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A Class of Quasilinear Equations with Distributed Gerasimov–Caputo Derivatives

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Abstract: Quasilinear equations in Banach spaces with distributed Gerasimov–Caputo fractional derivatives, which are defined by the Riemann–Stieltjes integrals, and with a linear closed operator A, are studied. The issues of unique solvability of the Cauchy problem to such equations are considered. Under the Lipschitz continuity condition in phase variables and two types of continuity over all variables of a nonlinear operator in the equation, we obtain two versions on a theorem on the nonlocal existence of a unique solution. Two similar versions of local unique solvability of the Cauchy problem are proved under the local Lipschitz continuity condition for the nonlinear operator. The general results are used for the study of an initial boundary value problem for a generalization of the nonlinear phase field system of equations with distributed derivatives with respect to time.

Keywords: distributed fractional derivative; fractional differential equation; Cauchy problem; quasiliner equation; fixed point theorem; initial boundary value problem

MSC: 34G20; 35R11; 34A08; 47D99

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1. Introduction

Various classes of fractional differential equations are the subjects of intensive research by many scientists in recent decades. Such equations are of interest both because of their increasing importance in applied investigations [1–4], and from the point of view of the development of theory [5–8]. A special class consists of equations with distributed derivatives (or so-called continual derivatives, mean derivatives), which, in partial, are applied to the research of some real phenomena and processes, when an order of a fractional derivative in a model depends on the process parameters: in the theory of viscoelastic media [9], in modeling dielectric induction and diffusion [10,11], in the kinetic theory [12], and in other scientific fields [13–16]. These works initiated other investigations of the equations with distributed derivatives from the point of view of the qualitative theory of differential equations [17–22].

The main aim of the present work is to investigate the Cauchy problem for a class of abstract quasilinear equations with distributed derivatives. Let $\mathcal{Z}$ be a Banach space, D^β be the fractional Gerasimov–Caputo derivative for $\beta > 0$ and the fractional Riemann–Liouville integral for $\beta \leq 0$, A be a linear closed densely defined in $\mathcal{Z}$ operator. Consider the Cauchy problem

$$D^k z(t_0) = z_k, \quad k = 0, 1, \dots, m-1, \tag{1}$$

for the quasilinear equation

$$\int_b^c D^\alpha z(t)d\mu(\alpha) = Az(t) + B\left(t, \int_{b_1}^{c_1} D^\alpha z(t)d\mu_1(\alpha), \dots, \int_{b_n}^{c_n} D^\alpha z(t)d\mu_n(\alpha)\right), \tag{2}$$

where $b < c$, $m - 1 < c \leq m \in \mathbb{N}$, $\mu \in BV((b, c]; \mathbb{C})$ (i. e., μ is a function of a bounded variation), c is a variation point of the measure $d\mu(\alpha)$, $b_l < c_l$, $c_1 \leq c_2 \leq \cdots \leq c_n < c$, $\mu_l \in BV((b_l, c_l]; \mathbb{C})$, c_l is a variation point of the measure $d\mu_l(\alpha)$, $l = 1, 2, \dots, n$. Equality (2) contains the Riemann–Stieltjes integrals.

Linear equations with a distributed order derivative

$$\int_b^c \omega(\alpha)D^\alpha z(t)d\alpha = Az(t) \tag{3}$$

were studied in works [23–25], where $\omega : (b, c) \to \mathbb{C}$ and A is a bounded operator, or a generator of an analytic resolving family of a fractional equation. For $b = 0$, $c \in (0, 1]$, a criteria in terms of conditions on a linear closed operator A for the existence of an analytic resolving family of operators for Equation (3) were obtained in paper [26]. In the work [27], these criteria were generalized to the case $c > 1$ and a perturbations theorem on generators of analytic resolving operators families for (3) was obtained. Analogous results for the equation with a discretely distributed Gerasimov–Caputo derivative

$$\sum_{k=1}^n \omega_k D^{\alpha_k} z(t) = Az(t).$$

were obtained in [28]. All these results were generalized and combined in general formulations with the Riemann–Stieltjes integral in the definition of the distributed derivative [29]. Recall that an arbitrary function μ with a bounded variation has the form $\mu = \mu_c + \mu_d$, where μ_c is a continuous function with a bounded variation, and μ_d is a jumps function. Consequently, the left-hand side of (2) has the form

$$\int_b^c D^\alpha z(t)d\mu(\alpha) = \int_b^c \mu_c'(\alpha)D^\alpha z(t)d\alpha + \sum_{k=1}^n \omega_k D^{\alpha_k} z(t),$$

if there exists an appropriate derivative μ_c', α_k are points of jumps of the function μ_d, ω_k are values of jumps, $k = 1, 2, \dots, n$.

Each result in the listed works [23–29] on the linear homogeneous equation is accompanied by theorems on the solvability of the corresponding linear inhomogeneous equation. Here, such theorems are used for the study of the Cauchy problem (1) to quasilinear Equation (2). Note that the above-mentioned papers concern equations with distributed order derivatives in finite-dimensional spaces, or in the linear case, or with a bounded operator A in a Banach space (see [25]). In the present paper, we have studied for the first time a quasilinear equation with distributed derivatives and an unbounded A operator in an infinite-dimensional space.

In the second section of the present work, the main definitions and results on the solvability of the inhomogeneous equation are formulated. The third section contains the definition of special functional spaces, statements and proofs of their properties and properties of operators of distributed Gerasimov–Caputo fractional derivatives, which are acting in these spaces. In the fourth section, theorems on nonlocal solvability of Cauchy problem (1) and (2) are proved under the condition $B \in C([t_0, T] \times \mathcal{Z}^n; D_A)$, where D_A is the domain of A with its graph norm, or with $B \in C([t_0, T] \times \mathcal{Z}^n; \mathcal{Z})$, but in a slightly narrower functional space. In the fifth section, analogous results were obtained on the local unique solvability of problem (1) and (2) with $B \in C(U; D_A)$, or $B \in C(U; \mathcal{Z})$, where U is an open set in $\mathbb{R} \times \mathcal{Z}^n$. The last section contains an application of abstract results to the research of an initial boundary value problem for some generalization of the phase field system of equations with the distributed order Gerasimov–Caputo time-derivatives.

2. Linear Equation and Resolving Families

Let $\mathcal{Z}$ be a Banach space, denote for $\beta > 0$, $h : (t_0, \infty) \to \mathcal{Z}$ the Riemann–Liouville fractional integral of an order $\beta > 0$

$$J^\beta h(t) := \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} h(s)ds, \quad t > t_0.$$

Let $m - 1 < \alpha \leq m \in \mathbb{N}$, D^m be the derivative of the m-th order, then

$$D^\alpha h(t) := D^m J^{m-\alpha} \left(h(t) - \sum_{k=0}^{m-1} h^{(k)}(t_0) \frac{(t-t_0)^k}{k!} \right)$$

is the Gerasimov–Caputo derivative of the order α [1,2,30]. It will be assumed that $D^\alpha h(t) := J^{-\alpha} h(t)$ for $\alpha < 0$.

For a function $h : \mathbb{R}_+ \to \mathcal{Z}$, the Laplace transform is denoted by $\widehat{h}$ or Lap$[h]$, if the expression for h is too large. For the Gerasimov–Caputo derivative of an order $\alpha \in (m-1, m]$, it is known the equality (see, e.g., [6])

$$\widehat{D^\alpha h}(\lambda) = \lambda^\alpha \widehat{h}(\lambda) - \sum_{k=0}^{m-1} h^{(k)}(0) \lambda^{\alpha-1-k}. \tag{4}$$

The notations $S_{\theta,a} := \{\mu \in \mathbb{C} : |\arg(\mu - a)| < \theta, \mu \neq a\}$ for $\theta \in [\pi/2, \pi]$, $a \in \mathbb{R}$, $\Sigma_\psi := \{t \in \mathbb{C} : |\arg t| < \psi, t \neq 0\}$ for $\psi \in (0, \pi/2]$ will be used later. Besides, the Banach space of all linear continuous operators from $\mathcal{Z}$ to $\mathcal{Z}$ will be denoted by $\mathcal{L}(\mathcal{Z})$, and denote the set of all linear closed operators, densely defined in $\mathcal{Z}$, acting in the space $\mathcal{Z}$, by $\mathcal{C}l(\mathcal{Z})$. The domain D_A of an operator $A \in \mathcal{C}l(\mathcal{Z})$ endows by its graph norm $\|\cdot\|_{D_A} := \|\cdot\|_{\mathcal{Z}} + \|A\cdot\|_{\mathcal{Z}}$. Hence, D_A is a Banach space with this norm due to the closedness of A.

