







Article

Divide and Conquer: A Location-Allocation Approach to Sectorization

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Abstract: Sectorization is concerned with dividing a large territory into smaller areas, also known as sectors. This process usually simplifies a complex problem, leading to easier solution approaches to solving the resulting subproblems. Sectors are built with several criteria in mind, such as equilibrium, compactness, contiguity, and desirability, which vary with the applications. Sectorization appears in different contexts: sales territory design, political districting, healthcare logistics, and vehicle routing problems (agrifood distribution, winter road maintenance, parcel delivery). Environmental problems can also be tackled with a sectorization approach; for example, in municipal waste collection, water distribution networks, and even in finding more sustainable transportation routes. This work focuses on sectorization concerning the location of the area's centers and allocating basic units to each sector. Integer programming models address the location-allocation problems, and various formulations implementing different criteria are compared. Methods to deal with multiobjective optimization problems, such as the ϵ -constraint, the lexicographic, and the weighted sum methods, are applied and compared. Computational results obtained for a set of benchmarking instances of sectorization problems are also presented.

Keywords: sectorization; multiobjective optimization; integer programming; districting problems; lexicographic method; ϵ -constraint method; weighted sum method

MSC: 90-10; 90B-06; 90B-50; 90B-80; 90C-10; 90C-20; 90C-29



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1. Introduction

Current territory design problems rely on dividing a large region into smaller areas to facilitate reaching convenient solutions. The division can be achieved through sectorization (or districting) and according to specified criteria [1]. Sectorization is a broadly applicable “procedure”, with applications that range from political districting to waste collection and health logistics, and it has been used to answer several social problems. In vehicle routing problems, sectorization is a procedure that can greatly impact routing decisions, allowing the decision-makers to obtain long-term savings. Sectorization is a divide-and-conquer strategy: by breaking a large problem into smaller subproblems they become more straightforward to manage.

Within a sectorization framework, this paper will deal with location-allocation matters, namely the location of the centers of each area and the allocation of basic units for each sector. We emphasize that the problems will be solved in an integrated way only with integer programming models. Two criteria are addressed, the compactness of the design of the sectors and the equilibrium between them regarding some activity measure.

Given that more than one objective is relevant to finding the best sectorization, methods to solve multiobjective optimization problems (MOPs) are needed.

This paper has several contributions. It addresses an understudied topic of sectorization, which has many diverse applications. The criteria considered are very commonly found in location problems and are regarded as the most relevant ones. The proposed formulations assign the basic units to the sectors and simultaneously find the sector centroids. This is performed in an integrated way that does not require prior knowledge of sector centroids, which is common in the literature on exact approaches. Moreover, two exact methods are compared, the ϵ -Constraint/Lexicographic and weighted sum methods. The paper also uses open instances that can be useful for other researchers.

Multiobjective optimization methods are discussed in Section 2, with some elements about Pareto front, Nadir and ideal points. Section 3 defines sectorization problems and points to possible applications. The adopted methodology and the formulation of the models are presented in Section 4. The computational results are discussed in Section 5, and Section 6 highlights the conclusions of this work.

2. Multiobjective: Definitions and Generalities

This section introduces the multiobjective optimization problem (MOP) and presents fundamental concepts and some solution methods. Special attention will be given to the methods relevant to the accomplished work. The MOP can be defined as:

$$\begin{aligned} &\min(f_1(x), f_2(x), \dots, f_N(x)) && (1) \\ &s.t. \\ &g_t(x) \leq 0, t \in \{1, \dots, n_g\} \\ &h_p(x) = 0, p \in \{1, \dots, n_h\} \end{aligned}$$

where $N \geq 2$ represents the number of objective functions and $x = (x_1, x_2, \dots, x_n)$ is an n -dimensional solution vector of the space of admissible solutions (\mathcal{S}) defined by n decision variables. \mathcal{S} is a subset of R^n .

Without loss of generality, all objective functions will be considered as minimization. The image of each feasible solution vector is represented in the objective space (\mathcal{O}) as

$$\begin{aligned} &F : R^n \rightarrow R^N \\ &x = (x_1, x_2, \dots, x_n) \mapsto F(x) = (f_1(x), f_2(x), \dots, f_N(x)) \end{aligned}$$

Figure 1 presents an example with three decision variables, x_1, x_2 and x_3 ($n = 3$), and two objective functions, f_1 and f_2 ($N = 2$). On the left, the cube represents the set of feasible solutions \mathcal{S} . In this case, the space involves three continuous decision variables. \mathcal{S} will be a set of discrete points if the decision variables are integer or binary. Each feasible solution $A = (a_1, a_2, a_3)$ corresponds to a bi-dimensional objective vector $(f_1(a_1, a_2, a_3), f_2(a_1, a_2, a_3))$ on objective space \mathcal{O} , on the right.

Note the difference between the feasible solution space (or feasible design space or feasible decision space) \mathcal{S} , which is the set of values for the decision variables that satisfy all constraints of the model, and the feasible objective space (or criterion space) \mathcal{O} , which is the set of values for the objective functions resulting from a point in the decision space. Each point in the solution space \mathcal{S} maps to a point in the objective space \mathcal{O} , but the reverse may not be true; a point in the objective space does not necessarily correspond to a single point in the solution space [2].

In most MOPs, the criteria can be contradictory; therefore, no single decision best satisfies all the efficiency criteria. Hence, a key aspect of solving MOPs is finding several compromise decisions that can only be improved by worsening the effectiveness of other efficiency criteria [3]

Scalarization methods for solving MOPs deal with conflicting criteria by reducing them to one single scalar objective function [3]. In iterative methods, the decision-maker actively selects decisions a priori or a posteriori [2]. Evolutionary algorithms such as NSGA

II e NSGA III rely on genetic metaphors for finding good solutions for MOPs very fast [4,5]. A survey of nonlinear multiobjective optimization methods is found in [2]. The authors advocate that the type of information provided in the problem, the solution requirements, and the user’s preferences and availability of software may lead to selecting different optimization methods to solve MOPs.

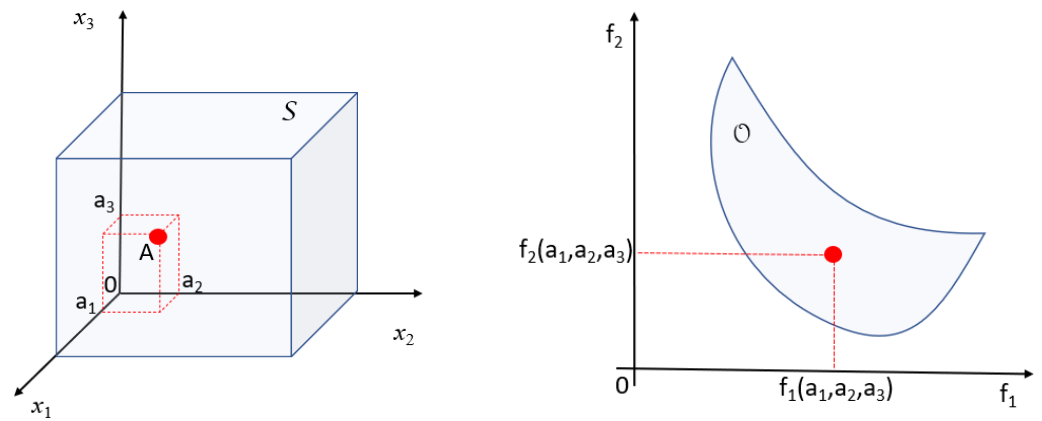


Figure 1. Solution space S of a MOP and the corresponding objective space O .

