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Higher Monotonicity Properties for Zeros of Certain Sturm-Liouville Functions

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Abstract: In this paper, we consider the differential equation $y'' + \omega^2\rho(x)y = 0$, where ω is a positive parameter. The principal concern here is to find conditions on the function $\rho^{-1/2}(x)$ which ensure that the consecutive differences of sequences constructed from the zeros of a nontrivial solution of the equation are regular in sign for sufficiently large ω. In particular, if $c_{\nu k}(\alpha)$ denotes the kth positive zero of the general Bessel (cylinder) function $C_\nu(x;\alpha) = J_\nu(x)\cos\alpha - Y_\nu(x)\sin\alpha$ of order ν and if $|\nu| < 1/2$, we prove that $(-1)^m\Delta^{m+2}c_{\nu k}(\alpha) > 0 \quad (m = 0,1,2,\ldots; k = 1,2,\ldots)$, where $\Delta a_k = a_{k+1} - a_k$. This type of inequalities was conjectured by Lorch and Szego in 1963. In addition, we show that the differences of the zeros of various orthogonal polynomials with higher degrees possess sign regularity.

Keywords: Sturm–Liouville equations; differences; zeros; completely monotonic functions; Bessel functions; orthogonal polynomials

MSC: 34B24; 33C10

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1. Introduction

We consider the differential equation

$$y'' + \omega^2\rho(x)y = 0, \quad a \le x \le b, \tag{1}$$

where ω is a positive parameter and $\rho(x)$ is a positive C^∞-function on the interval $[a,b]$. By a Sturm–Liouville function, we mean a nontrivial real solution of (1). Let $\{x_k(\omega)\}$ denote the ascending sequence of the zeros of a Sturm–Liouville function in the interval $[a,b]$. The Sturm comparison theorem (see, e.g., p. 314 of [1] or p. 56 of [2]) states that the second differences of the sequence $\{x_k(\omega)\}$ are all positive if $\rho'(x) < 0$ and are all negative if $\rho'(x) > 0$. Our main purpose here is to move beyond the second differences and to show that higher consecutive differences of sequences constructed from $\{x_k(\omega)\}$ are regular in sign. Lorch and Szego [2] initiated the study of the sign regularity of higher differences of the sequences associated with Sturm–Liouville functions. In particular, if $c_{\nu k}(\alpha)$ denotes the kth positive zero of the general Bessel (cylinder) function

$$C_\nu(x;\alpha) = J_\nu(x)\cos\alpha - Y_\nu(x)\sin\alpha,$$

they proved that

$$(-1)^m\Delta^{m+1}c_{\nu k}(\alpha) > 0 \quad (m = 0,1,2,\ldots; k = 1,2,\ldots), \tag{2}$$

for $|\nu| > 1/2$, and conjectured (p. 71 of [2]) that, on the basis of numerical evidence

$$(-1)^m\Delta^{m+2}c_{\nu k}(\alpha) > 0 \quad (m = 0,1,2,\ldots; k = 1,2,\ldots). \tag{3}$$

for $|\nu| < 1/2$.

The symbol $\Delta^m a_k$ means, as usual, the mth (forward) difference of the sequence $\{a_k\}$:

$$\Delta^0 a_k = a_k, \quad \Delta^m a_k = \Delta^{m-1} a_{k+1} - \Delta^{m-1} a_k \quad (m = 1, 2, \dots; k = 1, 2, \dots).$$

Note that $C_\nu(x; \alpha)$ is a solution of the equation

$$y'' + q(x)y = 0, \quad x \in (0, +\infty), \tag{4}$$

with $q(x) = 1 - (\nu^2 - (1/4))x^{-2}$. Because $q'(x) = 2(\nu^2 - (1/4))x^{-3}$, we can see that the Sturm comparison theorem provides the results (2) for $m = 1$ and (3) for $m = 0$. It is mentioned in [2] that the signs of the first M differences of zeros of a Sturm–Liouville function of (4) could be inferred from the signs of $q^{(m)}(x)$, $m = 1, 2, \dots, M$. Muldoon [3] made progress in (3), proving that (3) holds when $1/3 \le |\nu| < 1/2$ ([3], Corollary 4.2).

Our approach here is based on the ideas and results of [4], where the string equation $y'' + \lambda\rho(x)y = 0$ with $y(0) = y(1) = 0$ was considered. Using the eigenvalues and the nodal points, we constructed a sequence of piecewise continuous linear functions which converges to $\rho^{-1/2}(x)$ uniformly on $[0, 1]$. Moreover, we obtained a formula for derivatives of $\rho^{-1/2}(x)$ in terms of the eigenvalues and the differences of the nodal points.

The rest of this paper is organized as follows. In Section 2, we use the zeros $x_k(\omega)$ of a Sturm–Liouville function as nodes to obtain a difference-derivative theorem (Lemma 1). In addition, we provide asymptotic estimates for $\rho^{-1/2}(x_k(\omega))$ as $\omega \to +\infty$ (Lemma 3). Then, we are able to express the higher differences $\Delta^{m+1} x_k(\omega)$ in terms of the derivatives of $\rho^{-1/2}(x)$ at those zeros. Moreover, the expression can be used to determine the regular manner of these differences (Theorems 1 and 2). In addition, we construct sequences from $x_k(\omega)$, where all the mth differences have the same sign (Corollary 1). The proofs of Lemmas 1 and 3 rely on a system of interlaced inductions, which is presented in Section 5. In Section 3, we use an approximation process for the zeros of the general Bessel function to prove the conjecture of Lorch and Szego (Theorem 3). In Section 4, the zeros of various orthogonal polynomials with higher degrees are shown to share similar sign regularity (Theorems 4 and 5).

The notation used throughout is standard. A function $\varphi(x)$ is said to be M-monotonic (resp., absolutely M-monotonic) on an interval I if

$$(-1)^m \varphi^{(m)}(x) \ge 0 \quad (resp., \varphi^{(m)}(x) \ge 0), \quad (x \in I; m = 0, 1, \dots, M). \tag{5}$$

If (5) holds for $M = \infty$, then $\varphi(x)$ is said to be completely (resp., absolutely) monotonic on I. A sequence $\{a_k(\omega)\}$ depending on a positive parameter ω is said to be asymptotically M-monotonic (resp., asymptotically absolutely M-monotonic) if

$$(-1)^m \Delta^m a_k(\omega) \ge 0 \quad (resp., \Delta^m a_k(\omega) \ge 0), \quad (m = 0, 1, 2, \dots, M; k = 1, 2, \dots)$$

for sufficiently large ω.

Here, we should mention a number of recent studies related to this paper. In the proofs of Lemmas 1 and 3, we use the standard Taylor expansion of a function at the nodes. In fact, there have many different types of Taylor expansion; many interesting applications can be found in [5,6] and the references therein. The continuity of the coefficient function $\rho(x)$ ensures that the zeros of the solution of (1) have a regular asymptotic distribution. Readers interested in uniform distribution sequences can refer to [7]. Completely monotonic functions and sequences have specific representations, and arise in many research areas, such as moment problems and harmonic mappings. Interested readers can refer to [8–10] and the references therein.

