for any complex sequence $\{\tilde{\lambda}_n\}_{n \ge 1}$ close to the spectrum $\{\lambda_n\}_{n \ge 1}$ of the problem (60)–(61) in the sense (58), there exists a complex-valued function $\tilde{q} \in L_2(0, 2\pi)$ such that $q(x) = \tilde{q}(x)$ a.e. on $(\pi, 2\pi)$ and $\{\tilde{\lambda}_n\}_{n \ge 1}$ is the spectrum of the problem (60)–(61) with the potential $\tilde{q}$. Moreover, $\|q - \tilde{q}\|_{L_2(0,\pi)} \le C\Xi$, where the constant C depends only on q.*

It is worth noting that, since the potential $q(x)$ in (60) is complex-valued, a finite number of eigenvalues can be multiple. In this case, Corollaries 1 and 2 remain valid, and Method 2 is also valid with necessary technical modifications (see [42,43] for details). Therefore, to the best of the author's knowledge, Theorem 27 provides the first results on the local solvability and stability of the Hochstadt–Lieberman problem in the general case of a complex-valued potential with eigenvalues that are not necessarily simple. Theorem 25 can also be transferred to the Hochstadt–Lieberman problem.

An analogous reduction can be applied to Hochstadt–Lieberman-type inverse problems with the discontinuity conditions

$$y(d+) = ay(d-), \quad y'(d+) = a^{-1}y(d-) + by(d-),$$

and/or polynomial dependence on the spectral parameter in the boundary conditions (see, e.g., [9,81]). If all the discontinuities and the polynomial dependence lie on the "known" part of the interval, then such a partial inverse problem can be similarly reduced to Problem 21 for (45) and (46). The opposite case requires a separate investigation, which can be implemented analogously.

Proceeding to the McLaughlin–Polyakov problem (Problem 5), the reduction of this problem to Problem 21 was briefly described in [43]. We present it here in more detail.

Suppose that $a \in [0, 1) \cup (1, 3]$. Let $y_j(x, \lambda)$, $j = 1, 2$, denote the solutions of Equation (18) satisfying the initial conditions

$$y_1(1, \lambda) = y_2'(1, \lambda) = 0, \quad -y_1'(1, \lambda) = y_2(1, \lambda) = 1.$$

Obviously, the function

$$\xi(x, \lambda) := y_2(x, \lambda) \frac{\sin \rho a}{\rho} - y_1(x, \lambda) \cos \rho a. \tag{65}$$

for each $\lambda \in \mathbb{C}$ is the only solution (up to a constant multiplier) of Equation (18) satisfying the boundary condition (19) at $x = 1$. Therefore, for every eigenvalue λ_n of the boundary value problem (18)–(19), the corresponding eigenfunction has the form $S(x, \lambda_n) = c_n \xi(x, \lambda_n)$, where c_n is a constant. Consequently, the transmission eigenvalues coincide with the zeros of the characteristic function

$$\Delta(\lambda) := \begin{vmatrix} S(x, \lambda) & \xi(x, \lambda) \\ S'(x, \lambda) & \xi'(x, \lambda) \end{vmatrix}.$$

For $x = \alpha$, we have

$$\Delta(\lambda) = S(\alpha, \lambda)\xi'(\alpha, \lambda) - S'(\alpha, \lambda)\xi(\alpha, \lambda).$$

Comparing this relation with (47), one can conclude that the transmission eigenvalue problem can be represented as a Sturm-Liouville problem on the interval $(0, \alpha)$ with the entire functions

$$f_1(\lambda) := -\xi(\alpha, \lambda), \quad f_2(\lambda) := \xi'(\alpha, \lambda) \tag{66}$$

in the right-hand boundary condition. The only difference from the problem (45)–(46) is the interval length α instead of π. With this technical difference in mind, the McLaughlin–Polyakov problem is equivalent to Problem 21 with the functions $f_j(\lambda)$, $j = 1, 2$, defined by (66) and with an almost real subspectrum $\{\lambda_n\}_{n\ge1}$. The number $\omega = \frac{1}{2}\int_0^\alpha q(x)\,dx$ can be found using the asymptotics (21) and the known potential q on the subinterval $(\alpha, 1)$:

$$\omega = \lim_{n\to+\infty}((1-a)\sqrt{\lambda_n} - \pi n)\pi n - \frac{1}{2}\int_\alpha^1 q(x)\,dx.$$

