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Mathematical Model for Fault Handling of Singular Nonlinear Time-Varying Delay Systems Based on T-S Fuzzy Model

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Abstract: In this paper, a mathematical model based on the T-S fuzzy model is proposed to solve the fault estimation (FE) and fault-tolerant control (FTC) problem for singular nonlinear time-varying delay (TVD) systems with sensor fault. TVD is extremely difficult to solve and the Laplace transform is devised to build an equal system free of TVD. Additionally, the sensor fault is changed to actuator fault by the developed coordinate transformation. A fuzzy learning fault estimator is first built to estimate the detailed sensor fault information. Then, a PI FTC scheme is suggested aiming at minimizing the damage caused by the fault. Simulation results from multiple faults reveal that the FE and FTC algorithms are able to estimate the fault and guarantee the system performance properly.

Keywords: T-S fuzzy model; time-varying delay; singular systems; sensor fault; FE; FTC

MSC: 93C10; 93C42; 93C43

1. Introduction

Singular systems, also known as generalized systems, are more complex systems than general systems, consisting of faster-changing differential systems and slower-changing algebraic equations [1–3]. There are still many unsolved problems waiting to be studied, and the study of singular systems is very meaningful. The stability analysis and PD controller design methods for nonlinear singular systems are given in [4]. Reference [5] provides the design of a predictive sliding mode controller for the control of singular systems. Meanwhile, nonlinear problems are widely found in real systems [6–8]. The reference in [9] constructs an explicit controller for systems with stochastic nonlinearities to make the closed-loop system globally bounded. The neuro-fuzzy and state feedback control scheme for the problem of nonlinear control systems is presented in [10]. For fuzzy control, the selection of the membership function is very important. In [11], a membership function of interval type-2 is given and relies on the characteristics of the function for an analytical approach to analyze the system stability. An enhanced membership function transformation method is proposed in [12] to approximate the membership function, and stability analysis of a polynomial fuzzy system is performed. The problem of time delay has also received the attention of researchers [13,14]. Reference [15] presents a consistency analysis of a system with a distributed delay adopted PI controller. A nonlinear mixed reaction diffusion dynamics model for prostate cancer cells with time delay is given in [16], and the properties of the positive solutions of the model are investigated. Therefore, the problem addressed in this paper is a very worthy research direction.

When a fault occurs in a normal system, the output of the system can be severely affected or even destroy the system directly, so fault estimation algorithms should be proposed to obtain fault information for subsequent processing [17,18]. Firstly, for the data-based fault estimation aspect, a multi-scale bidirectional diversity-entropy-based FE algorithm is addressed in [19] to improve the extraction of nonlinear dynamic fault features. Various adaptive principal component analysis methods are proposed in [20] to perform early fault detection by comparing their monitoring metrics. There are also algorithms
that perform fault estimation by monitoring changes in the system model through an estimator [21–23]. In [24], a memory-state feedback control algorithm is designed for time-delay interval type-2 T-S fuzzy systems to mitigate the effects of external disturbances and actuator fault. A time-based sliding mode estimator for a quadrotor UAS with multiple actuator faults is suggested in [25] and the output residuals are applied in the estimation algorithm. Tests demonstrate that the algorithm is able to identify multiple faults even under the influence of perturbations. In [26], a reduced-order estimation observer is developed for an uncertain system and equipped with an online adaptive FE method, which is capable of identifying multiple faults simultaneously. In addition, a batch-type least squares projection method is constructed to measure the degree of the faults. Multiple fault detection methods based on directed unknown input observers are designed in [27] for networked control systems and validated in power systems. However, the FE research results of singular systems are relatively few.

Once the fault information is acquired, control algorithms are needed to mitigate the effects of the fault [28,29]. Reference [30] addresses the problem that bounded nonlinear systems are affected by sensor and link fault, and proposes a distributed state estimation and active FTC scheme to ensure the stable operation of the communication. An output feedback FTC scheme is suggested in [31] for nonlinear uncertain systems with multiple faults. In [32], a data-driven distributed formation FTC method is developed for quadrotors with nonlinearity, uncertainty, and multiple faults. A data-based distributed iterative FTC method is designed in [33] for multi-input systems, which does not require all the system information, reduces the computational burden, and the simulation verifies the performance of the algorithm. However, there has been relatively little focus on the research of FTC algorithms for generalized delay systems.

