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Endomorphism Type of $P(3m + 1, 3)$

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Abstract: There are six different classes of endomorphisms for a graph. The sets of these endomorphisms always form a chain under the inclusion of sets. In order to study these different endomorphisms more systematically, Böttcher and Knauer proposed the concept of the endomorphism type of a graph in 1992. In this paper, we explore the six different classes of endomorphisms of graph $P(3m + 1, 3)$. In particular, the endomorphism type of $P(3m + 1, 3)$ is given.

Keywords: endomorphism; half-strong endomorphism; isomorphism; endomorphism type

MSC: 05C25; 20M20

1. Introduction

The endomorphism monoids of graphs are generalizations of automorphism groups of graphs. They have been studied for a long time and many interesting results concerning graphs and their endomorphism monoids have been discovered (cf. [1–5]). The main goal of this research is to develop further relations between graph theory and algebraic theory of semigroups and to apply the theory of semigroups to graph theory. The endomorphism monoids of graphs have valuable applications (cf. [6]) and are related to automata theory (cf. [7,8]). In [9], Böttcher and Knauer gave the definition of endomorphism spectrum and endomorphism type of graphs. The endomorphism spectrum of a graph is a six-tuple related to the endomorphism of a graph. The endomorphism type of a graph refers to an integer from 0 to 31. The endomorphism types of bipartite graphs with diameter 3 and girth 6 were studied by Fan in [10]. Hou, Fan and Luo in [11] explored six classes of endomorphisms of generalized polygons and endomorphism types of these graphs were given. Hou, Luo and Cheng in [12] characterized the endomorphism monoid of $P_n$, and the endomorphism spectrum and the endomorphism type of them were given. In this paper, we try to explore the six different classes of endomorphisms of a class of graphs $P(3m + 1, 3)$. Particularly, the endomorphism types of $P(3m + 1, 3)$ was determined.

2. Preliminary Concepts

The graphs we considered here are undirected, finite and simple. Denote by $K_n$ the complete graph with $n$ vertices and by $K^n_n$ the graph obtained by deleting an edge from $K_n$. A clique of a graph $X$ is the maximal complete subgraph of $X$. The clique number of $X$, denoted by $\omega(X)$, is the maximal order among the cliques of $X$. Let $v \in V(X)$. Set $N(v) = \{x \in V(X) | \{x, v\} \in E(X)\}$. If $n$ and $d$ are both positive integers such that $n \geq 2d$, the circulant complete graph $K(n, d)$ is defined as a graph with vertex set $V = Z_n$, in which $\{i, j\} \in E(K(n, d))$ if and only if $d \leq |i - j| \leq n - d$. It is not difficult to see that $K(n, 2) = C_n$ and their endomorphism monoids were explored in [2]. Denote by $P(3m + 1, 3)$ a graph obtained by adding an edge $\{1, 3m + 1\}$ to the graph $K(3m + 1, 3)$ (see Figure 1).
Lemma 2. Let \( f \in \text{End}(P(3m + 1, 3)) \). Then

1. If \( f(x_1) = f(x_2) \) for some \( x_1, x_2 \in V(P(3m + 1, 3)) \), then \( |x_1 - x_2| \in \{1, 2\} \), or \( |x_1 - x_2| = 3m - 1 \).

2. There are no \( x_1, x_2, x_3, x_4 \in V(P(3m + 1, 3)) \) such that \( f(x_1) = f(x_2) = f(x_3) = f(x_4) \).
Proof. (1) By the definition of $P(3m + 1, 3)$, $\{x_1, x_2\} \in E(P(3m + 1, 3))$ if and only if $3 \leq |x_1 - x_2| \leq 3m - 2$ or $|x_1 - x_2| = 3m$. If $f(x_1) = f(x_2)$, then $\{x_1, x_2\} \notin E(P(3m + 1, 3))$. Hence, $|x_1 - x_2| \in \{1, 2\}$, or $|x_1 - x_2| = 3m - 1$.

(2) This follows directly from (1). \qed

Lemma 3. Let $S = \{1, 4, 7, \ldots , 3m - 2, 3m + 1\}$. Then, the subgraph of $P(3m + 1, 3)$ induced by $S$, namely $\langle S \rangle$, is the only clique whose order is $m + 1$.