It can be shown that, in the case of the simple subspectrum $\{\lambda_n\}_{n\ge1}$, the conditions (Basis C), (Separation), (Asymptotics), and (Estimates) of Section 4.1 hold. For the case of multiple eigenvalues, all the results are valid with some technical modifications. Consequently, the uniqueness theorem of McLaughlin and Polyakov (Theorem 12) can be easily deduced as a corollary of Theorems 22 and 24. The solution of the McLaughlin–Polyakov problem can be found using Method 2, taking the relation (66) into account and replacing π with α. Theorem 25 implies the following corollary on the global solvability of the McLaughlin–Polyakov problem:

Corollary 3. *Let numbers $a \in [0,1) \cup (1,3]$, $\omega \in \mathbb{R}$, and a real-valued function $\tilde{q} \in L_2(\alpha, 1)$ be fixed. For a sequence $\{\lambda_n\}_{n\ge1}$ to be an almost real subspectrum of the transmission eigenvalue problem (18)–(19) with a potential $q \in L_2(0,1)$ such that $q(x) = \tilde{q}(x)$ a.e. on $(\alpha, 1)$ and $\frac{1}{2}\int_0^1 q(x)\,dx = \omega_0$, the following conditions are necessary and sufficient:*

1. *$\{\lambda_n\}_{n\ge1}$ satisfies the asymptotics (21).*
2. *The zeros $\{\theta_{nj}\}_{n\ge1, j=1,2}$ of the functions $\eta_j(\lambda)$, $j = 1, 2$, defined by (49) and (50) using the functions $K, N \in L_2(0, \alpha)$, which are constructed by Method 2, are real and interlace in the sense of (57).*

Note that the global solvability of the inverse transmission eigenvalue problem was also investigated by Buterin et al. [69]. However, in [69], another problem statement was considered. The potential was not assumed to be known a priori on the subinterval $(\alpha, 1)$.

In addition, one can apply Theorem 27 to obtain the local solvability and stability of the McLaughlin–Polyakov problem. However, this result would be weaker than that of Theorem 13 proposed by Bondarenko and Buterin [22], because Theorem 27 does not allow one to take perturbations of the potential $q(x)$ on $(\alpha, 1)$ into account.

It is worth mentioning that the transmission eigenvalue problem (18)–(19) can be represented as the following boundary value problem on the three-edge graph in Figure 11:

$$-y_j''(x_j) + q_j(x_j)y_j(x_j) = \lambda y_j(x_j), \quad x_j \in (0, T_j), \quad j = 1, 2, 3,$$
$$y_1(0) = 0, \quad y_1(T_1) = y_2(0), \quad y_1'(T_1) = y_2'(0),$$
$$y_2(T_2) = y_3(0), \quad y_2'(T_2) = -y_3'(0), \quad y_3(T_3) = 0,$$
$$T_1 := \alpha, \quad T_2 := 1 - \alpha, \quad T_3 := a, \quad q_1(x) := q(x), \quad q_2(x) := q(x + \alpha), \quad q_3(x) := 0.$$

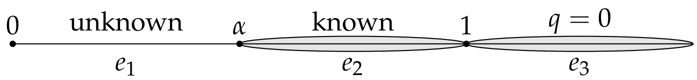

Figure 11. Graph representation of the transmission eigenvalue problem.

In order to model the condition $y'(1)\cos\rho a - y'(1)\dfrac{\sin\rho a}{\rho} = 0$, one can add a dummy edge of length a with a zero potential. Note that the matching conditions at the vertex joining e_2 and e_3 are non-standard and irregular. Nevertheless, the methods for partial inverse problems on graphs can also be used for the McLaughlin–Polyakov problem.

