**Global Boundedness in a Logarithmic Keller–Segel System**

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Abstract: In this paper, we propose a user-friendly integral inequality to study the coupled parabolic chemotaxis system with singular sensitivity under the Neumann boundary condition. Under a low diffusion rate, the classical solution of this system is uniformly bounded. Our proof relies on the construction of the energy functional containing \( \int_{\Omega} |v|^q \) with \( q > 0 \). It is noteworthy that the inequality used in the paper may be applied to study other chemotaxis systems.

Keywords: chemotaxis model; energy functional; integral inequality; global uniform boundedness

MSC: 35A01; 35A02

1. Introduction

Our work considers the coupled parabolic chemotaxis system with singular sensitivity

\[
\begin{aligned}
 u_t &= \nabla \cdot (\nabla u - \chi u \nabla v), \quad x \in \Omega, t > 0, \\
v_t &= k \Delta v - v + u, \quad x \in \Omega, t > 0, \\
\partial_n u &= \partial_n v = 0, \quad x \in \partial \Omega, t > 0, \\
u(x, 0) &= u_0, \quad v(x, 0) = v_0, \quad x \in \Omega,
\end{aligned}
\]

(1)

for parameters \( \chi, k > 0 \) with the Neumann boundary condition, where \( \Omega \subset \mathbb{R}^2 \) is a bounded domain with a smooth boundary. \( u \) and \( v \) are the cell density and concentration of chemical stimulus with respect to time \( t \) and \( x \), respectively. \( k \) represents the diffusion rate of the chemical signal. The initial functions \( u_0 \in C^0(\Omega) \) and \( v_0 \in W^{1,\infty}(\Omega) \) satisfy \( u_0 \geq 0 \) and \( v_0 > 0 \).

In 1970, Keller and Segel [1] originally introduced the system

\[
\begin{aligned}
 u_t &= \nabla \cdot (\nabla u - u \chi(v) \nabla v), \quad x \in \Omega, t > 0, \\
\tau v_t &= k \Delta v - a v + \beta u, \quad x \in \Omega, t > 0,
\end{aligned}
\]

(2)

to describe chemotaxis, the oriented movement of cells in response to the concentration of chemical signal produced by themselves and self-diffusion, where \( \tau, k, \alpha, \beta > 0 \) are parameters. The chemical signal experiences random diffusion and decay. Particular cases and derivatives of chemotaxis models have been developed extensively, such as the parabolic–elliptic case [2–5], the fully parabolic case [6–10] and other extensive versions [11–13]. Some studies have focused on the problem of whether the solution to the respective model undergoes a chemotactic collapse in the sense that the cell density becomes unbounded in finite or infinite time [3,6,7,12].
If \( \chi(v) = \chi \) with \( \tau = k = \alpha = \beta = 1 \), Osaki and Yagi \([14]\) showed the global boundedness of solutions to (2) for \( n = 1 \) and Nagai et al. \([15]\) proved the results if \( \int_{\Omega} u_0 < 4\pi \) for \( n = 2 \). For \( n \geq 3 \), if \( \| u_0 \|_{L^\frac{n}{2} (\Omega)} \) is small enough, there exist global weak solutions \([16]\). Another form of sensitivity function is

\[
\chi(v) = \frac{\chi_0}{(c + \alpha v)^k}
\]

for \( c, \chi_0 > 0, k > 1 \) and \( \alpha > 0 \), which is non-singular. In this case, the global existence is established for \( k = 2, c = 1 \) by \([17]\) and for \( k = 1, \alpha = 1 \) by \([12]\). Furthermore, if \( \chi(v) = \frac{v^p}{v^p + 1} \) for \( k > 1, \chi_0 > 0 \), there exist global classical solutions to (2) \([18]\).

The logarithmic sensitivity function \( \chi(v) = \frac{v}{\ln(v)} \) with \( \chi > 0 \) is commonly considered because it is in compliance with the Weber–Fechner law \([19]\). Taking this form with \( \tau = k = \alpha = \beta = 1 \), the chemotaxis model becomes the classical version:

\[
\begin{align*}
\dot{u}_t &= \nabla \cdot (\nabla u - \chi u \nabla v), & x \in \Omega, t > 0, \\
\dot{v}_t &= \Delta v - v + u, & x \in \Omega, t > 0, \\
\partial_n u &= \partial_n v = 0, & x \in \partial \Omega, t > 0, \\
u(x, 0) &= u_0, & v(x, 0) = v_0, & x \in \Omega.
\end{align*}
\]

