Second-Order Neutral Differential Equations with Distributed Deviating Arguments: Oscillatory Behavior

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Abstract: In this paper, new criteria for a class oscillation of second-order delay differential equations with distributed deviating arguments were established. Our method mainly depends on making sharper estimates for the non-oscillatory solutions of the studied equation. By using the Ricatitechnique and comparison theorems that compare the studied equations with first-order delay differential equations, we obtained new and less restrictive conditions that ensure the oscillation of all solutions of the studied equation. Further, we give an illustrative example.

Keywords: neutral; second-order; distributed deviating arguments; oscillation

MSC: 34C10; 34K11

1. Introduction

In this paper, we consider the second-order neutral delay differential equation

\[
\left( h(T) \left( (x(T) + \Omega(T)x(\xi(T)))' \right) \right)' + \int_{a}^{b} q(T,s) f(x(\theta(T,s))) ds = 0, \quad T \geq T_0. \tag{1}
\]

where \( \beta \) is a ratio of odd positive integers. We assume that

(A1) \( h, \Omega, \xi, \zeta, \varphi, q, \theta \in C(T_{T_0}, \mathbb{R}_+) \), \( \zeta \in C(T_{T_0}, \mathbb{R}_+) \), \( q \in C(T_{T_0} \times [a,b], \mathbb{R}_+) \), \( \theta \in C(T_{T_0} \times [a,b], \mathbb{R}) \),

where \( T_j, j = 0, 1, 2, \ldots \).

(A2) \( \varphi(T,s) \leq T, \xi(T) \leq T, \Omega(T) < \min \{ \xi(T) / \Gamma(\zeta(T)) \}, \) \( q(T,s) \geq 0, \) \( \lim_{T \to \infty} \xi(T) = \lim_{T \to \infty} \varphi(T,s) = \infty \) and

\[
\Gamma(T_0) := \int_{T_0}^{\infty} \frac{1}{h^{\beta}(\nu)} d\nu < \infty.
\]

(A3) \( f \in C(\mathbb{R}, \mathbb{R}) \) and \( f(x) > ku^i \) \( \forall x \neq 0, k > 0 \) and \( i \) is a quotient of odd positive integers.

Definition 1 ([1]). By a solution of (1), we mean \( x \in C([T_0, \infty), \mathbb{R}) \) with \( T_m = \min \{ \xi(T_b), \varphi(T_b) \} \), for \( T_b > T_0 \), which has the property \( h(\nu)^{\beta} \in C([T_m, \infty), \mathbb{R}) \) and satisfies (1) on \( [T_b, \infty) \). In fact, we focus on solutions of (1) that exist in some half-line \( [T_b, \infty) \) where

\[
\sup \{ |x(T)| : T_c \leq T < \infty \} > 0,
\]

for any \( T_c \geq T_b \).
Definition 2 ([1]). We call \( x(T) \) an oscillatory solution if the solution \( x \) is neither \( x < 0 \) nor \( x > 0 \) eventually. If not, it is a nonoscillatory solution.

One of the most important areas that has attracted the attention of researchers in the past few decades is studying functional differential equations, but finding solutions to most nonlinear differential equations is very difficult, so it is interesting to study the qualitative behavior of these solutions. The study of oscillation criteria for differential equations of different orders was and still is the focus of attention of mathematicians because of their theoretical and scientific importance in interpreting and solving different life phenomena; for example, studying the elastic band connected to vibrating blocks and lossless transmission lines related to high-speed electrical networks used to connect switch circuits. The development of discrete fractional differences and fractional calculus has become useful and effective in many diverse engineering and research sectors, including viscoelasticity, electromagnetics, electrochemistry, etc. For more details, see [2–4].

Although there are many results of oscillation or non-oscillation for special cases of (1), most of them shed light on Equation (1) in canonical form; for instance, [5–14]. In addition, we find that the available results about (1) in noncanonical form provide conditions that guarantee that the solutions of Equation (1) are either oscillatory or approaching zero eventually; see, for example, [15,16].

It is known that, when studying the oscillation of neutral differential equations of the second order, we need to classify the non-oscillatory solutions by determining the sign of the derivatives of the corresponding function to the solution; then, we can ensure the oscillation of all the solutions of the studied equation by finding criteria that guarantee the exclusion of any possible nonoscillatory solution. Thus, the emergence of two basic conditions to ensure the oscillation of Equation (1) or its special cases is usual in most studies. For example, the authors in [17,18] presented some conditions that ensure that all solutions of (1) are oscillatory. In addition, Han et al. [19] tested the oscillation of the differential equation

\[
\left( h(T) \left( (x(T) + \Omega(T) x(\zeta(T)))' \right) \right)' + q(T) x^\beta(\theta(T)) = 0,
\]

where

\[
\theta(T) \leq T - \xi_0 = \zeta(T) \quad \text{and} \quad \Omega(T) < 0
\]

and \( \exists \rho \in C^1([\xi_0, \infty), (0, \infty)) \). They showed that Equation (2) is oscillatory under the conditions

\[
\limsup_{T \to \infty} \int_{T_0}^{T} \left( \rho(s) q(s) (1 - \Omega(\theta(T))) \right)^{\beta} - \left( \frac{\beta(\zeta(s))}{\rho(s) \xi'(s)} \right) \frac{h(\zeta(s))}{(\rho(s) \xi'(s))^{\beta+1}} \right) ds = \infty
\]

and

\[
\limsup_{T \to \infty} \int_{T_0}^{T} \left( \frac{q(s) \Gamma^b(T)}{(1 + \Omega(s))^b} \right)^{\beta} - \left( \frac{\beta}{b + 1} \right)^{\beta+1} \frac{1}{\Gamma(T) h(\zeta(s))} \right) ds = \infty,
\]

or

\[
\limsup_{T \to \infty} \int_{T_1}^{T} \left( \int_{T_1}^{T} q(s) \left( \frac{\Gamma(s)}{1 + \Omega(s)} \right)^{\beta} ds \right)^{\frac{1}{b}} d\xi = \infty.
\]