Consider the Cauchy problem

$$D^k z(0) = z_k, \quad k = 0, 1, \ldots, m-1, \tag{5}$$

for the distributed order equation

$$\int_b^c D^\alpha z(t) d\mu(\alpha) = Az(t), \ \ t > 0, \tag{6}$$

where $b, c \in \mathbb{R}$, $b < c$, $m - 1 < c \leq m \in \mathbb{N}$, $\mu : (b, c] \to \mathbb{C}$ is a function with a bounded variation, briefly $\mu \in BV((b,c]; \mathbb{C})$, c is a variation point of the measure $d\mu(\alpha)$. Equality (6) contains the Riemann–Stieltjes integral. A solution of problem (5) and (6) is a function $z \in C^{m-1}(\overline{\mathbb{R}}_+; \mathcal{Z}) \cap C(\mathbb{R}_+; D_A)$, such that $\int_b^c D^\alpha z(t) d\mu(\alpha) \in C(\mathbb{R}_+; \mathcal{Z})$ and equalities (5) and (6) for $t \in \mathbb{R}_+$ are fulfilled. Hereafter, $\overline{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{0\}$.

Under the conditions of this section, consider the analytic on $S_{\pi,0}$ functions

$$W(\lambda) := \int_b^c \lambda^\alpha d\mu(\alpha) \quad W_k(\lambda) := \int_k^c \lambda^\alpha d\mu(\alpha), \ k = 0, 1, \ldots, m-1,$$

also defined by Riemann–Stieltjes integrals. Here and further, the main branch of the power function is considered.

Lemma 1 ([29]). *Let $b, c \in \mathbb{R}$, $b < c$, $m-1 < c \leq m \in \mathbb{N}$, $\mu \in BV((b,c];\mathbb{C})$, c be a variation point of the measure $d\mu(\alpha)$. Then for $k, l = 0, 1, \dots, m-1$, $k > l$,*

$$\forall \varepsilon \in (0,c) \quad \exists C, \varrho > 0 \quad \forall \lambda \in S_{\pi,0} \setminus \{\lambda \in \mathbb{C} : |\lambda| < \varrho\} \quad |W_k(\lambda)| \geq C|\lambda|^{c-\varepsilon};$$

$$\forall \varepsilon \in (0,c) \quad \exists C, \varrho > 0 \quad \forall \lambda \in S_{\pi,0} \setminus \{\lambda \in \mathbb{C} : |\lambda| < \varrho\} \quad |W(\lambda)| \geq C|\lambda|^{c-\varepsilon};$$

$$\exists C, \varrho > 0 \quad \forall \lambda \in S_{\pi,0} \setminus \{\lambda \in \mathbb{C} : |\lambda| < \varrho\} \quad |W_k(\lambda) - W_l(\lambda)| \leq C|\lambda|^k;$$

$$\exists C, \varrho > 0 \quad \forall \lambda \in S_{\pi,0} \setminus \{\lambda \in \mathbb{C} : |\lambda| < \varrho\} \quad |W_k(\lambda) - W(\lambda)| \leq C|\lambda|^k.$$

Definition 1 ([29]). *A family of operators $\{S_k(t) \in \mathcal{L}(\mathcal{Z}) : t \geq 0\}$, $k \in \{0, 1, \dots, m-1\}$, is called k-resolving for Equation (6), if:*

(i) $S_k(t)$ is strongly continuous for $t \geq 0$;
(ii) $S_k(t)[D_A] \subset D_A$, $S_k(t)Az = AS_k(t)z$ for all $z \in D_A$, $t \geq 0$;
(iii) $S_k(t)z_k$ is a solution of the Cauchy problem

$$D^l z(0) = 0, \quad l \in \{0, 1, \dots, m-1\} \setminus \{k\}, \quad D^k z(0) = z_k \tag{7}$$

to Equation (6) for any $z_k \in D_A$.

Remark 1. *Thus, a k-resolving family $\{S_k(t) \in \mathcal{L}(\mathcal{Z}) : t \geq 0\}$ consists of operators, such that $S_k(t)$ for $t \geq 0$ maps arbitrary $z_k \in D_A$ into the value $z(t) = S_k(t)z_k$ at the point t of a solution of Cauchy problem (6) and (7). Therefore, the families $\{S_k(t) \in \mathcal{L}(\mathcal{Z}) : t \geq 0\}$, $k = 0, 1, \dots, m-1$, entirely describe the solution of the complete Cauchy problem (5) and (6).*

A resolving family of operators is called *analytic* if it has an analytic continuation to a sector Σ_{ψ_0} for some $\psi_0 \in (0, \pi/2]$. An analytic resolving family of operators $\{S(t) \in \mathcal{L}(\mathcal{Z}) : t \geq 0\}$ has a type (ψ_0, a_0) for some $\psi_0 \in (0, \pi/2]$, $a_0 \in \mathbb{R}$, if for any $\psi \in (0, \psi_0)$, $a > a_0$ there exists $C(\psi, a) > 0$, such that for every $t \in \Sigma_\psi$ the inequality $\|S(t)\|_{\mathcal{L}(\mathcal{Z})} \leq C(\psi, a)e^{a\mathrm{Re}\, t}$ holds.

Remark 2. *Similar notions of analytic resolving families of operators are used in the study of integral evolution equations [31] and fractional differential equations [32]. They generalize the notion of an analytic resolving semigroup of operators for the first order equation $D_t^1 z(t) = Az(t)$ (see [33–35]).*

Denote $\rho(A) := \{\lambda \in \mathbb{C} : (\lambda I - A)^{-1} \in \mathcal{L}(\mathcal{Z})\}$ for an operator $A \in \mathcal{C}l(\mathcal{Z})$, i. e., $\rho(A)$ is the resolvent set of A. Define a class $\mathcal{A}_W(\theta_0, a_0)$ (see [29]) of all operators $A \in \mathcal{C}l(\mathcal{Z})$, such that:

(i) there exist $\theta_0 \in (\pi/2, \pi]$, $a_0 \geq 0$, such that $W(\lambda) \in \rho(A)$ for every $\lambda \in S_{\theta_0,a_0}$;
(ii) for every $\theta \in (\pi/2, \theta_0)$, $a > a_0$ there exists $K(\theta, a) > 0$, such that for all $\lambda \in S_{\theta,a}$

$$\|(W(\lambda)I - A)^{-1}\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{|\lambda| K(\theta, a)}{|W(\lambda)||\lambda - a|}.$$

Remark 3. *The classes $\mathcal{A}_W(\theta_0, a_0)$ in works [26–28] are partial cases of this class with the same denotation $\mathcal{A}_W(\theta_0, a_0)$ due to the more general construction of the distributed derivative in the present work. If μ is a constant, excluding a unique jump in the point $\alpha = c$, class $\mathcal{A}_W(\theta_0, a_0)$ coincides with the class $\mathcal{A}_c(\theta_0, a_0)$, defined in [32]. For $c = 1$, this class contains generators of analytic operator semigroups [33–35].*

Remark 4. *If $A \in \mathcal{L}(\mathcal{Z})$, then $A \in \mathcal{A}_W(\theta_0, a_0)$ for some $\theta_0 \in (\pi/2, \pi)$, $a_0 \geq 0$ (see [29]).*

For an operator $A \in \mathcal{A}_W(\theta_0, a_0)$, the operators

$$Z_k(t) := \frac{1}{2\pi i} \int_\Gamma \frac{W_k(\lambda)}{\lambda^{k+1}} (W(\lambda)I - A)^{-1} e^{\lambda t} d\lambda, \quad k = 0, 1, \dots, m-1,$$

are defined for $t > 0$, where $\Gamma = \Gamma_+ \cup \Gamma_- \cup \Gamma_0$, $\Gamma_\pm = \{\mu \in \mathbb{C} : \mu = a + re^{\pm i\theta}, r \in (\delta, \infty)\}$, $\Gamma_0 = \{\mu \in \mathbb{C} : \mu = a + \delta e^{i\varphi}, \varphi \in (-\theta, \theta)\}$ for some $\delta > 0$, $a > a_0$, $\theta \in (\pi/2, \theta_0)$.

Theorem 1 ([29]). *Let $b, c \in \mathbb{R}$, $b < c$, $m - 1 < c \leq m \in \mathbb{N}$, $\mu \in BV((b, c]; \mathbb{C})$, c be a variation point of the measure $d\mu(\alpha)$. Then, there exists an analytic 0-resolving family of operators of the type $(\theta_0 - \pi/2, a_0)$ for Equation (6), if and only if $A \in \mathcal{A}_W(\theta_0, a_0)$. In this case, there exists a unique k-resolving family of operators for every $k = 0, 1, \dots, m-1$, and it has the form $\{Z_k(t) \in \mathcal{L}(\mathcal{Z}) : t \geq 0\}$.*

Remark 5. *The theorem shows that the condition $A \in \mathcal{A}_W(\theta_0, a_0)$ is not only sufficient, but also necessary for the analytic resolving families existence, in other words, for the unique solvability of problem (5) and (6) in the considered sense.*

Theorem 2 ([29]). *Let $b, c \in \mathbb{R}$, $b < c$, $2 < c$, $\mu \in BV((b, c]; \mathbb{C})$, c be a variation point of the measure $d\mu(\alpha)$, $\mu(\alpha) \in \mathbb{R}$ for all α from some left neighborhood of c, $A \in \mathcal{A}_W(\theta_0, a_0)$ for some $\theta_0 \in (\pi/2, \pi)$, $a_0 \geq 0$. Then, $A \in \mathcal{L}(\mathcal{Z})$.*

Denote for $t > 0$

$$Z(t) := \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (W(\lambda)I - A)^{-1} d\lambda.$$

Recall that $C^\gamma([0, T]; \mathcal{Z})$ with $\gamma \in (0, 1]$ is the class of functions $f : [0, T] \to \mathcal{Z}$, such that for all $t, s \in [0, T]$ the Hölder condition $\|f(t) - f(s)\|_\mathcal{Z} \leq C|t - s|^\gamma$ is satisfied with some $C > 0$.