Global bounded solutions to (3) are provided by Osaki and Yagi \([14]\) in a one-dimensional case. As for \( n = 2 \), Lankeit \([7]\) introduced an energy functional and proved that the solutions are uniformly bounded in a convex domain with the range of \( \chi \) extending to slightly more than one. Moreover, Winkler \([20]\) proved that there exist global classical solutions if \( 0 < \chi < \sqrt{\frac{2}{n}} \), and Fujie \([6]\) showed the solutions are uniformly time bounded. In \([21]\), global bounded solutions are constructed under the the condition of \( \chi \leq \frac{4}{n} \) with \( \Omega \subset \mathbb{R}^n \) being the convex domain. Furthermore, (3) employs global weak solutions when \( \chi < \sqrt{\frac{n+2}{3n-4}} \) \([20]\). In the radially symmetric setting, weak solutions are constructed by \([22]\) under the condition \( \chi < \sqrt{\frac{n^2}{n^2-2}} \). These results imply that there is a balance between \( \chi \) and dimension \( n \) for the establishment of global solutions to classic models (3). The work to extend both \( \chi \) and \( n \) is laborious without giving any condition of (3). Lankeit and Winkler \([23]\) extended the range of \( \chi \) to

\[
\chi < \begin{cases} 
\infty & \text{if } n = 2, \\
\sqrt{\frac{8}{3}} & \text{if } n = 3, \\
\frac{n}{n-2} & \text{if } n \geq 4
\end{cases}
\]

under the definition of the generalized solution, which is constructed on the basis of the global weak solution.

There are also other results established on the changing of parameters, referring to \([9,24]\). Indeed, the parameters in (2) have an impact on the aggregation of cell density. Xiang-dong \([25]\) constructed global solutions to (1) with \( n \leq 8 \) under some conditions, where the relationship between \( k \) and \( \chi \) is established. However, if \( n = 2 \), the diffusion rate of the concentration of chemicals \( k \) does not work, since \( \chi \) is still less than one, as in \([25]\).

In \([26]\), the estimates containing \( \int_{\Omega} |\nabla v|^2 \) are established to study the system where the chemotactic sensitivity is a constant and the source of the signal is modeled by \( v \). In the work of Winkler \([27]\), the only evident global quasi-dissipative structure involving \( \int_{\Omega} |\nabla v|^2 (v > 0) \) is established to address the difficulty brought about by the nonlinear source of signal. However, the system with logarithmic sensitivity presents more challenges,
and the structure of \( \int_{\Omega} f(v) |\nabla v|^{n} \) (n is even) is essential to the estimates. Hence, motivated by Lankeit [7] and Nagai [15], we establish an energy-type functional containing
\[
\int_{\Omega} \frac{|\nabla v|^{4}}{v^{2}}.
\]

The fractional term of \( v \) in the energy-type functional may alleviate the difficulty of preventing the aggregation caused by nonlinear kinetics in some derivative systems such as [27,28], where the source of the signal is modeled by \( uv \).

In this paper, the global existence and uniform boundedness of the classical solutions of (1) are established as follows:

**Theorem 1.** Let \( \Omega \) be a bounded domain with a smooth boundary \( \partial \Omega \) on \( \mathbb{R}^{2} \), initial data \( v_{0} > 0 \) and \( u_{0} \geq e \) in \( \Omega \) with \( u_{0} \in C^{0}(\bar{\Omega}) \) and \( v_{0} \in W^{1,\infty}(\Omega) \). For all \( \chi > 0 \), there exists a constant \( C_{k} \) that depends on \( u_{0}, v_{0}, \Omega \) and \( \chi \), such that whenever

\[
k \geq C_{k},
\]

then (1) admits a unique classical solution \((u, v) \in C^{0}(\Omega \times [0, \infty)) \cap C^{1}(\Omega \times (0, \infty))\). Moreover, there exist constants \( \delta, C > 0 \) such that \( \delta \leq v < C \) and \( 0 \leq u < C \) for all \( t \in (0, \infty) \).

Intuitively, this shows that the large diffusion rate of chemical signals can prevent the aggregation of cell density resulting from a large \( \chi \).

In the paper, we first demonstrate the local existence of and recall some inequalities in the preliminaries. Then, we prove our key integral inequality in the Section 3 and give some useful a priori estimates in the Section 4. Finally, we prove the uniform boundedness of the solutions.

2. Preliminaries

2.1. Local Existence

The local existence of classical solutions to chemotaxis systems has been well-established using the methods of standard parabolic regularity theory and an appropriate fixed-point framework, which is shown in the following. Details of proof can be seen in Theorem 2.1 of [7] or [20].