2.1. Pareto Front

Contrary to optimization problems with a single objective, having a single optimal solution in MOPs is difficult. In MOPs, a set of solutions stands out by being better than the others in at least one of the objectives.

A solution in the objective space $u = (u_1, u_2, \dots, u_N)$ dominates (Pareto-dominates) another one $v = (v_1, v_2, \dots, v_N)$ if and only if [6]:

$$\forall i \in \{1, 2, \dots, N\} : u_i \leq v_i \quad \text{and} \quad \exists i \in \{1, 2, \dots, N\} : u_i < v_i$$

In Figure 2, with point Q as a reference, Q dominates R , Q is dominated by P and there is an indifference relation between Q and all the grey points. T and M present lower values about f_1 , so they are better; but they present higher values when f_2 is considered, so they are worse in this perspective. The situation with W is the opposite. W is better than Q in f_2 but worse in f_1 . In conclusion, whatever the preferences of a decision agent, s/he will always prefer the solution represented by P over the one represented by Q . The solution represented by P has lower values of both f_1 and f_2 compared to Q .

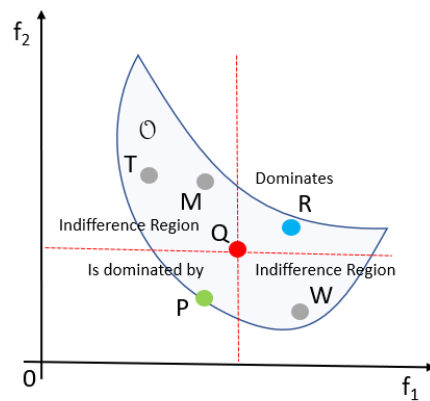


Figure 2. Pareto dominance relationships regarding point Q : Q dominates all blue points in the upper right quadrant, Q is dominated by all green points in the lower left quadrant, and indifferent regarding all the grey points in the other quadrants.

A solution $y \in \mathcal{S}$ is a Pareto optimal solution if and only if:

$$\forall x \in \mathcal{S}, \quad y \text{ is not dominated by } x$$

The Pareto set, PS , is the set of all Pareto optimal solutions in \mathcal{S} for a certain MOP. The Pareto front, PF , is the image of this set in the objective space \mathcal{O} .

Figure 3 presents three elements of the Pareto front: A , B , and C . None of them are dominated by others. The red line represents the Pareto front.

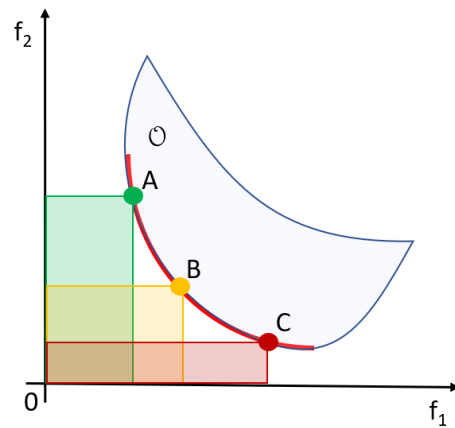


Figure 3. Pareto front in the objective space containing three points A , B and C , corresponding to three Pareto optimal solutions.

All Pareto optimal points lie on the boundary of the feasible objective space [7,8].

The term “efficiency” typically refers to a vector of design variables in the solution space, whereas dominance refers to a vector of functions in the objective space [2].

2.2. Notable Points in the Objective Space

The ideal point, F_{ideal} , is a point in the objective space that, for each of the N objective functions, presents the minimum value [9], that is,

$$F_{ideal} = \left(\min_{x \in \mathcal{S}} f_1(x), \min_{x \in \mathcal{S}} f_2(x), \dots, \min_{x \in \mathcal{S}} f_N(x) \right)$$

Generally, the ideal point does not represent a feasible solution.

The ideal point is sometimes called shadow minimum [10] and it can only be attained in the rare case when a single x minimizes all the objective functions (the case where the Pareto front is reduced to just one point). In practice, it is desired to get as close as possible to this ideal point containing the individual global minima of the objectives and assure an agreeable trade-off among the multiple objectives [10].

On the opposite side is the Nadir Point, F_{nadir} . A Nadir point corresponds to a vector whose components comprise the worst points of each of the multiple objectives among the non-dominated points [11]. Considering the Pareto set, F_{nadir} is defined as the worst point that maximizes each objective function [9] and is defined as:

$$F_{nadir} = (f_1^+, f_2^+, \dots, f_N^+), \text{ where } f_i^+ = \max_{x^* \in PS} f_i(x^*) \quad \forall i \in \{1, \dots, N\}$$

The ideal point and the Nadir Point are illustrated in Figure 4. In the bi-objective example,

$$F_{ideal} = (f_1^*, f_2^*) \quad F_{nadir} = (f_1^+, f_2^+)$$

such that

$$f_1^* = \min_{x \in \mathcal{S}} f_1(x)$$

$$f_2^* = \min_{x \in \mathcal{S}} f_2(x)$$

$$f_1^+ = \max f_1(x), x \in PS$$

$$f_2^+ = \max f_2(x), x \in PS$$

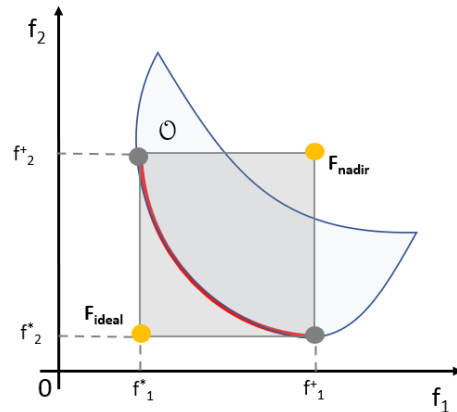


Figure 4. Nadir and ideal points in the objective space.

In a bi-objective problem, the Nadir point is easy to obtain [12]. Knowing the range of values for each objective is an important point for many multiobjective approaches [11].

Together, the Nadir and the ideal values define the limits of each criterion, which are of major interest to size the MOP and to make a graphical representation of non-dominated points and relative interpretation of objective values [12].