Theorem 3 ([29]). *Let $b, c \in \mathbb{R}$, $b < c$, $m - 1 < c \leq m \in \mathbb{N}$, $\mu \in BV((b, c]; \mathbb{C})$, c be a variation point of the measure $d\mu(\alpha)$, $\theta_0 \in (\pi/2, \pi]$, $a_0 \geq 0$, $A \in \mathcal{A}_W(\theta_0, a_0)$, $g \in C([0, T]; D_A) \cup C^\gamma([0, T]; \mathcal{Z})$, $\gamma \in (0, 1]$, $z_k \in D_A$, $k = 0, 1, \dots, m-1$. Then, the function*

$$z(t) = \sum_{k=0}^{m-1} Z_k(t) z_k + \int_0^t Z(t - s) g(s) ds$$

is a unique solution of Cauchy problem (5) for the equation

$$\int_b^c D^\alpha z(t) d\mu(\alpha) = Az(t) + f(t).$$

3. Some Properties of Distributed Derivatives

For $t_0, T, \beta \in \mathbb{R}$, $t_0 < T$, denote the space $C^{m-1,\beta}([t_0, T]; \mathcal{Z}) := \{z \in C^{m-1}([t_0, T]; \mathcal{Z}) : D^\beta z \in C([t_0, T]; \mathcal{Z})\}$ with the norm

$$\|z\|_{C^{m-1,\beta}([t_0,T];\mathcal{Z})} := \|z\|_{C^{m-1}([t_0,T];\mathcal{Z})} + \|D^\beta z\|_{C([t_0,T];\mathcal{Z})}.$$

It is evident, that $C^{m-1,\beta}([t_0, T]; \mathcal{Z}) = C^{m-1}([t_0, T]; \mathcal{Z})$, if and only if $\beta \leq m - 1$. It can be proved directly that even for $\beta > m - 1$ the space $C^{m-1,\beta}([t_0, T]; \mathcal{Z})$ is complete.

Lemma 2. *Let $m-1<\beta\le m\in\mathbb{N}$, $z\in C^{m-1,\beta}([t_0,T];\mathcal{Z})$. Then, for every $\alpha\in[0,\beta]$ $D^\alpha z\in C([t_0,T];\mathcal{Z})$, moreover, there exists $C>0$, such that for all $\alpha\in[0,\beta]$, $z\in C^{m-1,\beta}([t_0,T];\mathcal{Z})$*

$$\|D^\alpha z\|_{C([t_0,T];\mathcal{Z})}\le C\|z\|_{C^{m-1,\beta}([t_0,T];\mathcal{Z})}.$$

Proof. If $m-1<\alpha<\beta<m\in\mathbb{N}$, we have for $y\in C^m([t_0,T];\mathcal{Z})$

$$D^mJ^{\beta-\alpha}y(t)=D^m\int_0^{t-t_0}\frac{s^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}y(t-s)ds=$$

$$=D^{m-1}\left(\frac{(t-t_0)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}y(t_0)+\int_0^{t-t_0}\frac{s^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}D^1y(t-s)ds\right)=$$

$$=D^{m-2}\left(\frac{(t-t_0)^{\beta-\alpha-2}}{\Gamma(\beta-\alpha-1)}y(t_0)+\frac{(t-t_0)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}D^1y(t_0)+\int_0^{t-t_0}\frac{s^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}D^2y(t-s)ds\right)=$$

$$=\cdots=\sum_{l=0}^{m-1}\frac{(t-t_0)^{\beta-\alpha-m+l}}{\Gamma(\beta-\alpha-m+l+1)}D^ly(t_0)+\int_{t_0}^{t}\frac{(t-s)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}D^my(s)ds=$$

$$=\sum_{l=0}^{m-1}\frac{(t-t_0)^{\beta-\alpha-m+l}}{\Gamma(\beta-\alpha-m+l+1)}D^ly(t_0)+J^{\beta-\alpha}D^my(t). \tag{8}$$

Therefore, for $z\in C^{m-1,\beta}([t_0,T];\mathcal{Z})$

$$D^\alpha z(t)=D^mJ^{\beta-\alpha}J^{m-\beta}\left(z(t)-\sum_{k=0}^{m-1}D^kz(t_0)\frac{(t-t_0)^k}{k!}\right)=$$

$$=\sum_{l=0}^{m-1}\frac{(t-t_0)^{\beta-\alpha-m+l}}{\Gamma(\beta-\alpha-m+l+1)}D^ly(t_0)+J^{\beta-\alpha}D^my(t)=J^{\beta-\alpha}D^\beta z(t),$$

where

$$y(t)=J^{m-\beta}\left(z(t)-\sum_{k=0}^{m-1}D^kz(t_0)\frac{(t-t_0)^k}{k!}\right)\in C^m([t_0,T];\mathcal{Z}),$$

$$D^ly(t_0)=0,\ l=0,1,\dots,m-1.$$

Hence, $D^\alpha z\in C([t_0,T];\mathcal{Z})$ and

$$\|D^\alpha z\|_{C([t_0,T];\mathcal{Z})}\le\frac{(T-t_0)^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}\|D^\beta z\|_{C([t_0,T];\mathcal{Z})}\le\max_{s\in[0,\beta]}\frac{(T-t_0)^s}{\Gamma(s+1)}\|D^\beta z\|_{C([t_0,T];\mathcal{Z})}.$$

Let $m-1<\alpha<m=\beta$, then $D^\alpha z(t)=J^{m-\alpha}D^mz(t)$, and we have the same result.

In the case $n-1<\alpha<n\le m-1<\beta<m$, for $z\in C^{m-1,\beta}([t_0,T];\mathcal{Z})$ we can obtain similarly

$$D^\alpha z(t)=D^nJ^{n-\alpha}\left(z(t)-\sum_{k=0}^{n-1}D^kz(t_0)\frac{(t-t_0)^k}{k!}\right)=$$

$$=D^mJ^{m-n}J^{n-\alpha}\left(z(t)-\sum_{k=0}^{n-1}D^kz(t_0)\frac{(t-t_0)^k}{k!}\right)=$$

$$=D^mJ^{\beta-\alpha}J^{m-\beta}\left(z(t)-\sum_{k=0}^{n-1}D^kz(t_0)\frac{(t-t_0)^k}{k!}\right)=J^{\beta-\alpha}D^\beta z(t)$$

due to (8), since for $k = 0, 1, \dots, m-1$

$$\|D^k J^{m-\beta} z(t)\|_{\mathcal{Z}} \le \frac{(t-t_0)^{\beta-m}}{\Gamma(\beta-m+1)} \|z\|_{C^{m-1}([t_0,t];\mathcal{Z})} \to 0, \quad t \to t_0.$$

Consequently,

$$\|D^\alpha z\|_{C([t_0,T];\mathcal{Z})} \le \max_{s\in[0,\beta]} \frac{(T-t_0)^s}{\Gamma(s+1)} \|D^\beta z\|_{C([t_0,T];\mathcal{Z})}.$$

If $\beta = m$ here, then

$$D^\alpha z(t) = D^m J^{m-\alpha}\left(z(t) - \sum_{k=0}^{n-1} D^k z(t_0) \frac{(t-t_0)^k}{k!}\right) =$$

$$= \sum_{l=n}^{m-1} \frac{(t-t_0)^{l-\alpha}}{\Gamma(l-\alpha+1)} D^l z(t_0) + J^{m-\alpha} D^m y(t)$$

and

$$\|D^\alpha z\|_{C([t_0,T];\mathcal{Z})} \le m \max_{s\in[0,\beta]} \frac{(T-t_0)^s}{\Gamma(s+1)} \|z\|_{C^{m-1,\beta}([t_0,T];\mathcal{Z})}.$$

Finally, in the case $\alpha \in \{0, 1, \dots, m-1\}$, we have the estimate $\|D^\alpha z\|_{C([t_0,T];\mathcal{Z})} \le \|z\|_{C^{m-1}([t_0,T];\mathcal{Z})}$. □

Corollary 1. *Let $z \in C^{m-1}([t_0,T];\mathcal{Z})$. Then, $D^\alpha z \in C([t_0,T];\mathcal{Z})$ for all $\alpha \in [0, m-1]$, besides, there exists $C > 0$, such that for all $\alpha \in [0, m-1]$, $z \in C^{m-1}([t_0,T];\mathcal{Z})$*

$$\|D^\alpha z\|_{C([t_0,T];\mathcal{Z})} \le C\|z\|_{C^{m-1}([t_0,T];\mathcal{Z})}.$$