**Proposition 1.** Let \( \Omega \subset \mathbb{R}^{n} \) be a bounded domain with a smooth boundary, and \( u_{0} \in C^{0}(\bar{\Omega}) \) and \( v_{0} \in W^{1,q}(\Omega), q > n + 1 \) are non-negative; then, for any \( k, \chi > 0 \), there exists \( T_{\max} \in (0, \infty) \) and a pair of unique non-negative solutions satisfying

\[
\begin{cases}
\quad u \in C^{0}(\Omega \times [0, T_{\max})) \cap C^{2,1}(\Omega \times (0, T_{\max})), \\
\quad v \in C^{0}(\Omega \times [0, T_{\max})) \cap C^{2,1}(\Omega \times (0, T_{\max})) \cap L^{\infty}_{\loc}([0, T_{\max}); W^{1,q}(\Omega)),
\end{cases}
\]

such that \((u, v)\) solves (1) classically in \( \Omega \times [0, T_{\max}) \) and, moreover, if \( T_{\max} < \infty \), then

\[
\lim_{t \to T_{\max}} \|u(\cdot, t)\|_{L^{\infty}(\Omega)} + \|v(\cdot, t)\|_{W^{1,q}(\Omega)} = \infty.
\]

2.2. The Positive Lower Boundedness of \( v \)

In order to prove the lower boundedness of \( v \) in (1), we first prove the boundedness of \( \|u\|_{L^{1}} \) and \( \|v\|_{L^{1}} \). Integrating the first and the second PDE in (1), we have the mass identities

\[
\int_{\Omega} u = \int_{\Omega} u_{0} =: m, \quad t > 0
\]

and

\[
\int_{\Omega} v = \int_{\Omega} v_{0} + \left( \int_{0}^{t} \int_{\Omega} v_{0} - \int_{\Omega} u_{0} \right) e^{-t}, \quad t > 0.
\]
Based on these facts, one can deduce the non-negative lower boundedness of $v$ from the abstract representation formula of the $v$ equation. Copying Lemma 2.2 of [7], we write it as follows:

**Lemma 1.** Let $(u, v)$ satisfy Proposition 1; then, there exists $T_{\text{max}} > 0$ and a positive constant $\delta$ depending on $v_0$ such that

$$v(x, t) \geq \delta > 0, \forall (x, t) \in \bar{\Omega} \times [0, T_{\text{max}}). \quad (4)$$

**Proof.** Firstly, by the comparison principle and the fact of $v_0 > 0$ on $\bar{\Omega}$, we have for a small $t$

$$v(x, t) \geq \min_{x \in \bar{\Omega}} v_0 \cdot e^{-t} > 0.$$

Let us fix $\tau = \tau(u_0, v_0)$. Then, it follows that

$$v(x, t) \geq \min_{x \in \bar{\Omega}} v_0 \cdot e^{-\tau} := \delta_1 > 0, \forall t \in [0, \tau).$$

Now, from the well-known Neumann heat semigroup estimate for $e^{t\Delta}$ (see Lemma 1.3 in [29] and Lemma 2.2 in [20]), we denote by $d$ the diameter of the $\Omega$ and have for $\Omega \subset \mathbb{R}^2$

$$(e^{t\Delta} \omega) \geq \frac{1}{4\pi t} e^{-\frac{d^2}{4t}} \cdot \int_{\Omega} \omega > 0, \omega \in C^0(\Omega).$$

Then, the abstract representation formula of $v$ shows

$$v(\cdot, t) = e^{t(\Delta-1)}v_0 + \int_{\Omega} e^{(t-s)(\Delta-1)}u(\cdot, t)ds$$

$$\geq \int_{0}^{t} \frac{1}{4\pi (t-s)} e^{-((t-s)+\frac{d^2}{4(t-s)})} \left( \int_{\Omega} u(\cdot, t)ds \right)$$

$$\geq m \int_{0}^{t} \frac{1}{4\pi r} e^{-r+\frac{d^2}{4r}} dr := \delta_2 > 0, \forall t \in [\tau, \infty),$$

where $r := t - s$. Choosing $\delta = \min\{\delta_1, \delta_2\}$, we deduce (4). \(\square\)

### 2.3. Recall of Useful Theorems

The well-known general Young’s inequality [30] is recalled.

**Lemma 2.** Let $f, g \geq 0$ be the continuous function with $p, q > 0$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, then

$$f g \leq \epsilon f^p + \frac{1}{q} (\epsilon p)^{-\frac{q}{p}} g^q$$

holds for all $\epsilon > 0$. Moreover, for continuous $h > 0$ and any $\epsilon_1, \epsilon_2 > 0$, taking $p = 2, q = 3, r = 6$ such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, we have

$$f g h \leq \epsilon_1 f^2 + \frac{\epsilon_2}{4\epsilon_1} g^3 + \frac{\sqrt{6}}{36\epsilon_1 \sqrt{\epsilon_2}} h^6.$$  

**Proof.** In (7) is given the result of the straightforward calculation of the well-known inequality (6). \(\square\)
Lemma 3. Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$ be a smooth bounded domain. Any function $f \in C^2(\Omega)$ satisfies

1. $\nabla |\nabla f|^2 = 2\nabla f \cdot D^2 f$, \hspace{1cm} (8)
2. $(\Delta f)^2 \leq n|D^2 f|^2$, \hspace{1cm} (9)
3. $\nabla f \cdot \nabla \Delta f = \frac{1}{2} \Delta |\nabla f|^2 - |D^2 f|^2$. \hspace{1cm} (10)

All the identities and inequalities in the above lemma can be obtained from straightforward calculation. One can see \([7,31]\) and Lemma 3.1 in \([8]\) for their application. We could not find a precise reference in the literature that covers all that is necessary for our purpose; therefore, we conclude with a short lemma here.