Proof. It is easy to see that the subgraph of $P(3m + 1, 3)$ induced by $S$ is a clique of order $m + 1$. Let $K$ be a clique of order $3m + 1$. Then, it must contain 1 and $3m + 1$. Otherwise, $K$ is a clique of $K(3m + 1, 3)$. This is impossible since $\omega(K(3m + 1, 3)) = m$ as shown in [3]. Now, 4, 7, $\ldots$ , $3m - 2$ are the only $m - 1$ vertices in $V(P(3m + 1, 3))$ adjacent to both 1 and $3m + 1$. Hence $K = \langle S \rangle$ and $\langle S \rangle$ is the only clique whose order is $m + 1$. \qed

Lemma 4. $P(3m + 1, 3)$ does not contain a subgraph isomorphic to $K_m^{+2}$.

Proof. Suppose $P(3m + 1, 3)$ contains a subgraph isomorphic to $K_m^{+2}$. Then, it contains more than one clique whose order is $m + 1$. This is a contraction to Lemma 3.

Note that $\langle S \rangle$ is a complete graph with order $m + 1$. We may identify $\langle S \rangle$ with $K_{m+1}$.

Lemma 5. Let $f \in \text{End}(P(3m + 1, 3))$. Then, $f(K_{m+1}) = K_{m+1}$.

Proof. As any endomorphism $f$ maps a clique to a clique of the same size and $K_{m+1}$ is the only clique of size $m + 1$ in $P(3m + 1, 3)$, $f(K_{m+1}) = K_{m+1}$. \qed

Lemma 6. Let $f \in \text{End}(P(3m + 1, 3))$. If $f(x_1) = f(x_2) = f(x_3)$ for three distinct vertices $x_1, x_2, x_3 \in V(P(3m + 1, 3))$, then $x_1, x_2$ and $x_3$ are three consecutive integers in $V(P(3m + 1, 3))$. In this case, there exists an integer $i \in [1, 3m - 1]$ such that $f(i) = f(i + 1) = f(i + 2)$.

Proof. As $\{1, 3m + 1\} \in E(P(3m + 1, 3))$, $\{1, 3m + 1\} \notin \{x_1, x_2, x_3\}$. Assume $1 \in \{x_1, x_2, x_3\}$. In the following, we prove that $\{x_1, x_2, x_3\} = \{1, 2, 3\}$.

Firstly, $3m \notin \{x_1, x_2, x_3\}$. Otherwise, we have $3m \in \{x_1, x_2, x_3\}$ and $f(3m) = f(1)$. Let $x_3 = (x_1, x_2, x_3) \setminus \{1, 3m\}$. As $f(x_1) = f(1)$ and $f(x_2) = f(3m)$, $\{x_1, x_2\} \notin E(P(3m + 1, 3))$ and $\{x_1, x_2, 3m\} \notin E(P(3m + 1, 3))$. Denote $A = V(P(3m + 1, 3)) \setminus N(1)$ and $B = V(P(3m + 1, 3)) \setminus N(3m)$. Clearly, $x_1 \in A \cap B$. It is easy to see that $A = \{3m - 2, 3m - 3, 3m - 4, \ldots , 1\}$ and $B = \{3m - 2, 3m - 3, 3m - 4, \ldots , 1\}$. Since $m \geq 2$ is an integer, $A \setminus B = \emptyset$. This is a contradiction. Denote by $\{x_5, x_6\} = \{x_1, x_2, x_3\} \setminus \{1\}$. Then, $x_5 \in A$ and $x_6 \in A$. Note that $3m \notin \{x_1, x_2, x_3\}$. Then, $\{x_5, x_1\} = \{2, 3\}$. Therefore, $\{x_1, x_2, x_3\} = \{1, 2, 3\}$.

Similarly, if $3m + 1 \in \{x_1, x_2, x_3\}$, then we can show that $\{x_1, x_2, x_3\} = \{3m - 1, 3m, 3m + 1\}$.