In this paper, a mathematical model for the FE and FTC of singular nonlinear TVD systems with sensor fault is addressed based on a T-S fuzzy model. The T-S fuzzy model is implemented to approach the system’s nonlinear dynamics. A Laplace transform method is devised to construct an equivalent system without explicit TVD. The sensor fault is changed to actuator fault by the developed coordinate transformation. A fuzzy learning fault estimator is built to estimate the detailed sensor fault information. When the system output drifts away to the expected value, a PI FTC scheme is suggested with the aim of reducing the damage of the fault. Simulation results from multiple faults indicate that the FE and FTC algorithms are capable of estimating the fault and guaranteeing the system performance appropriately.

Among the main contributions of this paper are:

1. The nonlinear dynamics is approached through a T-S fuzzy model, Laplace transform is adopted to tackle the TVD issue, and coordinate transformation is utilized to simplify the sensor fault handling challenge.
2. A novel fuzzy learning fault estimator is addressed to capture detailed fault information.
3. A fuzzy PI FTC scheme is introduced to mitigate the impact of fault on system performance to the maximum extent possible.

This paper is structured as follows: In Section 2, the system T-S fuzzy modelling is presented. In Section 3, the coordinate transformation algorithms and the design of a fuzzy learning fault estimator are given. A PI FTC controller is introduced in Section 4. Next, some examples of comparative simulations are provided in Section 5. Eventually, Section 6 presents the conclusion.
2. Model Description

The T-S fuzzy model is the most powerful tool to process system nonlinearity, so the T-S fuzzy system is deployed to approach the system dynamics in this paper. The T-S model is characterized via IF-THEN fuzzy rules, and under each rule is a linear subsystem whose set represents an approximation of the nonlinear system. The ith fuzzy rule for a class of singular nonlinear systems is represented as

\[ \text{rule } i: \text{ IF } \zeta_1(k) \text{ is } \delta_{i1}, \zeta_2(k) \text{ is } \delta_{i2}, \ldots, \text{ and } \zeta_{k_i}(k) \text{ is } \delta_{i_{k_i}}, \text{ THEN } \]

\[
\begin{align*}
\dot{x}(t) & = A_i x(t) + B_i u(t - \tau) + N_i d(t), \\
y(t) & = D_i x(t) + S_i f(t).
\end{align*}
\]

where \( x(t), u(t), y(t) \) denote the state vector, control input vector, and output vector, respectively. Matrices \( A_i, B_i, D_i, N_i, S_i \) represent the system parameters with the required dimensions. The TVD \( \tau \) obeys the exponential distribution of parameter \( \alpha \). \( f(t) \) denotes the vector of possible sensor fault. \( ||d(t)|| \leq K_f \) is bounded extraneous noise. \( \zeta_j(k) (j = 1, 2, \ldots, k_1) \) is the antecedent variable, \( \delta_{ij}(i = 1, 2, \ldots, k_2; j = 1, 2, \ldots, k_1) \) is the vague collection, characterized with the membership function, and \( k_1 \) and \( k_2 \) are the number of If-Then rules and the antecedent variables.

**Assumption 1** ([34]). The system is regular, namely, \(|sE - A| \neq 0\).

**Assumption 2** ([35]). The system is pulseless, i.e., \( \text{rank}E = \text{deg}[sE - A] \).

**Assumption 3.** For fault \( f_s \), the condition is satisfied \( ||f_s|| \leq K_f \) with unknown positive scalar \( K_f \).

When assumptions 1 and 2 hold, there are two non-singular matrices and so that the following formula is established:

\[
\begin{align*}
\Psi_1 E \Psi_2 & = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}, \\
\Psi_1 A \Psi_2 & = \begin{bmatrix} A^1 & 0 \\ 0 & I_{n-p} \end{bmatrix}, \\
\Psi_1 B & = \begin{bmatrix} B^1 \\ B^2 \end{bmatrix}, \\
\Psi_1 N & = \begin{bmatrix} N^1 \\ N^2 \end{bmatrix}, \\
D \Psi_2 & = \begin{bmatrix} D^1 & D^2 \end{bmatrix}.
\end{align*}
\]

where \( \Psi_1, \Psi_2 \in R^{n \times n}, A^1 \in R^{p \times p}, I_p \) is \( i \)-dimensional unit matrix, then System (1) can be transformed by coordinate transformation \( \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \Psi_2^{-1} x(t) \) as

\[ \text{rule } i: \text{ IF } \zeta_1(k) \text{ is } \delta_{i1}, \zeta_2(k) \text{ is } \delta_{i2}, \ldots, \text{ and } \zeta_{k_i}(k) \text{ is } \delta_{i_{k_i}}, \text{ THEN } \]

\[
\begin{align*}
\Psi_1 E \Psi_2 \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} & = \Psi_1 A \Psi_2 \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \Psi_1 B_i u(t - \tau) + \Psi_1 N_i d(t), \\
y(t) & = D_i \Psi_2 \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + S_i f(t).
\end{align*}
\]