3. A User-Friendly Integral Inequality

The proof of Theorem 1 is based on the extension and application of an integral inequality, which is generated within one dimension by Q. Wang \([28]\). The following theorem has a multidimensional form. It is worth noting that the integral inequality connects the fraction of the gradient and the second derivative. A similar inequality can be found in \([7]\). Furthermore, the explicit coefficient in the integral inequality is easy to use for readers.

Theorem 2. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain with $w > 0$ satisfying $w \in C^2(\bar{\Omega})$ and $\frac{\partial w}{\partial n} = 0$ on $\partial \Omega$. Then,

$$\int_{\Omega} \frac{|\nabla w|^{2p+2}}{w^{q+2}} \leq \frac{n + 4pe}{2q + 1 - \frac{p}{e}} \int_{\Omega} \frac{|D^2 w|^2 |\nabla w|^{2p-2}}{w^q}$$

(11)

for all $p \geq 1$, $q > -\frac{1}{2}$ and $e > \frac{p}{2q+1} > 0$.

Proof. Let $J := \int_{\Omega} |\Delta \log w|^2 \frac{|\nabla w|^{2p-2}}{w^{q+2}} > 0$ for $p \geq 1$. Directly calculating $|\Delta \log w|^2$ leads to

$$J = \int_{\Omega} \frac{|\Delta w|^2 |\nabla w|^{2p-2}}{w^q} - 2 \int_{\Omega} \frac{|\nabla w|^{2p-2} |\Delta w|}{w^{q+1}} + \int_{\Omega} \frac{|\nabla w|^{2p+2}}{w^{q+2}}. \hspace{1cm} (12)$$

Since $\frac{\partial w}{\partial n} = 0$ on $\partial \Omega$, integration by parts gives

$$J_0 = 2 \int_{\Omega} \nabla |\nabla w|^{p} \cdot \nabla w = 2(q + 1) \int_{\Omega} \frac{|\nabla w|^{2p+2}}{w^{q+2}}$$

$$= 2p \int_{\Omega} \frac{|\nabla w|^{2p-2} |\nabla w|^{2} \cdot \nabla w}{w^{q+1}} - 2(q + 1) \int_{\Omega} \frac{|\nabla w|^{2p+2}}{w^{q+2}}. \hspace{1cm} (13)$$

By (8) of Lemma 3 and (6), we have for $e > 0$ that

$$J_0 = 4p \int_{\Omega} \frac{|\nabla w|^{2p} \cdot D^2 w}{w^{q+1}} - 2(q + 1) \int_{\Omega} \frac{|\nabla w|^{2p+2}}{w^{q+2}}$$

$$\leq 4pe \int_{\Omega} \frac{|\nabla w|^{2p-2} |D^2 w|^2}{w^q} - (2q + 1) - \frac{p}{e} \int_{\Omega} \frac{|\nabla w|^{2p+2}}{w^{q+2}}. \hspace{1cm} (13)$$

By (9), substituting (13) into (12) gives

$$J \leq (n + 4pe) \int_{\Omega} \frac{|\nabla w|^{2p-2} |D^2 w|^2}{w^q} - ((2q + 1) - \frac{p}{e}) \int_{\Omega} \frac{|\nabla w|^{2p+2}}{w^{q+2}}. \hspace{1cm} (11)$$

Due to $q > -\frac{1}{2}$, $e > \frac{p}{2q+1} > 0$; thus, $(2q + 1) - \frac{p}{e} > 0$, and we conclude with (11).
Remark 1. Letting $\Omega \subset \mathbb{R}^2$ and taking $q = p = 2, \epsilon > \frac{2}{5}$, then $\frac{n+4\epsilon}{2q+1-\epsilon} = \frac{2+8\epsilon}{5-\epsilon}$. Note that $\frac{2+8\epsilon}{5-\epsilon}$ achieves its global minimum over $(\frac{2}{5}, \infty)$ at $\epsilon = \frac{4+\sqrt{26}}{10}(\approx 0.9099)$ with the value $-\frac{2}{21-4\sqrt{26}}(\approx 3.3117)$. Therefore,

$$\int_{\Omega} \frac{|\nabla w|^6}{w^4} \leq \frac{2}{21-4\sqrt{26}} \int_{\Omega} \frac{|D^2 w|^2 |\nabla w|^2}{w^2}. \quad (14)$$

4. Some Useful A Priori Estimates

Let us first give an inequality to estimate the boundary integration.