Next, consider Problem 9 for the Sturm-Liouville problem L of the form (22)–(24) on a star-shaped graph. In contrast to [33], we suppose that the potentials $\{q_j\}_{j=1}^m$ are complex-valued. Recall that the characteristic function of problem L is given by Formula (25):

$$\Delta(\lambda) := \sum_{j=1}^{m} S_j'(\pi, \lambda) \prod_{\substack{k=1 \\ k \neq j}}^{m} S_k(\pi, \lambda).$$

Comparing (25) with (47), one can easily see that the eigenvalue problem L on the star-shaped graph is equivalent to the problem (45) and (46) with $q = q_1$ and with the following entire functions in the boundary condition:

$$f_1(\lambda) := \prod_{k=2}^{m} S_k(\pi, \lambda), \quad f_2(\lambda) := \sum_{j=2}^{m} S_j'(\pi, \lambda) \prod_{\substack{k=2 \\ k \neq j}}^{m} S_k(\pi, \lambda).$$

Suppose that a subspectrum $\Omega = \{\lambda_{nk}\}_{n \geq 1, k=1,2}$ satisfying the asymptotics (28) and (29) and Condition 1 is given together with the potentials $\{q_j\}_{j=2}^m$. Then, Condition 1 implies the separation condition $f_1(\lambda_{nk}) \neq 0$ or $f_2(\lambda_{nk}) \neq 0$ for $n \geq 1$, $k = 1, 2$. The number $\omega = \omega_1$ can be found from the asymptotics (28). The functions $f_1(\lambda)$ and $f_2(\lambda)$ can be constructed using the potentials $\{q_j\}_{j=2}^m$. Thus, Problem 9 is reduced to Problem 21 by the subspectrum Ω. In [43], the results of Section 4.1 were applied to this problem, and so the results of [30,33] were generalized to the case of complex-valued potentials. Certain other conditions of [30,33] were weakened. In particular, the local solvability and stability theorem (generalizing Theorem 16) was proved in the following form:

Theorem 28 ([43])**.** *Let $\{q_j\}_{j=1}^m$ be fixed complex-valued functions of $L_2(0, \pi)$, and let $\{\lambda_{nk}\}_{n \geq 1, k=1,2}$ be eigenvalues of the problem L satisfying the asymptotic relations (28) and (29). Suppose that Condition 1 holds and $z_2 \neq \omega_j$, $j = \overline{2, m}$. Then, there exists $\varepsilon > 0$ (depending on $\{q_j\}_{j=1}^m$ and $\{\lambda_{nk}\}_{n \geq 1, k=1,2}$) such that, for any sequence $\{\tilde{\lambda}_{nk}\}_{n \geq 1, k=1,2}$ satisfying the estimate*

$$\Xi := \left(\sum_{n=1}^{\infty} \sum_{k=1}^{2} (|\lambda_{nk}| + 1) |\sqrt{\lambda_{nk}} - \sqrt{\tilde{\lambda}_{nk}}|^2 \right)^{1/2} \leq \varepsilon,$$

there exists a unique complex-valued function $\tilde{q}_1 \in L_2(0, \pi)$ such that $\{\lambda_{nk}\}_{n \geq 1, k=1,2}$ is a subspectrum of the problem $\tilde{L}$ with $\tilde{q}_1$ instead of q_1. Moreover, $\|q_1 - \tilde{q}_1\|_{L_2(0,\pi)} \leq C\Xi$, where the constant C depends only on $\{q_j\}_{j=1}^m$ and $\{\lambda_{nk}\}_{n \geq 1, k=1,2}$.

An analogous reduction was applied to Problem 20 on an arbitrary graph with an unknown potential on a boundary edge in [41]. The characteristic function for the corresponding boundary value problem in (41) is given by Formula (43):

$$\Delta(\lambda) = S_1(T_1, \lambda)\Delta^K(\lambda) + S_1^{[1]}(T_1, \lambda)\Delta^\Pi(\lambda).$$

Consequently, the problem in (41) can be represented in the form

$$\left.\begin{aligned} -(y^{[1]}(x))' - \sigma(x)y^{[1]}(x) - \sigma^2(x)y(x) = \lambda y(x), \quad x \in (0, T), \\ y(0) = 0, \quad f_1(\lambda)y^{[1]}(T) + f_2(\lambda)y(T) = 0, \end{aligned}\right\} \tag{67}$$

where $\sigma := \sigma_1$, $y^{[1]} = y' - \sigma y$, $T := T_1$, $f_1(\lambda) := \Delta^\Pi(\lambda)$, $f_2(\lambda) := \Delta^K(\lambda)$.