2. Main Results

In this section, we consider the differential equation

$$y'' + \omega^2 \rho(x)y = 0, \quad a \le x \le b, \tag{6}$$

where ω is a positive parameter. We assume throughout that $\rho(x)$ is a positive C^∞-function on the interval $[a,b]$. The notation $f(x)$ is reserved for the function $\rho^{-1/2}(x)$. Let $y(x;\omega)$ be a nontrival real solution of (6) and let $x_1(\omega) < x_2(\omega) < \cdots$ be the zeros of $y(x;\omega)$ in the interval $[a,b]$. For $a \le x < b$, we denote by $k(x;\omega)$ the smallest positive integer k such that $x \le x_k(\omega)$. It is well known (see, e.g., [4,11]) that

$$\min_{[x_k(\omega),x_{k+1}(\omega)]} f \le \frac{\omega}{\pi}\Delta x_k(\omega) \le \max_{[x_k(\omega),x_{k+1}(\omega)]} f. \tag{7}$$

It follows that $\pi \min_{[a,b]} f \le \omega\Delta x_k(\omega) \le \pi \max_{[a,b]} f$. In particular, we have

$$\Delta x_k(\omega) = O(\omega^{-1}) \quad as \quad \omega \to +\infty. \tag{8}$$

Thus, by (7) and the continuity of f, we obtain $f(x) = \lim_{\omega\to+\infty} \frac{\omega}{\pi}\Delta x_{k(x;\omega)}(\omega)$, and for any fixed l,

$$\lim_{\omega\to+\infty} \frac{\Delta x_{k(x;\omega)+l}(\omega)}{\Delta x_{k(x;\omega)}(\omega)} = 1. \tag{9}$$

Note that (9) means that, because $\omega \to +\infty$, the sequence $x_k(\omega)$ behaves as if equally distributed.

If φ is m-times differentiable in $(t, t+md)$ and the lower derivatives of φ are continuous on $[t, t+md]$, a mean-value theorem ([12] p. 52, no. 98) for differences and derivatives states that there exists a δ such that

$$\Delta_d^m \varphi(t) = d^m \varphi^{(m)}(t+\delta md),$$

where $\Delta_d\varphi(t) = \varphi(t+d) - \varphi(t)$. It is interesting to look for a difference-derivative theorem which can express the differences of a smooth function on the sequence $\{x_k(\omega)\}$ in terms of its derivatives at this sequence. The following lemma provides such a result.

Lemma 1. *Let $x_k = x_k(\omega)$. If φ is a C^∞-function on $[a,b]$, then for $m = 1,2,\ldots,$*

$$\Delta^m \varphi(x_k) = O(\omega^{-m}). \tag{10}$$

Moreover,

$$\Delta^m \varphi(x_k) = \sum_{q=1}^{m} A_{q,k}^{(m)} \varphi^{(q)}(x_{k+m-q}) + O(\omega^{-m-1}), \tag{11}$$

where the coefficients $A_{q,k}^{(m)}$ satisfy the recurrence relation

$$A_{1,k}^{(m)} = \Delta^m x_k, \quad A_{q,k}^{(m)} = \sum_{r=q-1}^{m-1} \binom{m-1}{r} A_{q-1,k+m-1-r}^{(r)} \Delta^{m-r} x_k, \tag{12}$$

for $q = 2,3,\ldots,m$.

To prove Lemma 1, a more detailed investigation into the behaviour of $x_k(\omega)$ is required. We use the Prüfer method to achieve this purpose. For each nontrivial solution $y(x;\omega)$ of (6), we define the Prüfer angle $\theta(x;\omega)$ as follows:

$$\omega\rho^{1/2}(x)\cot\theta(x;\omega) = \frac{y'(x;\omega)}{y(x;\omega)}.$$

Then, $\theta(x;\omega)$ satisfies the differential equation

$$\theta'(x;\omega) = \omega\rho^{1/2}(x) + \frac{\rho'(x)}{4\rho(x)}\sin 2\theta(x;\omega). \tag{13}$$

If we specify the initial condition for $\theta(x;\omega)$ to be $\theta(a;\omega) = \theta_a(\omega)$ with $0 \leq \theta_a(\omega) < \pi$, then, by the standard results (see, e.g., [1] p. 315), we have

$$\theta(x_k(\omega);\omega) = k\pi, \tag{14}$$

and $k\pi \leq \theta(x;\omega) \leq (k+1)\pi$, $x \in [x_k(\omega), x_{k+1}(\omega)]$. Let $x_k = x_k(\omega)$. When integrating both sides of (13) from x_k to x_{k+1} and using (14), we find that

$$\pi = \omega \int_{x_k}^{x_{k+1}} \rho^{1/2}(x)dx + \int_{x_k}^{x_{k+1}} \frac{\rho'(x)}{4\rho(x)} \sin 2\theta(x;\omega)dx. \tag{15}$$

Taking the Taylor expansion of $(1/f)(x)$ at x_k and using (8), we obtain

$$\int_{x_k}^{x_{k+1}} \rho^{1/2}(x)dx = \sum_{r=0}^{m} \frac{(1/f)^{(r)}(x_k)}{(r+1)!}(\Delta x_k)^{r+1} + O(\omega^{-m-2}). \tag{16}$$

The estimate of the second integral in (15) is stated as the following lemma. Its proof consists of a reducible system of integrals which is provided in Appendix A.

Lemma 2. *Let $x_k = x_k(\omega)$. Then, for $m = 2, 3, \ldots$, we have*

$$\int_{x_k}^{x_{k+1}} \frac{\rho'(x)}{4\rho(x)} \sin 2\theta(x;\omega)dx = \sum_{r=0}^{m-2} \Delta F_r(x_k)\omega^{-r-1} + R_{m-2}(x_k), \tag{17}$$

where the functions F_r depend on $f = \rho^{-1/2}$ and

$$R_{m-2}(x_k) = O(\omega^{-m-1}). \tag{18}$$

Note that the first two functions F_r appearing in (17) are of the forms

$$F_0 = \frac{f'}{4} - \int \frac{(f')^2}{8f}dx \quad \text{and} \quad F_1 = 0. \tag{19}$$

For $m = 2, 3, \ldots$, using the estimates (16), (17) and (18), and multiplying (15) by $f(x_k)/\pi$, we find the estimate for $f(x_k)$:

$$f(x_k) = \frac{\omega}{\pi} \sum_{r=0}^{m} \frac{g_r(x_k)}{(r+1)!}(\Delta x_k)^{r+1} + \frac{1}{\pi} \sum_{r=0}^{m-2} (f\Delta F_r)(x_k)\omega^{-r-1} + O(\omega^{-m-1}), \tag{20}$$

where the functions $g_r = f(1/f)^{(r)}$ and $r = 0, 1, 2, \ldots, m$. Note that $g_0 = 1$. Moreover, if we apply the mth order difference operator to (20), we can find the estimates for differences of the function $f(x)$ at those zeros. Indeed, we have the following lemma.

