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Transitive Deficiency One Parallelisms of $PG(3,7)$

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Abstract: Consider the n-dimensional projective space $PG(n,q)$ over a finite field with q elements. A spread in $PG(n,q)$ is a set of lines which partition the point set. A parallelism is a partition of the set of lines by spreads. A deficiency one parallelism is a partial parallelism with one spread less than the parallelism. A transitive deficiency one parallelism corresponds to a parallelism possessing an automorphism group which fixes one spread and is transitive on the remaining spreads. Such parallelisms have been considered in many papers. As a result, an infinite family of transitive deficiency one parallelisms of $PG(n,q)$ has been constructed for odd q, and it has been proved that the deficiency spread of a transitive deficiency one parallelism must be regular, and its automorphism group should contain an elation subgroup of order q^2. In the present paper we construct parallelisms of $PG(3,7)$ invariant under an elation group of order 49 with some additional properties, and thus we succeed to obtain all (46) transitive deficiency one parallelisms of $PG(3,7)$. The three parallelisms from the known infinite family are among them. As a by-product, we also construct a much bigger number (55,022) of parallelisms which have the same spread structure, but are not transitive deficiency one.

Keywords: finite projective space; parallelism; automorphism; transitive group; deficiency one

MSC: 51E23

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1. Introduction

We consider the n-dimensional projective space $PG(n,q)$ over the finite field $GF(q)$. A *spread* in this finite projective space is a set of lines which partition the point set. Spreads exist when $n+1$ is even. Two spreads are isomorphic if there is an automorphism (collineation) of the projective space which maps one to the other.

A partition of the set of lines by spreads is a *parallelism*. Two parallelisms are isomorphic if there is an automorphism of $PG(n,q)$ which maps the spreads of one parallelism to spreads of the other. An automorphism of a parallelism is an automorphism of $PG(n,q)$ which preserves the parallelism, namely it maps each of its spreads to a spread of the same parallelism.

Spreads and parallelisms of $PG(n,q)$ are of interest for problems from projective geometry [1], design theory [2], network coding [3], error-correcting codes [4], and cryptography [5]. The relation of spreads of $PG(3,q)$ to translation planes of order q^2 is well known [6]. Every regular parallelism in $PG(3,q)$ is connected to a spread in $PG(7,q)$, and, hence, to a translation plane of order q^4 [7,8] and vice versa [9]. Recent intensive research is also motivated by the relation to subspace codes [10] and Grassmannian codes [11] involved in random network coding [3]. Modern data centers use PIR codes to reduce the storage overhead and one of the ways they can be constructed is based on parallelisms of $PG(n,q)$ [12]. These links of the research topic to problems from other research fields make it very important. More details and data on spreads and parallelisms can be found in [13–15].

Projective spaces of dimension $n=3$ are the most studied ones. The existence of parallelisms of $PG(3,q)$ follows from the constructions of Denniston [16] and Beutelspacher [17].

The first constructed and best known parallelisms are in $\mathrm{PG}(3,2)$ [18]. Recently, Betten [19] classified all parallelisms of $\mathrm{PG}(3,3)$ by computer. For bigger parameters, n and q, the classification of all parallelisms of the corresponding projective space becomes infeasible. During the years many authors have studied parallelisms and a lot of theoretical results have been obtained, for example [7,13,17,20,21]. There are many results found by computational methods too, for instance [19,22–25]. Authors often restrict the search space by assuming admissible automorphism groups or additional properties of the constructed objects. Examples of such properties follow.

A parallelism is *transitive* if it has an automorphism group which is transitive on the spreads. It can be defined by one spread and the corresponding automorphism group. A transitive parallelism is *cyclic* if there is an automorphism which permutes its spreads in one cycle. The easiest case for construction of parallelisms is to consider cyclic parallelisms. White [26] proved the non-existence of cyclic parallelisms of $\mathrm{PG}(2i-1,q)$ with $gcd(2i-1,q-1)>1$ which holds for $\mathrm{PG}(3,7)$ too.

A *regulus* of $\mathrm{PG}(3,q)$ is a set of $q+1$ mutually skew lines, such that any line intersecting three elements of the regulus intersects all its elements. Such a line is called a *transversal*. All the transversals of a regulus form its *opposite regulus*. A spread of $\mathrm{PG}(3,q)$ is *regular* if it contains the unique regulus determined by any three of its elements. The regular spread is unique up to isomorphism [20]. A spread is called *Hall spread* if it can be obtained from a regular spread by a replacement of one regulus by its opposite.