If $1 \notin \{x_1, x_2, x_3\}$ and $3m + 1 \in \{x_1, x_2, x_3\}$, then $2 \leq x_i \leq 3m$ for any $x_i \in \{x_1, x_2, x_3\}$. Suppose that $x_1, x_2, x_3$ are not three consecutive integers. Then, there exists $x_s, x_t \in \{x_1, x_2, x_3\}$ such that $|x_s - x_t| \geq 3$ it contradicts to Lemma 2(1).

Therefore, $x_1, x_2$ and $x_3$ are three consecutive integers in $V(P(3m + 1, 3))$. Let $i = \min \{x_1, x_2, x_3\}$. Then, $f(i) = f(i + 1) = f(i + 2)$, where $1 \leq i \leq 3m - 1$. \qed

Lemma 7. $\text{End}(P(7, 3)) = h\text{End}(P(7, 3))$.

Proof. Let $f \in \text{End}(P(7, 3))$. By Lemma 1, we need to show that $I_f$ is an induced subgraph of $P(7, 3)$.

Since $P(7, 3)$ is connected, $I_f$ is connected. Note that $P(7, 3)$ has an only clique isomorphic to $K_3$. It is induced by $S = \{1, 4, 7\}$. Since any endomorphism $f$ maps a clique to a clique of the same size, $S \subseteq I_f$. So $3 \leq |I_f| \leq 7$. There are 5 cases.

Case 1. Assume that $|I_f| = 3$. Clearly, $I_f$ is an induced subgraph of $P(7, 3)$.
Case 2. Assume that $|I_f| = 4$. Then, $2, 6 \notin V(I_f)$ since $I_f$ is connected. Thus $V(I_f) = V(K_3) \cup \{3\}$ or $V(I_f) = V(K_3) \cup \{5\}$. Note that 3 is only adjacent to 7 in $V(I_f)$ and 5 is only adjacent to 1 in $V(I_f)$. Since $I_f$ is connected, $I_f$ is an induced subgraph of $P(7, 3)$.

Case 3. Assume that $|I_f| = 5$. If $6 \in V(I_f)$, then $3 \in V(I_f)$ (Otherwise, $I_f$ is not connected). Thus, $V(I_f) = \{3, 6\} \cup V(K_3)$. Similarly, if $2 \in V(I_f)$, then $5 \in V(I_f)$. Thus, $V(I_f) = \{2, 5\} \cup V(K_3)$. If $2, 6 \notin V(I_f)$, then $V(I_f) = \{3, 5\} \cup V(K_3)$. Since $I_f$ is connected, $I_f$ is an induced subgraph of $P(7, 3)$.

Case 4. Assume that $|I_f| = 6$. Then, $V(I_f)$ is one of $S \cup \{2, 3, 6\}$, $S \cup \{2, 3, 5\}$, $S \cup \{3, 5, 6\}$ and $S \cup \{2, 5, 6\}$. In above cases, $I_f$ is an induced subgraph of $P(3m + 1, 3)$ since $I_f$ is connected.

Case 5. Assume that $|I_f| = 7$, then $f \in Aut(P(3m + 1, 3))$. Hence, $I_f$ is an induced subgraph of $P(7, 3)$.

Lemma 8. $End(P(3m + 1, 3)) \neq hEnd(P(3m + 1, 3))$ for any $m \geq 3$.

Proof. Let

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots & 3m-1 & 3m & 3m+1 \\ 1 & 1 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots & 3m+1 & 3m+1 \\ \end{pmatrix}$$

It is easy to check that $f \in End(P(3m + 1, 3))$. Note that $3, 6 \in V(I_f)$, $\{3, 6\} \in E(P(3m + 1, 3))$, $f^{-1}(3) = \{3\}$, $f^{-1}(6) = \{5\}$, but $\{3, 5\} \notin E(P(3m + 1, 3))$. Thus, $f \notin hEnd(P(3m + 1, 3))$. Hence $End(P(3m + 1, 3)) \neq hEnd(P(3m + 1, 3))$.

Lemma 9. $hEnd(P(3m + 1, 3)) \neq lEnd(P(3m + 1, 3))$ for any $m \geq 2$.