By plugging the block matrix from (2) into the above equation, the following can be obtained:

\[ \text{rule } i: \text{ IF } \zeta_1(k) \text{ is } \delta_{i1}, \zeta_2(k) \text{ is } \delta_{i2}, \ldots, \text{ and } \zeta_{k_i}(k) \text{ is } \delta_{i_{k_i}}, \text{ THEN } \]

\[
\begin{align*}
\dot{x}_1(t) & = A^1_1 x_1(t) + B^1_1 u(t - \tau) + N^1_1 d(t), \\
\dot{x}_2(t) & = -B^2_1 u(t - \tau) - N^2_1 d(t), \\
y(t) & = D^1_1 x_1(t) + D^2_1 x_2(t) + S_i f(t).
\end{align*}
\]
By fusing the local models guided by all IF-THEN rule through fuzzy blending, the global fuzzy model below is obtained:

\[ \begin{align*}
\dot{x}_1(t) &= \sum_{i=1}^{s_2} h_i(\zeta(t)) [A_i^1 x_1(t) + B_i^1 u(t - \tau) + N_i^1 d(t)], \\
x_2(t) &= \sum_{i=1}^{s_2} h_i(\zeta(t)) [-B_i^2 u(t - \tau) - N_i^2 d(t)], \\
y(t) &= \sum_{i=1}^{s_2} h_i(\zeta(t)) [D_i^1 x_1(t) + D_i^2 x_2(t) + S_i f_i(t)].
\end{align*} \]  

(5)

where \( \zeta(k) = [\zeta_1(k), \zeta_2(k), \ldots, \zeta_{s_2}(k)] \), \( \omega_i(\zeta(k)) = \prod_{j=1}^{s_2} \delta_{ij}(\zeta_j(k)) > 0, h_i(\zeta(k)) = \frac{\omega_i(\zeta(k))}{\sum_{i=1}^{s_2} \omega_i(\zeta(k))} > 0, \sum_{i=1}^{s_2} h_i(\zeta(k)) = 1. \)

**Lemma 1.** For real matrices \( X, Y \), the following inequality holds [31]:

\[ X^T Y + Y^T X \leq X^T X + Y^T Y. \]  

(6)

### 3. Fault Estimation

The system is affected by both sensor fault and TVD, which are very difficult to handle directly. Two transformation methods are presented below to deal with sensor fault and TVD.

Introducing a new state variable

\[ \dot{x}_s = -A_s x_s(t) + A_s y(t). \]  

(7)

where \( A_s \) is a Hurwitz matrix.

The new system structure is represented below

\[ \begin{align*}
\dot{x}_{s1}(t) &= \sum_{i=1}^{s_2} h_i(\zeta(t)) [A_i^1 x_{s1}(t) + B_i^1 u(t - \tau) + S_i f_i(t) + N_i^1 d(t)], \\
y_{s1}(t) &= \sum_{i=1}^{s_2} h_i(\zeta(t)) [D_i^1 x_{s1}(t)],
\end{align*} \]  

(8)

where \( x_{s1}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \), \( A_i^s = \begin{bmatrix} A_i^1 & 0 \\ A_s D_i^1 & -A_s \end{bmatrix} \), \( B_i^s = \begin{bmatrix} B_i^1 \\ -A_s D_i^1 B_i^2 \end{bmatrix} \), \( S_i^s = \begin{bmatrix} 0 \\ A_s S \end{bmatrix} \), \( D_i^s = \begin{bmatrix} 0 & I_p \end{bmatrix} \).

After the above conversion, the sensor fault is transformed into actuator fault, which is convenient for subsequent processing.