2.3. The ϵ -Constraint and the Lexicographic Methods

The ϵ -constraint method first appeared in [13] for a bi-criterion problem. In the ϵ -constraint method, one of the objective functions is selected to be minimized while the others are transformed to constraints and are not allowed to deviate from an upper bound for each of them [14]. This method reduces a MOP to a single objective problem by considering just one objective and the rest, the other $N - 1$ objectives, as constraints [14]. The ϵ -constraint method will transform the Model (1) into:

$$\begin{aligned} \min f_j(x) & \tag{2} \\ \text{s.t.} & \\ g_t(x) \leq 0, & \quad t \in \{1, \dots, n_g\} \\ h_p(x) = 0, & \quad p \in \{1, \dots, n_h\} \\ f_i \leq \epsilon_i, & \quad i \in \{1, \dots, N\} \text{ and } i \neq j \end{aligned}$$

In this case, only the most important criterion is used for optimization, while the maximum permissible value is set for other criteria (i.e., all other criteria are transformed into constraints) [3]. A systematic variation of ϵ_i yields a set of Pareto optimal solutions. However, improper selection of ϵ can result in a formulation with no feasible solution [2].

In [15], some weaknesses of the ϵ -constraint method are identified, namely the lack of easy-to-check conditions for properly efficient solutions and the inflexibility of the constraints.

Using the bi-objective example presented before, Figure 5 illustrates a single objective $\min f_2(x)$ with all previous constraints plus $f_1(x) \leq \epsilon$.

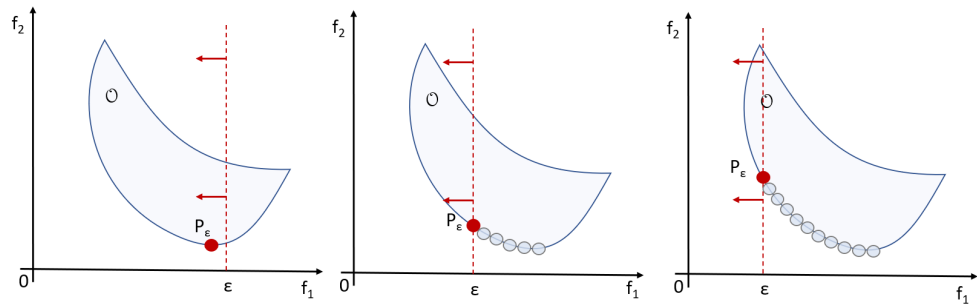


Figure 5. Solutions obtained with the ϵ -constraint method shaping the Pareto front.

The lexicographic method consists of arranging the objective functions by order of importance and then solving, one at a time, the single objective optimization problems for $i = 1, \dots, N$:

$$\begin{aligned} & \min f_i(x) \\ & \text{s.t. } f_j(x) \leq f_j(x_j^*), \quad j = 1, 2, \dots, i - 1, \quad \forall i > 1 \end{aligned} \tag{3}$$

The constraints can be replaced with equalities [2].

The lexicographic optimization method offers noteworthy advantages, such as: not requiring the normalization of the objective functions and always providing a Pareto optimal solution [16]. However, the method has some drawbacks. Different orders will show different sections of the Pareto front. Moreover, the solutions obtained are very sensitive to the predefined order of the importance of the objective functions. A decision-maker may have difficulty selecting the order of importance of objective functions, and lexicographic ordering does not allow a small increment of an important objective function to be traded off with a great decrement of a less important one [17].

For problems in which one cost factor can be identified as the most important one, but in which one or more other cost factors are not to be completely disregarded, Ref. [18] proposes a variation of the lexicographic approach. In the relaxed lexicographic scalarization method, the secondary criteria optimization is carried out within some engineering tolerance ϵ of the primary criterion best value. With this approach, the constraints are relaxed by adding positive tolerances:

$$f_j(x) \leq f_j(x_j^*) + \epsilon_j$$

Notice the connection between this tolerance and the upper bound presented in model (2): $\epsilon_j = f_j(x_j^*) + \epsilon_j$. As the tolerances increase, the feasible region dictated by the objective functions expands [2]. By relaxing all the additional constraints of the lexicographic problem, the influences of all the objective functions are considered. By changing the parameters, all the efficient solutions can be obtained [17].

In this study, the relaxed lexicographic scalarization is used. This approach combines characteristics of the “pure” lexicographic method and the ϵ -constraint approach.

2.4. The Weighted Sum Method

The weighted sum method (or global criterion method) consists of combining all the multiple objectives into a single objective function and minimizing the single objective over various values of the weight parameters used to combine the objectives [10,19].

A discrete set of Pareto optimal points can be obtained by minimizing a convex combination of the objectives [10] and by performing this minimization for different choices of the weights $w_i \geq 0$ such that $\sum_{i=1}^N w_i = 1$:

$$\min \sum_{i=1}^N w_i f_i(x) \tag{4}$$

$$s.t.$$

$$g_t(x) \leq 0, t \in \{1, \dots, n_g\}$$

$$h_p(x) = 0, p \in \{1, \dots, n_h\}$$

The relative value of the weights reflects the relative importance of the objectives. Under the convexity assumption on f and x [14], all non-inferior solutions can be found by solving formulation (4) (Figure 6).

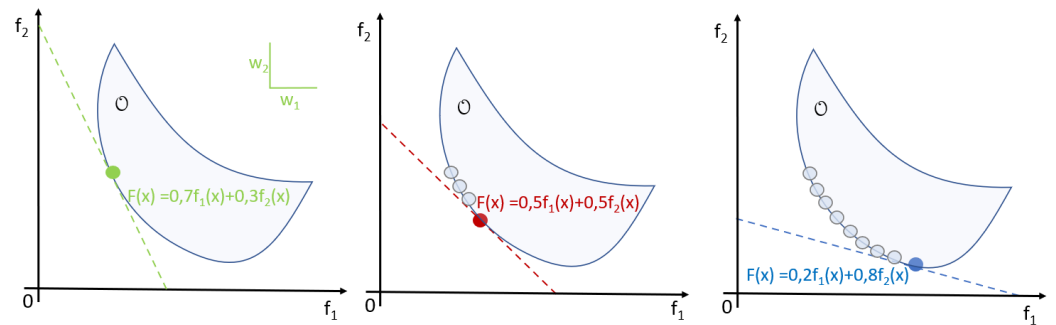


Figure 6. Different weights in a bi-objective example to obtain the Pareto front ($F(x) = w_1f_1(x) + w_2f_2(x)$, $w_1, w_2 \geq 0$ and $w_1 + w_2 = 1$).

This is one of the most common general scalarization methods for multiobjective optimization. However, it has several drawbacks. A satisfactory a priori selection of weights does not necessarily guarantee that the final solution will be acceptable, and sometimes the problem must be solved again with new weights [2]. Another issue with the weighted sum approach is that it is impossible to obtain points on non-convex portions of the Pareto front in the objective space, as illustrated in Figure 7. Moreover, even if varying the weights consistently and continuously, it may not necessarily result in an even distribution of Pareto optimal points and in a complete representation of the Pareto optimal set [2].