Lemma 4. Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain. If $v \in C^2(\Omega)$ satisfies $\frac{\partial v}{\partial \nu} = 0$, the for any $\hat{\epsilon} > 0$, there exists $C(\hat{\epsilon})$ depending on $\Omega$ such that

$$\int_{\partial \Omega} \frac{|\nabla v|^2}{v} \frac{\partial (|\nabla v|^2)}{\partial v} \leq \hat{\epsilon} \int_{\Omega} \frac{|\nabla v|^2}{v} \frac{\partial (|\nabla v|^2)}{\partial v} + C(\hat{\epsilon}) \quad (15)$$

for all $t \in (0, T_{\text{max}})$ and $n \geq 1$.

Proof. Firstly, we show that

$$\int_{\partial \Omega} \frac{|\nabla v|^2}{v} \frac{\partial (|\nabla v|^2)}{\partial v} = 16 \int_{\partial \Omega} \frac{|\nabla \sqrt{v}|^2}{v} \frac{\partial (|\nabla \sqrt{v}|^2)}{\partial v}. \quad (16)$$

From the Neumann boundary condition, we calculate the right-hand side, obtaining

$$16 \int_{\partial \Omega} \frac{|\nabla \sqrt{v}|^2}{v} \frac{\partial (|\nabla \sqrt{v}|^2)}{\partial v} = \int_{\partial \Omega} \frac{|\nabla v|^2}{v} \frac{\partial (|\nabla v|^2)}{\partial v}$$

$$= \int_{\partial \Omega} \frac{|\nabla v|^2}{v} \frac{\partial (\sqrt{v}|\nabla \sqrt{v}|^2)}{\partial v} \cdot \frac{\partial \sqrt{v}}{v^2} = \int_{\partial \Omega} \frac{|\nabla v|^2}{v} \frac{\partial (|\nabla v|^2)}{\partial v}$$

for all $t \in (0, T_{\text{max}})$.

Now, according to (3.17) in [11], we have for any $\epsilon > 0$ and constant $C_\epsilon > 0$ depending on $\Omega$ that

$$\int_{\partial \Omega} \frac{|\nabla \sqrt{v}|^2}{v} \frac{\partial (|\nabla \sqrt{v}|^2)}{\partial v} \leq \frac{\epsilon}{v} \int_{\Omega} \frac{|\nabla \sqrt{v}|^2}{v} + C_\epsilon \quad \text{for all } t \in (0, T_{\text{max}}). \quad (17)$$

By straightforward calculation, we have

$$\int_{\Omega} \frac{|\nabla \sqrt{v}|^2}{v} \frac{\partial (|\nabla \sqrt{v}|^2)}{\partial v} = \frac{1}{16} \int_{\Omega} \frac{|\nabla \sqrt{v}|^2}{v} \left( \frac{|\nabla v|^2}{v} - \frac{\nabla v |\nabla v|^2}{v^2} \right)$$

$$= \frac{1}{16} \int_{\Omega} \frac{\left( \frac{|\nabla |\nabla v|^2 |\nabla \sqrt{v}|^2}{v^2} - 2 \frac{|\nabla \sqrt{v}|^2 |\nabla \sqrt{v}|^2}{v^2} + \frac{|\nabla v|^6}{v^4} \right)}{\left( 2\epsilon_1 + 1 \right) \frac{|\nabla \sqrt{v}|^2}{v^2} + \left( \frac{1}{2\epsilon_1} + 1 \right) \frac{|\nabla v|^6}{v^4}}$$

for $\epsilon_1 > 0$. Then, we have from (14) that

$$\int_{\Omega} \frac{|\nabla \sqrt{v}|^2}{v} \frac{\partial (|\nabla \sqrt{v}|^2)}{\partial v} \leq \frac{1}{16} \left( 2\epsilon_1 + 1 + \frac{\epsilon}{8\epsilon_1} + \frac{\epsilon}{4} \right) \int_{\Omega} \frac{|\nabla \sqrt{v}|^2}{v^2}$$

(18)
for all \( t \in (0, T_{\text{max}}) \), where \( \hat{\epsilon} = \frac{2}{21 - 4\sqrt{26}} \) for simplicity. Combining (18) and (17) with (16), we can obtain that
\[
\int_{\partial \Omega} \frac{\partial (|\nabla v|^2)}{\partial t} \leq \epsilon \left( 2\epsilon_1 + 1 + \frac{\epsilon}{8\epsilon_1} + \frac{\epsilon}{4} \right) \int_{\Omega} \frac{|\nabla |\nabla v|^{2}|}{v^2} + 16C_r.
\]

Denoting \( \hat{\epsilon} = \epsilon (2\epsilon_1 + 1 + \frac{\epsilon}{8\epsilon_1} + \frac{\epsilon}{4}) \) and \( C(\hat{\epsilon}) = 16C_r \), we prove (15) for any \( \hat{\epsilon} > 0 \). \( \Box \)

In preparation for the construction and estimation of energy-type functionals, some important a priori estimates are provided and collected into two lemmas in the following.