To deal with TVD, the Laplace transform is used to solve the delays and convert System (8) into an equivalent system without significant delays. The delay module in this paper satisfies the exponential distribution function and yields the following results:

\[ F_T(t) = 1 - e^{-\alpha t}. \]  

(9)

The probability density function (PDF) of the exponential distribution is \( f(\alpha, t) = \frac{\alpha}{\alpha + s} e^{-\alpha t}, \) and taking the Laplace transform on the PDF yields

\[ F_T(s) = \int_0^\infty \frac{\alpha}{\alpha + s} e^{-\alpha t} d\tau = \frac{\alpha}{\alpha + \sigma}. \]  

(10)

The expected value of the output response of a random delay block is equivalent to the Laplace transform of the phase-synchronized signal with the same sampling period [36], and it leads to

\[ E[u(t - \tau)] = L^{-1} \left\{ \frac{\alpha}{\alpha + s} u(s) \right\}. \]  

(11)
Furthermore,
\[ x_{x2}(t) = \frac{\alpha}{a + s} u(s). \]  

(12)

Then the following system is available:
\[ \dot{x}_{x2}(t) = -a x_{x2}(t) + a u(t). \]  

(13)

Merging Systems (8) and (13), the novel system model is given below:
\[ \begin{cases} \dot{x}(t) = \sum_{i=1}^{K} h_i(\zeta(t)) [A_i \dot{x}(t) + B_i u(t) + S_i \dot{f}_s(t) + \bar{N}_i d(t)], \\ \dot{y}(t) = \sum_{i=1}^{K} h_i(\zeta(t)) [D_i \dot{x}(t)], \end{cases} \]  

(14)

where \( \bar{x} = \begin{bmatrix} x_{s1} \\ x_{s2} \end{bmatrix}, \bar{A}_i = \begin{bmatrix} A_i^s & B_i^s \\ 0 & -\alpha \end{bmatrix}, \bar{B}_i = \begin{bmatrix} 0 \\ \alpha \end{bmatrix}, \bar{S}_i = \begin{bmatrix} S_i^s \end{bmatrix}, \bar{N}_i = \begin{bmatrix} N_i^s \\ 0 \end{bmatrix}, \bar{D}_i = \begin{bmatrix} D_i^s & 0 \end{bmatrix}. \)

After the above transformation, the system has been transformed into a system without significant time delay affected by actuator fault. To estimate detailed fault information, the structure of the fuzzy learning fault estimator is given below:
\[ \begin{cases} \dot{\hat{x}}(t) = \sum_{i=1}^{K} h_i(\zeta(t)) [\bar{A}_i \hat{x}(t) + \bar{B}_i u(t) + \bar{S}_i \dot{f}_s(t) + G_i \epsilon(t)], \\ \dot{\hat{y}}(t) = \sum_{i=1}^{K} h_i(\zeta(t)) [\bar{D}_i \hat{x}(t)], \\ Z(t) = \sum_{i=1}^{K} h_i(\zeta(t)) [K_{1i} Z(t - \tau) + K_{2i} \epsilon(t - \tau)], \\ \dot{f}_s(t) = W Z(t), \end{cases} \]  

(15)

where \( \hat{x}(t) \) and \( \dot{f}_s(t) \) are the estimated state and sensor fault, \( \epsilon(t) = \dot{y}(t) - \hat{y}(t) \) is the residual signal, and \( G_i, K_{1i}, K_{2i} \) are gain matrices remaining undetermined.

The estimated error is specified below:
\[ \begin{align*} 
\epsilon_x(t) &= \bar{x}(t) - \hat{x}(t), \\
\epsilon_{f_s}(t) &= \dot{f}_s(t) - \dot{\hat{f}}_s(t). 
\end{align*} \]  

(16)

Combining System (14) and the learning estimator (15), the estimation error system is gained below:
\[ \dot{\epsilon}_x(t) = \sum_{i=1}^{K} h_i(\zeta(t)) [(\bar{A}_i - G_i \bar{D}_i) \epsilon_x(t) + \bar{S}_i \dot{f}_s(t) - \bar{S}_i Z(t) + \bar{N}_i d(t)]. \]  

(17)

**Theorem 1.** For System (14) and the learning estimator (15), the error system (17) is considered convergent when there exist positive definite symmetric matrices (PDSMs) \( P_1, R_1 \) and \( K_{1i}, K_{2i} > 0 \) to make the following inequalities valid:
\[ \begin{align*} 
(A_i - G_i D_i)^T P_1 + P_1 (A_i - G_i D_i) + R_1 + P_1 S_i S_i^T P_1 + Q_1 &\leq 0, \\
0 &< (6 + 3\sigma) K_{1i}^T K_{1i} \leq I, \\
0 &< (6 + 3\sigma) (K_{2i} D)^T (K_{2i} D) \leq R_1. 
\end{align*} \]  