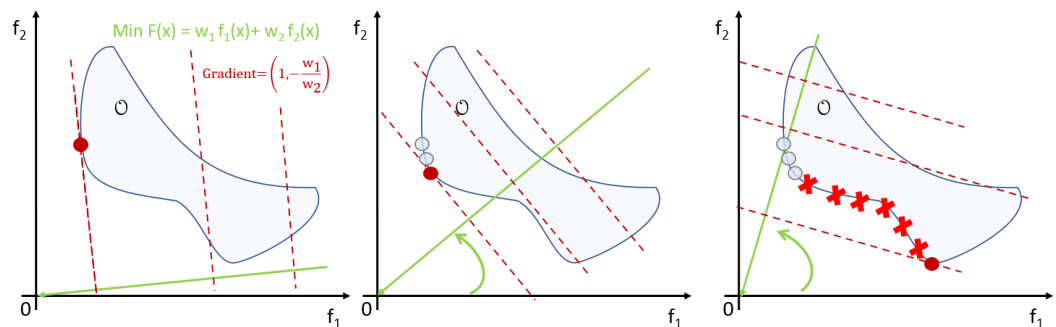


Figure 7. Not all of the Pareto front can be found using the weighted sum method. In this example, due to the lack of convexity, part of the Pareto frontier (red crosses on the right) cannot be obtained.

Both the lexicographic and the weighted sum methods are methods with a priori articulation of preferences. These require the user to specify preferences only in terms of objective functions. With a priori articulation of preferences, the decision maker must quantify opinions and the relative importance of different objective functions, before actually viewing points in the criterion space [2]. Alternatively, methods with a posteriori articulation of preferences allow the user to view potential solutions in the objective space (and/or in the solution space) before selecting an acceptable solution.

The main advantage of the ϵ -constraint problem over other scalarizations methods, such as the weighted sum method, is that every efficient solution can be found as the optimal solution for some ϵ [14] and this result is independent of the structure of X , i.e., it

is also true for nonconvex and discrete optimization problems (and the weighted sum method is restricted to convex MOPs).

In [20], connections between the two approaches are established; if the objective functions are linear and the lexicographic model has an optimal solution, then it is possible (although not easy) to find weights such as the weighted sum approach obtains the same solutions as the lexicographic approach.

3. Sectorization as a Multiobjective Problem

The sectorization problems (SP), also known as districting problems or territory design, consist in dividing a whole (often a geographical area) into smaller pieces according to a group of constraints and satisfying certain objectives.

The basic units (BU) are considered indivisible and are the smallest areas that constitute the territory.

Thus, sectorization can be seen as the partition into k parts (S_1, S_2, \dots, S_k) of a certain number n of basic units forming a set $V = \{BU_1, BU_2, \dots, BU_n\}$. Each sector $S_i, i \in \{1, \dots, k\}$ is defined by

$$S_i = \bigcup_{j=1}^{n_i} BU_i^j$$

where BU_i^j is the j th basic unit of sector i , such that

$$\bigcup_{i=1}^k S_i = V, S_i \cap S_j = \emptyset (i \neq j) \text{ and } \sum_{j=1}^k n_j = n$$

Although any such partition of V is a feasible solution to a sectorization problem, these problems usually require the solution to be “good” concerning one or multiple criteria. Sectors are built with several criteria in mind, such as balancing the workload, short travel distances within the sectors, the desirability of certain points belonging to the same sector, and other criteria that vary with the applications. Generally, in SP, as in many real-world problems, different and contradictory objectives compete for the best solutions. Objectives related to equilibrium, compactness, integrity, and contiguity of sectors (or districts) are traditional examples in most SP [21].

These problems appear in different contexts and have vast applications. Common examples of applications of sectorization problems in the public management sector are political districting [22], the design of police zones, school regions [23], and home healthcare services districting [24,25]. Popular applications of sectorization in commerce arise with the definition of sales territories [1], and parcel delivery [26]. Sectorization is a strong ally in Logistics challenges such as in the agrifood distribution [27], air traffic management areas [28], winter road maintenance, positioning tower cranes on construction sites [29], last mile delivery [30], and in many vehicle routing problems. Environmental problems can also be tackled with a sectorization approach; for example, in municipal waste collection [31,32], water distribution networks [33], and to find more sustainable transportation routes.

Although there are a large number of publications on this topic, SP continues to have the attention of researchers due to its great relevance, diversity, and difficulty in resolution. [34] presents a comprehensive overview of the main application areas and methods for solving this type of problem.

4. Two Methods to Sectorization

To deal with sectorization problems, two different approaches are proposed through a quadratic integer programming model that involves the location and the allocation of the basic units to the sectors. The first approach uses a lexicographic ordering of the objectives, which are optimized through the ϵ -constraint method. The second alternate approach deals with ideal and Nadir points to transform a bicriteria model into a weighted single objective model.

The concept of “sector center”, or centroid, appears in most sectorization problems. It usually represents the point of origin from which the distribution of activities is made to the remaining points in the sector. Depending on the type of problem, the centroids of the sectors can either be predefined [27] or obtained through various methods. For instance, in problems involving distribution centers, there are fixed potential locations out of which a few centroids must be selected. In designing school regions, the location of the schools is predefined. In a perspective of having a general approach that can be applied in different contexts, we present a mathematical model for obtaining the centroids from the data. We also assume that the centroids must be selected from the known points of the dataset.

Sets and Indexes

The sets and indexes considered in the following models are:

- V is a set of basic units (points);
- i, j and c are basic units;
- c is the basic unit that represents the centroid of the sector.

Parameters

The parameters considered in the following models are:

- n is the number of basic units in V ;
- k is the number of sectors in which V is to be partitioned;
- d_{ij} is the Euclidean distance between each two points i and j of V ;
- q_i is the quantity of activity assigned to point i .

Decision variables

The decision variables of the integer programming formulation for choosing the sectors’ centroids are based on the representative formulation for the graph coloring problem [35]. These variables are capable of defining, in an integrated way, both the location of the centroids and the allocation of the basic units to the sectors.

The decision variables are defined as such:

$$x_{ic} = \begin{cases} 1, & \text{if point } i \text{ is assigned to the sector with centroid } c \\ 0, & \text{otherwise} \end{cases} \quad \forall i, c \in V$$

Notice that these variables assign the basic units to the sectors. However, the same variables will also decide which points are chosen as representative for the centroids of the sectors, with the special case of x_{cc} having the following meaning:

$$x_{cc} = \begin{cases} 1, & \text{if point } c \text{ represents the centroid of a sector} \\ 0, & \text{otherwise} \end{cases} \quad \forall c \in V$$

The present work deals with two common criteria in SP, namely compactness and equilibrium. The main idea behind these two criteria is the following:

- Compactness—The sectors should have geographically concentrated activity; rounded shapes are preferred instead of thin long shapes. According to the scenario of the problem, higher density in sectors means less travel, easier accessibility, and more sales or service time.
- Equilibrium—The sectors should be balanced regarding a certain activity measure, containing similar parts of the whole; that is, ideally, each sector should be assigned the same amount of inhabitants, voters, or workload, depending on the application.

