

Article

Optimal Fault-Tolerant Resolving Set of Power Paths

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Abstract: In a simple connected undirected graph G , an ordered set R of vertices is called a resolving set if for every pair of distinct vertices u and v , there is a vertex $w \in R$ such that $d(u, w) \neq d(v, w)$. A resolving set F for the graph G is a *fault-tolerant resolving set* if for each $v \in F$, $F \setminus \{v\}$ is also a resolving set for G . In this article, we determine an optimal fault-resolving set of r -th power of any path P_n when $n \geq r(r-1) + 2$. For the other values of n , we give bounds for the size of an optimal fault-resolving set. We have also presented an algorithm to construct a fault-tolerant resolving set of P_m^r from a fault-tolerant resolving set of P_n^r where $m < n$.

Keywords: resolving set; metric dimension; fault-tolerant metric dimension; power of path

MSC: 05C12; 05C20; 05C90

1. Introduction

An undirected simple connected graph G has a vertex set $V(G)$ and an edge set $E(G)$, and the distance between the vertices u and v is called $d_G(u, v)$. This distance is equal to the length of the shortest $u - v$ path within G . With an understanding of G , we use $d(u, v)$ instead of $d_G(u, v)$. A vertex w *resolves* two vertices u and v if $d(u, w) \neq d(v, w)$. The *code* of a vertex u with respect to an ordered set $R = \{r_1, r_2, \dots, r_k\} \subset V(G)$ is the k -tuple $c(u|R) = (d(u, r_1), d(u, r_2), \dots, d(u, r_k))$. The set R is called a *resolving set* (RS) if for every two distinct vertices u, v in G , $c(u|R) \neq c(v|R)$. Equivalently, R is said to be a resolving set for G if for every two distinct vertices u and v in $V(G)$, there is a vertex w in R that resolves u and v . The minimum cardinality of a resolving set of G is called the *metric dimension* of G and it is denoted by $\beta(G)$.

Harry and Melter [1] and Slater [2] presented metric dimension concepts independently. The metric basis for a graph G is the resolving set of G containing the minimum number of vertices. Elements of metric bases were called sensors in an application given in [3]. For a general graph, finding the metric dimension is an NP-hard problem. In [4], Khuller et al. provided a construction that proves that the graph's metric dimension is NP-hard. It is important to note that metric bases can be utilized in a wide variety of applications, including robot navigation, network optimization, and sensor networks, all of which are heavily used by different government organizations. However, they still have some reservations because if some detectors (elements of the metric basis) are faulty, it will not be possible to identify the nodes (or vertices) uniquely. A malfunctioning sensor will not give us enough information to deal with an intruder (fire, thief, etc). Hernando et al. introduced a fault-tolerant metric dimension in order to address these kind of problems and in order to improve the accuracy of detection or robustness of the system [5]. A resolving set F of a graph G is a *fault-tolerant resolving set* (FTRS) if for each $v \in F$, $F \setminus \{v\}$ is also a resolving set for G . The *fault-tolerant metric dimension* of G , denoted by $\beta'(G)$, is the minimum cardinality of a fault-tolerant resolving set. A fault-tolerant resolving set of order $\beta'(G)$ is called a *fault-tolerant metric basis*. Fault-tolerant metric dimensions are NP-hard



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problems. Determining the fault-tolerant metric dimension of a graph is an interesting yet difficult combinatorial problem with potential applications to sensor networks. So far, it has been explored for a few basic families of graphs. In their introductory paper [5], Hernendo et al. have characterized the fault-tolerant resolving sets in a tree T . They have also shown that the fault-tolerant metric dimension is bounded by a function of the metric dimension (independent of any graph) as $\beta'(G) \leq \beta(G)(1 + 2 \cdot 5^{\beta(G)-1})$ for every graph's G . In [6], Javaid et al. determined the fault-tolerant metric dimension for a cycle C_n . Basak et al. [7] determine the fault-tolerant metric dimension of C_n^3 .

Due to their interesting properties and wide range of applications, the power graph has been widely studied in the past. Power graphs can be applied in different fields such as routing in networks, quantum random walk in physics, etc. The problem of the metric dimension or fault-tolerant metric dimension of the power of graphs has also been considered in the past. For the power of paths, Alholi et al. [8] have determined some exact value and upper bound for the metric dimension of the power of paths as follows: $\beta(P_n^k) \leq k$, $\beta(P_n^3) = 3$, $\beta(P_n^4) = 4$. Basak et al. [7] studied and envisioned the fault-tolerant metric dimension for the circulant graphs $C_n(1, 2, 3)$, which are basically the 3rd power of cycles. Saha et al. in [9] calculated the metric dimension of the power of finite paths and also characterized all metric bases for the same. There are several interesting recent results on this topic; see [10–12]. Due to the widespread applications of power graphs and motivated by the above results, in this article, we study an optimal fault-tolerant resolving set of P_n^r .