(18)

**Proof.** The following is a definition of the Lyapunov function:
\[ \begin{align*} 
\Lambda_1(t) &= \epsilon_x^T(t) P_1 \epsilon_x(t) + \int_{t-\tau}^{t} \epsilon_x^T(\xi) R_1 \epsilon_x(\xi) d\xi + \int_{t-\tau}^{t} Z^T(\xi) Z(\xi) d\xi. 
\end{align*} \]  

(19)
The derivative of $\Lambda_1(t)$ yields

$$\dot{\Lambda}_1(t) \leq \sum_{i=1}^{k_2} h_i(\xi(t)) \sum_{j=1}^{k_2} h_j(\xi(t)) [\varepsilon^T_i(t) (P_1 (\tilde{A}_i - G_i D_j)) + (\tilde{A}_i - G_i D_j)^T P_1] \varepsilon_s(t)$$

$$+ (\tilde{A}_i - G_i D_j)^T P_1 \varepsilon_s(t) + 2K_2^3 + 2K_2^3 + 6\sigma Z^T(t) Z(t)$$

$$+ 6\sigma Z^T(t) Z(t)$$

For the learning estimator (15), the following inequalities are established:

$$2Z^T(t) Z(t) \leq \sum_{i=1}^{k_2} h_i(\xi(t)) \sum_{j=1}^{k_2} h_j(\xi(t)) [2Z^T(t - \tau) K_1^T K_1 Z(t - \tau) + Z^T(t - \tau) K_2^T K_1 Z(t - \tau) + Z^T(t - \tau) K_2^T K_1 Z(t - \tau) + Z^T(t - \tau) K_2^T K_1 Z(t - \tau) + Z^T(t - \tau) K_2^T K_1 Z(t - \tau) + Z^T(t - \tau) K_2^T K_1 Z(t - \tau)]$$

$$= \sum_{i=1}^{k_2} h_i(\xi(t)) \sum_{j=1}^{k_2} h_j(\xi(t)) [6Z^T(t - \tau) K_1^T K_1 Z(t - \tau) + 6\sigma Z^T(t) Z(t)]$$

Using Lemma 1, one can obtain

$$2\varepsilon^T_i(t) P_1 \varepsilon_s(t) \leq \varepsilon^T_i(t) (P_1 \tilde{A}_i - G_i D_j) + (\tilde{A}_i - G_i D_j)^T P_1 + R_1$$

Substituting the above equation into Formula (20) and rearranging yields

$$\dot{\Lambda}_1(t) \leq \sum_{i=1}^{k_2} h_i(\xi(t)) \sum_{j=1}^{k_2} h_j(\xi(t)) [\varepsilon^T_i(t) (P_1 (\tilde{A}_i - G_i D_j)) + (\tilde{A}_i - G_i D_j)^T P_1] \varepsilon_s(t)$$

$$+ 2P_1 \tilde{A}_i - G_i D_j P_1 + 2K_2^3 + 2K_2^3$$

$$+ 6\sigma Z^T(t) Z(t)$$

When Theorem 1 holds, the following result is obtained:

$$\dot{\Lambda}_1(t) \leq \sum_{i=1}^{k_2} h_i(\xi(t)) \sum_{j=1}^{k_2} h_j(\xi(t)) [\varepsilon^T_i(t) (P_1 (\tilde{A}_i - G_i D_j)) + (\tilde{A}_i - G_i D_j)^T P_1 + R_1$$

$$+ 2P_1 \tilde{A}_i - G_i D_j P_1 + 2K_2^3 + 2K_2^3$$

$$+ 6\sigma Z^T(t) Z(t)$$

$$= \sum_{i=1}^{k_2} h_i(\xi(t)) \sum_{j=1}^{k_2} h_j(\xi(t)) [\varepsilon^T_i(t) Q_1 \varphi(t) + 2K_2^3 + 2K_2^3]$$

$$\leq \sum_{i=1}^{k_2} h_i(\xi(t)) \sum_{j=1}^{k_2} h_j(\xi(t)) [\varepsilon^T_i(t) Q_1 \varphi(t) + 2K_2^3 + 2K_2^3]$$
where \( \lambda_{\text{min}}(\cdot) \) represents the minimum eigenvalue of the matrix.