The way these two criteria are defined and evaluated by different authors is not universal; that is, there is no unique way to evaluate whether a solution is more or less balanced, or whether it is more or less compact. In this work, the compactness of a solution for a sectorization problem is measured with the total distance from the centroid of the sector to the points assigned to it, and equilibrium is measured with a function similar to the variance of the weight of the sectors [36]. The definition of each criterion is presented next.

- **Compactness:**

$$f_1 = \sum_{i=1}^n \sum_{c=1}^n d_{ic} \cdot x_{ic} \tag{5}$$

where i and c are basic units of V .

- **Equilibrium:**

$$f_2 = \frac{1}{(k-1)} \sum_{c=1}^n (Q_c^t - \bar{q})^2 \tag{6}$$

where $Q_c^t = \sum_{i=1}^n q_i \cdot x_{ic}$ is the total weight of the sector represented by basic unit c , and $\bar{q} = \frac{\sum_{c=1}^n Q_c^t}{k}$ is the average weight of sectors.

4.1. Lexicographic Model for Sectorization

In the lexicographic approach, the optimization is made in two phases. The first phase, where the first objective function, compactness, is minimized, followed by the second phase where the second objective function, equilibrium, is minimized while restricting compactness to a given tolerance. It is intended to obtain the points of the Pareto front by adjusting this tolerance.

First phase of the lexicographic approach:

$$\text{Min } f_1 = \sum_{c=1}^n \sum_{i=1}^n d_{ic} \cdot x_{ic} \tag{7}$$

$$\sum_{c=1}^n x_{cc} = k \tag{8}$$

$$\sum_{i=1}^n x_{ic} \geq x_{cc}, \quad \forall c = 1, \dots, n \tag{9}$$

$$\sum_{c=1}^n x_{ic} = 1, \quad \forall i = 1, \dots, n \tag{10}$$

$$x_{ic} \leq x_{cc}, \quad \forall i = 1, \dots, n \tag{11}$$

$$x_{ic} \in \{0, 1\}, \quad \forall i, c = 1, \dots, n. \tag{12}$$

The objective function (7) minimizes compactness. Constraint (8) ensures that exactly k centroids are defined. Constraints (9) ensure that a sector must have at least one point assigned to it (besides the centroid itself). The set of constraints (10) ensures that each point is assigned to exactly one sector (with centroid c). Constraints (11) make it so that points are assigned to the sector with centroid c only if c is, in fact, selected as a centroid. Finally, constraints (12) specify the domain of the decision variables.

The optimal solution of this first phase model corresponds to a compactness value of f_1^* which is used in the next phase.

Second phase of the lexicographic approach:

Replace the objective function (7) by the minimization of f_2 (equilibrium) and add the ϵ -constraint (18) to the model, to allow selecting solutions in a vicinity of the optimal compactness value f_1^* . This gives the following quadratic programming model for the lexicographic approach:

$$\text{Min } f_2 = \frac{1}{(k-1)} \sum_{c=1}^n (Q_c^t - \bar{q})^2 \tag{13}$$

$$\sum_{c=1}^n x_{cc} = k \tag{14}$$

$$\sum_{i=1}^n x_{ic} \geq x_{cc}, \quad \forall c = 1, \dots, n \tag{15}$$

$$\sum_{c=1}^n x_{ic} = 1, \quad \forall i = 1, \dots, n \tag{16}$$

$$x_{ic} \leq x_{cc}, \quad \forall i = 1, \dots, n \tag{17}$$

$$f_1 \leq f_1^* + \epsilon \tag{18}$$

$$Q_c^t = \sum_{i=1}^n q_i \cdot x_{ic} \quad \forall c = 1, \dots, n \tag{19}$$

$$\bar{q} = \frac{\sum_{c=1}^n Q_c^t}{k} \tag{20}$$

$$x_{ic} \in \{0, 1\}, \quad \forall i, c = 1, \dots, n. \tag{21}$$

By running this model while varying the values of ϵ in (18), several solutions are obtained to build the Pareto front.

4.2. Weighted Sum Model for Sectorization

Both the Nadir and ideal points were obtained from running the model in lexicographic form, alternating the two criteria. Running the first phase model, which minimizes compactness, gave the minimum value for compactness and a possible maximum value for equilibrium (one of the gray points represented in Figure 4). Then, running the model with the objective function minimizing equilibrium, without bounding the compactness, gave the minimum value for equilibrium and a possible maximum value for compactness (another gray point in Figure 4). The coordinates of these two points (*A* and *B*) were used as parameters in this approach.

$$A = (f_1^*, f_2^+)$$

$$B = (f_1^+, f_2^*)$$

Normalization of the two criteria was necessary, given that the two functions had very different ranges of possible values. Additionally, since points *A* and *B* are obtained empirically, there is no guarantee they are the actual endpoints of the Pareto front. Hence, a 10% safety margin was considered in normalizing the two criteria. In line with [6], the two objective values are normalized between 0.9 times the ideal points and 1.1 times the values of Nadir points. Complementary weights were used for the two criteria:

$$w_{equi} = 1 - w_{comp},$$

where $w_{equi}, w_{comp} \in [0, 1]$.

With this approach, the objective function is the minimization of the weighted sum of the two normalized criteria:

$$f_3 = w_{comp} \frac{f_1 - 0.9f_1^*}{1.1f_1^+ - 0.9f_1^*} + w_{equi} \frac{f_2 - 0.9f_2^*}{1.1f_2^+ - 0.9f_2^*} \tag{22}$$

The previous constraints (14)–(21) were used, but the epsilon constraint (18) was removed. By varying the weights in small steps within the interval 0–1, solutions were obtained to construct the Pareto front.

5. Results

The models in Section 4 were implemented in IBM ILOG CPLEX Optimization Studio 12.9 with Intel Core i5 CPU at 1.80 GHz and 8 GB RAM. Computational tests were conducted on five instances of different sizes, ranging from 43 to 250 points, for 10 sectors. They belong to a set of randomly generated instances for sectorization problems available at <https://drive.inescotec.pt/s/QfgAopoYXSCH9Ej> (accessed on 14 October 2022). The coordinates and weights of each basic unit were generated using the normal distribution with a two-step procedure to create unbiased and neutral data. Each instance consists of groups of nodes with different mean and variance. The five instances were selected from this set to constitute a variety of sizes regarding the number of basic units, without any other selection criteria in mind. The focus was to observe and compare the many solutions that could be obtained with each method for the same particular instance. Therefore, only the results of five instances are shown in this paper.

The coordinates of the ideal and Nadir points obtained with the optimization models for each instance are detailed in Table 1.