Proof. Let

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots & 3m-2 & 3m-1 & 3m & 3m+1 \\ 1 & 1 & 1 & 4 & 4 & 4 & 7 & 7 & 7 & \cdots & 3m-2 & 3m-2 & 3m-2 & 3m+1 \\ \end{pmatrix}$$

It is not difficult to check that $f \in hEnd(P(3m + 1, 3))$. Note that $\{3m + 1, 1\} \in E(I_f)$, $f^{-1}(3m + 1) = \{3m + 1\}$, $f^{-1}(1) = \{1, 2, 3\}$ and the vertex 2 is isolated. Thus, $f \notin lEnd(P(3m + 1, 3))$. Hence $hEnd(P(3m + 1, 3)) \neq lEnd(P(3m + 1, 3))$.

Lemma 10. $lEnd(P(3m + 1, 3)) \neq qEnd(P(3m + 1, 3))$ for any $m \geq 2$.

Proof. Let

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots & 3m-3 & 3m-2 & 3m-1 & 3m & 3m+1 \\ 1 & 1 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots & 3m-3 & 3m-2 & 3m-1 & 3m+1 \\ \end{pmatrix}$$

It is not difficult to check that $f \in lEnd(P(3m + 1, 3))$. Note that $\{3m + 1, 1\} \in E(I_f)$, $f^{-1}(3m + 1) = \{3m, 3m + 1\}$, $f^{-1}(1) = \{1, 2\}$. Thus, there is no vertex $x \in f^{-1}(1)$ such that $x$ is adjacent to every vertex of $f^{-1}(3m + 1)$, and so $f \notin qEnd(P(3m + 1, 3))$. Hence, $lEnd(P(3m + 1, 3)) \neq qEnd(P(3m + 1, 3))$.

Lemma 11. $qEnd(P(3m + 1, 3)) = Aut(P(3m + 1, 3))$ for any $m \geq 2$.

Proof. Let $f \in qEnd(P(3m + 1, 3))$. Firstly, we show that $[1]_{\rho_f} = \{1\}$. Otherwise, there exists $i \in V(P(3m + 1, 3)) \setminus \{1\}$ such that $f(1) = f(i)$. Thus $\{1, i\} \notin E(P(3m + 1, 3))$. Hence, $i \in \{2, 3, 3m\}$.

If $i = 3m$, then $f(1) = f(3m)$. Since $\{3m + 1\} \in E(P(3m + 1, 3))$, $\{f(3m + 1), f(1)\} \in E(P(3m + 1, 3))$. Note that $f$ is quasi-strong. There exists $y \in [3m + 1]_{\rho_f}$ such that $\{3m, y\} \in E(P(3m + 1, 3))$ and $\{1, y\} \in E(P(3m + 1, 3))$. Note that $[3m + 1]_{\rho_f} \subseteq$
\{3m - 1, 3m, 3m + 1, 2\}. This is a contradiction. Therefore, \(f(1) \neq f(3m)\). By symmetry of \(P(3m + 1, 3), f(2) \neq f(3m + 1)\).

If \(i = 3\), then \(f(1) = f(3)\). Since \(\{1, 4\} \in E(P(3m + 1, 3)), \{f(1), f(4)\} \in E(P(3m + 1, 3))\). Note that \(f\) is quasi-strong. There exists \(x \in [4]_\rho\) such that \(\{1, x\} \in E(P(3m + 1, 3))\) and \(\{3, x\} \in E(P(3m + 1, 3))\). Note that \([4]_\rho \subseteq \{2, 3, 4, 5, 6\}\). Then, \(x = 6\) and \(f(4) = f(6)\). Similarly, \(f(7) = f(9), \cdots, f(3m - 2) = f(3m)\). Since \(\{3m + 1, 3m - 2\} \in E(P(3m + 1, 3)), \{f(3m + 1), f(3m - 2)\} \in E(P(3m + 1, 3))\). Then, there exists \(y \in [3m + 1]_\rho\) such that \(\{3m - 2, y\} \in E(P(3m + 1, 3))\) and \(\{3m, y\} \in E(P(3m + 1, 3))\). Note that \([3m + 1]_\rho \subseteq \{3m - 1, 3m, 3m + 1, 2\}\). Thus, \(y = 2\). Hence \(f(2) = f(3m + 1)\). This is a contradiction. Therefore, \(f(1) \neq f(3)\). By symmetry of \(P(3m + 1, 3), f(3m - 1) \neq f(3m + 1)\).