Therefore, when Theorem 1 and \( \| \varphi(t) \| > \sqrt{2 \frac{K_1^2 + 2K_2^2}{\lambda_{\text{min}}(Q_1)}} \) are satisfied, then \( \dot{\Lambda}_1(t) < 0 \). Depending on the Lyapunov theory, the error system (17) is convergent. \( \square \)

4. Fault-Tolerant Control

System performance is severely affected when fault is not handled in a timely manner, so when fault information is obtained through the designed fuzzy learning estimator, a fault-tolerant control algorithm needs to be designed to compensate for fault to the maximum extent possible to ensure that expected output is followed even with fault. In this paper, a fault-tolerant controller is constructed based on a PI control strategy.

A new state variable is introduced as follows:

\[
\omega(t) = [x^T(t), \int_0^t (\bar{g}(\xi) - y_c)^T d\xi]^T.
\]  \( (25) \)

where \( y_c \) is a reference output.

The corresponding estimation error of \( \omega(t) \) is defined as

\[
\dot{e}_{\omega}(t) = \dot{\omega}(t) - \dot{\hat{\omega}}(t) = \begin{bmatrix}
\bar{x}(t) \\
\int_0^t (\bar{g}(\xi) - y_c) d\xi
\end{bmatrix}
- \begin{bmatrix}
\bar{x}(t) \\
\int_0^t (\hat{g}(\xi) - y_c) d\xi
\end{bmatrix}.
\]  \( (26) \)

On the basis of system (14), the new dynamic system is expressed as

\[
\dot{\omega}(t) = \sum_{i=1}^{K_2} h_i(\zeta(t)) \left[ \bar{A}_i \dot{\omega}(t) + \bar{B}_i u(t) + \bar{S}_i f \right] + \bar{N}_id(t) - Iy_c,
\]  \( (27) \)

where \( \bar{A}_i = \begin{bmatrix} \bar{A}_i & 0 \\ D_i & 0 \end{bmatrix}, \bar{B}_i = \begin{bmatrix} \bar{B}_i \\ 0 \end{bmatrix}, \bar{S}_i = \begin{bmatrix} \bar{S}_i \\ 0 \end{bmatrix}, \bar{N}_i = \begin{bmatrix} \bar{N}_i \\ 0 \end{bmatrix}, I = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \).

To ensure that the system output continues tracking the intended output when fault occurs, the PI compensating controller is provided below:

\[
u(t) = \sum_{i=1}^{K_2} h_i(\zeta(t)) \bar{B}_i^p \left[ G_{pi} \bar{x}(t) + G_{li} \int_0^t (\bar{g}(\tau) - y_c) d\tau - f_s(t) \right]
= \sum_{i=1}^{K_2} h_i(\zeta(t)) \bar{B}_i^p \left[ \begin{bmatrix} G_{pi} & G_{li} \end{bmatrix} \begin{bmatrix} \bar{x}(t) \\
\int_0^t (\bar{g}(\tau) - y_c) d\tau
\end{bmatrix} - f_s(t) \right] \]  \( (28) \)

where \( \bar{B}_i^p \) is the pseudo inverse of matrix \( \bar{B} \), and \( G_{pi} = \begin{bmatrix} G_{pi} & G_{li} \end{bmatrix} \) is the pending controller gain.

The error system yields the following results:

\[
\dot{e}_{\omega}(t) = \dot{\omega}(t) - \dot{\hat{\omega}}(t)
= \sum_{i=1}^{K_2} h_i(\zeta(t)) \left[ (\bar{A}_i + \bar{B}_i G_{pi}) \omega(t) + (\bar{S}_i - \bar{B}_i) f_s(t) + \bar{N}_id(t) - Iy_c \right]
- \left[ (\bar{A}_i + \bar{B}_i G_{pi}) \hat{\omega}(t) + (\bar{S}_i - \bar{B}_i) f_s(t) - Iy_c \right]
= \sum_{i=1}^{K_2} h_i(\zeta(t)) \left[ (\bar{A}_i + \bar{B}_i G_{pi}) e_{\omega}(t) + (\bar{S}_i - \bar{B}_i) e_{\phi_s}(t) \right].
\]  \( (29) \)