Table 1. Coordinates of the Nadir and ideal points in tested instances.

Instance	Number of Basic Units n	Min	Max	Min	Max
		Compactness f_1^*	Compactness f_1^+	Equilibrium f_2^*	Equilibrium f_2^+
Normal3	43	212.8309	372.1082	451.8700	524.3144
Normal1	67	559.4322	676.1220	1641.7633	1682.8744
Normal16	81	703.8095	1702.2454	3187.2900	3243.7300
Normal38	100	798.8770	944.8303	5953.8667	6096.7600
Normal30	250	1677.3754	3989.0312	105,840.6667	105,967.3333

The first and second phase models for the lexicographic approach and the model for the weighted sum approach were run with a time limit of 3600 s. In the lexicographic approach, ϵ was incremented in steps of 10, 50, or 100, while in the weighted sum approach, weights were incremented in steps of 0.05. In both approaches, the optimal solutions were inspected for dominance relationships. An analysis of the solutions obtained separately with the two methods led to the removal of the dominated solutions in order to draw the shape of each Pareto front.

The results of the non-dominated solutions obtained by the lexicographic approach are presented in Table 2. Note that ϵ is the tolerance considered for increasing the best compactness value while minimizing the equilibrium function. The last columns represent the amplitude of the range for the number of basic units in each sector, and the last column is the amplitude for the total weight of the basic units assigned to each sector. When compactness is allowed to be worsened by a large tolerance value, sectors can be more balanced and hence have less weight amplitudes.

The results obtained by the weighted sum approach are presented in Table 3. The values considered for the weights of the criteria in the compound objective function are shown. Larger weights for compactness produce solutions where the sectors are more compact and the distance of the points to their sector center is smaller. In this table, for the larger instance, only the first solution is a non-dominated one, and some of the dominated solutions obtained with this method are shown. In this case, the time limit of 3600 s prevented the model from finding better solutions (note that most of the solutions found were the same).

Figures 8–10 show the Pareto front solutions in the objective space for each instance.

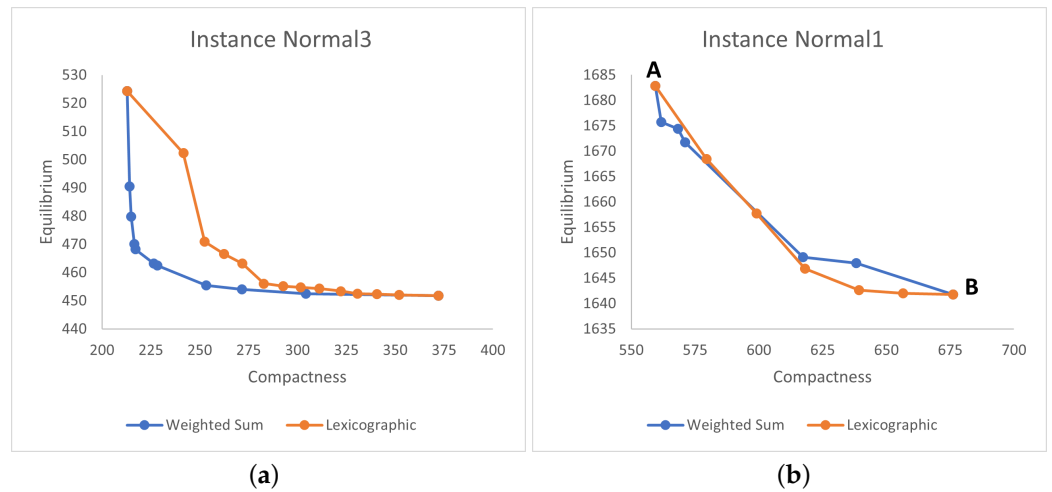


Figure 8. Pareto front solutions obtained with the lexicographic models and with the weighted sum model: (a) Instance Normal3 with 43 basic units; (b) Instance Normal1 with 67 basic units.

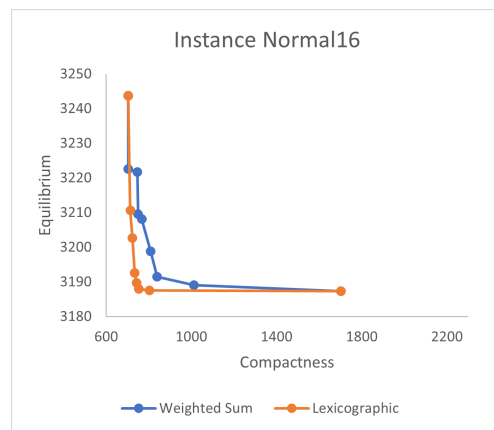


Figure 9. Pareto front solutions obtained with the lexicographic models and with the weighted sum model: Instance Normal16 with 81 basic units.

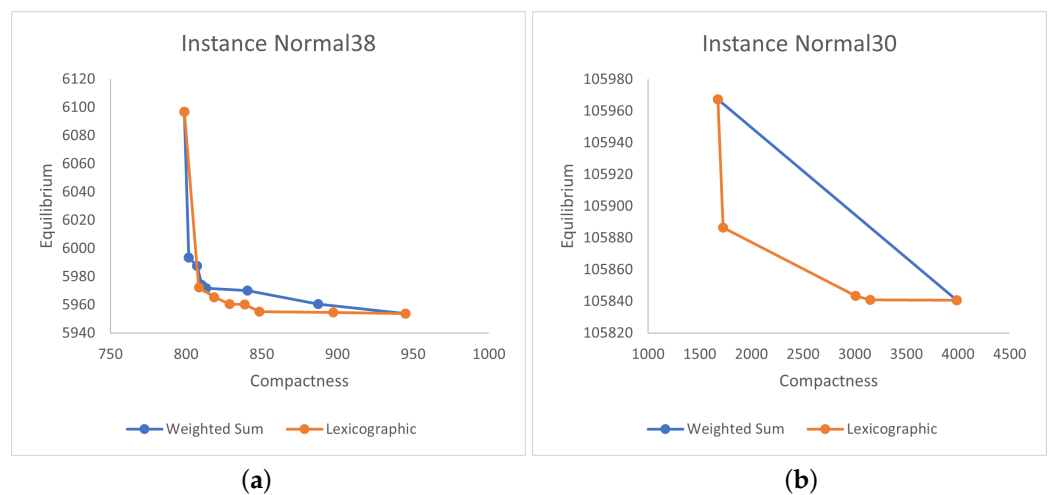


Figure 10. Pareto front solutions obtained with the lexicographic models and with the weighted sum model: (a) Instance Normal38 with 100 basic units; (b) Instance Normal30 with 250 basic units.

In the smallest instance Normal3 (Figure 8a), the weighted sum method produced better results than the lexicographic approach. However, in the instances with a larger number of basic units, the lexicographic approach performs better in the right region of the

curve obtaining sectorization solutions with better equilibrium, while the weighted sum models trace better the Pareto front in the left region obtaining more compact sectors. In the largest instance Normal30 (Figure 10b), all the solutions obtained with the weighted sum model were dominated by the two solutions of the endpoints of the Pareto front that were obtained when computing the Nadir and ideal points.