Lemma 3. *Let $f(x)$ and $x_k = x_k(\omega)$ be the same as above. Then, for $m = 1, 2, 3, \ldots$, we have*

$$\Delta^m x_k = O(\omega^{-m}). \tag{21}$$

Moreover,

$$\Delta^m f(x_k) = \frac{\omega}{\pi}\Delta^{m+1} x_k + O(\omega^{-m-1}). \tag{22}$$

The proofs of Lemmas 1 and 3 are provided in Section 5.

Now, if we apply Lemma 1 to the function $f(x)$, then by (22), we have the estimate for the higher differences of $x_k = x_k(\omega)$:

$$\frac{\omega}{\pi}\Delta^{m+1} x_k = \sum_{q=1}^{m} A_{q,k}^{(m)} f^{(q)}(x_{k+m-q}) + O(\omega^{-m-1}). \tag{23}$$

Moreover, by using (8) and (12), and iterating (23) for m from 1 to M, then choosing a sufficiently large ω, we can ensure the monotonicity of the sequence $\{\Delta x_k(\omega)\}$ by f.

Theorem 1. *Let $x_k = x_k(\omega)$ and $f(x) = \rho^{-1/2}(x)$ be the same as above. If $f(x)$ is M-monotonic on the interval $[a, b]$, then the sequence $\{\Delta x_k(\omega)\}$ is asymptotically M-monotonic.*

Proof. Because

$$(-1)^m f^{(m)}(x) \geq 0 \quad (x \in [a, b]; m = 0, 1, 2, \ldots, M), \tag{24}$$

it suffices to show that

$$(-1)^{m-q} A_{q,k}^{(m)} \geq 0 \quad (q = 1, 2, \ldots, m; m = 1, 2, \ldots, M), \tag{25}$$

as $\omega \to +\infty$ to conclude that

$$(-1)^m \Delta^{m+1} x_k(\omega) \geq 0, \quad (m = 0, 1, 2, \ldots, M). \tag{26}$$

We prove (25) by induction on M. When $M = 1$, (25) reduces to $A_{1,k}^{(1)} \geq 0$, which is true because $A_{1,k}^{(1)} = \Delta x_k$, by (12). Now, suppose that (25) is true for N, with $1 \leq N < M$. By (23) for $m = N$, we have

$$\frac{\omega}{\pi}(-1)^N \Delta^{N+1} x_k = \sum_{q=1}^{N} [(-1)^{N-q} A_{q,k}^{(N)}][(-1)^q f^{(q)}(x_{k+N-q})] + O(\omega^{-N-1}),$$

which is nonnegative, as $\omega \to +\infty$ by the induction hypothesis, (24) and (21) for $m = N + 1$. Thus, by (12) for $m = N + 1$, $(-1)^N A_{1,k}^{(N+1)} = (-1)^N \Delta^{N+1} x_k \geq 0$ and for $q = 1, 2, \ldots, N + 1$,

$$(-1)^{N+1-q} A_{q,k}^{(N+1)} = \sum_{r=q-1}^{N} \binom{N}{r} [(-1)^{r-q+1} A_{q-1,k+N-r}^{(r)}][(-1)^{N-r} \Delta^{N+1-r} x_k] \geq 0,$$

again following the induction hypothesis. This proves (25) for $N + 1$, and thereby proves the theorem. □

Note that if the factors $(-1)^m$ are deleted from the assumptions (24), followed by making the obvious changes in the above proof, conclusion (26) remains valid with amendation by eliminating the factors $(-1)^m$. Thus, we have the following theorem.

Theorem 2. *Let $x_k = x_k(\omega)$ and $f(x) = \rho^{-1/2}(x)$ be the same as mentioned above. If $f(x)$ is absolutely M-monotonic on the interval $[a, b]$, then the sequence $\{\Delta x_k(\omega)\}$ is asymptotically absolutely M-monotonic.*

As consequence of Lemma 1 and Theorems 1 and 2, we can use the zeros of a solution of (6) to construct sequences in which all mth differences have the same sign.

Corollary 1. *(a) Let $f(x)$ be M-monotonic on $[a, b]$. If $\varphi(x)$ is also M-monotonic on $[a, b]$, then the sequence $\{\varphi(x_k)\}$ is asymptotically M-monotonic.*

(b) Let $f(x)$ be absolutely M-monotonic on $[a, b]$. If $\varphi(x)$ is also absolutely M-monotonic on $[a, b]$, then the sequence $\{\varphi(x_k)\}$ is asymptotically absolutely M-monotonic.

Proof. Because $f(x)$ is M-monotonic on $[a, b]$, it can be seen from the proof of Theorem 1 that (25) holds. On the other hand, the M-monotonicity of $\varphi(x)$ on $[a, b]$ means that

$$(-1)^m \varphi^{(m)}(x) \geq 0 \quad (x \in [a, b]; m = 0, 1, 2, \ldots, M). \tag{27}$$

It now follows from (11), (25), (27) and (10) that

$$(-1)^m \Delta^m \varphi(x_k) = \sum_{q=1}^{m} [(-1)^{m-q} A_{q,k}^{(m)}][(-1)^q \varphi^{(q)}(x_{k+m-q})] + O(\omega^{-m-1}) \geq 0,$$

for all k and $m = 0, 1, 2, \ldots, M$, as $\omega \to +\infty$. The proof of part (b) is similar to that of part (a). □

Note that by the definition of the function $f(x) = \rho^{-1/2}(x)$, the conclusion of Theorem 1 (resp., Theorem 2) can be inferred directly from the assumptions on $\rho(x)$. In fact, $(-1)^m \rho^{(m+1)}(x) \geq 0$ (resp., $\rho^{(m+1)}(x) \leq 0$) on $[a, b]$ for $m = 0, 1, 2, \ldots, M-1$, implying $(-1)^m f^{(m)}(x) \geq 0$ (resp., $f^{(m)}(x) \geq 0$) on $[a, b]$ for $m = 1, 2, \ldots, M$. To examine these assertions, we can proceed by induction on M. For $M = 1$, per $f(x) = \rho^{-1/2}(x)$ and $f'(x) = (-1/2)\rho^{-3/2}(x)\rho'(x)$, the assertion is valid. For higher derivatives of $f(x)$, a general term of $f^{(m)}(x)$ would appear as

$$S_m = C[\rho]^{\alpha_0}[\rho']^{\alpha_1}[\rho'']^{\alpha_2} \cdots [\rho^{(m)}]^{\alpha_m}$$

with exponentials α_0 being a negative half-integer and $\alpha_1, \alpha_2, \ldots, \alpha_m$ all non-negative integers. The induction is carried through by differentiating S_m. We have

$$S_m' = C\alpha_0[\rho]^{\alpha_0-1}[\rho']^{\alpha_1+1}[\rho'']^{\alpha_2} \cdots [\rho^{(m)}]^{\alpha_m} + C\alpha_1[\rho]^{\alpha_0}[\rho']^{\alpha_1-1}[\rho'']^{\alpha_2+1} \cdots [\rho^{(m)}]^{\alpha_m}$$
$$+ \cdots + C\alpha_m[\rho]^{\alpha_0}[\rho']^{\alpha_1}[\rho'']^{\alpha_2} \cdots [\rho^{(m)}]^{\alpha_m-1}[\rho^{(m+1)}],$$

and under the conditions $(-1)^m \rho^{(m+1)}(x) \geq 0$ (resp., $\rho^{(m+1)}(x) \leq 0$) and the negative α_0, each term in the last sum has opposite sign (resp., the same sign) as S_m. Thus, $f^{(m)}(x)$ and $f^{(m+1)}(x)$ have alternating signs (resp., the same sign), completing the induction. Hence, we obtain the following corollary.