In [41], an inverse spectral theory for the problem (67) was created analogously to the theory in Section 4.1. Consequently, the results for the partial inverse problem on an arbitrary graph (Problem 20), which were described in Section 3.3, were obtained.

5. Other Types of Operators

This section deals with partial inverse problems for other classes of operators different from Sturm-Liouville differential operators and pencils. Namely, we consider the known results for the following types of operators:

- Integro-differential operators;
- Functional differential operators with a constant delay;
- Higher-order differential operators;
- Matrix Sturm-Liouville operators.

The most complete results in this direction have been obtained for integro-differential operators with an integral term in the form of convolution. Wang and Wei [47] studied a partial inverse problem for the integro-differential equation

$$-y'' + q(x)y + \int_0^x M(x-t)y(t)\,dt = \lambda y, \quad x \in (0,\pi), \tag{68}$$

with the Robin boundary conditions

$$y'(0) - hy(0) = 0, \quad y'(\pi) + Hy(\pi) = 0, \tag{69}$$

where $q(x)$ and $M(x)$ are real-valued functions of $L_2(0,\pi)$, and h and H are real constants. The spectrum of the problem (68)–(69) is denoted by $\sigma(L) = \{\lambda_n\}_{n\ge 0}$.

The following Gestezy–Simon-type uniqueness theorem was proved for the problem (68)–(69):

Theorem 29 ([47]). *Suppose that $a \in [0,\pi)$, $h = \tilde{h}$, $M(x) = \tilde{M}(x)$ a.e. on $(0,a)$, and $q(x) = \tilde{q}(x)$ on $(0,\pi)$. Then, for any $\varepsilon > 0$, if a subspectrum $S \subseteq \sigma(L) \cap \sigma(\tilde{L})$ satisfies*

$$\#\{\lambda_n \in S \colon |\lambda_n| \le t\} \ge \Big(1 - \frac{a}{\pi}\Big)\#\{\lambda_n \in \sigma(L) \colon |\lambda_n| \le t\} + \frac{a}{2\pi} - \frac{1}{2} + \varepsilon,$$

where $t \ge t_0$, t_0 is a positive constant, then $H = \tilde{H}$ and $M(x) = \tilde{M}(x)$ a.e. on (a,π).

However, the results of [47] are limited to uniqueness. Later on, Buterin and Sat [48] studied not only uniqueness but also reconstruction and subspectrum characterization for an integro-differential operator half-inverse problem. In [48], the integro-differential Equation (68) was considered subject to the Dirichlet boundary conditions

$$y(0) = y(\pi) = 0. \tag{70}$$

The functions $q(x)$ and $(\pi - x)M(x)$ were assumed to be complex-valued and belong to $L_2(0,\pi)$.

Buterin and Sat [48] studied the following inverse problem:

Problem 24 ([48]). *Given the even subspectrum $\{\lambda_{2n}\}_{n\ge 1}$, find the function $M(x)$ on $(\pi/2,\pi)$, provided that $M(x)$ on $(0,\pi/2)$ and the potential $q(x)$ are known.*

Buterin and Sat also proved the following theorem, which provides the uniqueness of the solution and the even subspectrum characterization of Problem 24.

Theorem 30 ([48]). *Let arbitrary complex-valued functions $q(x) \in L_2(0,\pi)$ and $f(x) \in L_2(0,\pi/2)$ be given and fixed. Then, for any sequence of complex numbers $\{\mu_n\}_{n\ge 1}$ of the form*

$$\mu_n = \Big(2n + \frac{A}{2n} + \frac{\varkappa_n}{n}\Big)^2, \quad A = \frac{1}{2\pi}\int_0^\pi q(x)\,dx, \quad \{\varkappa_n\} \in l_2, \quad n \ge 1, \tag{71}$$

there exists a unique (up to a set of measure zero) function $M(x)$ such that $(\pi - x)M(x) \in L_2(0,\pi)$, $M(x) = f(x)$ on $(0,\pi/2)$, and $\{\mu_n\}_{n\ge1}$ is the even subspectrum (i.e., $\lambda_{2n} = \mu_n$) of the boundary value problem (68)–(70).

Moreover, Buterin and Sat [48] provided a constructive algorithm for solving Problem 24. The method of [48] was based on the technique created by Buterin for solving inverse problems for integro-differential operators (see [82] and the references therein).