**Lemma 5.** Let \( k > 0 \) and \((u,v)\) be the solutions of (1) satisfying Proposition 1. Then, we have for any \( \hat{\epsilon} > 0 \) that
\[
\frac{d}{dt} \int_{\Omega} \frac{|\nabla v|^4}{v^2} + \int_{\Omega} \frac{|\nabla v|^4}{v^2} \leq - \left( \frac{4k}{3} - 2\hat{\epsilon} \right) \int_{\Omega} \frac{|\nabla v|^2 |D^2 v|^2}{v^2} - 2 \int_{\Omega} \frac{|\nabla v|^4 u}{v^3} + 4 \int_{\Omega} \frac{|\nabla v|^2 |\nabla v| u}{v^2} + C(\hat{\epsilon}).
\]

**Proof.** Through straightforward calculation, we can show
\[
\frac{d}{dt} \int_{\Omega} \frac{|\nabla v|^4}{v^2} = 4 \int_{\Omega} \frac{|\nabla v|^2 |\nabla \cdot \nabla v|}{v^2} - 2 \int_{\Omega} \frac{|\nabla v|^4 u}{v^3}
\]
\[
= 4k \int_{\Omega} \frac{|\nabla v|^2 |\nabla \cdot \nabla v|}{v^2} - 2 \int_{\Omega} \frac{|\nabla v|^4 u}{v^3} + 4 \int_{\Omega} \frac{|\nabla v|^2 |\nabla v| u}{v^2}.
\]

In light of (10), we have from (15) that
\[
I_1 = 2k \int_{\Omega} \frac{|\nabla v|^2 |\nabla v|^2}{v^2} - 4k \int_{\Omega} \frac{|\nabla v|^2 |D^2 v|^2}{v^2}
\]
\[
= 2k \int_{\partial \Omega} \frac{\partial (|\nabla v|^2)}{\partial t} - 2k \int_{\Omega} \nabla \left( \frac{|\nabla v|^2}{v^2} \right) \cdot \nabla |\nabla v|^2 - 4k \int_{\Omega} \frac{|\nabla v|^2 |D^2 v|^2}{v^2}
\]
\[
= 2k \int_{\partial \Omega} \frac{\partial (|\nabla v|^2)}{\partial t} - 2k \int_{\Omega} \frac{(|\nabla v|^2)^2}{v^2} \biggr|_{l_1}^{l_2} + 4k \int_{\Omega} \frac{|\nabla v|^2 |\nabla v| |\nabla v|^2}{v^2} - 4k \int_{\Omega} \frac{|\nabla v|^2 |D^2 v|^2}{v^2}
\]
\[
\leq - (12k - 2\hat{\epsilon}) \int_{\Omega} \frac{|\nabla v|^2 |D^2 v|^2}{v^2} + 4k \int_{\Omega} \frac{|\nabla v|^2 |\nabla v|^2 |\nabla v|^2}{v^2} + C(\hat{\epsilon}).
\]

Similarly, we calculate that
\[
I_2 = 2k \int_{\Omega} \frac{(|\nabla v|^4)^2}{v^3} = 4k \int_{\Omega} \frac{|\nabla v|^2 |\nabla v| |\nabla v|^2}{v^3} - 6k \int_{\Omega} \frac{|\nabla v|^6}{v^3}.
\]
Given by the sum of $I_2$ and $I_3$ and taking $\epsilon = \frac{1}{3}$, (6) implies that

$$I_2 + I_3 = 8k \int_\Omega \frac{|\nabla v|^2 \nabla \cdot \nabla v|}{v^2} - 6k \int_\Omega \frac{|\nabla v|^6}{v^4} \leq 8k \epsilon \int_\Omega \frac{(\nabla |\nabla v|)^2}{v^2} + \left( \frac{2k}{\epsilon} - 6k \right) \int_\Omega \frac{|\nabla v|^6}{v^4}$$

(22)

Substituting (22) and (21) into (20), we finish the proof by taking the first identity of Lemma 3.