Figures 11 and 12 show the two sectorization maps corresponding to solution point A and solution point B from the Pareto front shown in Figure 8b. In these sectorization maps, points with the same color are assigned to the same sector, and the center of the sector is represented by the crossed point. The diameter of the points varies according to their demand quantity. Both are non-dominated solutions of the same instance, but while solution A privileges the compactness of the sectors, solution B privileges the equilibrium between the sectors. Note that in solution A every point is close to its center, but there are some sectors with many heavy points (high demand quantity) and other sectors with just a few light points (low demand quantity). In solution B, the number and weight of the basic units are more balanced between sectors, but there are points situated far from the point chosen as the sector center.

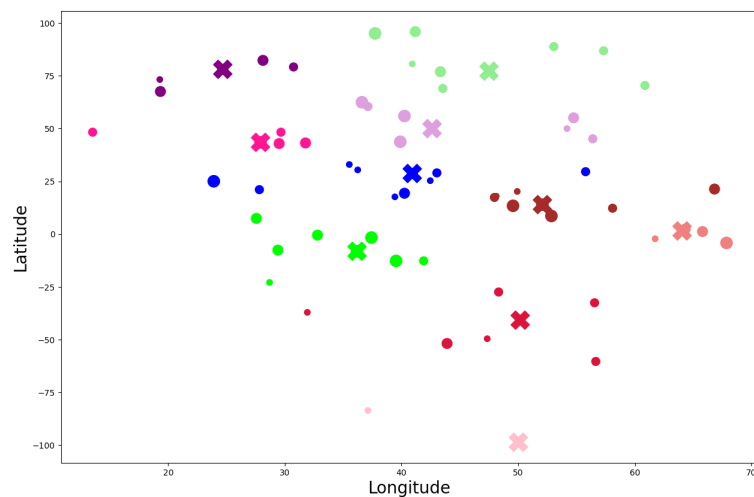


Figure 11. Sectorization map of the solution corresponding to point A for Instance Normal1 where compactness is privileged. Each color is a sector and crossed points are the points chosen to be the representatives of the sector centers.

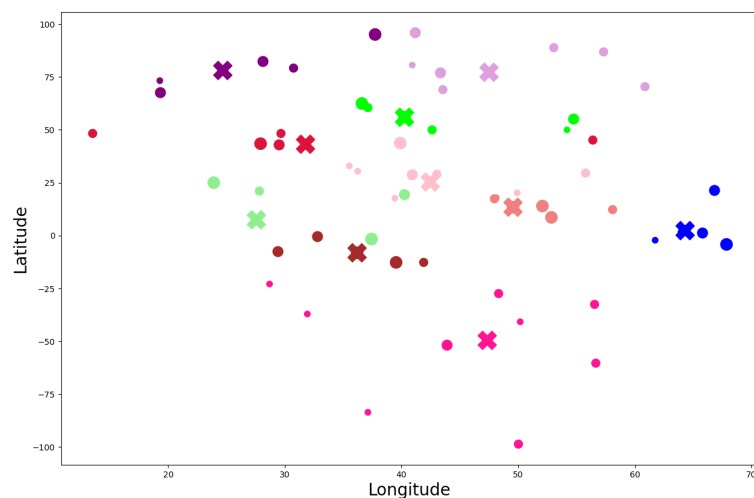


Figure 12. Sectorization map of the solution corresponding to point B for Instance Normal1 where the equilibrium of the sectors is privileged. Each color is a sector and crossed points are the points chosen to be the representatives of the sector centers.

Table 2. Results of the Lexicographic model for 10 sectors. Note that ϵ is the tolerance considered for increasing the best compactness value while minimizing the equilibrium function.

Name	Size	ϵ Tolerance	f_1 comp	f_2 equi	Time	Gap	Sector Amplitude	Sector Weight Amplitude
normal3	43	0	212.8309	524.3144	0.062	0%	10	31
normal3	43	30	241.6525	502.3144	0.281	0%	8	25
normal3	43	40	252.4852	470.9811	0.422	0%	6	14
normal3	43	50	262.4119	466.5367	0.922	0%	5	11
normal3	43	60	271.8500	463.2033	1.437	0%	3	12
normal3	43	70	282.8023	456.0922	1.547	0%	2	6
normal3	43	80	292.6765	455.2033	3.156	0%	1	5
normal3	43	90	301.6234	454.7589	27	0%	1	5
normal3	43	100	311.1992	454.3144	51	0%	3	4
normal3	43	110	322.2521	453.4256	97	0%	2	4
normal3	43	120	330.6537	452.5367	166	0%	3	2
normal3	43	130	340.6303	452.3144	275	0%	3	2
normal3	43	140	352.0862	452.0922	401	0%	3	2
normal3	43	160	372.1082	451.8700	593	0%	3	1
normal1	67	0	559.4322	1682.8744	0.656	0%	8	21
normal1	67	20	579.4108	1668.4300	1512	0%	6	19
normal1	67	40	599.1294	1657.7633	3600	2%	5	12
normal1	67	60	618.0832	1646.8744	3600	8%	5	5
normal1	67	80	639.2240	1642.6522	3600	10%	4	3
normal1	67	100	656.4681	1641.9856	3600	12%	3	2
normal1	67	120	676.1220	1641.7633	3600	12%	5	1
normal16	81	0	703.8095	3243.7300	1.375	0%	8	21
normal16	81	10	713.4498	3210.6233	36	0%	7	15
normal16	81	20	723.4829	3202.6233	3600	1%	6	14
normal16	81	30	733.3424	3192.6233	3600	5%	2	8
normal16	81	40	742.6534	3189.7344	3600	7%	3	5
normal16	81	50	752.9622	3187.9567	3600	9%	2	2
normal16	81	100	803.3282	3187.5122	3600	12%	4	2
normal16	81	1000	1702.2454	3187.2900	600	12%	3	1
normal38	100	0	798.8770	6096.7556	3.719	0%	14	40
normal38	100	10	808.4492	5972.3111	165	0%	4	11
normal38	100	20	818.5967	5965.4222	3600	1%	4	10
normal38	100	30	828.8572	5960.5333	3600	4%	4	9
normal38	100	40	838.8249	5960.3111	3600	7%	4	8
normal38	100	50	848.5137	5955.2000	3600	8%	4	4
normal38	100	100	897.0652	5954.5333	3600	9%	6	3
normal38	100	150	944.8303	5953.8667	3600	10%	5	1
normal30	250	0	1677.3754	105,967.3300	102	0%	18	34
normal30	250	50	1726.0805	105,886.4444	3600	4%	13	21
normal30	250	1350	3007.8490	105,843.3333	3600	4%	13	6
normal30	250	1500	3149.7725	105,840.8889	3600	4%	10	2
normal30	250	2350	3989.0312	105,840.6667	3600	4%	8	2

Table 3. Results of the weighted sum model for 10 sectors.