Proof. Take $\beta = m-1$ in the proof of Lemma 2. □

Remark 6. *If $z \in C([t_0,T];\mathcal{Z})$ and $\alpha < 0$, then it is obvious that $D^\alpha z \in C([t_0,T];\mathcal{Z})$.*

Corollary 2. *Let $m-1 < c \le m \in \mathbb{N}$, $b < c$, $\mu \in BV((b,c];\mathbb{C})$, c be a variation point of the measure $d\mu(\alpha)$, $z \in C^{m-1,c}([t_0,T];\mathcal{Z})$. Then, $\int_b^c D^\alpha z(t) d\mu(\alpha) \in C([t_0,T];\mathcal{Z})$, besides, there exists $C_1 > 0$, such that for all $z \in C^{m-1,c}([t_0,T];\mathcal{Z})$*

$$\left\|\int_b^c D^\alpha z(t) d\mu(\alpha)\right\|_{C([t_0,T];\mathcal{Z})} \le C_1 \|z\|_{C^{m-1,c}([t_0,T];\mathcal{Z})}.$$

Proof. Indeed, due to Lemma 2

$$\left\|\int_b^c D^\alpha z(t) d\mu(\alpha)\right\|_{C([t_0,T];\mathcal{Z})} \le C V_b^c(\mu) \|z\|_{C^{m-1,c}([t_0,T];\mathcal{Z})},$$

where $V_b^c(\mu)$ is the variation of μ on $(b,c]$. □

Lemma 3. *Let $\beta \in (0,1)$, $z, D^\beta z \in C([t_0,T];\mathcal{Z})$. Then, $z \in C^\beta([t_0,T]\mathcal{Z})$, moreover, there exists $C > 0$, such that for all $t, \tau \in [t_0,T]$*

$$\|h(t) - h(\tau)\| \le \frac{\|D^\beta z\|_{C([t_0,T];\mathcal{Z})}}{\Gamma(\beta+1)} |t-\tau|^\beta.$$

Proof. If $t_0 \le \tau < t \le T$, then

$$\|h(t) - h(\tau)\|_{\mathcal{Z}} = \|J^\beta D^\beta h(t) - J^\beta D^\beta h(\tau)\|_{\mathcal{Z}} \le$$
$$\le \frac{(t-t_0)^\beta - (\tau - t_0)^\beta}{\Gamma(\beta+1)} \|D^\beta h\|_{C([t_0,T];\mathcal{Z})} \le \frac{(t-\tau)^\beta}{\Gamma(\beta+1)} \|D^\beta h\|_{C([t_0,T];\mathcal{Z})},$$

since the function
$$\frac{(t-t_0)^\beta - (\tau - t_0)^\beta}{(t-\tau)^\beta}$$
decreases with respect to $\tau \in [t_0, t)$ at $\beta \in (0,1)$. □

Corollary 3. *Let $m-1 < c \le m \in \mathbb{N}$, $b < c < \beta$, $\mu \in BV((b,c];\mathbb{C})$, c be a variation point of the measure $d\mu(\alpha)$. Then, for every $z \in C^{m-1,\beta}([t_0,T];\mathcal{Z})$, $\varepsilon \in (0, \beta - c)$ we have $\int\limits_b^c D^\alpha z(t) d\mu(\alpha) \in C^{\beta - c - \varepsilon}([t_0,T];\mathcal{Z})$. Additionally, there exists $C > 0$, such that for all $z \in C^{m-1,c}([t_0,T];\mathcal{Z})$, $s, t \in [t_0, T]$*

$$\left\| \int\limits_b^c D^\alpha z(t) d\mu(\alpha) - \int\limits_b^c D^\alpha z(s) d\mu(\alpha) \right\|_{\mathcal{Z}} \le C\|z\|_{C^{m-1,\beta}([t_0,T];\mathcal{Z})} |t-s|^{\beta - c - \varepsilon}.$$

Proof. Indeed, due to Lemmas 2 and 3 for every s, t, such that $t_0 \le s < t \le T$, we have

$$\left\| \int\limits_b^c D^\alpha z(t) d\mu(\alpha) - \int\limits_b^c D^\alpha z(s) d\mu(\alpha) \right\|_{\mathcal{Z}} \le$$
$$\le \frac{|t-s|^{\beta-c-\varepsilon}}{\Gamma(\beta - c - \varepsilon + 1)} \left\| D^{\beta - c - \varepsilon} \int\limits_b^c D^\alpha z(t) d\mu(\alpha) \right\|_{C([t_0,T];\mathcal{Z})} \le C\|z\|_{C^{m-1,\beta}([t_0,T];\mathcal{Z})} |t-s|^{\beta-c-\varepsilon}.$$

□

4. Nonlocal Unique Solvability of Quasilinear Equation

A solution on a segment $[t_0, T]$ of the Cauchy problem

$$D^k z(t_0) = z_k, \quad k = 0, 1, \ldots, m-1, \tag{9}$$

for the equation

$$\int\limits_b^c D^\alpha z(t) d\mu(\alpha) = Az(t) + B\left(t, \int\limits_{b_1}^{c_1} D^\alpha z(t) d\mu_1(\alpha), \ldots, \int\limits_{b_n}^{c_n} D^\alpha z(t) d\mu_n(\alpha)\right), \tag{10}$$

where $b < c$, $m-1 < c \le m \in \mathbb{N}$, $b_l < c_l$, $m_l - 1 < c_l \le m_l \in \mathbb{Z}$, $c_1 \le c_2 \le \cdots \le c_n < c$, $\mu \in BV((b,c];\mathbb{C})$, $\mu_l \in BV((b_l,c_l];\mathbb{C})$, $l = 1, 2, \ldots, n$, $T > t_0$, $g \in C([t_0,T];\mathcal{Z})$, is a function $z \in C^{m-1}([t_0,T];\mathcal{Z}) \cap C((t_0,T];D_A)$, such that $\int\limits_b^c D^\alpha z(t) d\mu(\alpha) \in C((t_0,T];\mathcal{Z})$, $\int\limits_{b_l}^{c_l} D^\alpha z(t) d\mu_l(\alpha) \in C([t_0,T];\mathcal{Z})$, $l = 1, 2, \ldots, n$, and equalities (9) and (10) for $t \in (t_0, T]$ are fulfilled.

Lemma 4. *Let $m-1 < c \le m \in \mathbb{N}$, $b < c$, $\mu \in BV((b,c];\mathbb{C})$, c be a variation point of the measure $d\mu(\alpha)$, $n \in \mathbb{N}$, $c_1 \le c_2 \le \cdots \le c_n < c$, $\mu_l \in BV((b_l,c_l];\mathbb{C})$, c_l be a variation point of the measure $d\mu_l(\alpha)$, $l = 1, 2, \ldots, n$, $A \in \mathcal{A}_W(\theta_0, a_0)$ for some $\theta_0 \in (\pi/2, \pi)$, $a_0 \ge 0$, $z_k \in D_A$,*

$k = 0, 1, \ldots, m-1$, $B \in C([t_0, T] \times \mathcal{Z}^n; D_A)$. *Then a function* z *is a solution of problem* (9) *and* (10) *on the segment* $[t_0, T]$, *if and only if* $z \in C^{m-1,c_n}([t_0, T]; \mathcal{Z})$ *and for all* $t \in [t_0, T]$ *the equality*

$$z(t) = \sum_{k=0}^{m-1} Z_k(t - t_0) z_k + \int_{t_0}^{t} Z(t - s) B^z(s) ds \quad (11)$$

holds, where

$$B^z(s) := B\left(s, \int_{b_1}^{c_1} D^\alpha z(s) d\mu_1(\alpha), \int_{b_2}^{c_2} D^\alpha z(s) d\mu_2(\alpha), \ldots, \int_{b_n}^{c_n} D^\alpha z(s) d\mu_n(\alpha)\right).$$

Proof. If z is a solution of problem (9) and (10), then there exists $D^{c_n} z \in C([t_0, T]; \mathcal{Z})$, since c_n is a variation point of the measure $d\mu_n(\alpha)$. Therefore, $z \in C^{m-1,c_n}([t_0, T]; \mathcal{Z})$ and due to Corollary 2 the mapping

$$t \to B\left(t, \int_{b_1}^{c_1} D^\alpha z(t) d\mu(\alpha), \int_{b_2}^{c_2} D^\alpha z(t) d\mu(\alpha), \ldots, \int_{b_n}^{c_n} D^\alpha z(t) d\mu(\alpha)\right) \quad (12)$$

acts continuously from $[t_0, T]$ into D_A, since $B \in C([t_0, T] \times \mathcal{Z}^n; D_A)$. Consequently, by Theorem 3, equality (11) is valid.