The results of [48] showed the principal difference between differential and integro-differential operators. In half-inverse problems for integro-differential operators, the given mixed data (eigenvalues and operator coefficients on a subinterval) are independent of each other. In Problem 24, one can take arbitrary numbers satisfying the eigenvalue asymptotics (71) and an arbitrary function $M(x)$ on $(0,\pi/2)$ and reconstruct $M(x)$ on $(\pi/2,\pi)$. In Hochstadt–Lieberman-type problems for differential operators, the spectrum and the potential $q(x)$ on a subinterval are related to each other. This relationship implies hard-to-verify conditions in the characterization theorems (see, e.g., Theorems 9 and 11).

It is worth mentioning that Sat and Yilmaz [49] attempted to study a partial inverse problem of another kind for the integro-differential operator (68)–(70). Namely, they assumed that the kernel $M(x)$ is known on $(0,\pi)$ and the potential $q(x)$ is known on the half-interval $(\pi/2,\pi)$ and investigated the recovery of $q(x)$ on the interval $(0,\pi/2)$ from the spectrum. However, the results of [49] were wrong, and the proofs contained mistakes. Namely, the estimate $O\left(\frac{1}{\rho^2}\right)$ after Formula (2.10) in [49] was incorrect. Therefore, the problem of recovering $q(x)$ on a subinterval while $M(x)$ is known remains open.

Bondarenko and Yurko [50] studied the following partial inverse problem for a Sturm-Liouville-type operator with a constant delay. Let $\{\lambda_{n,j}\}_{n\ge1}$, $j = 0,1$, denote the eigenvalues of the corresponding boundary value problems

$$-y''(x) + q(x)y(x-a) = \lambda y(x), \quad 0 < x < \pi, \tag{72}$$

$$y(0) = y^{(j)}(\pi) = 0, \tag{73}$$

where $a \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right)$, $q(x)$ is a complex-valued potential of $L_2(0,\pi)$, and $q(x) = 0$ a.e. on $(0,a)$.

Problem 25 ([50])**.** *Assume that $q(x)$ is known a priori for $x \in \left[\frac{3a}{2}, \pi - \frac{a}{2}\right]$. Given subspectra $\{\lambda_{n_k,j}\}_{k\ge1}$, $j = 0,1$, find $q(x)$ on (a,π) (see Figure 12).*

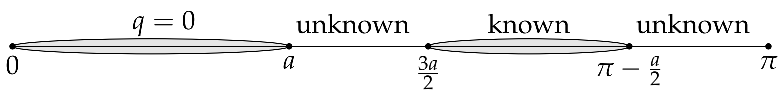

Figure 12. Partial inverse problem with delay.

Note that Problem 25 is different from the Hochstadt–Lieberman problem, since the potential $q(x)$ is given on an interior subinterval. However, for differential operators with a constant delay, the statement of Problem 25 appears to be natural.

Bondarenko and Yurko [50] proved the following uniqueness theorem and obtained a constructive algorithm for finding the solution of Problem 25.

Theorem 31 ([50])**.** *Suppose that the sequences $\{\cos n_k x\}_{k\ge0}$ $(n_0 := 0)$ and $\left\{\sin\left(n_k - \frac{1}{2}\right)x\right\}_{k\ge1}$ are complete in $L_2(0,\pi - a)$, $q(x) = \tilde{q}(x)$ a.e. on $\left[\frac{3a}{2}, \pi - \frac{a}{2}\right]$ and $\lambda_{n_k,j} = \tilde{\lambda}_{n_k,j}$, $k \ge 1$, $j = 0,1$. Then, $q(x) = \tilde{q}(x)$ a.e. on (a,π).*

Djurić and Vladičić [51] considered the boundary value problem (72)–(73) in the case $a \in \left(\frac{\pi}{3}, \frac{2\pi}{5}\right)$ and noticed that the two full spectra $\{\lambda_{n,j}\}_{n\ge1}$, $j = 0,1$, uniquely specify the potential on not only the boundary subintervals $\left(a, \frac{3a}{2}\right)$ and $\left(\pi - \frac{a}{2}, \pi\right)$, but also the interior subinterval $(\pi - a, 2a)$ (see Figure 13). In this case, knowledge of the potential on the subintervals $\left(\frac{3a}{2}, \pi - a\right)$ and $(\pi - a, 2a)$ is unnecessary.