In addition, by using generalized Riccati substitution, Li et al. [20,21] obtained similar results. Despite the large number of previous results about the equation

\[
\left( h(T) \left( (x'(T))' \right) \right)' + q(T) x^\beta(\theta(T)) = 0,
\]
a limited number of studies appeared that established some theorems to ensure the oscillation of all solutions under one condition, as we see in [22]. The authors found that all solutions of Equation (7) oscillate if

$$\int_{-\infty}^{\infty} \left( h^{-1}(z) \int_{T}^{T} q(s) \Gamma^\beta(\theta(s)) \, ds \right)^{1/\beta} \, dz = \infty \quad (8)$$

or

$$\limsup_{T \to \infty} T^{1/\beta} \int_{T_1}^{T} q(s) \, ds > 1. \quad (9)$$

In this work, we obtained new monotonous properties. By using these properties, we were able to improve the relationship between the solution $$x$$ and the corresponding function $$\upsilon$$. Moreover, we ensured that all the solutions of the studied equation oscillate under one condition, in contrast to the results in [17–19], which require two conditions (4) and (5) or (6); see Theorems 1 and 2. By a comparison result with a first-order delay differential equation, we obtained lower bounds for the solutions of Equation (1) in order to achieve qualitatively stronger results when $$\theta(T, s) < T$$. Moreover, obtaining conditions in the form (lim sup. >1) instead of the traditional form allows them to be applied to different equations that cannot be applied to the previous results mentioned above.

This paper is organized as follows. We begin with Section 2, where we define some notations as shortcuts to ease writing and display the results and relationships that we will use in the Section 3. Our results are then presented in Section 3: we prove the conditions assuring that every solution $$x$$ of (1) oscillates; in addition, we give an example. We end Section 4—conclusions and future action—with an interesting open-ended question.

2. Preliminaries

We present some of the lemmas that we will rely on to obtain the main results. For ease, let us define the following functions:

$$\upsilon(T) = x(T) + \Omega(T)x(\Theta(T));$$
$$\phi(T) = \int_{a}^{b} q(s) \left( 1 - \Omega(\theta(s)) \frac{\Gamma(\Theta(s))}{\Gamma^\beta(\theta(s))} \right) \, ds;$$
$$\hat{\phi}(T) = \left( \frac{k}{h^\beta(T)} \int_{T_1}^{T} \phi(v) \, dv \right)^{1/\beta};$$
$$\Phi(T) = \delta(T) + \frac{1}{h^{1/\beta}(T) \Gamma^\beta(T)};$$
$$\Psi(T) = \delta(T) \left( k\eta(T) \phi(T) + \frac{1 - \beta}{h^{1/\beta}(T) \Gamma^\beta + 1(T)} \right);$$
$$G(T) = k\eta(T) \phi(T)(1 - \Omega(\Theta(T)))\delta(T);$$

and

$$\eta(T) := \begin{cases} 1 & \text{if } \beta = \iota, \\ a_1 & \text{if } \beta > \iota, \text{ where } T_1 \in [\Theta_0, \infty) \text{ and } a_1, a_2 > 0, \\ a_2 \Gamma^{-\beta}(T) & \text{if } \beta < \iota \end{cases}$$

**Lemma 1** ([13]). Assume that $$x > 0$$ is a solution of (1) and $$\upsilon(T) \downarrow$$. Then,

$$\upsilon^{-\beta}(T) \geq \eta(T).$$

**Lemma 2.** Assume that $$x$$ is a positive solution of (1). If

$$\int_{T_1}^{\infty} \phi(v) \, dv = \infty,$$

then...
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**Proof.** Let $x > 0$ be a solution of (1) on $[T_0, \infty)$. Then, $\exists \top_1 \in T_{T_0}$ such that $x(T) > 0$, $x(\gamma(T)) > 0$ and $x(\phi(T)) > 0$ for all $T \in T_{T_1}$. Clearly, $v(T) \geq x(T)$ and

$$
\left( h(v')^\beta \right)'(T) = - \int_{a}^{b} q(T, s) f(x(\phi(\phi(T), s))) ds \leq 0.
$$

Thus, $v' > 0$ or $v' < 0$. Let $v' > 0$ on $T_{T_1}$. Then,

$$
\frac{x(T)}{v(T)} \geq (1 - \Omega(T)).
$$

Since $\frac{\Gamma(\gamma(\phi(T), s))}{\Gamma(\phi(T), s)} \geq 1$, we see that

$$
1 - \Omega(\phi(T), s) \geq 1 - \Omega(\phi(T), s) \frac{\Gamma(\gamma(\phi(T), s))}{\Gamma(\phi(T), s)}.
$$

Using (13) and (14) in (12), we find

$$
\left( h(v')^\beta \right)'(T) \leq -kv'(\phi(T), a) \int_{a}^{b} q(T, s) \left( 1 - \Omega(\phi(T), s) \frac{\Gamma(\gamma(\phi(T), s))}{\Gamma(\phi(T), s)} \right)' ds
$$

$$
= -k\phi(T)v'(\phi(T), a) .
$$

Integrating (15) from $T_1$ to $T$ and by (14), we obtain

$$
h(T)(v'(T))\beta \leq h(T_1)(v'(T_1))\beta - k \int_{T_1}^{T} \phi(T)v'(\phi(T), a) dv
$$

$$
\leq h(T_1)(v'(T_1))\beta - kv'(\phi(T), a) \int_{T_1}^{T} \phi(T) dv.
$$

According to (10) and since $\Gamma(T) \downarrow$, we note that

$$
\int_{T_1}^{\infty} \phi(v) dv \to \infty \text{ as } T \to \infty,
$$

which contradicts the positivity of $v'(T)$. The proof is complete.  