**Theorem 2.** The convergence of the closed-loop dynamical system (29) is guaranteed when there exist \( G_{pi}, \) PDSM \( P_2 \) and \( Q_2 \) to make the following inequality hold:

\[
(\bar{A}_i + \bar{B}_i G_{pi})^T P_2 + P_2 (\bar{A}_i + \bar{B}_i G_{pi}) + P_2 (\bar{S}_i - \bar{B}_i) (\bar{S}_i - \bar{B}_i)^T P_2 + Q_2 \leq 0.
\]  \( (30) \)
**Proof.** The Lyapunov function is defined below:

$$\Lambda_2(t) = e^T(t)P_2e(t),$$

Taking the interval derivative of $\Lambda_2(t)$ and employing Lemma 1, this leads to

$$\dot{\Lambda}_2(t) = \sum_{i=1}^{\tilde{N}} h_i(\tilde{z}(t))((\dot{A}_i + A_i G_{pf})e_{\alpha}(t) + (\dot{S}_i - \tilde{S}_i) e_{fs}(t))^T P_2 e_{\alpha}(t)$$

$$+ \sum_{i=1}^{\tilde{N}} h_i(\tilde{z}(t))e^T_{\alpha}(t)((\dot{A}_i + A_i G_{pf})e_{\alpha}(t) + (\dot{S}_i - \tilde{S}_i) e_{fs}(t))$$

$$+ 2e^T_{\alpha}(t)P_2(\tilde{S}_i - \dot{\tilde{S}}_i) e_{fs}(t)$$

$$\leq \sum_{i=1}^{\tilde{N}} h_i(\tilde{z}(t))e^T_{\alpha}(t)((\dot{A}_i + A_i G_{pf})e_{\alpha}(t) + (\dot{S}_i - \tilde{S}_i) e_{fs}(t))$$

$$+ e^T_{\alpha}(t)P_2(\tilde{S}_i - \dot{\tilde{S}}_i)(\dot{S}_i - \tilde{S}_i)^T P_2 e_{\alpha}(t) + e^T_{\alpha}(t)e_{fs}(t)$$

$$\leq \sum_{i=1}^{\tilde{N}} h_i(\tilde{z}(t))e^T_{\alpha}(t)((\dot{A}_i + A_i G_{pf})e_{\alpha}(t) + (\dot{S}_i - \tilde{S}_i) e_{fs}(t))$$

$$+ P_2(\dot{S}_i - \tilde{S}_i)(\dot{S}_i - \tilde{S}_i)^T P_2 e_{\alpha}(t) + 2K_f^2$$

$$= \sum_{i=1}^{\tilde{N}} h_i(\tilde{z}(t))e^T_{\alpha}(t)Q_2 e_{\alpha}(t) + 2K_f^2$$

$$\leq \sum_{i=1}^{\tilde{N}} h_i(\tilde{z}(t))[-\lambda_{\min}(Q_2)]\|e_{\alpha}(t)\|^2 + 2K_f^2.$$

Thus, in case Theorem 2 and $\|e_{\alpha}(t)\| \geq K_f \sqrt{\frac{2}{\lambda_{\min}(Q_2)}}$ are valid, it is evident that $\dot{\Lambda}_2 < 0$, which implies that System (29) is stable.

**5. Simulation and Discussion**

In this paper, MATLAB software was applied for simulation verification. To verify the efficacy of the suggested FE and FTC schemes, some simulation procedures are given and the desired results are presented.

The parameter matrix of the initial system is given below:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 1 & 0 \\ -0.7 & 0.5 \\ -1 \end{bmatrix}, A_2 = \begin{bmatrix} -2 & 1 & 0 \\ 3 & -2.8 & 0 \\ 0 & 0 & -0.16 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.15 \\ 0.3 \\ 0.21 \end{bmatrix}, B_2 = \begin{bmatrix} 0.21 \\ 0.29 \\ 0.5 \end{bmatrix}, N_1 = \begin{bmatrix} 1.2 \\ 0.9 \\ 0.5 \end{bmatrix}, N_2 = \begin{bmatrix} 1.6 \\ 0.1 \end{bmatrix}, D_1 = \begin{bmatrix} 1 & 0.2 & 0 \\ 0.5 & -1 & 1 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.6 \end{bmatrix}, S_1 = \begin{bmatrix} 0.3 \\ 1 \end{bmatrix}, S_2 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}.$$

To validate the algorithm, the TVD module is set to satisfy an exponential distribution whose rate parameter is $\alpha = 10$. In this paper, the coupling between the TVD and the system is reduced, so the TVD is relatively independent, and the initial state of the TVD has very little influence on the system, so the initial state of time delay is taken as 0.05.