A parallelism is *uniform* if all its spreads are isomorphic. A parallelism is *regular* if all its spreads are regular. There exists an infinite class of cyclic regular parallelisms of $\mathrm{PG}(3,q)$ for each $q\equiv 2\ (mod\ 3)$ due to Pentilla and Williams [21]. It incorporates regular parallelisms in $\mathrm{PG}(3,8)$ [27] and in $\mathrm{PG}(3,5)$ [23]. Parameters of $\mathrm{PG}(3,7)$ do not match to this class.

The number of points equals the number of hyperplanes of $\mathrm{PG}(n,q)$. The dual space of $\mathrm{PG}(3,q)$ can be obtained by reversing the inclusion relation, namely, in the dual space hyperplanes become points, points become hyperplanes, and lines stay lines. A spread is a spread and a parallelism is a parallelism in the dual space of $\mathrm{PG}(3,q)$ [28]. A parallelism which is isomorphic to its dual is *self-dual*.

A *deficiency one parallelism* is a partial parallelism with one spread less than the parallelism. Each deficiency one parallelism can be uniquely extended to a parallelism. A *transitive (partial) parallelism* possesses an automorphism group acting transitively on its spreads. A *transitive deficiency one parallelism* is a parallelism with an automorphism group that fixes the deficiency spread and is transitive on the remaining spreads. Therefore, a transitive deficiency one parallelism can be completely defined by the deficiency spread, one spread from the transitive part, and the corresponding automorphism group.

Biliotti, Jha, and Johnson [29] and Diaz, Johnson, and Montinaro [30] determine the properties of transitive deficiency one parallelisms of finite projective spaces. They prove that the deficiency spread must be regular and the automorphism group should contain an elation subgroup (a subgroup which fixes all points of one line) of order q^2 ([14] Theorem 267).

There is an infinite class of transitive deficiency one parallelisms in $\mathrm{PG}(3,q)$ for $q=p^r$, p an odd prime, due to Johnson. Their deficiency spread is regular, the remaining spreads are Hall, and the parallelisms are invariant under the full central collineation group (all automorphisms of the projective space which fix the points of one line) ([14] Theorem 174). Let us call them Johnson type parallelisms. They comprise part of all transitive deficiency one parallelisms in $\mathrm{PG}(3,q)$. The rest are invariant under subgroups of the full central collineation group [31].

Johnson-type parallelisms have, in general, the same structure as the parallelisms of Denniston [16] and Beutelspacher [17], but the construction used by Johnson allows to consider the isomorphism classes of the parallelisms too. In ([32] Corollary 26), Johnson and Pomareda show that for $q=p$, an odd prime, the number of nonisomorphic Johnson-type parallelisms is exactly $(p-1)/2$. Theorem 27 in the same paper shows that the order of their full automorphism groups is 2 or 4 times the order of the full central collineation group of the regular spread.

All transitive deficiency one parallelisms of $\mathrm{PG}(3,q)$ for $q = 3,4,5$ are known [19,24,25]. The transitive deficiency one parallelisms of $\mathrm{PG}(3,5)$ are 12 and two of them are of the Johnson type (which complies with the formula given in [32]).

The next open problem is the classification of transitive deficiency one parallelisms of $\mathrm{PG}(3,7)$. This is the main aim of our work. We use the above mentioned group-theoretic characterization ([14] Theorem 267) to construct all transitive deficiency one parallelisms of $\mathrm{PG}(3,7)$. A more general purpose of our work is to construct new examples of parallelisms of $\mathrm{PG}(3,7)$. It is of importance for the study of the properties of these objects.

Our approach to the problem is different from the one which was used in [25] to obtain all the transitive deficiency one parallelisms of $\mathrm{PG}(3,5)$ because the construction of all parallelisms of $\mathrm{PG}(3,7)$ with automorphisms of order $q^2 = 49$ is presently infeasible by our method. That is why we impose additional restrictions which follow from the transitivity requirement, and we obtain all parallelisms for which they hold. We then check which of them are transitive deficiency one.

The paper consists of five sections. Section 2 considers the construction of the parallelisms, and Section 3 the backtrack search algorithms that we use. The obtained results are presented in Section 4, and additional comments on their impact are made in Section 5.