**Lemma 3.** Let $x$ be a positive solution of (1). If

$$
\int_{T_1}^{\infty} \left( \frac{1}{h(s)} \int_{T_2}^{s} \phi(v) dv \right)^{1/\beta} ds = \infty, \text{ for } T_1 \geq T_0,
$$

then property (40) holds and

$$
\lim_{T \to \infty} v(T) = \infty.
$$

**Proof.** Let $x > 0$ be a solution of (1) on $T_{T_0}$. From $(\Gamma(T_0) < \infty)$ and (16), (10) is satisfied. From Lemma 2, we note that $v(T) \downarrow$; hence, $\lim_{T \to \infty} v(T) = c$, where $c \geq 0$. Let $c > 0$. Then, there exists $T_2 \geq T_1$ such that $v(\phi(T), a) \leq c$. By (14) and (15), we see that

$$
\left( h(v')^\beta \right)'(T) + kc\phi(T) \leq 0, \text{ for } T \geq T_2.
$$

for $T_1 \geq T_0$, then

$$
v \downarrow, h(v')^\beta \downarrow.
$$

(11)
Integrating (18) twice from $T_0$ to $T$, we obtain
\[ v(T) \leq v(T_0) - (vk)^{1/\beta} \int_{T_0}^{T} \left( \frac{1}{h(u)} \int_{T_0}^{u} \phi(v) dv \right)^{1/\beta} du. \]

The proof is complete. \(\square\)

3. Main Results

In this section, we present some different theories that ensure the oscillation of the solutions of Equation (1).

**Theorem 1.** If
\[ \int_{T_1}^{\infty} \left( \frac{1}{h(u)} \int_{T_1}^{u} \phi(v) \Gamma'(\theta(T, v)) dv \right)^{1/\beta} du = \infty, \text{ for } T_1 \geq T_0, \] (19)
then (1) is oscillatory.

**Proof.** Let $x > 0$ be a solution of (1) on $T_{T_0}$. Then, there exists $T_1 \in T_{T_0}$ such that $x(T) > 0$, $x(\zeta(T)) > 0$ and $x(\theta(T, s)) > 0$ for all $T \in T_{T_1}$. Since $(\Gamma(T) < \infty)$ and (19) hold, $\int_{T_1}^{\infty} \Gamma'(\theta(v)) \phi(v) dv = \infty$ and, by using the fact that $\Gamma(T) \downarrow$, we note that (10) is satisfied. By Lemma 2, $v(T) \downarrow$ and (12) holds. Since
\[ v(T) \geq - \int_{T}^{\infty} v'(v) dv = - \int_{T}^{\infty} \frac{1}{h^{1/\beta}(v)} h^{1/\beta}(v) v'(v) dv \geq - \Gamma(T) h^{1/\beta}(T) v'(T); \]
that is,
\[ v(T) + \Gamma(T) h^{1/\beta}(T) v'(T) \geq 0 \] (20)

By (20), we have
\[ \left( \frac{v(T)}{\Gamma(T)} \right)' = \frac{\Gamma(T) h^{\frac{1}{\beta}}(T) v'(T) + v(T)}{h^{\frac{1}{\beta}}(T) \Gamma^2(T)} \geq 0. \]

In addition, we obtain
\[ x(T) = v(T) - \Omega(T)(x(\zeta(T))) \geq v(T) - \Omega(T)(v(\zeta(T))) \]
\[ \geq v(T) \left( 1 - \Omega(T) \frac{\Gamma(\zeta(T))}{\Gamma(T)} \right). \]

In (12), we find that
\[ \left( h(v')^\beta \right)'(T) \leq - kq(T, s) \left( 1 - \Omega(T) \frac{\Gamma(\zeta(T), s)}{\Gamma(T)} \right) v'(\theta(T, s)) \]
\[ = - k\phi(T) v'(\phi(T, b)). \] (21)

By the monotonicity of $h(T)(v'(T))^\beta$, we obtain
\[ - h(T)(v'(T))^\beta \geq - h(T_1)(v'(T_1))^\beta =: L > 0; \]
that is, (20) implies
\[ v'(T) \geq L^{\frac{1}{\beta}} \Gamma'(T). \] (22)

In (21) and (22), we see that
\[ \left( h(v')^\beta \right)'(T) \leq - kL^{\frac{1}{\beta}} \phi(T) \Gamma'(\phi(T, s)). \] (23)
Integrating from $T_1$ to $T$, we have
\[
h(\tau)(v'(\tau))^{\beta} \leq h(\tau_1)(v'(\tau_1))^{\beta} - kL^{\beta} \int_{\tau_1}^{\tau} \phi(v)\Gamma'(\theta(T, v))dv
\]
\[
\leq -kL^{\beta} \int_{\tau_1}^{\tau} \phi(v)\Gamma'(\theta(T, v))dv. \tag{24}
\]
Integrating from $T_1$ to $T$ and by using (19), we see that
\[
v(\tau) \leq v(\tau_1) - k^{\beta}L^{\beta} \int_{\tau_1}^{\tau} \frac{1}{h^\beta(u)} \left( \int_{\tau_1}^{u} \phi(v)\Gamma'(\theta(T, v))dv \right)^{\frac{1}{\beta}} du.
\]
According to (19), this contradicts the positivity of $v(\tau)$, and this completes the proof. \(\square\)

**Theorem 2.** Assume that $\theta(T, s)$ is an increasing function. If
\[
\limsup_{T \to \infty} \Gamma^{\beta}(\tau)\eta(\tau) \int_{T_1}^{\tau} \phi(v)dv > 1, \text{ for } T_1 \geq T_0,
\]
then (1) is oscillatory.