Let $z \in C^{m-1,c_n}([t_0, T]; \mathcal{Z})$ and for all $t \in [t_0, T]$ equality (11) holds. Then, by Corollary 2, mapping (12) belongs to the class $C([t_0, T]; D_A)$ in the case $B \in C([t_0, T] \times \mathcal{Z}^n; D_A)$. By Theorem 3, z is a solution of problem (9) and (10). □

A mapping $B : [t_0, T] \times \mathcal{Z}^n \to \mathcal{Z}$ is called Lipschitz continuous, if there exists $C_L > 0$, such that for all $t \in [t_0, T]$, $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in \mathcal{Z}$

$$\|B(t, x_1, x_2, \ldots, x_n) - B(t, y_1, y_2, \ldots, y_n)\|_{\mathcal{Z}} \leq C_L \sum_{l=1}^{n} \|x_l - y_l\|_{\mathcal{Z}}.$$

Theorem 4. *Let* $m - 1 < c \leq m \in \mathbb{N}$, $b < c$, $\mu \in BV((b, c]; \mathbb{C})$, c *be a variation point of the measure* $d\mu(\alpha)$, $n \in \mathbb{N}$, $c_1 \leq c_2 \leq \cdots \leq c_n < c$, $b_l < c_l$, $\mu_l \in BV((b_l, c_l]; \mathbb{C})$, c_l *be a variation point of the measure* $d\mu_l(\alpha)$, $l = 1, 2, \ldots, n$, $A \in \mathcal{A}_W(\theta_0, a_0)$ *for some* $\theta_0 \in (\pi/2, \pi)$, $a_0 \geq 0$, $z_k \in D_A$, $k = 0, 1, \ldots, m-1$, *a mapping* $B \in C([t_0, T] \times \mathcal{Z}^n; D_A)$ *be Lipschitz continuous. Then, problem* (9) *and* (10) *have a unique solution on the segment* $[t_0, T]$.

Proof. Due to Lemma 4, it is sufficient to prove that the integro-differential Equation (11) has a unique solution in the Banach space $C^{m-1,c_n}([t_0, T]; \mathcal{Z})$.

For $z \in C^{m-1,c_n}([t_0, T]; \mathcal{Z})$ define the operator

$$G(z)(t) := \sum_{k=0}^{m-1} Z_k(t - t_0) z_k + \int_{t_0}^{t} Z(t - s) B^z(s)\, ds, \quad t \in [t_0, T].$$

Since mapping (12) belongs to $C([t_0, T]; D_A)$, due to Theorem 3, we find that $G(z) \in C^{m-1}([t_0, T]; \mathcal{Z})$, $D^k G(z)(t_0) = z_k$ for $k = 0, 1, \ldots, m-1$.

If $c_n < k$, then the form of Z_k implies that by (4)

$$\text{Lap}[D^{c_n} Z_k(t) z_k](\lambda) = \lambda^{c_n - 1 - k} W_k(\lambda) R_{W(\lambda)}(A) z_k,$$

$$\left\|\lambda^{c_n - 1 - k} W_k(\lambda) R_{W(\lambda)}(A) z_k\right\|_{\mathcal{Z}} \leq \frac{C\|z_k\|_{\mathcal{Z}}}{|\lambda|^{k+1-\varepsilon-c_n}}$$

for some $\varepsilon \in (0, k - c_n)$ due to Lemma 1. Hence, $D^{c_n} Z_k(0) z_k = 0$ and $D^{c_n} Z_k(t - t_0) z_k \in C([t_0, T]; \mathcal{Z})$. It is known that $D^k Z_k(t - t_0) z_k \in C([t_0, T]; \mathcal{Z})$. In the case $c_n > k$, we have due to equality (4)

$$\text{Lap}[D^{c_n} Z_k(t) z_k](\lambda) = \lambda^{c_n - 1 - k} W_k(\lambda) R_{W(\lambda)}(A) z_k - \lambda^{c_n - 1 - k} z_k =$$

$$= \lambda^{c_n - 1 - k} [W_k(\lambda) - W(\lambda)] R_{W(\lambda)}(A) z_k + \lambda^{c_n - 1 - k} R_{W(\lambda)}(A) A z_k,$$

$$\left\| \lambda^{c_n - 1 - k} [W_k(\lambda) - W(\lambda)] R_{W(\lambda)}(A) z_k \right\|_{\mathcal{Z}} \le \frac{C \|z_k\|_{\mathcal{Z}}}{|\lambda|^{c - \varepsilon + 1 - c_n}},$$

$$\left\| \lambda^{c_n - 1 - k} R_{W(\lambda)}(A) A z_k \right\|_{\mathcal{Z}} \le \frac{C \|z_k\|_{D_A}}{|\lambda|^{c - \varepsilon + 1 - c_n + k}},$$

for some $\varepsilon \in (0, c - c_n)$ by Lemma 1. Therefore, $D^{c_n} Z_k(0) z_k = 0$ and $D^{c_n} Z_k(t - t_0) z_k \in C([t_0, T]; \mathcal{Z})$.

Due to [29] Lemma 4 $D^k Z(0) = 0$, $k = 0, 1, \dots, m - 2$, $\|D^{m-1} Z(t)\|_{\mathcal{L}(\mathcal{Z})} = O(t^{c - \varepsilon - m})$ as $t \to 0+$. Therefore,

$$\left\| \int_{t_0}^{t} Z(t - s) B^z(s) ds \right\|_{\mathcal{Z}} = O((t - t_0)^{c - \varepsilon}), \ t \to t_0+,$$

$$D^k|_{t = t_0} \int_{t_0}^{t} Z(t - s) B^z(s) ds = 0, \quad k = 0, 1, \dots, m - 1,$$

since B^z is continuous on $[t_0, T]$ for $z \in C^{m-1, c_n}([t_0, T]; \mathcal{Z})$ by Corollary 2. We have

$$\|\text{Lap}[D^{m_n} J^{m_n - c_n} Z(t)](\lambda)\|_{\mathcal{L}(\mathcal{Z})} = \|\lambda^{c_n} R_{W(\lambda)}(A)\|_{\mathcal{L}(\mathcal{Z})} \le \frac{C}{|\lambda|^{c - \varepsilon - c_n}}$$

with $\varepsilon \in (0, c - c_n)$, consequently, $\|D^{m_n} J^{m_n - c_n} Z(t)\|_{\mathcal{L}(\mathcal{Z})} = O(t^{c - \varepsilon - c_n - 1})$ as $t \to 0+$. Since,

$$D^{c_n} \int_{t_0}^{t} Z(t - s) B^z(s) ds = D^{m_n} J^{m_n - c_n} \int_{t_0}^{t} Z(t - s) B^z(s) ds = \int_{t_0}^{t} D^{m_n} J^{m_n - c_n} Z(t - s) B^z(s) ds,$$

we have

$$\left\| \int_{t_0}^{t} D^{m_n} J^{m_n - c_n} Z(t - s) B^z(s) ds \right\|_{\mathcal{L}(\mathcal{Z})} = O(t^{c - \varepsilon - c_n}).$$

Thus, $G(z) \in C^{m-1, c_n}([t_0, T]; \mathcal{Z})$.

Let G^j be the j-th degree of the operator G, $j \in \mathbb{N}$. For the sake of certainty, we consider that $T - t_0 \ge 1$. In the case $T - t_0 < 1$, further reasoning will remain valid after the replacement $T - t_0$ by 1.

Arguing as before, we can find that for $k = 0, 1, \dots, m - 1$ and for small $\varepsilon > 0$ the inequality $\|D^k Z(t)\|_{\mathcal{L}(\mathcal{Z})} \le C t^{c - \varepsilon - 1 - k}$ is valid. Consequently, for $x, y \in C^{m-1, c_n}([t_0, T]; \mathcal{Z})$, we have in the case $c_n > m - 1$

$$\|D^k G(x)(t) - D^k G(y)(t)\|_{\mathcal{Z}} \le C \int_{t_0}^{t} (t - s)^{c - \varepsilon - 1 - k} \|B^x(s) - B^y(s)\|_{\mathcal{Z}} ds \le$$

$$\le C_1 \|x - y\|_{C^{m-1, c_n}([t_0, t]; \mathcal{Z})} (t - t_0)^{c - \varepsilon - k} \le$$

$$\le C_2 \|x - y\|_{C^{m-1, c_n}([t_0, t]; \mathcal{Z})} [(t - t_0)^c + (t - t_0)^{c - \varepsilon - c_n}],$$

$$\|D^{c_n}G(x)(t)-D^{c_n}G(y)(t)\|_{\mathcal{Z}}\leq C\int_{t_0}^{t}(t-s)^{c-\varepsilon-1-c_n}\|B^x(s)-B^y(s)\|_{\mathcal{Z}}ds\leq$$

$$\leq C_2\|x-y\|_{C^{m-1,c_n}([t_0,t];\mathcal{Z})}[(t-t_0)^c+(t-t_0)^{c-\varepsilon-c_n}].$$