The following two types of sensor faults are given:

- **Constant fault:**
  $f_s(t) = \begin{cases} 0 & 0 \leq t < 120 \\ 0.7 & 120 \leq t < 300 \end{cases}$

- **Time-varying fault:**
  $f_s(t) = \begin{cases} 0 & 0 \leq t < 120 \\ 0.2 \sin(0.05t) + 0.4 & 120 \leq t < 300 \end{cases}$
The intended output is given as \( y_c = [1.2, 4.6] \). By solving the inequalities in Theorems 1 and 2, the FE and FTC parameters are obtained below:

\[
K_{11} = 0.29, K_{21} = \begin{bmatrix} -5.5 & -0.04 & 0.02 & 0.5 \end{bmatrix}, \\
K_{12} = 0.53, K_{22} = \begin{bmatrix} -3.8 & 0.15 & 0.6 & 0.09 \end{bmatrix}, \\
G_{P1} = \begin{bmatrix} -0.366 & 0.453 & -0.648 & 0.048 \end{bmatrix}, \\
G_{P2} = \begin{bmatrix} 0.109 & -0.043 & 0.214 & -0.513 \end{bmatrix}, \\
G_{I1} = \begin{bmatrix} -0.371 & 0.266 & 0.012 & 0.743 \end{bmatrix}, \\
G_{I2} = \begin{bmatrix} -1.292 & -0.021 & 0.125 & -1.503 \end{bmatrix}.
\]

Regarding a constant fault, the residual signal depicted in Figure 1 displays a noticeable variation at 120 s, indicating the occurrence of the fault at that moment. The results of FE in Figure 2 demonstrate that sensor fault is effectively estimated quite well. The proposed FE algorithm is highly effective for constant fault. The FTC result is depicted in Figure 3. Obviously, the system output deviates from the target output at 120 s. Nevertheless, by adopting the FTC method suggested in Section 4, the system output can continue tracking the intended output. To illustrate the comparison, Figure 4 depicts the system output without FTC, indicating that the system is not well controlled and its performance is seriously affected.

![Figure 1](image1.png)

**Figure 1.** Residual signal of constant fault.

![Figure 2](image2.png)

**Figure 2.** Sensor fault and its estimation of constant fault.

To more comprehensively verify the algorithm’s feasibility, simulation results for time-varying fault are displayed in Figures 5–8. The residual signal in Figure 5 additionally confirms that the fault occurred at 120 s. The FE algorithm can accurately track the changes
in the time-varying fault, as evident from the FE results in Figure 6. When detailed fault information is obtained, applying the designed controller, the system output results displayed in Figure 7 show that system output tracking of the expected output can be ensured even in the event of fault, thus maintaining system performance. The results in Figure 8 without FTC reveal that the system output failed to track the desired output.

![Figure 3. The system output with FTC of constant fault.](image)

![Figure 4. The system output without FTC of constant fault.](image)

![Figure 5. Residual signal of time-varying fault.](image)
Therefore, for different types of faults, the algorithm designed in this paper yields very good results.

6. Conclusions

In this paper, the mathematical modelling problem for the FE and FTC of singular nonlinear TVD systems with sensor fault is solved based on the T-S fuzzy model. The
Laplace transform is utilized to build an equal system free of TVD to solve the TVD issue. The coordinate transformation is developed to change sensor fault to actuator fault for processing. A fuzzy learning fault estimator is built to estimate sensor fault. As shown in the results in Figures 2 and 6, the fault estimation error at steady state is much less than 0.01 for constant or time-varying faults, indicating that the fault diagnosis algorithm has a good estimation effect. When fault occurs, the PI-compensated FTC is engineered to decrease the possible effect of the fault. The simulation results show that the error of the system output before and after the fault is less than 0.3 when adopting the FTC algorithm designed in this paper, but the error of the system output before and after the fault is generally greater than 1 when there is no FTC algorithm. The algorithm of this paper has good results for both constant and time-varying faults. Singular systems are widely available in power grid systems, chemical processes, satellite attitude control systems, etc. The algorithms in this paper provide solutions for fault handling in these systems.

Further research may include research on mathematical models for the fault handling of singularly uncertain systems, singularly switching systems, and their applications.

Author Contributions: Conceptualization, J.C. and H.C.; Methodology, J.C.; Software, J.C.; Validation, J.C. and H.C.; Formal Analysis, J.C.; Investigation, J.C.; Resources, J.C.; Data Curation, J.C.; Writing—Original Draft Preparation, J.C.; Writing—Review and Editing, H.C.; Visualization, J.C.; Supervision, H.C.; Project Administration, H.C. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Major Scientific and Technological Projects of CNPC (ZD2019-184-001).

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

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