**Proof.** Suppose that $x > 0$ is a solution of (1). Then, $\exists T_1 \geq T_0, x(\tau(\tau)) > 0$ and $x(\theta(T, s)) > 0$ for all $T \geq T_1$. From (25) and ($\Gamma(\tau_0) < \infty$), we see that (10) is satisfied.

Now, by Lemma 2, $v' < 0$ on $T_1$. In the same way as the proof of Theorem 1, we note that
\[
v(\tau) = \Gamma(\tau)h^{1/\beta}(\tau)v'(\tau) \geq 0
\]
and
\[
\left( h(\tau)(v'(\tau))^{\beta} \right)' \leq -k\phi(\tau)v'(\theta(\tau, b)).
\]
Integrating (21) from $T_1$ to $T$, we have
\[
h(\tau)(v'(\tau))^{\beta} \leq h(\tau_1)(v'(\tau_1))^{\beta} - k \int_{\tau_1}^{\tau} \phi(v)v'(\theta(\tau, s))ds
\]
Since $v' < 0$, we obtain
\[
h(\tau)(v'(\tau))^{\beta} \leq -kv'(\theta(\tau, s)) \int_{\tau_1}^{\tau} \phi(s)ds
\]
\[
\leq -kv^{\beta}(\tau)v^{\beta}(\theta(\tau, s)) \int_{\tau_1}^{\tau} \phi(s)ds. \tag{26}
\]
From Lemma 1 and (20), we obtain
\[
-h(\tau)(v'(\tau))^{\beta} \geq -h(\tau)(v'(\tau))^{\beta}\Gamma^{\beta}(\tau)\eta(\tau) \int_{T_1}^{\tau} \phi(v)dv,
\]
i.e.,
\[
\Gamma^{\beta}(\tau)\eta(\tau) \int_{T_1}^{\tau} \phi(v)dv \leq 1.
\]
This ends the proof. \(\square\)

**Theorem 3.** Let (10) hold. If
\[
v'(\tau) + \phi(\tau)v^{\beta}(\theta(\tau, s)) = 0 \tag{27}
\]
is oscillatory, then (1) is oscillatory.

Proof. Suppose that \( x > 0 \) is a solution of (1). Then, \( \exists T_1 \geq T_0 \) such that \( x(\xi(T)) > 0 \) and \( x(\theta(T), s)) > 0 \) for all \( T \geq T_1 \). By (10) and using Lemma 2, we obtain that \( v' < 0 \) on \( T_{T_1} \). In the same way as the proof of Theorem 2, (26) holds. From (26), we have

\[
v'(T) + \hat{\phi}(T)v'^{\beta}(\theta(T), s)) \leq 0.
\]

(28)

Note that \( v \) is a positive solution of (28). By [11], (27) has a solution \( v > 0 \), which is a contradiction. □

Corollary 1. If (10) holds and one of the following statements is true:

(i) \( \beta = \iota \) and

\[
\liminf_{T \to \infty} \int_{T_1}^{T} \hat{\phi}(v)dv > \frac{1}{e}.
\]

(29)

(ii) \( \beta > \iota > 0 \) and

\[
\int_{T_0}^{\infty} \hat{\phi}(v)dv = \infty.
\]

(30)

(iii) \( \beta < \iota, \theta'(T) > 0 \). If there \( \exists \) a continuously differentiable function and \( \xi(T) \uparrow \),

\[
\lim_{T \to \infty} \xi(T) = \infty,
\]

\[
\limsup_{T \to \infty} \left[ \frac{1}{\xi'(T)} \hat{\phi}(T) e^{-\xi(T)} \right] > 0,
\]

(31)

then (1) is oscillatory.

Proof.

(i) Let \( \beta = \iota \). According to [23] (Theorem 2), (29) implies the oscillation of (27). In addition, \( \int_{T_0}^{\infty} \hat{\phi}(v)dv = \infty \) is sufficient to ensure (10).

(ii) According to [24] and [1] and since \( \iota/\beta \in (0, 1) \), if (30) holds, then all solutions of (27) oscillate. On the other hand, we see that (30) is necessary to ensure (10).

(iii) In view of [11] (Theorem 1), (31) implies the oscillation of (27). Therefore, the proof is complete. □

The following theorem is an improvement of Theorem 2 when \( \beta = \iota \) and

\[
\limsup_{T \to \infty} \Gamma^{\beta}(T)\eta(T) \int_{T_1}^{T} \hat{\phi}(v)dv < 1.
\]

Theorem 4. Assume that (16) holds and \( \beta = \iota \). If

\[
m := \liminf_{T \to \infty} \frac{k}{\Gamma(T)} \int_{T_0}^{\infty} \Gamma^{\beta+1}(s)\phi(s)ds > \beta
\]

(32)

or

\[
m \leq \beta \text{ and } M := \limsup_{T \to \infty} \left( k \int_{T_0}^{T} \phi(s)ds \right)^{1/\beta} > \frac{\beta - m}{\beta}.
\]

(33)

then (1) is oscillatory.
Proof. Suppose that Equation (1) has a nonoscillatory solution \( x \) on \( T_{\tau_0} \). Then, \( \exists x(T) > 0 \) and \( x(\theta(T)) > 0 \) for \( T \geq \tau_1 \geq \tau_0 \). Let
\[
g'(T) = \left( v(T) + h^{1/\beta}(T)v'(T)\Gamma'(T) \right)' = \Gamma(T) \left( h^{1/\beta}(T)v'(T) \right)' \\
= \frac{1}{\beta} \Gamma(T) \left( \left( h^{1/\beta}(T)v'(T) \right)^\beta \right)'. \tag{34}\]