If \(i = 2\), then \(f(1) = f(2)\). Since \(\{3m + 1, 1\} \in E(P(3m + 1, 3)), \{f(3m + 1), f(1)\} \in E(P(3m + 1, 3))\). Note that \(f\) is quasi-strong. There exists \(x \in [3m + 1]_\rho\) such that \(\{1, x\} \in E(P(3m + 1, 3))\) and \(\{2, x\} \in E(P(3m + 1, 3))\). Note that \([3m + 1]_\rho \subseteq \{3m - 1, 3m, 3m + 1, 2\}\). Then, \(x = 3m - 1\) and \(f(3m + 1) = f(3m - 1)\). This is a contradiction. Therefore, \(f(1) \neq f(2)\).

Now, we have \([1]_\rho = \{1\}\). By symmetry of \(P(3m + 1, 3), [3m + 1]_\rho = \{3m + 1\}\).

Secondly, we show that \([2]_\rho = \{2\}\). Otherwise, there exists \(j \in V(P(3m + 1, 3)) \setminus \{2\}\) such that \(f(2) = f(j)\). Thus, \(j \in \{1, 3, 4, 3m + 1\}\). Note that \([1]_\rho = \{1\}\) and \([3m + 1]_\rho = \{3m + 1\}\). Then, \(j \in \{3, 4\}\). Since \(\{3m + 1, j\} \in E(P(3m + 1, 3)), \{f(3m + 1), f(j)\} \in E(P(3m + 1, 3))\). Note that \(f\) is quasi-strong. There exists \(x \in [3m + 1]_\rho\) such that \(\{2, x\} \in E(P(3m + 1, 3))\) and \(\{j, x\} \in E(P(3m + 1, 3))\). Note that \([3m + 1]_\rho = \{3m + 1\}\) and \([3m + 1, 2] \notin E(P(3m + 1, 3))\). This is a contradiction.

Lastly, we show that \([3]_\rho = \{3\}\). Otherwise, there exists \(k \in V(P(3m + 1, 3)) \setminus \{3\}\) such that \(f(3) = f(k)\). Clearly, \(k \in \{1, 2, 4, 5\}\). Note that \([1]_\rho = \{1\}\) and \([2]_\rho = \{2\}\). Then, \(k \in \{4, 5\}\). Since \(\{1, k\} \in E(P(3m + 1, 3)), \{f(1), f(k)\} \in E(P(3m + 1, 3))\). Note that \(f\) is quasi-strong. There exists \(x \in [1]_\rho\) such that \(\{3, x\} \in E(P(3m + 1, 3))\) and \(\{k, x\} \in E(P(3m + 1, 3))\). Note that \([1]_\rho = \{1\}\) and \([3]_\rho \notin E(P(3m + 1, 3))\). This is a contradiction.

A similar argument will show that \([i]_\rho = \{i\}\) for any \(i = 5, 6, \cdots, 3m - 1, 3m\). Thus, \(f \in Aut(P(3m + 1, 3))\). Hence, \(\text{qEnd}(P(3m + 1, 3)) = Aut(P(3m + 1, 3))\).

**Theorem 1.** (1) If \(m = 2\), then \(\text{End}(P(3m + 1, 3)) = \text{hEnd}(P(3m + 1, 3)) \neq \text{lEnd}(P(3m + 1, 3)) \neq \text{qEnd}(P(3m + 1, 3)) \neq \text{sEnd}(P(3m + 1, 3)) = Aut(P(3m + 1, 3))\).

(2) If \(m \geq 3\), then \(\text{End}(P(3m + 1, 3)) \neq \text{hEnd}(P(3m + 1, 3)) \neq \text{lEnd}(P(3m + 1, 3)) \neq \text{qEnd}(P(3m + 1, 3)) \neq \text{sEnd}(P(3m + 1, 3)) = Aut(P(3m + 1, 3))\).

**Proof.** This follows directly from Lemmas 7–11. 

**Theorem 2.** (1) If \(m = 2\), then \(\text{End}(P(3m + 1, 3)) = 6\).

(2) If \(m \geq 3\), then \(\text{End}(P(3m + 1, 3)) = 7\).

**Proof.** This follows directly from Theorem 1.

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References


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