Corollary 2. *Let $x_k = x_k(\omega)$ be the same as above: (a) if $\rho'(x)$ is $(M-1)$-monotonic on $[a, b]$, then the sequence $\{\Delta x_k(\omega)\}$ is asymptotically M-monotonic, and*

(b) if $-\rho'(x)$ is absolutely $(M-1)$-monotonic on $[a, b]$, then the sequence $\{\Delta x_k(\omega)\}$ is asymptotically absolutely M-monotonic.

Although Corollary 2(a) is a partial result included in ([13], Theorem 3.3), the techniques employed in this section are independent of the methods in the series of papers [3,13,14] and the results of Hartman ([15], Theorems 18.1$_n$ and 20.1$_n$). It provides the connection of the quantities between the differences of the zeros and the coefficient function $\rho(x)$, and might have some numerical interest.

One can find similar results concerned with the critical points of a Sturm–Liouville function of (6). In fact, by letting $x_k'(\omega)$ denote the kth critical point of a solution $y(x; \omega)$ of (6) in the interval $[a, b]$ and noting the definition of the Prüfer angle

$$\theta(x_k'(\omega); \omega) = (k - \frac{1}{2})\pi,$$

the procedures employed in this section are all valid. Thus, if we replace $\{x_k(\omega)\}$ in Theorems 1 and 2 and Corollaries 1 and 2 with $\{x_k'(\omega)\}$, the conclusions in these Theorems and Corollaries continue to hold.

3. Applications to Bessel Functions

Let $c_{\nu k}(\alpha)$ be the kth positive zero of the general Bessel (cylinder) function

$$C_\nu(x; \alpha) = J_\nu(x) \cos \alpha - Y_\nu(x) \sin \alpha,$$

where $J_\nu(x)$ and $Y_\nu(x)$ denote the Bessel functions with order ν of the first and second kind, respectively. The main results in this section are stated as follows.

Theorem 3. *(a) For $|\nu| < 1/2$, we have*

$$(-1)^m \Delta^{m+2} c_{\nu k}(\alpha) > 0 \quad (m = 0, 1, 2, \ldots; k = 1, 2, 3, \ldots).$$

(b) For $0 < |\nu| < 1/2$, we have

$$(-1)^m \Delta^{m+1} c_{\nu k}^{2|\nu|}(\alpha) > 0 \quad (m = 0, 1, 2, \ldots; k = 1, 2, 3, \ldots).$$

The Airy functions (see, e.g., [16] p. 18) satisfy the differential equation $y'' + \frac{x}{3} y = 0$. Here, we consider a broader class of functions, including the Airy functions, which satisfy the differential equation (see, e.g., [17] p. 97)

$$z'' + \omega^2 x^\gamma z = 0, \quad x \in (0, +\infty), \tag{28}$$

where $0 < \gamma < +\infty$. These functions are closely related to Bessel functions. Indeed,

$$z(x; \omega) = x^{1/2} C_\nu(2\nu\omega x^{1/(2\nu)}; \alpha), \quad \textit{where} \quad \nu = 1/(\gamma + 2),$$

is a nontrivial real solution of (28). Note that for each $\omega > 0$, the kth positive zeros $\xi_k(\omega)$ of $z(x; \omega)$ satisfies the identities

$$2\nu\omega(\xi_k(\omega))^{1/(2\nu)} = c_{\nu k}(\alpha) \quad \textit{and} \quad (2\nu\omega)^{2\nu} \xi_k(\omega) = c_{\nu k}^{2\nu}(\alpha).$$

Moreover, for each $\omega > 0$ and for $m = 0, 1, 2, \ldots$, we have

$$\Delta^{m+2} c_{\nu k}(\alpha) = 2\nu\omega \Delta^{m+2} (\xi_k(\omega))^{1/(2\nu)} \tag{29}$$

and

$$\Delta^{m+1} c_{\nu k}^{2\nu}(\alpha) = (2\nu\omega)^{2\nu} \Delta^{m+1} \xi_k(\omega). \tag{30}$$

Here, the identities (29) and (30) are the key to the regularity behaviour of the Bessel zeros.

To prove Theorem 3, we consider the family of differential equations

$$y'' + \omega^2 (x + a)^\gamma y = 0 \quad (a > 0; 0 < \gamma < +\infty), \tag{31}$$

on the interval $[0, b]$. Let $y_a(x; \omega)$ be a nontrivial real solution of (31) and let the sequence $\{x_k(\omega; a)\}$ be the zeros of $y_a(x; \omega)$ with ascending order in $[0, b]$. Following Theorem 1 with $f(x) = (x + a)^{-\gamma/2}$ and Corollary 1(a) with the function $\varphi(x) = (x + a)^{-1/(2\nu)}$, we have

$$(-1)^m \Delta^{m+1} x_k(\omega; a) \geq 0 \quad (m = 0, 1, 2, \ldots, M) \tag{32}$$

and

$$(-1)^m \Delta^m (x_k(\omega; a) + a)^{-1/(2\nu)} \geq 0 \quad (m = 0, 1, 2, \ldots, M), \tag{33}$$

as $\omega \to +\infty$. If we specify the initial conditions for the solution $y_a(x; \omega)$ of (31) to be

$$y_a(0; \omega) = z(a; \omega) \quad \textit{and} \quad y_a'(0; \omega) = z'(a; \omega),$$

then it is easy to verify that $y_a(x; \omega) = z(x + a; \omega)$ for $x \in [0, b]$; hence, for each k, $x_k(\omega; a) + a$ converges to ξ_k as $a \to 0^+$. Thus, for each $\omega > 0$, by (29) and (30) we have

$$\Delta^{m+2} c_{\nu k}(\alpha) = \lim_{a \to 0^+} 2\nu\omega \Delta^{m+2} (x_k(\omega; a))^{1/(2\nu)} \tag{34}$$

and

$$\Delta^{m+1} c_{\nu k}^{2\nu}(\alpha) = \lim_{a \to 0^+} (2\nu\omega)^{2\nu} \Delta^{m+1} x_k(\omega; a). \tag{35}$$