Therefore,

$$\|G(x)-G(y)\|_{C^{m-1,c_n}([t_0,t];\mathcal{Z})}\leq C_2(m+1)\|x-y\|_{C^{m-1,c_n}([t_0,t];\mathcal{Z})}[(t-t_0)^c+(t-t_0)^{c-\varepsilon-c_n}].$$

Then, for $k=0,1,\dots,m-1$

$$\|D^kG^2(x)(t)-D^kG^2(y)(t)\|_{\mathcal{Z}}\leq C\int_{t_0}^{t}(t-s)^{c-\varepsilon-1-k}\|B^{G(x)}(s)-B^{G(y)}(s)\|_{\mathcal{Z}}ds\leq$$

$$\leq C_2(T-t_0)^c\int_{t_0}^{t}\|G(x)-G(y)\|_{C^{m-1,c_n}([t_0,s];\mathcal{Z})}ds\leq$$

$$\leq C_2^2(m+1)(T-t_0)^c\|x-y\|_{C^{m-1,c_n}([t_0,t];\mathcal{Z})}[(t-t_0)^{c+1}+(t-t_0)^{c-\varepsilon-c_n+1}],$$

$$\|D^{c_n}G^2(x)(t)-D^{c_n}G^2(y)(t)\|_{\mathcal{Z}}\leq C\int_{t_0}^{t}(t-s)^{c-\varepsilon-1-c_n}\|B^{G(x)}(s)-B^{G(y)}(s)\|_{\mathcal{Z}}ds\leq$$

$$\leq C_2^2(m+1)(T-t_0)^c\|x-y\|_{C^{m-1,c_n}([t_0,t];\mathcal{Z})}[(t-t_0)^{c+1}+(t-t_0)^{c-\varepsilon-c_n+1}],$$

$$\|G^2(x)-G^2(y)\|_{C^{m-1,c_n}([t_0,t];\mathcal{Z})}\leq$$

$$\leq C_2^2(m+1)^2(T-t_0)^c\|x-y\|_{C^{m-1,c_n}([t_0,t];\mathcal{Z})}[(t-t_0)^{c+1}+(t-t_0)^{c-\varepsilon-c_n+1}].$$

By the same way, we obtain

$$\|G^3(x)-G^3(y)\|_{C^{m-1,c_n}([t_0,t];\mathcal{Z})}\leq$$

$$\leq C_2^3(m+1)^3(T-t_0)^{2c}\|x-y\|_{C^{m-1,c_n}([t_0,t];\mathcal{Z})}\frac{(t-t_0)^{c+2}+(t-t_0)^{c-\varepsilon-c_n+2}}{2}.$$

Similarly, we obtain for $t\in[t_0,T]$, $j\in\mathbb{N}$, $x,y\in C^{m-1,c_n}([t_0,T];\mathcal{Z})$ that

$$\|G^j(x)-G^j(y)\|_{C^{m-1,c_n}([t_0,t];\mathcal{Z})}\leq$$

$$\leq\frac{C_0^j[(t-t_0)^{c+j-1}+(t-t_0)^{c-\varepsilon-c_n+j-1}]}{(j-1)!}\|x-y\|_{C^{m-1,c_n}([t_0,t];\mathcal{Z})}$$

with $C_0=C_2(m+1)(T-t_0)^c$. Consequently,

$$\|G^j(x)-G^j(y)\|_{C^{m-1,c_n}([t_0,T];\mathcal{Z})}\leq\frac{2C_0^j(T-t_0)^{c+j-1}}{(j-1)!}\|x-y\|_{C^{m-1,c_n}([t_0,T];\mathcal{Z})}.$$

Hence, for a large enough j, the mapping G^j is a contraction in the space $C^{m-1,c_n}([t_0,T];\mathcal{Z})$ and it has a unique fixed point in this space, which is known to be the unique fixed point in $C^{m-1,c_n}([t_0,T];\mathcal{Z})$ of the mapping G. Due to Lemma 4, z is the fixed point of G, if and only if it is a unique solution of problem (9) and (10).

If $c_n\leq m-1$, then we will omit the estimates for the derivatives of the order c_n. □

Lemma 5. *Let $m-1<c\leq m\in\mathbb{N}$, $b<c$, $\mu\in BV((b,c];\mathbb{C})$, c be a variation point of the measure $d\mu(\alpha)$, $n\in\mathbb{N}$, $c_1\leq c_2\leq\cdots\leq c_n<\beta<c$, $\mu_l\in BV((b_l,c_l];\mathbb{C})$, c_l be a variation point of the measure $d\mu_l(\alpha)$, $l=1,2,\dots,n$, $A\in\mathcal{A}_W(\theta_0,a_0)$ for some $\theta_0\in(\pi/2,\pi)$, $a_0\geq 0$,*

$z_k \in D_A$, $k = 0, 1, \dots, m-1$, $B \in C([t_0, T] \times \mathcal{Z}^n; \mathcal{Z})$ be Lipschitz continuous. Then, a function $z \in C^{m-1,\beta}([t_0, T]; \mathcal{Z})$ is a solution of problem (9) and (10) on the segment $[t_0, T]$, if and only if for all $t \in [t_0, T]$ it satisfies equality (11).

Proof. If $z \in C^{m-1,\beta}([t_0, T]; \mathcal{Z})$ is a solution of problem (9) and (10), then due to Lipschitz continuity of B and by Corollary 3 the function B^z satisfies the Hölder condition. Due to Theorem 3, equality (11) is valid.

Let $z \in C^{m-1,\beta}([t_0, T]; \mathcal{Z})$ and for all $t \in [t_0, T]$ equality (11) is valid. Then, by Corollary 3, the function B^z is Hölderian. By Theorem 3, the function z is a solution of problem (9) and (10). □

Theorem 5. *Let $m - 1 < c \le m \in \mathbb{N}$, $b < c$, $\mu \in BV((b, c]; \mathbb{C})$, c be a variation point of the measure $d\mu(\alpha)$, $n \in \mathbb{N}$, $c_1 \le c_2 \le \dots \le c_n < c$, $b_l < c_l$, $\mu_l \in BV((b_l, c_l]; \mathbb{C})$, c_l be a variation point of the measure $d\mu_l(\alpha)$, $l = 1, 2, \dots, n$, $A \in \mathcal{A}_W(\theta_0, a_0)$ for some $\theta_0 \in (\pi/2, \pi)$, $a_0 \ge 0$, $z_k \in D_A$, $k = 0, 1, \dots, m-1$, a mapping $B \in C([t_0, T] \times \mathcal{Z}^n; \mathcal{Z})$ be Lipschitz continuous. Then, problem (9) and (10) have a unique solution on the segment $[t_0, T]$.*

Proof. Choose $\beta \in (c_n, c)$ and for $z \in C^{m-1,\beta}([t_0, T]; \mathcal{Z})$ consider the operator

$$G(z)(t) := \sum_{k=0}^{m-1} Z_k(t - t_0) z_k + \int_{t_0}^{t} Z(t - s) B^z(s)\, ds, \quad t \in [t_0, T].$$

Since B is Lipschitz continuous and by Corollary 3 all the arguments of B satisfy the Hölder condition, hence, B^z is Hölderian also. Consequently, by Theorem 3, we have $G(z) \in C^{m-1}([t_0, T]; \mathcal{Z})$, $D^k G(z)(t_0) = z_k$ for $k = 0, 1, \dots, m-1$.

If $c_n \ge m - 1$, then $\beta > m - 1$ and, as in the proof of the previous theorem, it can be shown that $G(z) \in C^{m-1,\beta}([t_0, T]; \mathcal{Z})$, for sufficiently large j, the mapping G^j is a contraction in $C^{m-1,\beta}([t_0, T]; \mathcal{Z})$ and G has a unique fixed point in $C^{m-1,\beta}([t_0, T]; \mathcal{Z})$. Due to Lemma 5, the unique fixed point is a unique solution of problem (9) and (10).

If $c_n < m - 1$, we can take $\beta = m - 1$ and the proof will be simpler. □

5. Local Unique Solvbability of Quasilinear Equation

Now, the nonlinear operator B is defined on some open subset U of $\mathbb{R} \times \mathcal{Z}^n$. A solution on some segment $[t_0, t_1]$, $t_1 > t_0$, of Cauchy problem (9) for Equation (10) is a function $z \in C^{m-1}([t_0, t_1]; \mathcal{Z}) \cap C((t_0, t_1]; D_A)$, such that $\int_b^c D^\alpha z(t) d\mu(\alpha) \in C((t_0, t_1]; \mathcal{Z})$, $\int_{b_l}^{c_l} D^\alpha z(t) d\mu_l(\alpha) \in C([t_0, t_1]; \mathcal{Z})$, $l = 1, 2, \dots, n$, equalities (9), inclusion

$$\left(t, \int_{b_1}^{c_1} D^\alpha z(t) d\mu_1(\alpha), \int_{b_2}^{c_2} D^\alpha z(t) d\mu_2(\alpha), \dots, \int_{b_n}^{c_n} D^\alpha z(t) d\mu_n(\alpha) \right) \in U$$

for $t \in [t_0, t_1]$ and equality (10) for $t \in (t_0, t_1]$ are satisfied.