Using inequality (21), we have
\[
g'(T) = \frac{1}{\beta} \Gamma(T) \left( h^{1/\beta}(T)v'(T) \right)^{1-\beta} \left( \left( h(T) \left( v'(T) \right)^\beta \right) \right)' \\
\leq -\frac{k}{\beta} \Gamma(T) \left( h^{1/\beta}(T)v'(T) \right)^{1-\beta} \phi(T)v^\beta(\theta(T), s)). \tag{35}\]

Integrating from \( T \) to \( \infty \), we find
\[
g(T) \geq \frac{k}{\beta} \int_T^\infty \Gamma(s)\phi(s) \left( h^{1/\beta}(s)v'(s) \right)^{1-\beta} v^\beta(s) ds \\
\geq -\frac{k}{\beta} \int_T^\infty \Gamma(T)\phi(s) \left( h^{1/\beta}(s)v'(s) \right)^{1-\beta} \left( -\Gamma(T)h^{1/\beta}(s)v'(s) \right)^{\beta-1} v(s) ds \\
\geq \frac{k}{\beta} \int_T^\infty v(T) \int_T^\infty \Gamma^{\beta+1}(s)\phi(s) ds; \]
that is,
\[
v(T) + h^{1/\beta}(T)v'(T) - \Gamma(T) \frac{k}{\beta} \int_T^\infty \Gamma^{\beta+1}(s)\phi(s) ds \geq 0, \]
and so
\[
v(T) \left( 1 - \frac{k}{\beta} \frac{1}{\Gamma(T)} \int_T^\infty \Gamma^{\beta+1}(s)\phi(s) ds \right) \geq -h^{1/\beta}(T)v'(T)\Gamma(T) > 0. \tag{36}\]

Since (32) holds, \( \exists \epsilon > 0 \) such that \( m - \epsilon > \beta \). Thus, we see that
\[
1 - \frac{k}{\beta} \frac{1}{\Gamma(T)} \int_T^\infty \Gamma^{\beta+1}(s)\phi(s) ds \leq 1 - \frac{1}{\beta} (m - \epsilon) < 0.
\]

Now, let \( m \leq \beta \). In the same way as the proof of Theorem 2, we obtain (26). Thus, by (36), we obtain
\[
-h^{1/\beta}(T)v'(T) \left( 1 - \frac{k}{\beta} \frac{1}{\Gamma(T)} \int_T^\infty \Gamma^{\beta+1}(s)\phi(s) ds \right) \\
\geq kv(T) \left( 1 - \frac{k}{\beta} \frac{1}{\Gamma(T)} \int_T^\infty \Gamma^{\beta+1}(s)\phi(s) ds \right) \left( \int_{T_0}^T \phi(s) ds \right)^{1/\beta} \\
\geq -kr^{1/\beta}(T)v'(T)\Gamma(T) \left( \int_{T_0}^T \phi(s) ds \right)^{1/\beta}.
\]

In other words,
\[
\left( 1 - \frac{k}{\beta} \frac{1}{\Gamma(T)} \int_T^\infty \Gamma^{\beta+1}(s)\phi(s) ds \right) - k\pi(T) \left( \int_{T_0}^T \phi(s) ds \right)^{1/\beta} \geq 0,
\]
which implies
\[
\limsup_{\top \to \infty} \Gamma(\top) \left( k \int_{\top_0}^{\top} \phi(s) ds \right)^{1/\beta} \leq 1 - \liminf_{\top \to \infty} \frac{k}{\rho(\top)} \int_{\top}^{\infty} \Gamma^{\beta + 1}(s) \phi(s) ds.
\]

Hence,
\[
\alpha M \leq \beta - m.
\]

This ends the proof. \(\square\)

**Theorem 5.** Let (16) hold and \(\beta = \iota\). If
\[
\Gamma(\theta(\top, s)) \geq \Gamma(\top) \kappa, \text{ for all } \top \geq \top_0, \text{ where } \kappa > 1 \text{ is a constant}
\]
and
\[
M^M > 1,
\]
then (1) is oscillatory.

**Proof.** As proof of Theorem 2, from (26), we have
\[
-h(\top)(\nu'(\top))^\beta \geq k\nu^\beta(\top)\kappa^{N} \int_{\top_0}^{\top} \phi(s) ds.
\]

Now, by using Lemma 3, \(x(\top)\) satisfies (40) and (17). Similarly to the proof of Theorem 2, we see that (26) holds. Thus,
\[
h^{1/\beta}(\top)\nu'(\top) + \left( k \int_{\top_0}^{\top} \phi(s) ds \right)\nu(\top) \leq 0.
\]

On the other hand,
\[
\left( \frac{\nu(\top)}{\Gamma^{M-\epsilon}(\top)} \right)' = \frac{h^{1/\beta}(\top)\nu'(\top)\Gamma^{M-\epsilon}(\top) + (M - \epsilon)\nu(\top)\Gamma^{M-\epsilon - 1}(\top)}{h^{1/\beta}(\top)\Gamma^{2(M-\epsilon - 1)}(\top)} \leq \frac{\nu(\top)}{h^{1/\beta}(\top)\Gamma^{M-\epsilon + 1}(\top)} \left( M - \epsilon + \frac{h^{1/\beta}(\top)\nu'(\top)\Gamma(\top)}{\nu(\top)} \right).
\]

From (40) and (41), it is implied that
\[
\left( \frac{\nu(\top)}{\Gamma^{M-\epsilon}(\top)} \right) < 0, \epsilon > 0.
\]