Recalling (15) and (17) with the function $\rho(x) = (x+a)^\gamma$ and denoting $x_k = x_k(\omega; a)$, we have

$$\omega \int_{x_k}^{x_{k+1}} (x+a)^{\gamma/2} dx = \pi - \sum_{r=0}^{m+1} \Delta F_r(x_k) \omega^{-r-1} - R_{m+1}(x_k). \tag{36}$$

Note that $\nu = 1/(\gamma+2)$ and $f(x) = (x+a)^{(2\nu-1)/(2\nu)}$. By (19), we have

$$\Delta F_0(x_k) = \frac{4\nu^2 - 1}{16\nu} \Delta (x_k + a)^{-1/(2\nu)}.$$

Thus, (36) becomes

$$2\nu\omega \Delta (x_k + a)^{1/(2\nu)} = \pi + \frac{1 - 4\nu^2}{16\nu\omega} \Delta (x_k + a)^{-1/(2\nu)} - \sum_{r=1}^{m+1} \Delta F_r(x_k) \omega^{-r-1} - R_{m+1}(x_k). \tag{37}$$

If we apply the difference operator Δ^{m+1} to (37), by (10) in the case $m+2$ and (18) in the case $m+3$, we can find

$$2\nu\omega \Delta^{m+2} (x_k + a)^{1/(2\nu)} = \frac{1 - 4\nu^2}{16\nu\omega} \Delta^{m+2} (x_k + a)^{-1/(2\nu)} + O(\omega^{-m-4}). \tag{38}$$

Moreover, multiplying (38) by $(-1)^m \omega^{m+3}$, we have

$$2\nu\omega^{m+4} (-1)^m \Delta^{m+2} (x_k + a)^{1/(2\nu)} = \frac{1 - 4\nu^2}{16\nu} \omega^{m+2} (-1)^m \Delta^{m+2} (x_k + a)^{-1/(2\nu)} + O(\omega^{-1}). \tag{39}$$

By (39), (33), (10) in the case $m+2$ and $0 < \nu < 1/2$, we have

$$(-1)^m \Delta^{m+2} (x_k + a)^{1/(2\nu)} \geq 0 \quad as \quad \omega \to +\infty. \tag{40}$$

Now, for each $a > 0$, if we choose a sufficiently large $\omega = \omega(a)$ such that (40) and (32) hold, then by (34) and (35) we have

$$(-1)^m \Delta^{m+2} c_{\nu k}(\alpha) \geq 0 \quad (m = 0, 1, 2, \ldots; k = 1, 2, 3, \ldots), \tag{41}$$

and

$$(-1)^m \Delta^{m+1} c_{\nu k}^{2\nu}(\alpha) \geq 0 \quad (m = 0, 1, 2, \ldots; k = 1, 2, 3, \ldots). \tag{42}$$

Second, according to $Y_\nu(x) = (J_\nu(x) \cos \pi\nu - J_{-\nu}(x)) / \sin \pi\nu$ (see, e.g., [17] p. 64), it is easy to verify that $C_{-\nu}(x; \alpha) = C_\nu(x; \alpha + \pi\nu)$; hence,

$$c_{-\nu k}(\alpha) = c_{\nu k}(\alpha + \pi\nu).$$

Thus, for $0 < |\nu| < 1/2$, (41) holds and (42) holds in the modified form:

$$(-1)^m \Delta^{m+1} c_{\nu k}^{2|\nu|}(\alpha) \geq 0 \quad (m = 0, 1, 2, \ldots; k = 1, 2, 3, \ldots). \tag{43}$$

Third, for $\nu = 0$, any positive zero $c_{\nu k}(\alpha)$ of $C_\nu(x; \alpha)$ is definable as a continuously increasing function of the real variable ν (see, e.g., [17] p. 508), meaning that by an approximating process, (41) holds for all $|\nu| < 1/2$.

Finally, because neither $\{\Delta^2 c_{\nu k}(\alpha)\}$ nor $\{\Delta c_{\nu k}^{2|\nu|}(\alpha)\}$ are constant sequences, the results of Lorch, Szego, and Muldoon for completely monotonic sequences ([2] p. 72 or [18] Theorem 2) guarantee the strict inequalities of (41) and (43). This completes the proof of Theorem 3.

4. Applications to Classical Orthogonal Polynomials

Several important classical orthogonal polynomials are related to Sturm–Liouville functions, such as the Hermite and Jacobi polynomials. In ([2] p. 71), Lorch, Szego, and their coworkers conjectured on the basis of numerical evidence that the θ-zeros of the Legendre polynomials, the special cases of Jacobi polynomials, and the positive zeros of the Hermite polynomials form sequences with mth differences having constant signs. In this section, we apply the results in Sections 2 and 3 to obtain partial answers for these conjectures.

4.1. Positive Zeros of Hermite Polynomials

Let $H_n(t)$ be the Hermite polynomial (see, e.g., [16] p. 105 (5.5.3)), defined by

$$H_n(t) = (-1)^n e^{t^2} (\frac{d}{dt})^n e^{-t^2}. \tag{44}$$

We consider the Hermite differential equation

$$H_n'' - 2tH_n' + 2nH_n = 0,$$

and the related equation

$$u'' + [(2n+1) - t^2]u = 0. \tag{45}$$

A simple calculation shows that (see, e.g., [16] p. 105 (5.5.2))

$$u_n(t) = e^{-t^2/2} H_n(t)$$

is a nontrivial solution of (45). From the general theory of orthogonal polynomials, we know that $H_n(t)$ has precisely n real zeros. By (44), we see that for even n it is the case that $H_n(t)$ is an even function of t, while for odd n $H_n(t)$ is an odd function of t. Accordingly, all zeros of $H_n(t)$ are placed symmetrically with respect to the origin, and the same phenomenon is clearly true for $u_n(t)$. For each n, the positive zeros of $H_n(t)$ are named by $h_1^{(n)} < h_2^{(n)} < \cdots < h_{[n/2]}^{(n)}$, where $[\cdot]$ is the greatest integer function.

The main result concerned with Hermite polynomials is as follows.

Theorem 4. *Let $h_k^{(n)}$ be as above. Then, for each k we have*

$$\Delta^m h_k^{(n)} \geq 0 \quad (m = 1, 2, \ldots, M), \tag{46}$$

for sufficiently large n.