2. Construction

2.1. Preliminaries

We use $V(4,7)$-the 4-dimensional vector space over $GF(7)$ to construct $\mathrm{PG}(3,7)$. The points of $PG(3,7)$ are all 4-dimensional vectors (v_1, v_2, v_3, v_4) over $GF(7)$, such that $v_i = 1$ if i is the maximum index for which $v_i \neq 0$. We sort these 400 vectors in ascending lexicographic order and then assign them numbers, such that $(1,0,0,0)$ is number 1, and $(6,6,6,1)$ number 400. There are 2850 lines (1-dimensional subspaces) in $\mathrm{PG}(3,7)$. We sort them in lexicographic order defined on the numbers of the points they contain and assign to each line a number according to this order. The first line $l_1 = \{1,2,3,4,5,6,7,8\}$, while $l_{2850} = \{57,106,147,188,229,270,311,352\}$.

A spread in $\mathrm{PG}(3,q)$ has 50 lines which partition the point set and a parallelism has 57 spreads. Each invertible matrix $(a_{i,j})_{4\times 4}$ over $GF(7)$ defines an automorphism of this projective space by the map $v'_i = \sum_j a_{i,j} v_j$.

The deficiency spread of a transitive deficiency one parallelism of $\mathrm{PG}(3,7)$ must be regular [29]. In $\mathrm{PG}(3,7)$, there are 1347 spreads up to isomorphism. The regular spread is unique up to isomorphism and the construction of transitive deficiency one parallelisms can start with the deficiency spread S_R. Without loss of generality we choose for S_R the smallest in the considered lexicographic order regular spread which begins with the first line l_1.

2.2. Automorphism Groups

The authomorphism (collineation) group of $\mathrm{PG}(3,7)$ is isomorphic to the projective general semi-linear group $P\Gamma L(4,7)$. Let us denote it by $G \cong P\Gamma L(4,7)$. It is of order $2^{10}.3^4.5^2.7^6.19$. We use permutation representation of the groups. To find the generators of G we use the "Isomorphism and automorphism group" module of the Q-Extension program [33]. For the other computations on the needed groups and subgroups, we use the computer algebra system GAP [34] and the software of the first author where needed.

It follows from the investigations of Biliotti, Diaz, Jha, Johnson, and Montinaro ([14] Theorem 267) that a transitive deficiency one parallelism of $\mathrm{PG}(3,7)$ is invariant under an elation group of order $q^2 = 49$ which fixes a regular spread. That is why we construct parallelisms invariant under the elation group G_{49} of order 49 which fixes the regular spread (S_R).

We start with constructing G_{49}. We are interested in S_R to be the deficiency spread. Since G_{49} should act as an elation on S_R, it must fix one of its lines pointwise. Therefore, we find the generators of the automorphism group of S_R which fixes the line l_1 pointwise. This is the full central collineation group with axis l_1. It is of order $2352 = q^2(q^2 - 1)$ and has up to conjugacy only one subgroup of order 49. We let G_{49} be this subgroup, and try to construct transitive deficiency one parallelisms invariant under it. G_{49} can be generated by the two automorphisms of order 7 of $\mathrm{PG}(3,7)$ which are defined by the following two invertible matrices:

$$M_1 : \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad M_2 : \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We need the normalizer of G_{49} in G for the rejection of isomorphic solutions [35]. It is defined as $N(G_{49}) = \{g \in G | g G_{49} g^{-1} = G_{49}\}$ and $|N(G_{49})| = 1{,}843{,}968$. There are permutations in $N(G_{49})$ which do not preserve S_R. After their removal, a group N of order $|N| = 37{,}632$ remains. We want to construct parallelisms invariant only under this particular G_{49}, therefore we use the group N to find all isomorphism classes.

Theorem 267 [14] states that the subgroup of order q^2 of the full automorphism group of a transitive deficiency one parallelism in $\mathrm{PG}(3,q)$ is a normal subgroup. Hence, we can find the factor group

$$H = N/G_{49}, \quad |H| = 768. \tag{1}$$

2.3. *A Parallelism Invariant under G_{49}*

To obtain a parallelism of $\mathrm{PG}(3,7)$ we need 57 spreads. S_R and the group G_{49} which fixes it are already known. Next the remaining 56 spreads of the parallelism must be added. The group G_{49} acts on the points and lines of $\mathrm{PG}(3,7)$ as it is shown in Table 1. The first point is in l_1 and in eight other line orbits under G_{49}. Hence a parallelism fixing S_R and invariant under G_{49} can be presented by eight spreads (orbit leaders) each one with a spread orbit of length 7 under G_{49}. Let us denote the orbit leader by S_i, and its spread orbit by $O_{S_i}, i = 1, 2, \ldots, 8$.