As before, here $b < c$, $m - 1 < c \le m \in \mathbb{N}$, $\mu \in BV((b, c]; \mathbb{C})$,$b_l < c_l$, $m_l - 1 < c_l \le m_l \in \mathbb{Z}$, $c_1 \le c_2 \le \dots \le c_n < c$, $\mu_l \in BV((b_l, c_l]; \mathbb{C})$, $l = 1, 2, \dots, n$.

A mapping $B : U \to \mathcal{Z}$ is called locally Lipschitz continuous, if for every point $(t, x_1, x_2, \dots, x_n) \in U$ there exists its vicinity $V \subset U$ and a constant $C > 0$, such that for all $(s, y_1, y_2, \dots, y_n), (s, v_1, v_2, \dots, v_n) \in V$

$$\|B(s, y_1, y_2, \dots, y_n) - B(s, v_1, v_2, \dots, v_n)\|_{\mathcal{Z}} \le C \sum_{l=1}^{n} \|y_l - v_l\|_{\mathcal{Z}}. \tag{13}$$

Denote for $z_k \in D_A$, $k = 0, 1, \dots, m-1$, from initial conditions (9)

$$\widetilde{z}(t) = z_0 + z_1(t - t_0) + \dots + z_{m-1}\frac{(t-t_0)^{m-1}}{(m-1)!}, \ \widetilde{z}_l = \int_{b_l}^{c_l} D^\alpha \widetilde{z}(t_0) d\mu_l(\alpha), \ l = 1, 2, \dots, n.$$

Theorem 6. *Let $m - 1 < c \le m \in \mathbb{N}$, $b < c$, $\mu \in BV((b, c]; \mathbb{C})$, c be a variation point of the measure $d\mu(\alpha)$, $n \in \mathbb{N}$, $c_1 \le c_2 \le \dots \le c_n < c$, $b_l < c_l$, $\mu_l \in BV((b_l, c_l]; \mathbb{C})$, c_l be a variation point of the measure $d\mu_l(\alpha)$, $l = 1, 2, \dots, n$, $A \in \mathcal{A}_W(\theta_0, a_0)$ for some $\theta_0 \in (\pi/2, \pi)$, $a_0 \ge 0$, $z_k \in D_A$, $k = 0, 1, \dots, m-1$, $(t_0, \widetilde{z}_1, \widetilde{z}_2, \dots, \widetilde{z}_n) \in U$, a mapping $B \in C(U; D_A)$ be locally Lipschitz continuous. Then, there exists $t_1 > t_0$, such that problem (9) and (10) have a unique solution on the segment $[t_0, t_1]$.*

Proof. Take a sufficiently small $\delta > 0$, such that in the neighborhood

$$V := \{(t, x_1, x_2, \dots, x_n) \in \mathbb{R} \times \mathcal{Z}^n : |t - t_0| \le \delta, \ \|x_l - \widetilde{z}_l\|_{\mathcal{Z}} \le \delta\}$$

the inequality (13) holds with some $C > 0$. Define

$$\mathfrak{M}_{t_1} := \left\{ z \in C^{m-1, c_n}([t_0, t_1]; \mathcal{Z}) : \left\| \int_{b_l}^{c_l} D^\alpha z(t) d\mu_l(\alpha) - \widetilde{z}_l \right\|_{\mathcal{Z}} \le \delta, \right.$$

$$\left. t \in [t_0, t_1], \, l = 1, 2, \dots, n \right\}.$$

Due to Corollary 2 $\mathfrak{M}_{t_1}$ is a complete metric space with the metric

$$d(x, y) = \|x - y\|_{C^{m-1, c_n}([t_0, t_1]; \mathcal{Z})}.$$

For $z \in \mathfrak{M}_{t_1}$, define the operator

$$G(z)(t) := \sum_{k=0}^{m-1} Z_k(t - t_0) z_k + \int_{t_0}^{t} Z(t - s) B^z(s)\, ds, \ \ t \in [t_0, t_1].$$

Since B^z belongs to $C([t_0, t_1]; D_A)$, we have $G(z) \in C^{m-1}([t_0, t_1]; \mathcal{Z})$, $D^k G(z)(t_0) = z_k$ for $k = 0, 1, \dots, m-1$. As in the proof of Theorem 4, we have $G(z) \in C^{m-1, c_n}([t_0, t_1]; \mathcal{Z})$, therefore, $G(z) \in \mathfrak{M}_{t_1}$. If necessary, we can reduce t_1 here. Due to Corollary 2

$$\int_{b_l}^{c_l} D^\alpha G(z)(t) d\mu_l(\alpha) \in C([t_0, t_1]; \mathcal{Z}), \quad l = 1, 2, \dots, n.$$

Consequently, for small enough $t_1 - t_0$ $G(z) \in \mathfrak{M}_{t_1}$.

Arguing as in the proof of Theorem 4, we have for $k = 0, 1, \dots, m-1$ and small $\varepsilon > 0$ $\|D^k Z(t)\|_{\mathcal{L}(\mathcal{Z})} \le C t^{c - \varepsilon - 1 - k}$. Therefore, for $x, y \in \mathfrak{M}_{t_1}$

$$\|D^k G(x)(t) - D^k G(y)(t)\|_{\mathcal{Z}} \le C \int_{t_0}^{t} (t - s)^{c - \varepsilon - 1 - k} \|B^x(s) - B^y(s)\|_{\mathcal{Z}} ds \le$$

$$\le C_1 \|x - y\|_{C^{m-1, c_n}([t_0, t_1]; \mathcal{Z})} (t_1 - t_0)^{c - \varepsilon - k} \le \frac{\|x - y\|_{C^{m-1, c_n}([t_0, t_1]; \mathcal{Z})}}{2(m+1)}, \quad k = 0, 1, \dots, m-1,$$

$$\|D^{c_n} G(x)(t) - D^{c_n} G(y)(t)\|_{\mathcal{Z}} \le C_1 \|x - y\|_{C^{m-1, c_n}([t_0, t_1]; \mathcal{Z})} (t_1 - t_0)^{c - \varepsilon - c_n} \le$$

$$\leq \frac{\|x-y\|_{C^{m-1,c_n}([t_0,t_1];\mathcal{Z})}}{2(m+1)},$$

for sufficiently small $t_1 - t_0$, hence,

$$\|G(x)-G(y)\|_{C^{m-1,c_n}([t_0,t_1];\mathcal{Z})} \leq \frac{1}{2}\|x-y\|_{C^{m-1,c_n}([t_0,t_1];\mathcal{Z})}.$$

Thus, the mapping G is a contraction in the metric space $\mathfrak{M}_{t_1}$. By the Banach theorem on a fixed point, G has a unique fixed point z in this space. Due to Lemma 4, the fixed point z is a unique solution of problem (9) and (10) on $[t_0, t_1]$. □

Theorem 7. *Let $m-1 < c \leq m \in \mathbb{N}$, $b < c$, $\mu \in BV((b,c];\mathbb{C})$, c be a variation point of the measure $d\mu(\alpha)$, $n \in \mathbb{N}$, $c_1 \leq c_2 \leq \cdots \leq c_n < c$, $b_l < c_l$, $\mu_l \in BV((b_l,c_l];\mathbb{C})$, c_l be a variation point of the measure $d\mu_l(\alpha)$, $l = 1,2,\ldots,n$, $A \in \mathcal{A}_W(\theta_0, a_0)$ for some $\theta_0 \in (\pi/2,\pi)$, $a_0 \geq 0$, $z_k \in D_A$, $k = 0,1,\ldots,m-1$, $(t_0, \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n) \in U$, a mapping $B \in C([t_0,T]\times\mathcal{Z}^n;\mathcal{Z})$ be locally Lipschitz continuous. Then, there exists $t_1 > t_0$, such that problem (9) and (10) have a unique solution on the segment $[t_0, t_1]$.*

Proof. For a fixed $\beta \in (c_n, c)$ take a small enough $\delta > 0$, such that in

$$V := \{(t, x_1, x_2, \ldots, x_n) \in \mathbb{R}\times\mathcal{Z}^n : |t-t_0| \leq \delta,\ \|x_l - \tilde{z}_l\|_{\mathcal{Z}} \leq \delta\}.$$

the inequality (13) is satisfied with a constant $C > 0$. Define

$$\mathfrak{M}_{t_1} := \left\{ z \in C^{m-1,\beta}([t_0,t_1];\mathcal{Z}) : \left\| \int_{b_l}^{c_l} D^\alpha z(t)d\mu_l(\alpha) - \tilde{z}_l \right\|_{\mathcal{Z}} \leq \delta, \right.$$

$$\left. t \in [t_0,t_1],\ l = 1,2,\ldots,n \right\}.$$

For $z \in \mathfrak{M}_{t_1}$, define the operator

$$G(z)(t) := \sum_{k=0}^{m-1} Z_k(t-t_0)z_k + \int_{t_0}^{t} Z(t-s)B^z(s)\,ds, \quad t \in [t_0,t_1].$$

Due to the Lipschitz continuity of B by Corollary 3, B^z satisfies the Hölder condition. Due to Theorem 3, $G(z) \in C^{m-1}([t_0,t_1];\mathcal{Z})$, $D^k G(z)(t_0) = z_k$, $k = 0,1,\ldots,m-1$.