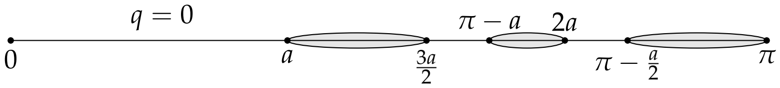

Figure 13. The potential recovered by Djurić and Vladičić.

Theorem 32 ([51])**.** *The spectra* $\{\lambda_{n,j}\}_{n\geq 1}$, $j = 0, 1$, *uniquely determine the potential* $q(x)$ *on the set* $\left(a, \frac{3a}{2}\right) \cup (\pi - a, 2a) \cup \left(\pi - \frac{a}{2}, \pi\right)$.

Moreover, the following uniqueness theorem for a partial inverse problem was proved.

Theorem 33 ([51])**.** *Assume that the potential is known on the set* $\left(\frac{3a}{2}, \frac{\pi}{2} + \frac{a}{4}\right)$ *as well as the integral* $\int_{\pi/2+a/4}^{\pi-a} q(x)\,dx$. *Then, the spectra* $\{\lambda_{n,j}\}_{n\geq 1}$, $j = 0, 1$, *uniquely determine the potential* $q(x)$ *on* (a, π).

In [52], Buterin et al. conducted a comprehensive study of inverse spectral problems for quadratic differential pencils with delays of the form

$$y''(x) + \rho^2 y(x) = q_0(x) y_0(x - a_0) + 2\rho q_1(x) y_1(x - a_1), \quad x \in (0, \pi), \tag{74}$$

where ρ is the spectral parameter, $a_0 \in \left[\frac{\pi}{3}, \pi\right)$, $a_1 \in \left[\frac{\pi}{2}, \pi\right)$, $a_0 + a_1 \geq \pi$, $q_\nu \in W_2^\nu[a_\nu, \pi]$, $q_\nu(x) = 0$ on $(0, a_\nu)$, and $\int_{a_1}^{\pi} q_1(x)\,dx = 0$. Let $\{\rho_{n,j}\}$ denote the spectra of the boundary value problems for Equation (74) subject to the boundary conditions $y(0) = y^{(j)}(\pi) = 0$, $j = 0, 1$. In particular, Buterin et al. [52] generalized Theorem 32 to the pencil in (74).

Theorem 34 ([52])**.** *Let both spectra* $\{\rho_{n,j}\}$, $j = 0, 1$, *be specified. Then, the function* $q_0(x)$ *is uniquely determined a.e. on* $\left(a, \frac{3a_0}{2}\right) \cup (\pi - a_0, 2a_0) \cup \left(\pi - \frac{a_0}{2}, \pi\right)$, *while* $q_1(x)$ *is uniquely determined on* $[a_1, \pi]$.

Theorem 33 was also generalized (see [52] for details).

Next, let us consider the higher-order differential equation

$$y^{(n)} + \sum_{k=0}^{n-2} p_k(x) y^{(k)} = \lambda y, \quad n > 2, \quad x \in (0, T), \tag{75}$$

on a finite interval ($T < \infty$) and the half-line ($T = \infty$). The general theory of inverse spectral problems for Equation (75) was created by Yurko [53]. In Section 4 of [53], Yurko considered partial inverse problems that consisted in the recovery of part of the coefficients $\{p_{\varkappa_j}\}_{j=1}^N$ ($\varkappa = \{\varkappa_j\}_{j=1}^N \subseteq \{0, 1, \dots, n-2\}$) from the Weyl functions $\{\mathfrak{M}_i(\lambda)\}_{i=1}^N$, which were defined using suitable boundary conditions (see [53] for details). The other coefficients $\{p_k\}_{k\notin\varkappa}$ were assumed to be known a priori and integrable on either the finite or infinite interval $(0, T)$. The unknown coefficients $\{p_k\}_{k\in\varkappa}$ were assumed to be piece-wise analytic functions. The partial inverse problem was considered under a specific information condition, which guaranteed its unique solvability. The solution was constructed by the method of standard models.