G_{49} has 8 subgroups $G_{7_i}, i = 1, 2, \ldots, 8$ of order 7 which are in one conjugacy class under G. The action of G_{7_i} on the points and lines of $\mathrm{PG}(3,7)$ is given in Table 1. A line orbit can entirely be included in a spread if it comprises pairwise disjoint lines. We call such an orbit *spread-like* orbit. The line orbits without this property are *non-spread-like*. The line orbits of length 49 are all spread-like. The number of the line orbits of length 7 are given as "spread-like + non-spread-like" in Table 1.

Table 1. Action of G_{49} and G_{7_i} on the lines of $\mathrm{PG}(3,7)$.

Group	Point Orbits			Line Orbits		
	fixed	length 7	length 49	fixed	length 7	length 49
G_{7_i}	8	56	0	57	343 + 56	0
G_{49}	8	0	8	1	0 + 64	49

G_{7_i} fixes l_1 pointwise and other 56 lines not pointwise. It also fixes all the 7 spreads of one of the spread orbits of length 7. Without loss of generality, we can assume that G_{7_i} fixes S_i and the other spreads of O_{S_i}. Figure 1 shows the structure of O_{S_i}. We assume that consecutive numbers have been assigned to the line orbits under G_{49} and to the elements of each line orbit.

Each row in Figure 1 presents one of the 7 spreads of O_{S_i}. Each spread has 8 lines which are fixed by G_{7_i} and are in different orbits of length 7 under G_{49}. We denote such a line by b_l^m where $m = 1, 2, \ldots, 8$ denotes its orbits of length 7 under G_{49}, and $l = 1, 2, \ldots, 7$ is the number of this line in its orbit under G_{49}. The lines b_l^m are all the 56 lines fixed by G_{7_i}.

Apart from the 8 fixed lines each spread of O_{S_i} contains 6 spread-like line orbits $\left\{c^j_{u_1}, c^j_{u_2}, c^j_{u_3}, c^j_{u_4}, c^j_{u_5}, c^j_{u_6}, c^j_{u_7}\right\}$ of length 7 under G_{7_i}, where $j = 1, 2, \ldots, 6$ is the line orbit of length 49 to which the line belongs, and $u_1, u_2, \ldots, u_7$ are the numbers of these lines in the line orbit of length 49. The action of G_{49} and its subgroups coincide with the theoretical results about the automorphism group of the transitive deficiency one parallelisms in $\mathrm{PG}(3, q)$ ([28] Theorem 13).