The rest of the paper is organized as follows: Firstly, the Preliminaries and Notations section presents a detailed explanation of the various terms and expressions we will use later on to establish the corresponding basic results for the fault-tolerant metric dimension of P_n^r . In Section 3, we have given some basic properties and results of the resolving set and fault-tolerant resolving set of P_n^r . We have proven certain essential lemmas that facilitate the determination of the resolving set and fault-tolerant set of P_n^r . Moreover, in this section, we have presented an algorithm to construct a fault-tolerant resolving set of P_m^r from a fault-tolerant resolving set of P_n^r when $m < n$. In Section 4, we determine fault-tolerant resolving sets of P_n^r and have given different examples in support of our results. In Section 5, we presented the lower bounds of $\beta'(P_n^r)$ for all n and r and then show the optimality of the FTRS presented in Theorems 1 and 2. In Section 6, we applied lower and upper bounds presented in Sections 4 and 6 to P_n^r , where $r \in \{2, 3, 4, 5, 6, 8\}$. Our bounds have been verified with some examples. We have used an algorithm (Algorithm 1) to construct an optimal fault-tolerant resolving set of P_n^r , when $r(r-1) + 2 \leq n \leq 2r(r-1) + 1$, from an optimal fault-tolerant resolving set of $P_{2r(r-1)+2}^r$. In the concluding remarks, we have announced an open problem.

2. Preliminaries and Notations

In this section, we give some definitions, notations, and lemmas that will be beneficial for a better understanding of the paper and its contents. We denote the vertices of P_n^r by $v_0, v_1, \dots, v_{n-1}$. The distance between two vertices v_i and v_j in P_n^r is given by $d(v_i, v_j) = \left\lceil \frac{|i-j|}{r} \right\rceil$ and hence the diameter of P_n^r is $\left\lceil \frac{n-1}{r} \right\rceil$. A vertex v_i is the left-side or right-side vertex of a vertex v_ℓ according to $i < \ell$ and $i > \ell$.

The following lemma gives the basic property of a fault-tolerant resolving set for an arbitrary graph.

Lemma 1 ([6]). *A set $F \subset V(G)$ is a fault-tolerant resolving set of G if and only if every pair of vertices in G is resolved by at least two vertices of F .*

Notation 1. We denote $W_{a,a+1}$ as the set of all vertices that resolve the consecutive vertices v_a and v_{a+1} .

Remark 1. If F is any fault-tolerant resolving set for P_n^r , then every pair of vertices must be resolved by at least two elements of F due to Lemma 1, that is, $|F \cap W_{a,a+1}| \geq 2$ for every fault-tolerant resolving set F of P_n^r and for every pair of vertices v_a and v_{a+1} .

Definition 1. For $m \in \{2, 3, \dots, r+1\}$, define K_m^j to be the complete subgraph of P_n^r induced by $\{v_j, v_{j+1}, \dots, v_{j+m-1}\}$. For the clique K_m^j , we call the vertices v_j, v_{j+m-1} as the end vertices and the others are intermediate vertices of K_m^j . For a clique K_m^i , we denote $I(K_m^i)$ as the set of all intermediate vertices of K_m^i .

Definition 2 (Distance-hereditary graph). A graph G is called a distance-hereditary graph if distances in any connected induced subgraph are the same as they are in the original graph G .

Definition 3. In P_n^r , a vertex v_j is called s -class element, if $j \equiv s \pmod{r}$, where $0 \leq s \leq r-1$. We shall denote $V^s(P_n^r)$ as the set of all s -th class elements in P_n^r . Note that $V(P_n^r) = \bigcup_{s=0}^{r-1} V^s(P_n^r)$. The largest s -th class element is the s -th class vertex v_ℓ with a maximum index ℓ . For example, in P_{16}^3 ,

$$\begin{aligned} V^0(P_{16}^3) &= \{v_0, v_3, v_6, v_9, v_{12}, v_{15}\}, \\ V^1(P_{16}^3) &= \{v_1, v_4, v_7, v_{10}, v_{13}\}, \\ V^2(P_{16}^3) &= \{v_2, v_5, v_8, v_{11}, v_{14}\}. \end{aligned}$$

The largest 0-class element is v_{15} in P_{16}^3 , whereas the largest 1-class element is v_{13} in P_{16}^3 . Note that $V(P_{16}^3) = V^0(P_{16}^3) \cup V^1(P_{16}^3) \cup V^2(P_{16}^3)$.