Recently, Chen et al. [54] attempted to study the Hochstadt–Lieberman-type inverse problem for the fourth-order differential equation

$$y^{(4)} + q(x)y = \lambda^4 y, \quad x \in (0,1), \quad q \in L_1(0,1), \tag{76}$$

subject to the boundary conditions

$$y(0) = y'(0) = 0, \quad y(1) = y'(1) = 0. \tag{77}$$

The inverse problem of [54] consists in the recovery of the potential $q(x)$ on the half-interval $(1/2, 1)$ from the eigenvalues $\{\lambda_k\}$ of (76) and (77), while the potential $q(x)$ on $(0, 1/2)$ is known a priori. However, the main result of [54] (Theorem 1.1) was wrong. In particular, the authors of [54] asserted that, for any sequence $\{\lambda_k\}_{k \in \mathbb{Z}\setminus\{0\}}$ satisfying the conditions $\lambda_{-k} = \lambda_k$, $0 < \lambda_1^4 \le \lambda_2^4 \le \dots \le \lambda_N^4 < \lambda_{N+1}^4 < \dots$, and the asymptotics

$$\lambda_k = \left(k - \frac{1}{2}\right)\pi + \beta_k, \quad \{\beta_k\} \in l_2, \tag{78}$$

there exists a corresponding potential q of class L_1. However, the asymptotics (78) are not precise. For example, Polyakov [83] recently obtained more precise eigenvalue asymptotics, implying that not every sequence satisfying (78) together with the other conditions of [54] can be a spectrum of the problem (76) and (77) with potential $q \in L_1(0,1)$. This was not the only mistake of [54]. Furthermore, it is surprising that, in the Hochstadt–Lieberman-type theorem in [54], the eigenvalues $\{\lambda_k\}$ and the potential $q(x)$ on $(0, 1/2)$ are not related to each other. For the second-order case, there is such a relationship (see, e.g., Theorems 9 and 11). Nevertheless, the problem stated in [54] is a challenging issue for future investigation.

Malamud [55,56] proved the following analog of the Hochstadt–Lieberman theorem for the matrix Sturm-Liouville equations

$$-y'' + Q(x)y = \lambda^2 y, \quad -\tilde{y}'' + \tilde{Q}(x)\tilde{y} = \lambda^2 \tilde{y}, \quad x \in (0,1), \tag{79}$$

where $Q(x)$ and $\tilde{Q}(x)$ are $(n \times n)$ matrix functions. Let I_n denote the $(n \times n)$ unit matrix.

Theorem 35 ([55,56])**.** *Let the entries of $Q(x)$ and $\tilde{Q}(x)$ be complex-valued functions of $L_1(0,1)$, and let $Q(x) = \tilde{Q}(x)$ for a.a. $x \in [1/2, 1]$. Let $Y(x,\lambda)$ and $\tilde{Y}(x,\lambda)$ be the $(n \times n)$ matrix solutions of the initial value problems*

$$Y(0,\lambda) = \tilde{Y}(0,\lambda) = I_n, \quad Y'(0,\lambda) = H_1, \ \tilde{Y}'(0,\lambda) = \tilde{H}_1$$

for the first and second equations in (79), respectively. If

$$Y'(1,\lambda) + H_2 Y(1,\lambda) = \tilde{Y}'(1,\lambda) + H_2 \tilde{Y}(1,\lambda) = 0, \quad \lambda \in \mathbb{C},$$

for some $(n \times n)$ complex matrix H_2, then $H_1 = \tilde{H}_1$ and $Q(x) = \tilde{Q}(x)$ for a.a. $x \in [0,1]$.

Theorem 35 shows that the monodromy matrix $Y'(1,\lambda) + H_1 Y(1,\lambda)$ uniquely determines the matrix potential $Q(x)$ on the half-interval $[0, 1/2]$ if $Q(x)$ is known on $[1/2, 1]$.

6. Conclusions

In this review, we considered selected results on partial inverse spectral problems for differential operators.

The most complete results were obtained for the Hochstadt–Lieberman problem. Several constructive methods were developed that allowed researchers to obtain numerical algorithms for solutions and the necessary and sufficient conditions for the solvability of half-inverse problems. The uniqueness of the inverse problem solution was studied fairly completely for cases in which the potential is known a priori on a subinterval $(0, a)$. Some results have also been obtained for the known potential on an interior subinterval

$(a,b) \subset (0,1)$. However, cases in which the potential is unknown on an interior subinterval and is known on some boundary subintervals remain open.