b^1_1	b^2_1	...	b^7_1	b^8_1	$c^1_1\, c^1_2\, c^1_3\, c^1_4\, c^1_5\, c^1_6\, c^1_7$	$c^2_1\, c^2_2\, c^2_3\, c^2_4\, c^2_5\, c^2_6\, c^2_7$	...	$c^6_1\, c^6_2\, c^6_3\, c^6_4\, c^6_5\, c^6_6\, c^6_7$
b^1_2	b^2_2	...	b^7_2	b^8_2	$c^1_8\, c^1_9\, c^1_{10} c^1_{11} c^1_{12} c^1_{13} c^1_{14}$	$c^2_8\, c^2_9\, c^2_{10} c^2_{11} c^2_{12} c^2_{13} c^2_{14}$	...	$c^6_8\, c^6_9\, c^6_{10} c^6_{11} c^6_{12} c^6_{13} c^6_{14}$
b^1_3	b^2_3	...	b^7_3	b^8_3	$c^1_{15} c^1_{16} c^1_{17} c^1_{18} c^1_{19} c^1_{20} c^1_{21}$	$c^2_{15} c^2_{16} c^2_{17} c^2_{18} c^2_{19} c^2_{20} c^2_{21}$	...	$c^6_{15} c^6_{16} c^6_{17} c^6_{18} c^6_{19} c^6_{20} c^6_{21}$
b^1_4	b^2_4	...	b^7_4	b^8_4	$c^1_{22} c^1_{23} c^1_{24} c^1_{25} c^1_{26} c^1_{27} c^1_{28}$	$c^2_{22} c^2_{23} c^2_{24} c^2_{25} c^2_{26} c^2_{27} c^2_{28}$	...	$c^6_{22} c^6_{23} c^6_{24} c^6_{25} c^6_{26} c^6_{27} c^6_{28}$
b^1_5	b^2_5	...	b^7_5	b^8_5	$c^1_{29} c^1_{30} c^1_{31} c^1_{32} c^1_{33} c^1_{34} c^1_{35}$	$c^2_{29} c^2_{30} c^2_{31} c^2_{32} c^2_{33} c^2_{34} c^2_{35}$	...	$c^6_{29} c^6_{30} c^6_{31} c^6_{32} c^6_{33} c^6_{34} c^6_{35}$
b^1_6	b^2_6	...	b^7_6	b^8_6	$c^1_{36} c^1_{37} c^1_{38} c^1_{39} c^1_{40} c^1_{41} c^1_{42}$	$c^2_{36} c^2_{37} c^2_{38} c^2_{39} c^2_{40} c^2_{41} c^2_{42}$	...	$c^6_{36} c^6_{37} c^6_{38} c^6_{39} c^6_{40} c^6_{41} c^6_{42}$
b^1_7	b^2_7	...	b^7_7	b^8_7	$c^1_{43} c^1_{44} c^1_{45} c^1_{46} c^1_{47} c^1_{48} c^1_{49}$	$c^2_{43} c^2_{44} c^2_{45} c^2_{46} c^2_{47} c^2_{48} c^2_{49}$	...	$c^6_{43} c^6_{44} c^6_{45} c^6_{46} c^6_{47} c^6_{48} c^6_{49}$

Figure 1. The structure of O_{S_i}.

A parallelism invariant under G_{49} has the following structure

$$\left\{S_R, O_{S_1}, O_{S_2}, O_{S_3}, O_{S_4}, O_{S_5}, O_{S_6}, O_{S_7}, O_{S_8}\right\}. \tag{2}$$

A transitive deficiency one parallelism can be considered as a parallelism invariant under G_{49} possessing an automorphism group which is transitive on the non-trivial spread orbits $O_{S_i}, i = 1, 2, \ldots, 8$.

2.4. The Isomorphism of Solutions

For the construction of transitive deficiency one parallelisms we use an exhaustive backtrack search which leads to lexicographically ordered objects. This allows the rejection of partial solutions which are not minimal with respect to the chosen lexicographic order, because they already have been constructed. Our way to achieve this is by applying a minimality test to some of the partial solutions and to all full solutions [36]. The minimality test checks if there exists an element of the group N which maps the current solution to a lexicographically smaller one. If such an element is found, the current partial solution is rejected.

3. Methods

3.1. Preliminaries

Computer-aided classification of combinatorial structures requires methods for their construction, as well as methods for the rejection of equivalent ones. Often these two stages interleave and the objects can be constructed in a way that makes it possible to reduce the appearance of equivalent solutions. Depending on the theoretical requirements imposed on the objects different algorithms can be incorporated. For example, in [37], Bouyukliev uses the concept of canonical augmentation; for the classification of parallelisms in $\mathrm{PG}(3, 3)$, Betten applies the Schmalz algorithm that proceeds along a chain of subobjects which are extended to larger subobjects until the target objects are classified [19].

The software used by the authors in the present work is based on the orderly generation technique [38] which implies exhaustive backtrack search, lexicographically ordered objects and a minimality test on partial solutions. The authors use their own programs written in C++. Each of the authors implements a slightly different construction algorithm.

Each point (the first one too) has to be in each spread of the parallelism. In the described lexicographic order point 1 is in the first 57 lines. Line l_1 is in S_R and without loss of generality we assume that S_1 contains line l_2. Thus, we start by searching for S_1 which begins with line l_2.

3.2. Method 1

All possibilities for S_1 are constructed in advance. For that purpose the subgroup G_{7_1} which fixes l_2 is used. It is defined by the matrix M_1. One of the 14 line orbits under G_{7_1} which S_1 contains (see Figure 1) is known (l_2). The rest are chosen in all possible ways from the 400 spread-like line orbits under G_{7_1} (see Table 1). This is performed by the function `S1Construct(2)`.

```
void S1Construct(int Orb)
{
  for(int i=1; i<=400; i++)
  {
    if(NotPossibleLineOrb(Orb, i)) continue;
    Put(Orb, i);
    if(Orb==14) WriteSpread();
    else S1Construct(Orb+1);
    Take(Orb);
  }
}
```

Here, `NotPossibleLineOrb` returns true if some of the lines of the considered *i*-th line orbit under G_{7_1} intersect lines of the already chosen `Orb-1` orbits, or if no more orbits of this length can be added (there must be 8 orbits of length 1 and 6 orbits of length 7—Figure 1). `Put` adds the orbit to the spread and `Take` removes it. If the spread is ready, it is saved by `WriteSpread`, and if more orbits have to be added, `S1Construct (Orb+1)` is called to choose the next orbit.