Proof. For each n, by introducing the variable $x = t/\sqrt{2n+1}$ and letting $z_n(x) = u_n(t)$, Equation (45) is transformed into

$$z_n'' + (2n+1)^2(1-x^2)z_n = 0.$$

We denote the kth positive zero of $z_n(x)$ by $\xi_k^{(n)}$, where $\xi_k^{(n)} = h_k^{(n)}/\sqrt{2n+1}$. Thus, we have

$$\Delta^m h_k^{(n)} = \sqrt{2n+1}\Delta^m \xi_k^{(n)}.$$

To prove (46), we consider the differential equation

$$y'' + (2n+1)^2(a - x^2)y = 0 \quad (a > 1; x \in [0,1]). \tag{47}$$

Let $\omega = 2n+1$, let $f(x) = (a-x^2)^{-1/2}$, let $y_n(x;a)$ be a nontrivial real solution of (47), and let $x_k^{(n)}(a)$ be the kth positive zero of $y_n(x;a)$. Then, from the following fact about $f^{(m)}(x)$

$$f^{(m)}(x) = \{a\, polynomial\, of\, x\, with\, nonnegative\, coefficients\}(a-x^2)^{-(2m+1)/2},$$

we know that $f^{(m)}(x) \geq 0$ on the interval $[0,1]$ for $m = 1,2,3,\ldots$. Thus, by Theorem 2, we obtain

$$\Delta^m x_k^{(n)}(a) \geq 0 \quad (m = 1,2,\ldots,M)$$

for sufficiently large n. If we specify the initial conditions for $y_n(x;a)$ to be

$$y_n(0;a) = z_n(0) \quad and \quad y_n'(0;a) = z_n'(0),$$

then it is easy to verify that $y_n(x;a)$ uniformly converges to $z_n(x)$ on the interval $[0,1]$ as $a \to 1^+$. Consequently, for $k = 1,2,\ldots,[\frac{n}{2}]$, the zero $x_k^{(n)}(a)$ converges to $\xi_k^{(n)}$ as $a \to 1^+$. Therefore, for fixed k,

$$\Delta^m \xi_k^{(n)} = \lim_{a\to 1^+} \Delta^m x_k^{(n)}(a) \geq 0,$$

and (46) holds. □

4.2. Zeros of Jacobi Polynomials

Considering $a > -1$ and $b > -1$, the Jacobi polynomial $P_n^{(a,b)}(x)$ (see, e.g., [16] p. 67 (4.3.1)) is defined by

$$(1-x)^a(1+x)^b P_n^{(a,b)}(x) = \frac{(-1)^n}{2^n n!}(\frac{d}{dx})^n\{(1-x)^{n+a}(1+x)^{n+b}\}.$$

Concerning the Jacobi polynomials $P_n^{(a,b)}(x)$ on the orthogonal interval $[-1,1]$, if we denote the zeros $x_k^{(n)} = x_k^{(n)}(a,b)$ of $P_n^{(a,b)}(x)$ with the descending order

$$1 > x_1^{(n)} > x_2^{(n)} > \cdots > x_n^{(n)} > -1,$$

then the θ-zeros $\theta_k^{(n)} = \theta_k^{(n)}(a,b)$ and $x_k^{(n)} = cos\theta_k^{(n)}$ of $P_n^{(a,b)}(cos\theta)$ behave as the order

$$0 < \theta_1^{(n)} < \theta_2^{(n)} < \cdots < \theta_n^{(n)} < \pi.$$

According to the uniform convergence theorem ([16] Theorem 8.1.1, p. 190)

$$\lim_{n\to+\infty} n^{-\alpha} P_n^{(a,b)}(cos\frac{x}{n}) = (\frac{x}{2})^{-a} J_a(x),$$

we know that

$$\lim_{n\to+\infty} n\theta_k^{(n)}(a,b) = j_{ak}.$$

Now, by Theorem 3(a), for $\nu = a$ and $\alpha = 0$, we have the following theorem.

Theorem 5. *For $|a| < 1/2$ and k fixed, we have*

$$(-1)^m \Delta^{m+2}\theta_k^{(n)}(a,b) \geq 0 \quad (m = 0,1,2,\ldots,M)$$

for sufficiently large n.

5. Proofs of Lemmas 1 and 3

In this section, we prove (10), (11), (12), (21), and (22) simultaneously by induction.

For $m = 1$, taking the Taylor expansion of φ at x_k

$$\varphi(x_{k+1}) = \varphi(x_k) + \varphi'(x_k)\Delta x_k + \varphi''(\xi_{k,2})\frac{(\Delta x_k)^2}{2}$$

where $x_k \le \xi_{k,2} \le x_{k+1}$ and using (8), we have

$$\Delta\varphi(x_k) = \varphi'(x_k)\Delta x_k + O(\omega^{-2}),$$

hence, (10), (11), and (12) are valid for $m = 1$. If we apply the first order difference operator to (20) and use (10) for $m = 1$ with $\varphi = F_0$, then we have

$$\Delta f(x_k) = \frac{\omega}{\pi}\Delta^2 x_k + \frac{\omega}{2!\pi}\Delta\{g_1(x_k)(\Delta x_k)^2\} + O(\omega^{-2}).$$

Because $\Delta\{\alpha_k\beta_k\} = \alpha_{k+1}(\Delta\beta_k) + (\Delta\alpha_k)\beta_k$, we have

$$\begin{aligned}\Delta f(x_k) &= \tfrac{\omega}{\pi}\Delta^2 x_k + \tfrac{\omega}{2!\pi}g_1(x_{k+1})\{\Delta x_{k+1}\Delta^2 x_k + \Delta^2 x_k \Delta x_k\} + O(\omega^{-2}) \\ &= \tfrac{\omega}{\pi}\Delta^2 x_k(1 + O(\omega^{-1})) + O(\omega^{-2}).\end{aligned}$$

Applying (10) for $m = 1$ again to the function $f(x)$, we find that $\Delta f(x_k) = O(\omega^{-1})$; now, we have

$$\Delta^2 x_k = O(\omega^{-2}),$$

hence,

$$\Delta f(x_k) = \frac{\omega}{\pi}\Delta^2 x_k + O(\omega^{-2}).$$

Thus, (21) for $m = 2$ and (22) for $m = 1$ are valid. The validity of (21) for $m = 2$ is the impetus of our induction argument.

Now, suppose that (10), (11), (12), (21), and (22) are fulfilled for $m = 1, 2, \ldots, N$. If we apply (10) for $m = N$ with $\varphi(x) = f(x)$ to (22) for $m = N$, then we have (21) for $m = N + 1$, that is,

$$\Delta^{N+1} x_k = O(\omega^{-N-1}).$$

Taking the Taylor expansion of φ at x_k

$$\varphi(x_{k+1}) = \varphi(x_k) + \sum_{p=1}^{N+1}\frac{\varphi^{(p)}(x_k)}{p!}(\Delta x_k)^p + \frac{\varphi^{(N+2)}(\xi_{k,N+2})}{(N+2)!}(\Delta x_k)^{N+2}, \tag{48}$$

where $x_k \le \xi_{k,N+2} \le x_{k+1}$, applying the Nth order difference operator to (48), and then using (21) for $m = 1$, we have

$$\Delta^{N+1}\varphi(x_k) = \sum_{p=1}^{N+1}\frac{1}{p!}\Delta^N\{\varphi^{(p)}(x_k)(\Delta x_k)^p\} + O(\omega^{-N-2}). \tag{49}$$