**Lemma 6.** Supposing that $(u, v)$ solves (1) and all conditions of Proposition 1 hold, then there exist small $\epsilon_1, \epsilon_2 > 0$ and $\delta > 0$ such that

$$\frac{1}{2} \frac{d}{dt} \int_\Omega u^2 \leq - \left( 1 - \chi \epsilon_1 \delta^* \right) \int_\Omega |\nabla u|^2 + \frac{\chi \epsilon_2}{4 \epsilon_1} \int_\Omega u^3 + \frac{\chi \sqrt{6}}{36 \epsilon_1 \sqrt{\epsilon_2}} \int_\Omega \frac{|\nabla v|^6}{v^4}.$$  

(23)

**Proof.** In light of the $u$ equation of (1) and integration by parts, we can show that

$$\frac{1}{2} \frac{d}{dt} \int_\Omega u^2 = \int_\Omega u \nabla \cdot (\nabla u - \chi \frac{\nabla v}{v} u) = - \int_\Omega |\nabla u|^2 + \int_\Omega \chi u \frac{\nabla u \cdot \nabla v}{v}.$$  

(24)

The employment of (7) implies

$$\int_\Omega \chi u \frac{\nabla u \cdot \nabla v}{v} \leq \chi \epsilon_1 \int_\Omega \frac{|\nabla u|^2}{v^5} + \frac{\chi \epsilon_2}{4 \epsilon_1} \int_\Omega u^3 + \frac{\chi \sqrt{6}}{36 \epsilon_1 \sqrt{\epsilon_2}} \int_\Omega \frac{|\nabla v|^6}{v^4}$$

(25)

for small $\epsilon_1, \epsilon_2 > 0$. Note that $v$ has the lower bound for any $t > 0$. Let $\delta^* := \delta^{-\frac{3}{2}}$ be the upper bound of $v^{-\frac{3}{2}}$ and substitute (25) into (24) to obtain (23).  

5. Uniform Boundedness

In this section, we shall finish the proof of Theorem 1. Firstly, we construct the energy functional and prove that each item of the functional is uniform bounded.

**Theorem 3.** For $\alpha > 0$, let $\mathcal{F}_\alpha(u, v)$ take the following form:

$$\mathcal{F}_\alpha(u, v) = \alpha \int_\Omega u^2 + \int_\Omega \frac{|\nabla v|^4}{v^2}.$$  

Then, for $\Omega \subset \mathbb{R}^2$ and any $\chi > 0$, there exists a constant $C_k(u_0, v_0, \Omega, \chi) > 0$ such that if $k > C_k(u_0, v_0, \Omega, \chi)$, then for some $C > 0$

$$\frac{d}{dt} \mathcal{F}_\alpha(u, v) + \mathcal{F}_\alpha(u, v) < C$$

for all $t \in (0, T_{\max})$.