There are 15,435 spreads with l_2 fixed by G_{7_1}. In this list remain 13,736 spreads if the spreads not disjoint to S_R are removed. Among them, there are 8084 spreads with a spread orbit of length 7 under G_{49}.

In a transitive deficiency one parallelism the spread S_1 is mapped to the orbit leaders S_i of O_{S_i}, $i = 2, 3, \ldots, 8$ (see (2)) under the action of a subgroup of the group H defined in (1). We apply all the elements of H to each of the 8084 possibilities for S_1 and remove the spreads which are mapped by these elements to less than 7 different spreads. At this step, isomorphism check on the partial solutions with two spread orbits is performed. As a result 15 non-isomorphic partial solutions for the first two spread orbits remain. For each of them we save a list L of all possible $S_i, i = 2, 3, \ldots, 8$ to which S_1 is mapped under H. To construct $S_2, S_3, \ldots, S_8$, a backtrack search on the elements of L is performed by `PConstr(2)`, where `allS=|L|` and `SpreadOK` is called to check if the *i*-th spread possibility does not have common lines with the obtained until this moment partial parallelism.

```
void PConstr(int Spr)
{
  for(int i=1; i<=allS; i++)
  {
    if(SpreadOK(Spr, i))
    {
      PutSpread(Spr, i);
      if(Spr==8) WriteParallelism();
      else PConstr(Spr+1);
      TakeSpread(Spr);
    }
  }
}
```

At the end isomorphism check is applied to the obtained parallelisms. The extension of one of the 15 partial solutions takes about a week on a 3 GHz PC. That is why the resources of the National Centre for High Performance and Distributed Computing were used.

3.3. Method 2

By this method, the backtrack search to obtain parallelisms is not applied on lists of spreads constructed in advance, but on the lines of the projective space. The lines of the spread orbit leaders $S_1, S_2, \ldots, S_8$ are constructed in consecutive order by exhaustive backtrack search on the lines that are not in the partial solution yet. Details on this 'line by line' construction approach can be found in [39]. The algorithm used here is generally the same as in [39], but more restrictions on the constructed spreads are added because our aim is to obtain transitive deficiency one parallelisms.

We know the first spread S_R, fix one line in each spread orbit leader $S_1, S_2, \ldots, S_8$, and choose their other spread lines by `MakePar(2, 1)`.

```
void MakePar(int Line, int Spr)
{
  int Point = FirstMissingPoint(Line, Spr);
  for(int i = FirstLine[Point]; i<=LastLine[Point]; i++)
  {
    if(Possible(Line, i, Spr))
    {
      PutLine(Line, i, Spr);
      if(Line==50)
      {
        if(RestrictionsOK(Spr)
        {
          if(Spr==8) WriteParallelism();
          else MakePar(2, Spr+1);
        }
      }
      else MakePar(Line+1, Spr);
      TakeLine(Line, Spr);
    }
  }
}
```

`FirstMissingPoint` returns the number of the first missing point in spread `Spr`. Each point must be in one line of the spread. Therefore, we try to add only lines containing the first missing point. Their numbers are between `FirstLine[Point]` and `LastLine[Point]`. `Possible` returns true if the considered line i has no common points with the lines of the current partial spread. If this is the case, `PutLine` adds it to the current solution. We continue to add lines until all points are covered (i.e., the number of lines is 50) and when this happens, `RestrictionsOK` checks if the imposed restrictions hold. If so, `MakePar(2, Spr+1)` starts adding the next spread, or `WriteParallelism` saves a ready parallelism.