Thus,
\[
\nu(\theta(\top, s)) \geq \nu(\top)\kappa^{M-\epsilon}.
\]

Using (20), we have
\[
-h(\top)(\nu'(\top))^\beta \geq -kr(\top)(\nu'(\top))^\beta \Gamma^{\beta}(\top)\kappa^{N} \int_{\top_0}^{\top} \phi(s) ds,
\]
i.e.,
\[
\kappa^{M-\epsilon}\Gamma(\top) \left( k \int_{\top_0}^{\top} \phi(s) ds \right)^{1/\beta} \leq 1.
\]

This is a contradiction. The proof is complete. \(\square\)
Theorem 6. Assume that (16) holds, \( i = \beta \) and (37) holds. If (32) or

\[
\frac{m}{\beta} \leq 1 \quad \text{and} \quad \frac{\alpha M \kappa M}{\beta - m} > 1,
\]

then (1) is oscillatory.

Proof. As in the proof of Theorem 4, replace (26) with (39); we have

\[
-h^{1/\beta}(\tau)(v'(\tau))\left(1 - \frac{k}{\beta} \int_{\tau}^{\infty} \frac{\Gamma^{\beta+1}(s)}{\Gamma(\tau)} = \sum \phi(s)ds \right) \geq \nu(\tau)k^{M-\epsilon} \left(1 - \frac{k}{\beta} \int_{\tau}^{\infty} \frac{\Gamma^{\beta+1}(s)}{\Gamma(\tau)} = \sum \phi(s)ds \right)^{1/\beta}
\]

and

\[
\left(1 - \frac{k}{\alpha \pi(\tau)} \int_{\tau}^{\infty} \Gamma^{\beta+1}(s)\phi(s)ds \right) \geq \Gamma(\tau)k^{M-\epsilon} \left(1 - \frac{k}{\alpha \pi(\tau)} \int_{\tau}^{\infty} \Gamma^{\beta+1}(s)\phi(s)ds \right) \leq \Gamma(\tau)k^{M-\epsilon} \left(1 - \frac{k}{\alpha \pi(\tau)} \int_{\tau}^{\infty} \Gamma^{\beta+1}(s)\phi(s)ds \right)^{1/\beta};
\]

that is, we obtain

\[
\limsup_{\tau \to \infty} \Gamma(\tau)k^{M-\epsilon} \left(1 - \frac{k}{\alpha \pi(\tau)} \int_{\tau}^{\infty} \Gamma^{\beta+1}(s)\phi(s)ds \right)^{1/\beta} \leq 1 - \liminf_{\tau \to \infty} \frac{k}{\alpha \pi(\tau)} \int_{\tau}^{\infty} \Gamma^{\beta+1}(s)\phi(s)ds.
\]

Therefore,

\[
\alpha \kappa^{M-\epsilon} M \leq \beta - m.
\]

Then, this ends the proof. \( \square \)

Theorem 7. Let \( \theta(\tau) \uparrow \) and \( \beta \in [1, \infty) \). If there \( \exists \) functions \( \varphi, \delta \in C^1([\tau_0, \infty), \mathbb{R}^+) \) such that

\[
\limsup_{\tau \to \infty} \int_{\tau_0}^{\tau} \left(\Psi(v) - \frac{1}{(\beta + 1)^{\beta+1}} G_1(v) \right) dv = \infty \quad \text{(44)}
\]

and

\[
\limsup_{\tau \to \infty} \int_{\tau_0}^{\tau} \left(\varphi(v)G(v) - \frac{1}{(\beta + 1)^{\beta+1}} G_2(v) \right) dv = \infty, \quad \text{(45)}
\]

where

\[
G_1(\tau) = \delta(v)h(v)(\Phi_+ (v))^{\beta+1}, \quad G_2(\tau) = \frac{h(\theta(\tau), v))((\phi'_u(v))^\beta+1}{(\Phi(\nu, \theta(\tau), v))^\beta}
\]

and \( H_+ (\tau) = \max \{ H(\tau), 0 \} \), then (1) is oscillatory.

Proof. Let \( x > 0 \) be a solution of (1). Thus, there \( \exists \) \( \tau_1 \geq \tau_0 \) such that \( x(\xi(\tau)) > 0 \) and \( x(\theta(\tau), s) > 0 \) for all \( \tau \geq \tau_1 \). Let \( v'(\tau) < 0 \) for \( \tau \geq \tau_1 \). As in the proof of Theorem 1, we have (21), and from Lemma 1, we obtain

\[
\left(\frac{h(\tau)v'(\tau))^{\beta}}{v^\beta(\tau)} \right) \leq -k\Phi(\tau) \frac{v'(\theta(\tau), s)}{v^\beta(\tau)} \leq -k\eta(\tau)\phi(\tau). \quad \text{(46)}
\]

Define \( \omega \) as follows:

\[
\omega(\tau) = \delta(\tau) \left[ h(\tau)\left(\frac{v'(\tau))^{\beta}}{v^\beta(\tau)} + \frac{1}{\Gamma^\beta(\tau)} \right) \right] \geq 0. \quad \text{(47)}
\]
By differentiating (47), we obtain
\[
\omega'(T) = \frac{\delta'(T)\omega(T)}{\delta(T)} + \frac{\delta(T)(h(T)(v'(T))^\beta)'}{v^\beta(T)} - \frac{a\delta(T)h(T)(v'(T))^\beta+1}{v^{\beta+1}(T)} + \frac{a\delta(T)}{h^{\beta}(T)\Gamma^{\beta+1}(T)}
\]

\[
= \frac{\delta'(T)\omega(T)}{\delta(T)} + \frac{\delta(T)(h(T)(v'(T))^\beta)'}{v^\beta(T)} + \frac{a\delta(T)}{h^{\beta}(T)\Gamma^{\beta+1}(T)}
\]

\[-a\delta(T)h(T)\left(\frac{\omega(T)}{\delta(T)h(T)} - \frac{1}{h(T)\Gamma^{\beta}(T)}\right)^{(\beta+1)/\beta}.
\]