Following the product rule for higher differences, we know that

$$\Delta^N\{\varphi^{(p)}(x_k)(\Delta x_k)^p\} = \sum_{r=0}^{N}\binom{N}{r}\Delta^r\varphi^{(p)}(x_{k+N-r})\Delta^{N-r}(\Delta x_k)^p.$$

If we replace $\varphi(x_k)$ with $\varphi^{(p)}(x_{k+N-r})$ in (10) for $m = r$, $r = 1, 2, \ldots, N$ and use (21) for $m = 1, 2, \ldots, N + 1$, then we obtain

$$\Delta^N\{\varphi^{(p)}(x_k)(\Delta x_k)^p\} = O(\omega^{-N-p}) \quad (p = 1, 2, \ldots, N + 1). \tag{50}$$

Thus, (49) and (50) imply (10) for $m = N+1$. Moreover, we have

$$\begin{aligned} \Delta^{N+1}\varphi(x_k) &= \Delta^N\{\varphi'(x_k)\Delta x_k\} + O(\omega^{-N-2}) \\ &= \textstyle\sum_{r=0}^{N} \binom{N}{r}\Delta^r\varphi'(x_{k+N-r})\Delta^{N+1-r}x_k + O(\omega^{-N-2}). \end{aligned} \tag{51}$$

Applying (11) for $m = r$ to (51) with $\varphi'(x_{k+N-r})$ instead of $\varphi(x_k)$ for $r = 1, 2, \ldots, N$, we find

$$\begin{aligned} \Delta^{N+1}\varphi(x_k) = \quad & \varphi'(x_{k+N})\Delta^{N+1}x_k \\ & + \textstyle\sum_{r=1}^{N} \binom{N}{r}\{\sum_{q=1}^{r} A_{q,k+N-r}^{(r)}\varphi^{(q+1)}(x_{k+N-q})\}\Delta^{N+1-r}x_k + O(\omega^{-N-2}). \end{aligned} \tag{52}$$

If we change the order of the summation in (52) and shift the q index, then we can find

$$\begin{aligned} \Delta^{N+1}\varphi(x_k) = \quad & \varphi'(x_{k+N})\Delta^{N+1}x_k \\ & + \textstyle\sum_{q=2}^{N+1} \varphi^{(q)}(x_{k+N+1-q})\{\sum_{r=q-1}^{N} \binom{N}{r}A_{q-1,k+N-r}^{(r)}\Delta^{N+1-r}x_k\} + O(\omega^{-N-2}). \end{aligned}$$

Thus, (11) and (12) are valid for $m = N+1$.

Finally, to prove (22) for $m = N+1$, by applying the $(N+1)$th order difference operator to (20) for $m = N+1$, we have

$$\begin{aligned} \Delta^{N+1}f(x_k) = \quad & \tfrac{\omega}{\pi}\textstyle\sum_{r=0}^{N+1} \frac{1}{(r+1)!}\Delta^{N+1}\{g_r(x_k)(\Delta x_k)^{r+1}\} \\ & + \tfrac{1}{\pi}\textstyle\sum_{r=0}^{N-1} \Delta^{N+1}\{f(x_k)\Delta F_r(x_k)\}\omega^{-r-1} + O(\omega^{-N-2}). \end{aligned} \tag{53}$$

Following the product rule for higher differences again, we have

$$\Delta^{N+1}\{g_r(x_k)(\Delta x_k)^{r+1}\} = \sum_{\beta=0}^{N+1}\binom{N+1}{\beta}\Delta^\beta g_r(x_{k+N+1-\beta})\Delta^{N+1-\beta}(\Delta x_k)^{r+1}.$$

Using (10) for $m = \beta$ with $g_r(x_{k+N+1-\beta})$ replacing $\varphi(x_k)$ for $\beta = 1, 2, \ldots, N+1$ and using (21) for $m = 1, 2, \ldots, N+1$, we obtain

$$\begin{aligned} & \Delta^{N+1}\{g_r(x_k)(\Delta x_k)^{r+1}\} \\ & = g_r(x_{k+N+1})\Delta^{N+1}(\Delta x_k)^{r+1} + \textstyle\sum_{\beta=1}^{N+1}\binom{N+1}{\beta}\Delta^\beta g_r(x_{k+N+1-\beta})\Delta^{N+1-\beta}(\Delta x_k)^{r+1} \\ & = g_r(x_{k+N+1})(\Delta^{N+2}x_k)O(\omega^{-r}) + O(\omega^{-N-r-2}). \end{aligned} \tag{54}$$

On the other hand, applying (10) to the functions $f(x)$ and $F_r(x)$ for $m = 1, 2, \ldots, N+1$, we have

$$\begin{aligned} & \Delta^{N+1}\{f(x_k)\Delta F_r(x_k)\} \\ & = f(x_{k+N+1})\Delta^{N+2}F_r(x_k) + \textstyle\sum_{\beta=1}^{N+1}\binom{N+1}{\beta}\Delta^\beta f(x_{k+N+1-\beta})\Delta^{N+2-\beta}F_r(x_k) \\ & = O(\omega^{-N-1}) + O(\omega^{-N-2}). \end{aligned} \tag{55}$$

Applying the estimates (54) and (55) to (53), we obtain

$$\begin{aligned} \Delta^{N+1}f(x_k) &= \tfrac{\omega}{\pi}\Delta^{N+2}x_k + (\tfrac{\omega}{\pi}\Delta^{N+2}x_k)O(\omega^{-1}) + O(\omega^{-N-2}) \\ &= \tfrac{\omega}{\pi}\Delta^{N+2}x_k(1 + O(\omega^{-1})) + O(\omega^{-N-2}). \end{aligned} \tag{56}$$

If we replace $\varphi(x_k)$ with $f(x_k)$ in (10) for $m = N+1$, then we have

$$\Delta^{N+1}f(x_k) = O(\omega^{-N-1}). \tag{57}$$

Note that (56) and (57) imply

$$\frac{\omega}{\pi}\Delta^{N+2}x_k = O(\omega^{-N-1}). \tag{58}$$

Then, by (56) and (58), we have (22) for $m = N+1$. This completes the proofs of Lemmas 1 and 3.

6. Conclusions

In this work, we consider the second-order differential equation $y'' + \omega^2\rho(x)y = 0$ on the interval $[a,b]$ associated with a positive parameter ω. When the function $\rho^{-1/2}(x)$ satisfies the (absolutely) M-monotonic condition on the interval $[a,b]$, we show that the difference of the zeros for a nontrivial solution of the equation satisfies the asymptotically (absolutely) M-monotonic property. As applications, we use an approximation process for the zeros of the Bessel function and prove the conjecture of Lorch and Szego. In addition, we show that the differences of the zeros of various orthogonal polynomials with higher degrees possess sign regularity.