For differential operators on geometrical graphs, the most simple situation occurs when the potential is unknown only on a boundary edge or even on part of a boundary edge. Such partial inverse problems can be reduced to inverse problems on an interval with entire functions in a boundary condition using the unified approach. These entire functions are constructed by the operator coefficients on the "known" part of the graph. Therefore, for this kind of problems, uniqueness, constructive solutions, global solvability, local solvability, and stability have been obtained even on graphs of an arbitrary geometrical structure. Analogous ideas can be applied to cases in which the potential is unknown on a boundary subgraph. Cases in which the potential is known on some interior edges of the graph have also been considered. For the unknown potential on an interior part of a graph, the question is open, as with the case of the interval.

In addition, there have been several attempts to study partial inverse problems for nonlocal operators, higher-order differential operators, and differential systems. However, the results of these studies are fragmentary, and they do not form a general picture. Some ideas are easily transferred from Hochstadt–Lieberman problems for differential operators to other types of operators. However, for functional differential operators with a delay, higher-order differential operators, and other types of operators, fundamentally new problem statements appear to be natural and, consequently, different methods are required for their investigation.

In conclusion, we formulated several open problems.

Problem 26. *Determine the potential $q(x)$ of the Sturm-Liouville equation $-y'' + q(x)y = \lambda y$ on an interior subinterval $(a,b) \subset (0,1)$ from fewer spectral data than are used for the complete inverse problem, while $q(x)$ is known on $(0,1) \setminus (a,b)$ (see Figure 3).*

Problem 27. *Investigate the solvability and stability of the inverse Sturm-Liouville partial inverse problem on the interval $(0,1)$ in the case of a known potential on an interior subinterval (see Figure 2) using the spectral data of Guo and Wei [63] or any other spectral data.*

Problem 28. *Study partial Sturm-Liouville inverse problems on graphs in case where the potentials are known on an interior subgraph. Determine the spectral data that are sufficient for the unique reconstruction of the potentials on the whole graph. This problem is open even for simple graphs (star-shaped graphs, lasso graphs, and trees).*

Problem 29. *Investigate the solvability and stability of partial inverse problems on graphs for cases in which the potentials are known on a boundary part of the graph. These issues have been studied only for an unknown potential on one edge.*

Problem 30. *Construct an inverse problem theory for the Sturm-Liouville equation with entire analytical functions in one of the boundary conditions* (45)–(46) *and discontinuity conditions of the form*

$$y(d+) = ay(d-), \quad y'(d+) = a^{-1}y(d-) + by(d-)$$

at one or several points inside the interval. Note that the investigation of this problem will open up the possibility of studying a wide class of partial inverse problems with discontinuities. Inverse Sturm-Liouville problems with discontinuities in interior points appear in electronics when constructing the parameters of heterogeneous electric lines with desirable technical charateristics [84] and in geophysical models of the Earth's oscillations [9].

Problem 31. *Study the reconstruction of the potential $q(x)$ of the integro-differential equation*

$$-y'' + q(x)y + \int_0^x M(x-t)y(t)\, dt = \lambda y$$

on a half-interval from spectral data under the assumption that $M(x)$ is known.

Problem 32. *Suppose that the coefficients $\{p_k\}_{k=0}^{n-2}$ of the higher-order differential equation*

$$y^{(n)} + \sum_{k=0}^{n-2} p_k(x) y^{(k)} = \lambda y, \quad n > 2, \quad x \in (0,1),$$

are known on the half-interval $(0, 1/2)$. How many spectral data are sufficient for the unique specification of these coefficients on $(1/2, 1)$? In particular, one can study this half-inverse problem for the fourth-order differential equation

$$y^{(4)} - (p(x)y')' + q(x)y = \lambda y. \tag{80}$$

Note that this equation is important for mechanical applications, since the Euler-Bernoulli equation $(a(x)u'')'' = \mu b(x) u$, which describes beam vibrations, can be reduced to the form of (80) (see [85]).

Thus, the theory of partial inverse spectral problems still poses many challenges.

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