Using inequality
\[A^{(\beta+1)/\beta} - (A - B)^{(\beta+1)/\beta} \leq \frac{1}{\beta}B^{1/\beta}[(1 + \beta)A - B], \quad \beta \geq 1, \ AB \geq 0\]
with
\[A := \frac{\omega(T)}{\delta(T)h(T)} \quad \text{and} \quad B := \frac{1}{h(T)\Gamma^{\beta}(T)},\]
we see that
\[
\left[\frac{\omega(T)}{\delta(T)h(T)} - \frac{1}{h(T)\Gamma^{\beta}(T)}\right]^{\delta+1/\beta} \geq \left(\frac{\omega(T)}{\delta(T)h(T)}\right)^{\beta+1/\beta} - \frac{1}{\alpha r(T)^{1/2}\Gamma(T)}\left[(\beta + 1)\omega(T) - \frac{1}{h(T)\Gamma^{\beta}(T)}\right] + \frac{a\delta(T)}{h^{\beta}(T)\Gamma^{\beta+1}(T)}
\]

From (46)–(50), we find
\[
\omega'(T) \leq \frac{\delta'(T)\omega(T)}{\delta(T)} - k\delta(T)\eta(T)\Phi(T) - a\delta(T)h(T)\left(\frac{\omega(T)}{\delta(T)h(T)}\right)^{\delta+1/\beta} - \frac{1}{ar(T)^{1/2}\Gamma(T)}\left[(\beta + 1)\omega(T) - \frac{1}{h(T)\Gamma^{\beta}(T)}\right] + \frac{a\delta(T)}{h^{\beta}(T)\Gamma^{\beta+1}(T)}
\]

\[
= \Phi(T)\omega(T) - \Psi(T) - \frac{a\omega(T)^{\delta+1/\beta}}{(\delta(T)h(T))^{\beta}}\cdot
\]

By inequality
\[DV - CV^{(\beta+1)/\beta} \leq \frac{\beta^\beta}{(\beta + 1)^{\beta+1}} \frac{D^{\beta+1}}{C^\beta}, \quad C > 0,
\]
with \(C := \beta(\delta(T)h(T))^{-1/\beta}, \ V := \omega(T)\) and \(D := \Phi(T),\) we have
\[
\omega'(T) \leq \delta(T)h(T)\left(\frac{\Phi(T)}{(\beta + 1)^{\beta+1}} - \Psi(T)\right).
\]

Integrating from \(T_2\) to \(T,\) we obtain
\[
\int_{T_2}^{T} \left(\frac{-\delta(v)h(v)(\Phi(v))^\beta+1}{(\beta + 1)^{\beta+1}} + \Psi(v)\right)dv \leq \omega(T) - \omega(T_2) \leq \omega(T).
\]
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Now, let \( \nu'(T) > 0 \) for all \( T \geq T_2 \). It is easy to see that \( x(T) \geq (1 - \Omega(T))v(T) \); thus,
\[
\left( h(T) \left( \frac{\nu'(T)}{v(T)} \right)^\beta \right)' + kq(T)(1 - \Omega(T,s))\nu'(\theta(T,s)) \leq 0. \tag{51}
\]

Since \( h(\nu)^\beta \downarrow \), we obtain
\[
\nu'(\theta(T,s))h^{1/\beta}(\theta(T,s)) \geq \nu'(T)h^{1/\beta}(T). \tag{52}
\]

Define the function
\[
R(T) = \varphi(T)h(T) \left( \frac{\nu'(T)}{v(\theta(T,s))} \right)^\beta \geq 0. \tag{53}
\]

By differentiating (53) and using (51) and (52), we obtain
\[
R'(T) \leq -\varphi(T)G(T) + \frac{\varphi'(T)R(T)}{\varphi(T)} - \frac{\alpha}{h^{1/\beta}(\theta(T,s))} \varphi^{1+1/\beta}(T).
\]

As in the proof of case \( \nu'(T) < 0 \), we have
\[
\int_{T_0}^T \left( \varphi(v)G(v) - \frac{h(\theta(T,v))(\nu'(v))^{\beta+1}}{(\beta + 1)^{\beta+1} \varphi(v) \varphi'(T,v)^\beta} \right)dv \leq R(T_2).
\]

This completes the proof. □

**Theorem 8.** Assume that \( \theta(T,s) \uparrow \). If (45) holds and there \( \exists \) functions \( \varphi, \delta \in C^1([T_0, \infty), \mathbb{R}^+) \) such that
\[
\limsup_{T \to \infty} \left( \frac{\psi(T)}{\delta(T)} \int_{T_0}^T \left( k\delta(v)\eta(v)\phi(v) - \frac{1}{(\beta + 1)^{\beta+1} \delta(T)} \right)dv \right) > 1, \tag{54}
\]
then (1) is oscillatory.