On the basis of numerical evidence, Lorch, Szego, and their coworkers conjectured that the θ-zeros of the Legendre polynomials, the special cases of Jacobi polynomials, and the positive zeros of the Hermite polynomials are able to form absolutely monotonic sequences, that is, sequences in which all consecutive differences of the zeros are non-negative. In Theorem 5, the x-zeros of Jacobi polynomials are arranged in descending order, and hence the θ-zeros are arranged in increasing order, while the mth differences and $(m+1)$th differences of the θ-zeros of Jacobi polynomials are sign-alternating.

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Appendix A

Recalling $f(x) = \rho^{-1/2}(x)$ and the differential Equation (13) for the Prüfer angle $\theta(x;\omega)$, we have

$$\theta'(x;\omega) = \frac{\omega}{f(x)}\{1 - \frac{f'(x)}{2\omega}\sin 2\theta(x;\omega)\}. \tag{A1}$$

Then,

$$\{-\frac{f'}{2f}\sin 2\theta\}\frac{\theta'}{\{1-(f'/2\omega)\sin 2\theta\}\omega/f} = -\sum_{r=0}^{\infty}\{\frac{\omega^{-1}}{2}f'\sin 2\theta\}^{r+1}\theta', \tag{A2}$$

hence,

$$\int_{x_k}^{x_{k+1}}\frac{\rho'}{4\rho}\sin 2\theta dx = -\sum_{r=0}^{m-1}\frac{\omega^{-r-1}}{2^{r+1}}\int_{x_k}^{x_{k+1}}(f')^{r+1}(sin^{r+1}2\theta)\theta' dx + O(\omega^{-m-1}), \tag{A3}$$

where $\theta = \theta(x;\omega)$, $\theta' = \theta'(x;\omega)$ and $x_k = x_k(\omega)$.

To prove Lemma 2, we introduce the following integrals for a C^∞-function φ which is defined on $[a,b]$:

$$P_r[\varphi] = \int_{x_k}^{x_{k+1}}\varphi\cdot sin^r(2\theta)\cdot\theta' dx,$$

$$Q_r[\varphi] = \int_{x_k}^{x_{k+1}}\varphi\cdot sin^r(2\theta)\cdot cos(2\theta)dx,$$

and

$$R_r[\varphi] = \int_{x_k}^{x_{k+1}} \varphi \cdot sin^{r+1}(2\theta)dx,$$

where $r = 0, 1, 2, \ldots$. Now, (A3) can be written as

$$\int_{x_k}^{x_{k+1}} \frac{\rho'}{4\rho} \sin 2\theta dx = -\sum_{r=0}^{m-1} \frac{\omega^{-r-1}}{2^{r+1}} P_{r+1}[(f')^{r+1}] + O(\omega^{-m-1}). \tag{A4}$$

Via integration by parts, we have the following reduced formula for $P_{r+1}[\varphi]$:

$$P_{r+1}[\varphi] = \frac{-\varphi \cdot sin^r 2\theta \cdot cos 2\theta}{2(r+1)}|_{x_k}^{x_{k+1}} + \frac{r}{r+1} P_{r-1}[\varphi] + \frac{1}{2(r+1)} Q_r[\varphi']. \tag{A5}$$

Introducing θ' in the same way as in (A2) and using integration by parts and (14), we have the following estimates for $Q_r[\varphi]$ and $R_r[\varphi]$:

$$Q_r[\varphi] = -\sum_{j=0}^{m-r-3} \frac{\omega^{-j-1}}{2^{j+1}(r+j+1)} R_{r+j}[(\varphi_j)'] + O(\omega^{-m+r}), \tag{A6}$$

and

$$R_r[\varphi] = \sum_{j=0}^{m-r-3} \frac{\omega^{-j-1}}{2^j} P_{r+j+1}[\varphi_j] + O(\omega^{-m+r+1}), \tag{A7}$$

where $\varphi_j = \varphi f (f')^j$. By applying the estimates (A6) and (A7) with suitable integrands to (A5) and then collecting the terms with the same order of ω in the sum together, we can find

$$\begin{aligned} P_{r+1}[\varphi] = \quad & \tfrac{-\varphi \cdot sin^r 2\theta \cdot cos 2\theta}{2(r+1)}|_{x_k}^{x_{k+1}} + \tfrac{r}{r+1} P_{r-1}[\varphi] \\ & - \textstyle\sum_{\beta=0}^{m-r-3} \tfrac{\omega^{-\beta-2}}{2^{\beta+2}(r+1)} \sum_{j=0}^{\beta} \tfrac{1}{r+j+1} P_{r+\beta+1}[(\varphi')_{j,\beta-j}] + O(\omega^{-m+r}), \end{aligned} \tag{A8}$$

where $\varphi_{j_1,j_2} = [(\varphi_{j_1})']_{j_2}$. By (A8) and (14), we have

$$P_1[\varphi] = \frac{-\Delta\varphi(x_k)}{2} - \sum_{\beta=0}^{m-3} \frac{\omega^{-\beta-2}}{2^{\beta+2}} \sum_{j=0}^{\beta} \frac{1}{j+1} P_{\beta+1}[(\varphi')_{j,\beta-j}] + O(\omega^{-m}), \tag{A9}$$

and

$$P_2[\varphi] = \frac{P_0[\varphi]}{2} - \sum_{\beta=0}^{m-4} \frac{\omega^{-\beta-2}}{2^{\beta+3}} \sum_{j=0}^{\beta} \frac{1}{j+2} P_{\beta+2}[(\varphi')_{j,\beta-j}] + O(\omega^{-m+1}). \tag{A10}$$

If we apply (A1) and (A7) to the integral $P_0[\varphi]$, then we have

$$P_0[\varphi] = \omega \int_{x_k}^{x_{k+1}} \frac{\varphi}{f} dx - \sum_{j=0}^{m-3} \frac{\omega^{-j-1}}{2^{j+1}} P_{j+1}[(\varphi f'/f)_j] + O(\omega^{-m+1}). \tag{A11}$$

Applying (A11) to (A10), we obtain

$$\begin{aligned} P_2[\varphi] = \quad & \tfrac{\omega}{2} \textstyle\int_{x_k}^{x_{k+1}} \tfrac{\varphi}{f} dx - \sum_{j=0}^{m-3} \tfrac{\omega^{-j-1}}{2^{j+2}} P_{j+1}[(\varphi f'/f)_j] \\ & - \textstyle\sum_{\beta=0}^{m-4} \tfrac{\omega^{-\beta-2}}{2^{\beta+3}} \sum_{j=0}^{\beta} \tfrac{1}{j+2} P_{\beta+2}[(\varphi')_{j,\beta-j}] + O(\omega^{-m-1}). \end{aligned} \tag{A12}$$

In (A4), if we apply (A8) to the function $\varphi = (f')^{r+1}$ and use (A9) and (A12) to collect the reductions of those integrals $P_{r-1}[(f')^{r+1}]$ and $P_{r+\beta+1}[((f')^{r+1})'_{j,\beta-j}]$, then all reduction processes are stopped after a finite number of steps, while the remainders behave as $O(\omega^{-m-1})$. This completes the proof of Lemma 2.

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