Name	Size	w_{comp}	w_{equi}	f_3 Weighted Sum	f_1 comp	f_2 equi	Time	Gap	Sector Amplitude	Sector Weight Amplitude
normal3	43	1	0	0.09773	212.8309	524.3144	0.062	0%	10	31
normal3	43	0.95	0.05	0.12303	214.0984	490.5367	0.75	0%	7	22
normal3	43	0.9	0.1	0.13999	215.0080	479.8700	1.109	0%	6	18
normal3	43	0.85	0.15	0.15374	216.6068	470.0922	2.109	0%	6	14
normal3	43	0.8	0.2	0.16625	217.0727	468.3144	4.234	0%	6	14
normal3	43	0.4	0.6	0.26371	226.5557	463.2033	3600	47%	6	13
normal3	43	0.35	0.65	0.27229	226.5557	463.2033	3600	65%	6	13
normal3	43	0.3	0.7	0.28057	228.3301	462.5367	3600	89%	6	13
normal3	43	0.25	0.75	0.28579	253.2497	455.4256	3600	103%	2	6
normal3	43	0.1	0.9	0.28766	271.6147	454.0922	3600	160%	3	5
normal3	43	0.05	0.95	0.28203	304.2661	452.5367	3600	187%	3	2
normal3	43	0	1	0.26571	1577.4426	451.8700	3600	221%	5	1
normal1	67	1	0	0.13739	559.4322	1682.8744	0.188	0%	8	21
normal1	67	0.6	0.4	0.29812	561.8041	1675.7633	3600	22%	8	21
normal1	67	0.5	0.5	0.34308	568.3318	1674.4300	3600	65%	6	21
normal1	67	0.45	0.55	0.36064	571.1399	1671.7633	3600	76%	6	21
normal1	67	0.25	0.75	0.41418	617.2633	1649.0967	3600	102%	5	8
normal1	67	0.2	0.8	0.43105	638.1550	1647.9856	3600	111%	4	7
normal1	67	0	1	0.43947	3377.1727	1641.7633	3600	148%	8	1
normal16	81	1	0	0.04024	703.8095	3243.7344	0.219	0%	8	21
normal16	81	0.95	0.05	0.06359	703.9094	3222.6233	24.547	0%	8	18
normal16	81	0.4	0.6	0.32878	746.5311	3221.7344	3600	107%	6	19
normal16	81	0.55	0.45	0.25578	749.3624	3209.5122	3600	96%	5	14
normal16	81	0.45	0.55	0.30137	766.9575	3208.1789	3600	104%	5	12
normal16	81	0.1	0.9	0.43492	808.1144	3198.8456	3600	119%	4	12
normal16	81	0.35	0.65	0.34124	839.1282	3191.5122	3600	109%	4	6
normal16	81	0.05	0.95	0.44610	1012.7198	3189.0678	3600	121%	3	4
normal16	81	0	1	0.45563	3050.1110	3187.2900	3600	123%	5	1
normal38	100	1	0	0.04936	798.8770	6096.7556	0.391	0%	14	40
normal38	100	0.95	0.05	0.07208	801.6657	5993.4222	2322	0%	7	19
normal38	100	0.85	0.15	0.11632	807.2151	5987.4222	3600	39%	5	19
normal38	100	0.75	0.25	0.15655	810.4458	5974.0889	3600	47%	4	11
normal38	100	0.55	0.45	0.23678	813.1855	5971.8667	3600	72%	5	11
normal38	100	0.4	0.6	0.30226	840.4876	5970.0889	3600	83%	5	12
normal38	100	0.3	0.7	0.34384	887.2802	5960.5333	3600	91%	3	9
normal38	100	0	1	0.44170	3143.4252	5953.8667	3600	100%	8	1
normal30	250	1	0	0.02887	1677.3754	105,967.3333	12.531	0%	18	34
normal30	250	0.9	0.1	0.07811	1689.1577	105,975.5556	3600	22%	12	34
normal30	250	0.8	0.2	0.12874	1710.0880	106,032.6667	3600	32%	17	42
normal30	250	0.7	0.3	0.79060	6813.1765	106,033.7778	3600	85%	14	48
normal30	250	0.5	0.5	0.70922	6813.1765	106,033.7778	3600	76%	14	48
normal30	250	0.1	0.9	0.54647	6813.1765	106,033.7778	3600	50%	14	48
normal30	250	0.05	0.95	0.49689	1943.4125	106,317.7778	3600	42%	24	53
normal30	250	0	1	0.50105	6027.5860	105,927.1111	3600	41%	11	26

6. Conclusions

The paper considered special sectorization problems and followed what was called a location-allocation approach of the area’s centers and basic units. The associated problems were modeled as integer programming. The multiobjective optimization methods utilized were the ϵ -constraint, the lexicographic, and the weighted sum methods.

Sectorization is concerned with dividing a large space into smaller areas, also known as sectors: assigning the points of a set S of size n to k sectors, according to certain criteria. The foremost strategy is simplifying a complex problem, leading to easier approaches to solving the resulting subproblems. Sectorization problems intrinsically involve diverse objectives and have plenty of applications, as mentioned in the introduction.

In the scope of this work, two objectives were employed to carry out the sectorization: equilibrium and compactness, as previously characterized. Working in a multidimensional space is always a challenge because of the difficulty in defining which solutions to select as being “the best”.

The multicriteria optimization methods were explored and evaluated on randomly generated instances. The lexicographic method performed better in larger instances, producing more balanced sectorization solutions concerning the activity measure of the sectors. The weighted sum method performed better in smaller instances, leading to sectorization solutions in which the sectors are more compact.

Both models were tested relative to their running times. For the weighted sum approach, when the weight of the equilibrium function is small and compactness is privileged, the model is solved almost instantaneously. When the weight of the equilibrium function is higher, and compactness is less significant, the time limit of 3600 seconds is always reached, even in the smaller instances. These gaps did not close, but in most cases, no new incumbent solutions appeared after the first minutes, revealing that although these models find good solutions quickly, proving optimality is difficult and time-consuming. Stronger lower bounds would benefit the models and shorten the solving times.

In conclusion, this paper makes significant contributions by handling a topic that has been little studied but with varied applications and by proposing integer programming models that, in a new way, assign the basic units to the sectors and simultaneously find the sector centroids. Based on the results obtained with the two exact methods, which were also compared, we understand that the multiobjective optimization procedures allow dealing with the designated sectorization problems.

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Abbreviations

The following abbreviations are used in this manuscript:

BU	Basic Unit
MOP	Multiobjective Optimization Problem
NSGA	Non-dominated Sorting Genetic Algorithm
PF	Pareto Front
PS	Pareto Set
SP	Sectorization Problems

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