**Proof.** As in the proof of Theorem 7, for the case where \( \nu'(T) < 0 \) for all \( T \geq T_1 \), as in (47), define the function \( \omega > 0 \). From (49), we obtain
\[
\omega'(T) = \frac{\delta'(T)\omega(T)}{\delta(T)} + \frac{\delta(T)\left( h(T) \left( \frac{\nu'(T)}{v(T)} \right)^\beta \right)'}{\delta(T)h(T)} + \frac{\alpha}{h^{1/\beta}(T)} \frac{\psi(T)}{\delta(T)h(T)} \left( \frac{\omega(T)}{\varphi(T)} \right)^{\beta+1/\beta}.
\]

Using inequality
\[
DV - CV^{(\beta+1)/\beta} \leq \frac{\beta^\beta}{(\beta + 1)^{\beta+1} C^\beta}, \quad C > 0,
\]
with \( C = \delta'(\tau) / \delta(\tau), D = \beta(\delta(\tau) h(\tau))^{1/\beta} \) and \( V = \delta(\tau) / \Gamma^\beta(\tau) \), we have

\[
\omega'(\tau) \leq -k \delta(\tau) \eta(\tau) \phi(\tau) + \frac{\delta'(\tau) + \frac{h(\tau)(\delta'(\tau))^{\beta+1}}{(\beta + 1)^{\beta+1} \delta^\beta(\tau)} + \frac{a \delta}{h^\tau(\tau) \Gamma^\beta(\tau)}}{(\beta + 1)^{\beta+1} \delta^\beta(\tau)} \tag{55}
\]

Integrating from \( T_2 \) to \( \tau \), we obtain

\[
\int_{T_2}^{\tau} \left( k \eta(v) \delta'(v) \phi(v) - \frac{h(v)(\delta'(v))^{\beta+1}}{(\beta + 1)^{\beta+1} \delta^\beta(v)} \right) dv - \frac{\delta(\tau)}{\Gamma^\beta(\tau)} + \frac{\delta(T_2)}{\Gamma^\beta(T_2)} \leq \omega(\tau) - \omega(T).
\]

By (47), we see that

\[
\int_{T_2}^{\tau} \left( k \eta(v) \delta'(v) \phi(v) - \frac{h(v)(\delta'(v))^{\beta+1}}{(\beta + 1)^{\beta+1} \delta^\beta(v)} \right) dv \leq \frac{\delta(T_2) h(T_2)(\delta'(T_2))^{\beta}}{\delta^\beta(T_2)} - \frac{\delta(\tau) h(\tau)(\delta'(\tau))^{\beta}}{\delta^\beta(\tau)}.
\]

From (20), it is easy to see that

\[
h(\tau)(\delta'(\tau))^{\beta} \Gamma^\beta(\tau) \geq -\nu^\beta(\tau).
\]

In (57), we have

\[
\frac{\Gamma^\beta(T)}{\delta(T)} \int_{T_2}^{\tau} \left( k \delta(v) \eta(v) \phi(v) - \frac{h(v)(\delta'(v))^{\beta+1}}{(\beta + 1)^{\beta+1} \delta^\beta(v)} \right) dv \leq 1.
\]

On the other hand, let \( \nu'(\tau) > 0 \). The proof for \( \nu'(\tau) > 0 \) is proof of Theorem 7. The proof is complete. \( \square \)

**Example 1.** Consider the second-order neutral differential equation

\[
\frac{d}{d\tau} \left( \tau^2 \frac{d}{d\tau} \left( x(\tau) + \Omega_0 \chi(\lambda t) \right) \right)^{\beta} + \int_0^1 \frac{q_0}{s^{1-\gamma}} x'(\gamma s) ds = 0, \lambda, \delta \in (0, 1), \tag{57}
\]

where \( \Omega_0 \in [0, \lambda) \) and \( \gamma = \max\{\beta, \lambda\} \).

When \( \beta > \lambda \), we see that \( \gamma = \beta \), by Corollary 1, if

\[
\int_{T_0}^{\infty} \frac{K}{s^{2-\gamma}} ds = \infty, \text{ where } K = \left( \frac{q_0}{\gamma} \left( \frac{\lambda - \Omega_0}{\lambda} \right) \right)^{1/\beta} > 0,
\]

then Equation (57) is oscillatory.

For \( \beta < \lambda \), we see that (25) holds and

\[
q_0 \left( 1 - \frac{\Omega_0}{\lambda} \right)^{1/2} > \lambda.
\]
We see that, as a result of constant $a_2$, this condition is not applicable. However, if we take $\gamma = 1 + 1$ (according to Theorem 2), then (25) holds; that is, (57) is oscillatory. For $i = 1$, we see that

\[ \text{Oscillation condition} \]

By Theorem 2, \[ q_0 \left( 1 - \frac{Q_0}{x} \right)^{\beta} > \beta. \]

By Corollary 1, \[ q_0^{1/\beta} \left( 1 - \frac{Q_0}{x} \right) \ln \frac{1}{\sigma} > 0. \]

By Theorem 6, \[ q_0 \left( 1 - \frac{Q_0}{x} \right)^{\left( 1 - Q_0 / \lambda \right)} \] is the most efficient condition for (58), and Condition (a2) gives $q_0 > 1.06$ and $q_0 > 0.18$ for (58) and (59), which is the most efficient condition for (59). Moreover, for (A1), we see that Condition (a3) provides an improvement of Condition (a1) and (a2), namely $q_0 > 0.8532$. By choosing $Q_0 = 0$ and $\beta = 1$, we obtain a Euler differential equation; that is, Condition (a4) leads us to a sharp result for oscillation $q_0 > 1/4$.

4. Conclusions

In this paper, we first classified the positive solutions of the studied equation based on the sign of their derivatives. Next, we presented some important relationships that we have recently used in the main results. Then, we introduced new criteria to guarantee the oscillation of all solutions of (1).

In this study, in contrast to most of the previous literature, we overcame condition (4) that was imposed in most of the previous results—see [17,19,25,26]—and some other additional conditions—see [14–18,20–22]—where Theorems 1 and 2 ensure that all solutions of the studied equation oscillate under one condition. Even when $\beta = i$, we see that Theorems 4–6 simplify and complement the previous results; see [17–19]. It would be interesting to extend our results to fractional differential equations.

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