Proposition 1. Let a, b and r be three positive integers, where both a and b are greater than r . Then, $\lceil \frac{a}{r} \rceil \neq \lceil \frac{b}{r} \rceil$ if and only if $\min\{a, b\}$ is a multiple of r .

The workflow of the article may be stated as follows. To prove a set W is an RS, we need to verify the globalized conditions in which each pair of vertices is resolved by some vertex of W . First, we transform these globalized conditions into localized conditions in Lemma 3. After that, we show that $\beta'(P_n^r)$ is a non-decreasing function with n (see Lemma 5 and Algorithm 1). Finally, we search for a revolving set F which resolves each consecutive pair vertices of P_n^r by at least two elements of F (see Theorem 1, Theorem 2). To test the optimality of the FTRS presented in Theorems 1 and 2, we give lower bounds in Section 5.

3. Some Basic Properties of RS and FTRS of P_n^r

In this section, we give some basic properties and results of the resolving set and fault-tolerant resolving set of P_n^r . We have proved certain essential lemmas that are beneficial for determining the resolving set and fault-tolerant set of P_n^r . We have also provided an algorithm to construct a fault-tolerant resolving system of P_m^r from a fault-tolerant resolving system of P_n^r when $m < n$.

Definition 4. For a fault-tolerant resolving set F , the location of an element $u \in F$, we mean the position of u in F . For every fault-tolerant resolving set F of P_n^r , there is an arrangement of vertices $v_{i_1}, v_{i_2}, \dots, v_{i_m}$ in F such that $i_1 < i_2 < \dots < i_m$ and location of v_{i_ℓ} is ℓ .

In the lemma below, we determine the vertices which resolve two consecutive vertices of P_n^r .

Lemma 2. For any two vertices v_a and v_{a+1} ,

$$W_{a,a+1} = \{v_j : j \equiv a \pmod{r}, 0 \leq j \leq a\} \cup \{v_j : j \equiv a+1 \pmod{r}, a+1 \leq j \leq n-1\}.$$

Proof. Let v_ℓ be a vertex that resolved v_a and v_{a+1} . The distances of v_a and v_{a+1} from v_ℓ are given by

$$d(v_a, v_\ell) = \left\lceil \frac{|a - \ell|}{r} \right\rceil, \quad d(v_{a+1}, v_\ell) = \left\lceil \frac{|a + 1 - \ell|}{r} \right\rceil.$$

Using Proposition 1, we obtain the following result: $\square$

Corollary 1. For two non-negative integers a and b with $a < b$,

$$W_{a,a+1} \cap W_{b,b+1} = \begin{cases} \{v_{a+1}\} & \text{if } b = a + 1, \\ \emptyset & \text{if } a + 2 \leq b \leq a + r - 1. \end{cases}$$

Lemma 3. A set $W \subset V(P_n^r)$ is a resolving set of P_n^r if and only if it resolves every pair of consecutive vertices of P_n^r .

Proof. If W is a resolving set, then it must resolve every pair of consecutive vertices of P_n^r . So, we assume W resolves every consecutive vertex of P_n^r . We show that W is a resolving set for P_n^r . Let v_i and v_j be two arbitrary non-consecutive vertices in P_n^r . With no loss of generality, we assume $i < j$. We consider the following two cases.

Case 1. $j - i \leq r$. Then, the vertices v_i and v_j are in the same clique, say K . Let v_ℓ resolve the pair (v_i, v_{i+1}) . Then, using Lemma 2, either $\ell < i$ with $\ell \equiv i \pmod{r}$ or $\ell \geq i + 1$ with $\ell \equiv i + 1 \pmod{r}$. Thus, we consider two sub-cases.

Case 1.1. $\ell < i$ with $\ell \equiv i \pmod{r}$. Here, we assume $\ell \equiv s \pmod{r}$. Then, $\ell = rm + s$ and $i = rq + s$ and so, we have

$$d(v_\ell, v_i) = q - m, \quad d(v_\ell, v_j) = \left\lceil \frac{j - \ell}{r} \right\rceil \geq \left\lceil \frac{i + 2 - \ell}{r} \right\rceil = \left\lceil \frac{r(q - m) + 2}{r} \right\rceil = q - m + 1.$$

Thus, v_ℓ resolves the pair (v_i, v_j) .

Case 1.2. $\ell \geq i + 1$ with $\ell \equiv i + 1 \pmod{r}$. First, we take $\ell \neq i + 1$. Then, $\ell > j$ as $j - i \leq r$. Let $\ell \equiv s \pmod{r}$. