Groups with Subnormal Deviation

Francesco de Giovanni 1,*,†, Leonid A. Kurdachenko 2 and Alessio Russo 3,‡

1 Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II, Via Cintia, 80138 Napoli, Italy
2 Department of Algebra, National University of Dnipro, 49000 Dnipro, Ukraine; ikurdachenko@i.ua
3 Dipartimento di Matematica e Fisica, Università della Campania Luigi Vanvitelli, Via Vivaldi, 81100 Caserta, Italy; alessio.russo@unicampania.it
* Correspondence: degiovan@unina.it
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Abstract: The structure of groups which are rich in subnormal subgroups has been investigated by several authors. Here, we prove that if a periodic soluble group has subnormal deviation, which means that the set of its non-subnormal subgroups satisfies a very weak chain condition, then either $G$ is a Černikov group or all its subgroups are subnormal. It follows that if a periodic soluble group has a subnormal deviation, then its subnormal deviation is 0.

Keywords: subnormal subgroup; subnormal deviation; minimal condition

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1. Introduction

In contrast to the finite case, a celebrated example by Heineken and Mohamed [1] shows that there exist infinite groups with a trivial centre in which all subgroups are subnormal. On the other hand, a relevant result of Möhres [2] states that groups with only subnormal subgroups are at least soluble, while Smith [3] proved that torsion-free groups of this type are nilpotent. The structure of infinite groups which are somehow rich in subnormal subgroups has been investigated by several authors, with special emphasis on the imposition of chain conditions. In particular, groups satisfying the minimal or the maximal condition on non-subnormal subgroups were described in [4,5], respectively. A further step was the investigation of groups satisfying the so-called weak minimal and weak maximal condition on non-subnormal subgroups (see [6,7]). We consider here the effect of the imposition of a further very general chain restriction on the set of non-subnormal subgroups.

Let $(\Lambda, \leq)$ be a partially ordered set and let $\Omega$ and $\Delta$ be non-empty subsets of $\Lambda$. We say that $\Omega$ has $\Delta$-deviation 0 if either $\Omega \subseteq \Delta$ or the set $\Omega \setminus \Delta$ satisfies the minimal condition; moreover, if $\delta > 0$ is any ordinal, we use induction to say that $\Omega$ has $\Delta$-deviation $\delta$ if every descending chain

$$a_1 > a_2 > \ldots > a_n > a_{n+1} > \ldots$$

of elements of $\Omega \setminus \Delta$ there exists a positive integer $t$ such that the interval $[a_n/a_{n+1}]$ has a $\Delta$-deviation strictly smaller than $\delta$ for each $n \geq t$, and $\delta$ is the smallest ordinal with such a property. In particular, if the set $\Omega \setminus \Delta$ satisfies the weak minimal condition (i.e., if for any descending chain of elements of $\Omega \setminus \Delta$ there is a positive integer $t$ such that $[a_n/a_{n+1}]$ is finite for all $n \geq t$), then $\Omega$ has $\Delta$-deviation at most 1.

Let $G$ be a group and let $\text{sn}(G)$ be the set of all subnormal subgroups of $G$. Then a non-empty set $\Omega$ of subgroups of $G$ is said to have a $G$-subnormal deviation if it has a $\text{sn}(G)$-deviation in the partially ordered set $\mathcal{L}(G)$ of all subgroups of $G$. In particular, if $X$ is a subgroup of $G$ and the set of all subgroups of $X$ has a $G$-subnormal deviation, we just say...
that $X$ has a $G$-subnormal deviation; the group $G$ is said to have a subnormal deviation if it has a $G$-subnormal deviation, which means that the set $\Sigma(G)$ has sn($G$)-deviation. Of course, if a group $G$ has a subnormal deviation, then every set of subgroups of $G$ has a $G$-subnormal deviation.

The aim of this paper is to give a further contribution to the theory of groups with many subnormal subgroups, characterizing periodic groups with a subnormal deviation, at least within the universe of soluble groups. The structure of locally soluble groups with subnormal deviation at most 1 was described in [8,9], where it was specifically proved that such groups are soluble. Our main result proves that if a periodic soluble group has a subnormal deviation, then only the extreme (and unavoidable) cases can occur.

**Theorem 1.** Let $G$ be a periodic soluble group with a subnormal deviation. Then either $G$ is Černikov or all its subgroups are normal. In particular, the subnormal deviation of $G$ is 0.

It follows from the above result that a soluble group with a positive subnormal deviation cannot be periodic; however, the study of soluble non-periodic groups with a subnormal deviation needs different methods and will be the subject of a forthcoming paper.

Recall, finally, that a subgroup $X$ of a group $G$ is said to be pronormal if $X$ and $X^g$ are conjugate in $\langle X, X^g \rangle$ for every element $g$ of $G$. A group $G$ is said to have a pronormal deviation if $\Sigma(G)$ has pn($G$)-deviation, where pn($G$) is the set of all pronormal subgroups of $G$. The replacement of pn($G$) by the set of all normal subgroups of $G$ gives the corresponding concept of normal deviation. Since a subgroup of an arbitrary group is normal if, and only if, it is subnormal and pronormal, the combination of our theorem with the results in [10] on groups with a pronormal deviation gives the following statement.

**Corollary 1.** Let $G$ be a periodic soluble group with a normal deviation. Then either $G$ is Černikov or all its subgroups are normal. In particular, the normal deviation of $G$ is 0.

Our notation is mostly standard and can be found in [11], and we refer to [12] for results concerning subnormal subgroups.

2. Proof of the Theorem

The proof of our main result will be accomplished in a series of lemmas. We recall that, if $G$ is a group and $X$ and $Y$ are subnormal subgroups of $G$, then also $X \cap Y$ is subnormal in $G$ (see [12], Proposition 1.1.2), while $\langle X, Y \rangle$ need not be subnormal in general; however, this is the case if $XY = YX$ (see [12], Theorem 1.2.5).

**Lemma 1.** Let $G$ be a group and let $Y \leq X$ be subgroups of $G$ such that $Y$ is normal in $X$ and $X/Y$ is the direct product of an infinite collection $(X_\lambda/Y)_{\lambda \in \Lambda}$ of non-trivial subgroups. If the interval $[X/Y]$ has G-subnormal deviation, then every $X_\lambda$ is subnormal in $G$.

**Proof.** Let $\lambda$ be an arbitrary element of $\Lambda$, and let $\Gamma$ and $\Omega$ be infinite subsets of $\Lambda$, such that $\Gamma \cup \Omega = \Lambda \setminus \{\lambda\}$ and $\Gamma \cap \Omega = \emptyset$. Choose in $\Gamma$ an infinite descending chain of subsets

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \ldots \supset \Gamma_n \supset \Gamma_{n+1} \supset \ldots,$$

such that $\Gamma_n \setminus \Gamma_{n+1}$ is infinite for all $n$, and write

$$V_n/Y = X_\lambda/Y \times \left( \bigcup_{\gamma \in \Omega_n} X_\gamma/Y \right)$$

for each non-negative integer $n$. Similarly, let

$$\Omega = \Omega_0 \supset \Omega_1 \supset \ldots \supset \Omega_n \supset \Omega_{n+1} \supset \ldots$$
be an infinite descending chain of subsets of $\Omega$, such that $\Omega_n \setminus \Omega_{n+1}$ is infinite for all $n$, and put
\[ W_n/Y = X_\lambda/Y \times \left( \prod_{\omega \in \Omega_n} X_\omega/Y \right) \]
for each non-negative integer $n$. Then
\[ V_0 > V_1 > \ldots > V_n > V_{n+1} > \ldots \]
and
\[ W_0 > W_1 > \ldots > W_n > W_{n+1} > \ldots \]
are infinite descending chains of elements of $[X/Y]$.

Let $\delta$ be the $G$-subnormal deviation of the interval $[X/Y]$ and assume first $\delta = 0$, so that $[X/Y]$ satisfies the minimal condition on subgroups which are not subnormal in $G$. Then there exist non-negative integers $h$ and $k$, such that $V_h$ and $W_k$ are subnormal in $G$. It follows that $X_\lambda = V_h \cap W_k$ is likewise subnormal in $G$, and so the statement is proved when $\delta = 0$.

Suppose now $\delta > 0$. By definition there exists a non-negative integer $r$, such that the interval $[V_n/V_{n+1}]$ has a $G$-subnormal deviation strictly smaller than $\delta$ for each $n \geq r$. Since
\[ V_r/V_{r+1} = Dr_{\gamma \in \Gamma_r \setminus \Gamma_{r+1}} (X_\gamma V_{r+1}/V_{r+1}) \]
is the direct product of an infinite collection of non-trivial subgroups, by induction on $\delta$ we have that $X_\gamma V_{r+1}$ is subnormal in $G$ for each $\gamma \in \Gamma_r \setminus \Gamma_{r+1}$. The same argument shows that there exists a non-negative integer $s$, such that $X_\omega W_{s+1}$ is subnormal in $G$ for all $\omega$ in $\Omega_s \setminus \Omega_{s+1}$. Then also $X_\lambda = X_\gamma V_{r+1} \cap X_\omega W_{s+1}$ is subnormal in $G$ and the proof is complete. $\square$

**Lemma 2.** Let $G$ be a group with a subnormal deviation and let $X$ be a subgroup of $G$. If $X$ contains a normal subgroup $Y$, such that $X/Y$ is the direct product of infinitely many non-trivial subgroups, then $X$ is subnormal in $G$.

**Proof.** Let
\[ X/Y = \prod_{\lambda \in \Lambda} X_\lambda/Y, \]
where the set $\Lambda$ is infinite and each $X_\lambda/Y$ is a non-trivial subgroup of $X/Y$. Obviously, we may consider two infinite subsets $\Gamma$ and $\Omega$ of $\Lambda$, such that $\Gamma \cup \Omega = \Lambda$ and $\Gamma \cap \Omega = \emptyset$. Put
\[ V/Y = \prod_{\gamma \in \Gamma} X_\gamma/Y \quad \text{and} \quad W/Y = \prod_{\omega \in \Omega} X_\omega/Y. \]

Application of Lemma 1 to the direct products
\[ (V/Y) \times \left( \prod_{\omega \in \Omega} X_\omega/Y \right) \quad \text{and} \quad (W/Y) \times \left( \prod_{\gamma \in \Gamma} X_\gamma/Y \right) \]
yields that the subgroups $V$ and $W$ are subnormal in $G$. Since $VW = WV$, it follows that also $X = VW$ is subnormal in $G$. $\square$

**Lemma 3.** Let $A$ be a periodic abelian group which is not Černikov. Then, $A$ contains a subgroup $B$ such that $A/B$ is the direct product of infinitely many non-trivial subgroups.

**Proof.** Of course, it can be assumed that the set $\pi(A)$ is finite, so that there exists a prime number $p$, such that the $p$-component $A_p$ of $A$ is not Černikov. Write $A_p = D_p \times R_p$, where $D_p$ is divisible and $R_p$ is reduced. Since $D_p$ is a direct factor of $A$, we may also suppose that $D_p$ is the direct product of only finitely many Prüfer subgroups. Then $R_p$ is infinite
and so also \( R_p / R_p^0 \) is infinite. It follows that \( A / R_p^0 \) is the direct product of infinitely many non-trivial subgroups. □

**Corollary 2.** Let \( G \) be a group with a subnormal deviation and let \( X \) be a subgroup of \( G \). If \( X \) contains a normal subgroup \( Y \) such that \( X/Y \) is periodic abelian but not Černikov, then \( X \) is subnormal in \( G \).

**Proof.** It follows from Lemma 3 that \( X/Y \) contains a subgroup \( Z/Y \), such that \( X/Z \) is the direct product of infinitely many non-trivial subgroups. Then \( X \) is subnormal in \( G \) by Lemma 2. □

**Lemma 4.** Let \( G \) be a group and let \( X, Y, Z \) be subgroups of \( G \), such that \( Y \leq X \) and \( Y \) is normal in \( ⟨X, Z⟩ \). \( X/Y \) is the direct product of an infinite collection \( (X_\lambda/Y)_\lambda \) of non-trivial \( Z \)-invariant subgroups and \( X \cap Z \leq Y \). If the interval \( [ZX/ZY] \) has a \( G \)-subnormal deviation, then the subgroup \( ZY \) is subnormal in \( G \).

**Proof.** We can repeat the argument of the proof of Lemma 1 with a few slight modifications. Use the same notation of that proof, but take the infinite subsets \( \Gamma \) and \( \Omega \) of \( \Lambda \) in such a way that \( \Gamma \cup \Omega = \Lambda \) and \( \Gamma \cap \Omega = \emptyset \). For each positive integer \( n \), put

\[
V_n/Y = (ZY/Y, \langle X_\gamma/Y \mid \gamma \in \Gamma_n \rangle) = (ZY/Y) \rtimes \left( \bigoplus_{\gamma \in \Gamma_n} X_\gamma/Y \right)
\]

and

\[
W_n/Y = (ZY/Y, \langle X_\omega/Y \mid \omega \in \Omega_n \rangle) = (ZY/Y) \rtimes \left( \bigoplus_{\omega \in \Omega_n} X_\omega/Y \right).
\]

The proof can now be completed as in Lemma 1, just replacing the interval \( [X/Y] \) by the interval \( [ZX/ZY] \). □

The following easy result is well known.

**Lemma 5.** Let \( G \) be a group and let \( A \) be an abelian normal subgroup of \( G \) of prime exponent. If \( G/C_G(A) \) is finite, then there exists a collection \( (A_n)_{n \in \mathbb{N}} \) of finite non-trivial \( G \)-invariant subgroups of \( A \), such that \( \langle A_n \mid n \in \mathbb{N} \rangle = \text{Dr}_{n \in \mathbb{N}} A_n \).

If \( G \) is any group, the **Baer radical** of \( G \) is the subgroup generated by all abelian subnormal subgroups of \( G \), and \( G \) is called a **Baer group** if it coincides with its Baer radical. Thus, any Baer group is locally nilpotent and all its finitely generated subgroups are subnormal. Our next lemma shows that if a group \( G \) has a subnormal deviation and contains a periodic abelian subgroup which is not Černikov, then all the elements of finite order of \( G \) belong to the Baer radical.

**Lemma 6.** Let \( G \) be a group with a subnormal deviation. If \( G \) contains a periodic abelian subgroup \( A \) which is not Černikov, then all finite cyclic subgroups of \( G \) are subnormal.

**Proof.** The subgroup \( A \) is subnormal in \( G \) by Corollary 2 and so it is contained in the Baer radical \( B \) of \( G \). Thus, the subgroup \( T \) of all elements of finite order of \( B \) is not Černikov. Let \( g \) be any element of finite order of \( G \). Then \( ⟨g, T⟩ \) is a periodic locally soluble group and, hence, it follows from a result of Zaicev [13] that it contains an abelian \( ⟨g⟩ \)-invariant subgroup \( C \) which is not Černikov. Of course, \( C \) can be chosen either of prime exponent or with infinitely many non-trivial primary components, and so by Lemma 5 in both cases there exists an infinite collection \( (C_n)_{n \in \mathbb{N}} \) of non-trivial \( ⟨g⟩ \)-invariant subgroups of \( C \), such that

\[
C_0 = \langle C_n \mid n \in \mathbb{N} \rangle = \text{Dr}_{n \in \mathbb{N}} C_n
\]
and \( \langle g \rangle \cap C_0 = \{1\} \). We can now apply Lemma 4 for \( X = C_0, Y = \{1\} \), and \( Z = \langle g \rangle \) to obtain that the subgroup \( \langle g \rangle \) is subnormal in \( G \). \( \square \)

It is well known that any divisible abelian normal subgroup of a periodic nilpotent group \( G \) lies in the centre \( \zeta(G) \) of \( G \) (see for instance [11] Part 1, Lemma 3.13). For our purposes, we need the following slight generalization of this result.

**Lemma 7.** Let \( G \) be a periodic Baer group and let \( D \) be a divisible abelian subnormal subgroup of \( G \). Then \( D \leq \zeta(G) \).

**Proof.** Let \( g \) be any element of \( G \). Since \( \langle g \rangle \) is subnormal in \( G \) and the normal closure \( D^G \) is abelian (see [11] Part 1, Lemma 4.46), the subgroup \( \langle D, g \rangle \leq D^G \langle g \rangle \) is nilpotent. Thus, \( D \leq \langle D^G, g \rangle \leq \zeta(\langle D, g \rangle) \) and hence \( [D, g] = \{1\} \). Therefore, \( D \leq \zeta(G) \). \( \square \)

The following result is crucial for our purposes.

**Lemma 8.** Let \( G \) be a group with a subnormal deviation, \( D \) a subgroup of \( G \) which is the direct product of finitely many Prüfer subgroups and \( A \) a periodic abelian \( D \)-invariant subgroup of \( G \). If \( A \) is not Černikov, then \( D \) is subnormal in \( G \).

**Proof.** Assume for a contradiction that the subgroup \( D \) is not subnormal in \( G \). Then \( A \) cannot be the direct product of an infinite collection of \( D \)-invariant subgroups, since otherwise \( D \) would be subnormal in \( G \) as an application of Lemma 4 for \( X = A, Y = D \cap A \), and \( Z = D \). In particular, the set of primes \( \pi(A) \) must be finite and, hence, there is a prime number \( p \), such that the \( p \)-component \( A_p \) of \( A \) is not Černikov; clearly, the socle \( S \) of \( A_p \) is likewise infinite and so the replacement of \( A \) by \( S \) allows us to suppose that \( A \) is an elementary abelian \( p \)-group.

The subgroup \( DA \) is a Baer group by Lemma 6. Assume that \( D \) is subnormal in \( DA \), so that \( D \leq \zeta(DA) \) by Lemma 7. Then \( DA \) is abelian and, hence, it is subnormal in \( G \) by Corollary 2, which is clearly impossible because \( D \) is not subnormal in \( G \). Therefore, \( D \) cannot be subnormal in \( DA \), so that also \( DA \) is a counterexample to the statement and, hence, we may suppose without loss of generality that \( G = DA \). Then \( G \) is a Baer group and, in particular, it is locally nilpotent. The intersection \( D \cap A \) is a finite normal subgroup of \( G \), and we may be replace \( G \) by the counterexample \( G/(D \cap A) \) so assume \( D \cap A = \{1\} \). Moreover, \( DC_A(D) \) is not subnormal in \( G \), while it follows from Lemma 4 that the centralizer \( C_A(D) \) is finite, so that we may further replace \( G \) by \( G/C_A(D) \) and assume \( C_A(D) = \{1\} \).

Of course, \( D \) can be written as the union of an ascending chain

\[
\{1\} = D_0 \leq D_1 \leq \ldots \leq D_n \leq D_{n+1} \leq \ldots
\]

of finite subgroups. If \( a \) is any non-trivial element of \( A \), the finite subgroup \( \langle D_n, a \rangle \) is nilpotent and, hence, \( A \cap \zeta(\langle D_n, a \rangle) \neq \{1\} \). It follows that \( A_n = C_A(D_n) \) is a non-trivial \( D \)-invariant subgroup for each \( n \). Consider now the descending chain

\[
A = A_0 \geq A_1 \geq \ldots \geq A_n \geq A_{n+1} \geq \ldots
\]

and note that

\[
\bigcap_{n \in \mathbb{N}} A_n \leq C_A(D) = \{1\}.
\]

In particular, the above chain cannot stop after finitely many steps and so each \( A_n \) is infinite.

Among all counterexamples choose the group \( G \) and its non-subnormal subgroup \( D \) in such a way that the set of all subgroups of \( G \) containing \( D \) has a minimal \( G \)-subnormal deviation \( \delta \). Suppose first \( \delta = 0 \). Then the set \( \{DA_n \mid n \in \mathbb{N} \} \) satisfies the minimal condition on its members which are not subnormal in \( G \) and, hence, there is a positive
integer $m$, such that $DA_n$ is subnormal in $G$ for all $n \geq m$. Application of Lemma 7 yields that $DA_n/A_n$ is contained in the centre of $DA/A_n$ for all $n \geq m$, so that

\[
[D, A] \leq \bigcap_{n \geq m} A_n = \{1\},
\]

which is, of course, impossible. Therefore, there exists a positive integer $t$ such that, for each $n \geq t$, the interval $[DA_n/DA_{n+1}]$ has a $G$-subnormal deviation strictly smaller than $\delta$. It follows that $DA_{n+1}$ is subnormal in $DA_n$ for each $n \geq t$. In fact, if $A_n/A_{n+1}$ is finite, we have that $DA_{n+1}$ has finite index in $DA_n$, while if $A_n/A_{n+1}$ is infinite the same conclusion follows from the minimal assumption on $\delta$. Thus, $DA_{t+k}$ is subnormal in $DA_t$ for each positive integer $k$ and hence $[D, A_t] \leq A_{t+k}$ by Lemma 7. Thus,

\[
[D, A_t] \leq \bigcap_{k \in \mathbb{N}} A_{t+k} = \{1\},
\]

a final contradiction that completes the proof. \(\square\)

**Lemma 9.** Let $G$ be a group with a subnormal deviation and let $N$ be a periodic soluble normal subgroup of $G$. If $N$ is contained in the Baer radical of $G$, then all subgroups of $N$ are subnormal in $G$.

**Proof.** Assume for a contradiction that the statement is false, and among all counterexamples for which $N$ has smallest derived length choose one such that $N$ contains a non-subnormal subgroup $X$ whose derived length is minimal possible. Let $A$ be the smallest non-trivial term of the derived series of $N$. It follows from the minimality of the derived length of $N$ that $XA$ is subnormal in $G$, so $X$ cannot be subnormal in $XA$. Moreover, the minimality of the derived length of $X$ yields that the subgroup $Y = X'$ is subnormal in $G$ of, say, defect $k$. The intersection $X \cap A$ is a normal subgroup of $XA$ and the factor group $XA/(X \cap A)$ is still a minimal counterexample, so that without loss of generality we may suppose $G =XA$ and $X \cap A = \{1\}$; in particular, $G$ is a Baer group.

Put $A_0 = A$ and

\[
A_i = [A, Y, \ldots, Y]
\]

for each positive integer $i \leq k$. Clearly every $A_i$ is a normal subgroup of $G$ and $A_k = \{1\}$, so that

\[
G = XA_0 \geq XA_1 \geq \ldots \geq XA_k = X
\]

is a chain running from $G$ to $X$. It follows that there exists a positive integer $j \leq k$, such that $XA_j$ is not subnormal in $XA_{j-1}$. Of course, the subgroup $YA_j$ is contained in $XA_j$ and normal in $XA_{j-1}$, whence the factor group $XA_{j-1}/YA_j$ is likewise a minimal counterexample. Thus, we can replace $G$ by $XA_{j-1}/YA_j$ and $X$ by $XA_j/YA_j$, which allows us to suppose that $X$ is abelian.

Since $X$ is not subnormal in $G$, it follows from Corollary 2 that $X$ is a Černikov group. Let $J$ be the largest divisible subgroup of $X$ and let $E$ be a finite subgroup, such that $X = JE$. Clearly, $E$ is subnormal in $G$, because $G$ is a Baer group, so that $J$ is not subnormal in $G$ and, hence, not even in the normal subgroup $JA$ of $G$. It follows that $JA$ is not Černikov, so neither $A$ is a Černikov group. However, in this case $J$ is subnormal in $G$ by Lemma 8, and this contradiction completes the proof. \(\square\)

We can now prove our main result.

**Proof of Theorem 1.** Suppose that $G$ is not a Černikov group, so that it contains an abelian subgroup $A$ which is not Černikov (see [11] Part 1, Theorem 3.45). Then, it follows from Lemma 6 that $G$ is a Baer group and, hence, all its subgroups are subnormal by Lemma 9. \(\square\)
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References
1. Heineken, H.; Mohamed, I.J. A group with trivial centre satisfying the normalizer condition. J. Algebra 1968, 10, 368–376. [CrossRef]

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