(26)
\textbf{Proof.} Combining (19) and (23), we achieve
\[
\frac{d}{dt} \left( \alpha \int_{\Omega} u^2 + \int_{\Omega} \frac{|\nabla u|^4}{\vartheta^2} \right) + \left( \alpha \int_{\Omega} u^2 + \int_{\Omega} \frac{|\nabla u|^4}{\vartheta^2} \right)
\leq - (2\alpha - 2\alpha \chi \epsilon_1 \delta^*) \int_{\Omega} |\nabla u|^2 + \frac{\alpha \chi \epsilon_2}{2\epsilon_1} \int_{\Omega} u \frac{1}{\vartheta^2} - \frac{\alpha \chi \sqrt{6}}{18 \epsilon_1 \sqrt{\epsilon_2}} \int_{\Omega} \frac{|\nabla u|^6}{\vartheta^4} \tag{27}
\]
\[
- \left( \frac{4k}{3} - 2\hat{\epsilon} \right) \int_{\Omega} \frac{|\nabla u|^2 |D^2 u|^2}{\vartheta^2} - 2 \int_{\Omega} \frac{|\nabla u|^4}{\vartheta^3} + 4 \int_{\Omega} \frac{|\nabla u|^2 |D u \cdot \nabla u|}{\vartheta^5} + C(\hat{\epsilon}).
\]
\[
\int_{\Omega} u^2 \leq \eta \int_{\Omega} |\nabla u|^2 + C
\]
for some small \( \eta > 0 \) and
\[
I_1 = \frac{\alpha \chi \epsilon_2}{2\epsilon_1} \|u\|_{L^3(\Omega)}^3 \leq \frac{\alpha \chi \epsilon_2}{2\epsilon_1} (C_1 \|\nabla u\|_{L^2(\Omega)} + C_2),
\]
where \( C_1, C_2 > 0 \), depending on \( \|u_0\|_{L^1(\Omega)} \) and \( \Omega \). For \( I_3 \), we employ Lemma 2 to obtain
\[
I_3 \leq \frac{1}{e_3} \int_{\Omega} \frac{|\nabla u|^6}{\vartheta^4} + 4e_3 \int_{\Omega} |\nabla u|^2 \tag{29}
\]
for any \( e_3 > 0 \). Combining the first item of (29) with \( I_2 \) and employing (14), we have
\[
\left( \frac{\alpha \chi \sqrt{6}}{18 \epsilon_1 \sqrt{\epsilon_2}} + \frac{1}{e_3} \right) \int_{\Omega} \frac{|\nabla u|^6}{\vartheta^4} \leq \left( \frac{\alpha \chi \sqrt{6}}{18 \epsilon_1 \sqrt{\epsilon_2}} + \frac{1}{e_3} \right) \hat{\epsilon} \int_{\Omega} \frac{|\nabla u|^2 |D^2 u|^2}{\vartheta^2}, \tag{30}
\]
where we denote \( \hat{\epsilon} = \frac{2}{21 - 4\sqrt{26}} \) for simplicity. Thus, substituting (28)–(30) into (27) gives
\[
\frac{d}{dt} \left( \alpha \int_{\Omega} u^2 + \int_{\Omega} \frac{|\nabla u|^4}{\vartheta^2} \right) + \left( \alpha \int_{\Omega} u^2 + \int_{\Omega} \frac{|\nabla u|^4}{\vartheta^2} \right)
\leq - (2\alpha(1 - \chi \epsilon_1 \delta^* - e_2 C_1 - \eta) - 4e_3) \int_{\Omega} |\nabla u|^2 \tag{31}
\]
\[
- \left( \frac{4k}{3} - 2\hat{\epsilon} - \frac{\alpha \chi \sqrt{6}}{18 \epsilon_1 \sqrt{\epsilon_2}} \hat{\epsilon} - \frac{1}{e_3} \right) \int_{\Omega} \frac{|\nabla u|^2 |D^2 u|^2}{\vartheta^2} + C.
\]
Let \( \epsilon_1, \epsilon_2 \) and \( \epsilon_3 \) be small, such that \( \kappa_1 = 0 \). Then, taking a small \( \hat{\epsilon} \) such that \( \frac{2}{3} > \hat{\epsilon} \), we denote
\[
C_{(\epsilon_1, \epsilon_2, \epsilon_3)} := \frac{\alpha \chi \sqrt{6}}{6 \epsilon_1 \sqrt{\epsilon_2}} \hat{\epsilon} + \frac{3}{e_3(4 - 6\hat{\epsilon})} \hat{\epsilon} > 0,
\]
and let \( C_k \) depending on \( u_0, v_0, \Omega, \chi \) be the lower bound of \( C_{(\epsilon_1, \epsilon_2, \epsilon_3)} \) provided \( \kappa_1 = 0 \). Therefore, for any \( k \geq C_k > 0 \), we have \( \kappa_2 \geq 0 \) and can then deduce (26). \( \square \)
Theorem 4. Let \((u,v)\) be the solutions of (1) satisfying all conditions in Proposition 1. Then,
\[
\int_\Omega u^2(\cdot, t) \leq C \quad \text{and} \quad \int_\Omega |\nabla v(\cdot, t)|^2 \leq C, \tag{32}
\]
with \(t \in (0, T_{\text{max}})\).

Proof. According to (26), there is \(C > 0\) such that
\[
\int_\Omega u^2(\cdot, t) \leq C \quad \text{and} \quad \int_\Omega |\nabla v(\cdot, t)|^4 \leq C
\]
for all \(t \in (0, T_{\text{max}})\). From Young’s inequality and the Gagliardo–Nirenberg inequality, there exist \(\epsilon_4, \epsilon_{\text{GN}} > 0\) and \(C > 0\) such that
\[
\int_\Omega |\nabla v(\cdot, t)|^2 \leq C_4 \int_\Omega \frac{|\nabla v|^4}{v^2}(\cdot, t) + \epsilon_4 \int_\Omega u^2(\cdot, t)
\]
\[
\leq C_4 \int_\Omega \frac{|\nabla v|^4}{v^2}(\cdot, t) + \epsilon_4 \epsilon_{\text{GN}} \int_\Omega |\nabla v(\cdot, t)|^2 + C \tag{33}
\]
for all \(t \in (0, T_{\text{max}})\). Taking \(\epsilon_4 < \frac{1}{2\epsilon_{\text{GN}}}\), then we have \(\epsilon_4 \epsilon_{\text{GN}} < \frac{1}{2}\) and prove (32).

Proof of Theorem 1. Using the well-known Moser’s technique [32], the \(L^\infty\) boundedness of \(u\) follows from Theorem 4. Indeed, one can follow the estimates of Nagai [15] or directly employ Lemma 2.3 in [7] to prove the theorem.

6. Conclusions

Our paper proves the uniform boundedness of solutions of the chemotaxis system with singular sensitivity under a small diffusion rate of the chemical signal. We prove a user-friendly inequality that has certain parameters, and construct a new energy functional that is applicable to the double Keller–Segel model with nonlinear sources.

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References


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