If $c_n \geq m-1$, then $\beta > m-1$. Reasoning by the same way as in the proof of Theorem 6, we can obtain that $G(z) \in C^{m-1,\beta}([t_0,T];\mathcal{Z})$ and the mapping G is a contraction in $\mathfrak{M}_{t_1}$ and has a unique fixed point in the metric space $\mathfrak{M}_{t_1}$. By Lemma 5, the fixed point is a unique solution of problem (9) and (10) on the segment $[t_0, t_1]$.

If $c_n < m-1$, we take $\beta = m-1$. □

6. Application to a Nonlinear Initial-Boundary Value Problem

In the framework of the Cauchy problem for a quasilinear equation in Banach space, we can investigate initial-boundary value problems for partial differential equations with time-distributed derivatives. For this aim, we need to choose an appropriate space $\mathcal{Z}$ and an operator A. Now, we will demonstrate this with the example of the following problem.

Consider a bounded region $\Omega \subset \mathbb{R}^d$ with a smooth boundary $\partial\Omega$, $\beta, \gamma, \nu \in \mathbb{R}$, $c \in (1,2)$, $b < c$, $\alpha_1 < \alpha_2 < \cdots < \alpha_n \leq c$, $\omega_k \in \mathbb{R}\setminus\{0\}$, $k = 1,2,\ldots,n$, $\omega \in C([b,c];\mathbb{R})$; if $\alpha_n < c$, then $\omega(c) \neq 0$ in a some left vicinity of c; $\beta_l < c$, $b_l < c_l < c$, $\mu_l \in BV((b_l,c_l];\mathbb{R})$, $l = 1,2$. Consider the initial-boundary value problem

$$u(s,0) = u_0(s), \quad v(s,0) = v_0(s), \quad s \in \Omega, \tag{14}$$

$$\frac{\partial u}{\partial t}(s,0) = u_1(s), \quad \frac{\partial v}{\partial t}(s,0) = v_1(s), \quad s \in \Omega, \tag{15}$$

$$u(s,t) = v(s,t) = 0, \quad (s,t) \in \partial\Omega \times (0,T], \tag{16}$$

for the nonlinear system of equations in $\Omega \times (0,T]$

$$\sum_{k=1}^{n} \omega_k D_t^{\alpha_k} u(s,t) + \int_b^c \omega(\alpha) D_t^{\alpha} u(s,t) d\alpha = \Delta u(s,t) - \Delta v(s,t) +$$

$$+ F_1\left(s, D^{\beta_1} u(s,t), \int_{b_1}^{c_1} D^{\alpha} v(s,t) d\mu_1(\alpha)\right), \tag{17}$$

$$\sum_{k=1}^{n} \omega_k D_t^{\alpha_k} v(s,t) + \int_b^c \omega(\alpha) D_t^{\alpha} v(s,t) d\alpha = \nu \Delta v(s,t) + \beta u(s,t) + \gamma v(s,t) +$$

$$+ F_2\left(s, D^{\beta_2} v(s,t), \int_{b_2}^{c_2} D^{\alpha} u(s,t) d\mu_2(\alpha)\right). \tag{18}$$

Remark 7. *If $\omega_2 = \omega_3 = \cdots = \omega_n = 0$, $\alpha_1 = 1$, $\omega(\alpha) \equiv 0$ for all $\alpha \in (b,c)$, $F_1 \equiv 0$, $\beta_2 = 0$, $\mu_2 \equiv 0$, after linear replacement of unknown functions $u(s,t) = \widetilde{u}(s,t) + \frac{\kappa}{2}\widetilde{v}(s,t)$, $v(s,t) = \frac{\kappa}{2}\widetilde{v}(s,t)$, $\kappa \in \mathbb{R}$, systems (17) and (18) are the linearization of the phase field system of equations [36,37].*

Define $\Lambda_1 z = \Delta z$, $D_{\Lambda_1} = H_0^{j+2}(\Omega) \subset H^j(\Omega)$. By $\{\varphi_k : k \in \mathbb{N}\}$, denote an orthonormal in the inner product $\langle \cdot, \cdot \rangle$ in $L_2(\Omega)$ eigenfunctions of Λ_1, which are enumerated in the non-increasing order of the eigenvalues $\{\lambda_k : k \in \mathbb{N}\}$ taking into account their multiplicities.

Take the Sobolev space $\mathcal{Z} = (H^j(\Omega))^2$ for some $j \in \mathbb{N} \cup \{0\}$, such that $j > d/2$,

$$A = \begin{pmatrix} \Lambda_1 & -\Lambda_1 \\ \beta I & \gamma I + \nu \Lambda_1 \end{pmatrix}, \quad D_A = (H_0^{j+2}(\Omega))^2, \tag{19}$$

where $H_0^{j+2}(\Omega) := \{z \in H^{j+2}(\Omega) : z(s) = 0,\ s \in \partial\Omega\}$. Consequently, $A \in \mathcal{Cl}(\mathcal{Z})$.

Theorem 8 ([29])**.** *Let $c \in (1,2)$, $\nu > 0$, $\beta, \gamma \in \mathbb{R}$, then there exist $\theta_0 \in (\pi/2, \pi)$, $a_0 \geq 0$, such that $A \in \mathcal{A}_W(\theta_0, a_0)$.*

Theorem 9. *Let $c \in (1,2)$, $b < c$, $\nu > 0$, $\beta, \gamma \in \mathbb{R}$, $\alpha_1 < \alpha_2 < \cdots < \alpha_n \leq c$, $\omega_k \in \mathbb{R} \setminus \{0\}$, $k = 1,2,\ldots,n$, $\omega \in C([b,c];\mathbb{R})$; if $\alpha_n < c$, then $\omega(c) \neq 0$ in the left vicinity of c; $\beta_l < c$, $b_l < c_l < c$, $\mu_l \in BV((b_l,c_l];\mathbb{R})$, c_l be a variation point of the measure $d\mu_l(\alpha)$, $l = 1,2$, $u_0, u_1, v_0, v_1 \in H_0^{j+2}(\Omega)$, $F_1, F_2 \in C^\infty(\mathbb{R}^n;\mathbb{R})$. Then, there exists a unique solution of problem (14)–(18) on a segment $[0,t_1]$ with some $t_1 > 0$. If the first order partial derivatives of functions F_1, F_2 with respect to the second and the third variables are bounded, then there exists a unique solution of problem (14)–(18) on a segment $[0,T]$ with every $T > 0$.*

Proof. We can consider problem (14)–(18) as Cauchy problem (9) and (10) in the space $\mathcal{Z} = (H^j(\Omega))^2$ with the operator A, which is defined by (19). Note that the left sides of Equations (17) and (18) are the same distributed derivative. By Theorem 8 $A \in \mathcal{A}_W(\theta_0, a_0)$ for some $\theta_0 \in (\pi/2, \pi)$, $a_0 \geq 0$ and it remains to show that the nonlinear operator $B(x,y)(\cdot) = (F_1(\cdot, x(\cdot), y(\cdot)), F_2(\cdot, x(\cdot), y(\cdot)))$ satisfies the conditions of Theorem 7. Due to [38] (Proposition 1 in Appendix B) for $x, y \in H^j(\Omega)$, we have $F_l(x,y) \in H^j(\Omega)$, $l = 1,2$, since $j > d/2$. Moreover, by [38] (Proposition 1 in Appendix B), $B \in C^\infty((H^j(\Omega))^2; H^j(\Omega))$. Hence, B is locally Lipschitz continuous and in the case of boundedness of the first order partial derivatives of functions F_1, F_2 with respect to the second and the third variables B is Lipschitz continuous. It remains to apply Theorem 7 or Theorem 5, respectively. □

7. Conclusions

Using the form of the unique solution for the Cauchy problem to the linear inhomogeneous equation in a Banach space with a distributed Gerasimov–Caputo fractional derivative and with a linear closed operator A, which generates an analytic resolving family, we reduce the Cauchy problem for an analogous quasilinear equation to an equation of the form $z = G(z)$, where the mapping $G(z)$ uses the forms of k-resolving families of operators of the initial linear equation. It allows us to prove the fulfillment of the conditions of the Banach theorem on a fixed point in a specially constructed spaces of functions. Thus, in this paper, it is shown how the linear theory of resolving families of operators made it possible to make the transition from the study of linear equations with a distributed derivative to the study of the corresponding quasilinear equations. The obtained results will allow us to study the unique solvability issues for new initial-boundary value problems for equations and systems of equations with distributed Gerasimov–Caputo partial derivatives.

Using the approach developed in this paper, we plan to investigate the initial problems for quasilinear equations with distributed Riemann–Liouville, Hilfer, φ-Hilfer fractional derivatives [39], as well as other integrodifferential operators.

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