We obtain 15 non-isomorphic partial solutions for the first two spread orbits after rejecting by `RestrictionsOK` solutions for S_1 for which there exists a line $l_e \notin S_R$ such that no element of H maps any line of O_{S_1} to l_e (because in this case transitivity on S_1 is not possible). We also save a list L of the different spreads to which S_1 is mapped by the elements of H. From the solutions for $S_i, i = 2, 3, \ldots, 8$ we reject by `RestrictionsOK` those which are not in L and those which are in L, but for which there exists a line l_f not contained in the partial solution and not contained in any of the spreads from L that can be added after S_i.

Method 2 was implemented by the first author, and Method 1 by the second one. The number of the transitive deficiency one parallelisms constructed by them is the same. With this technique we also construct some parallelisms which are not transitive deficiency one.

4. Results

4.1. The Transitive Deficiency One Parallelisms of PG(3, 7)

The main aim of this work is to construct transitive deficiency one parallelisms of PG(3, 7). We obtain 46 transitive deficiency one parallelisms of PG(3, 7). The distribution of the constructed parallelisms by the order of their automorphism groups is presented in Table 2, where **All** is the number of parallelisms with a full automorphism group of order **Auts**. The rows below the double line show how many of these parallelisms are invariant under elation groups of given orders, namely **E A** is the number of parallelisms invariant under an elation group of order **A**.

Table 2. Transitive deficiency one parallelisms of PG(3, 7)-order of the full automorphism groups and their elation subgroups.

Auts	2352	4702	9408	All
All	24	18	4	46
E 294	12	8	–	20
E 588	6	5	2	13
E 1176	6	3	1	10
E 2352	–	2	1	3

Three of the transitive deficiency one parallelisms are invariant under the full central collineation group of order 2352 and are, therefore, of Johnson type. The full automorphism group orders and the number of these parallelisms comply with the results obtained in [32]. The remaining 43 transitive deficiency one parallelisms have the same spread structure as the parallelisms from Johnson's infinite family, but do not belong to it.

We construct the duals of these 46 parallelisms. The dual parallelisms are transitive deficiency one parallelisms with the same order of the automorphism groups, but they are not isomorphic to the original parallelisms. The duals of the Johnson-type parallelisms are not of the Johnson type.

4.2. Uniform Deficiency One Parallelisms of PG(3, 7)

By our construction methods we obtain 55,022 non-isomorphic uniform deficiency one parallelisms. All transitive deficiency one parallelisms are among them, but not all the constructed parallelisms are transitive deficiency one. These parallelisms are of particular interest too, because not many parallelisms of PG(3, 7) are known. The order of their full automorphism groups and the number of their spread orbits are presented in Table 3, where **All** is the number of parallelisms with a full automorphism group of order **Auts**. The rows below the double line show how many of these parallelisms have a given number of spread orbits under their full automorphism groups, namely **k orbits** is the number of parallelisms whose spreads are in **k** orbits under the full automorphism group of the parallelism.

Table 3. Spread orbits and order of the full automorphism groups of the constructed uniform deficiency one parallelisms.

Auts	49	294	588	1176	2352	4704	9408	All
All	23,040	29,500	2258	168	34	18	4	55,022
9 orbits	23,040	29,500	2	–	–	–	–	52,542
6 orbits	–	–	1096	4	–	–	–	1100
5 orbits	–	–	1160	8	–	–	–	1168
4 orbits	–	–	–	88	2	–	–	90
3 orbits	–	–	–	68	8	–	–	76
2 orbits	–	–	–	–	24	18	4	46

The parallelisms are constructed assuming invariance under G_{49} and, thus, their spreads are in at most nine orbits (see Section 2.3). Under the full automorphism group the number of spread orbits can decrease. The last row in the table corresponds to the obtained transitive deficiency one parallelisms of $PG(3,7)$. In that case the deficiency spread comprises the first orbit, and the other spreads the second one.

The number of reguli in the regular spread of $PG(3,7)$ is

$$\binom{q^2+1}{3} / \binom{q+1}{3} = 350. \quad (3)$$

The number of reguli in the other spreads of $PG(3,7)$ is smaller. To recognize the type of the spreads in the constructed parallelisms, we use invariants based on the intersection of the spread with the reguli of the projective space as in [40]. These invariants can be represented by the triple (s,r,w), where s is the number of whole reguli in the spread and r (w) is the number of reguli which share exactly seven (six) lines (out of all $q+1=8$ lines) with the spread. For the regular spread of $PG(3,7)$ this invariant is $(s,r,w) = (350,0,0)$ and for the Hall spread it is $(s,r,w) = (106,336,384)$.

All the constructed parallelisms have the same structure: one regular spread and the remaining spreads are Hall spreads. They are available online and can be used for further investigations as well as in suitable applications.

5. Discussion on the Obtained Results

Only several parallelisms of $PG(3,7)$ were known before the present work, while their number is at least 55,022 now. They include all transitive deficiency one parallelisms of this projective space. All the parallelisms constructed in this work are available online and can be used for further investigations, as well as in suitable applications.

Our results comply with previous theoretical results, namely:

- We obtain three parallelisms of Johnson type and it is shown in [32] that for $q=p$, an odd prime, the number of non-isomorphic Johnson-type parallelisms is exactly $(p-1)/2$.
- The order of the full automorphism groups of the parallelisms of Johnson type that we construct is either 4704, or 9408 and it is shown in [32] that the order should be two or four times the order of the full central collineation group of the regular spread.
- None of the transitive deficiency one parallelisms of $PG(3,7)$ is self-dual, as shown in [28].

It is proved in previous papers on transitive deficiency one parallelisms that their deficiency spread is regular, and they are invariant under an elation group of order q^2. Our investigation on the transitive deficiency one parallelisms of $PG(3,7)$ is based on these results. We construct transitive deficiency one parallelisms with one regular spread which are invariant under an elation group of order 49.

Our computer-aided results show that:

- There are 46 transitive deficiency one parallelisms of $PG(3,7)$ which are invariant under an elation group of order 49 and have a regular deficiency spread. The three parallelisms from Johnson's infinite class are among them.
- All the spreads of the constructed transitive deficiency one parallelism of $PG(3,7)$ are Hall spreads and the deficiency spread is regular by assumption.
- The dual of a transitive deficiency one parallelism of $PG(3,7)$ is a transitive deficiency one parallelism with the same order of the full automorphism group as the original parallelism, but it is not isomorphic to the original parallelism.
- The duals of the Johnson type parallelisms of $PG(3,7)$ are not of the Johnson-type.

The observations from the last three items can be made on the computer-aided classification of transitive deficiency one parallelisms of PG$(3,5)$ too [25]. It is possible that they hold in each PG$(3,p)$ for an odd prime p, and this is an interesting open problem, which we think is a challenge for future theoretical considerations. If the duals of the Johnson-type parallelisms of PG$(3,p)$ are always transitive deficiency one, but always not of the Johnson-type, then Johnson's infinite family will yield a twice bigger number of transitive deficiency one parallelisms (the family members and their duals).

We believe that our present computer-aided investigation will be helpful to further theoretical and computer-aided considerations of these interesting objects and their applications.

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Data Availability Statement: The data obtained and presented in this study are openly available at http://www.moi.math.bas.bg/moiuser/~stela (accessed on 24 May 2023). Transitive deficiency one parallelisms are in a separate file with the point-line incidence matrix of PG$(3,7)$. For each parallelism the order of its full automorphism group, the order of the elation subgroup, and the name of its dual parallelism is given.

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Abbreviations

The following abbreviations and notations are used in this manuscript:

$GF(q)$	-	the finite field with q elements with addition and multiplication defined on them
PG(n,q)	-	the n-dimensional projective space over the finite field $GF(q)$
$P\Gamma L(m,q)$	-	the projective general semi-linear group
elation subgroup	-	a group of automorphisms of PG$(3,q)$ which fixes all points of one line
central collineation	-	elation (automorphism of PG$(3,q)$ which fixes all points of one line)
G	-	the group of automorphisms of the considered projective space PG$(3,7)$, $G \cong P\Gamma L(4,7)$ ($G \cong P\Gamma L(n+1,q)$)
G_{49}	-	an elation subgroup of G of order 49
$G_{7_1}, G_{7_2}, \ldots, G_{7_8}$	-	the eight subgroups of order 7 of G_{49}
S_R	-	the regular deficiency spread of the constructed parallelisms
$N(G_{49})$	-	the normalizer of G_{49} in G, $N(G_{49}) = \{g \in G \mid gG_{49}g^{-1} = G_{49}\}$
N	-	the subgroup of $N(G_{49})$ which preserves S_R
H	-	$H = N/G_{49}$
$S_1, S_2, \ldots, S_8$	-	representatives (orbit leaders) of the spread orbits of length 7
$O_{S_1}, O_{S_2}, \ldots, O_{S_8}$	-	the spread orbits of length 7 of a parallelism invariant under G_{49}

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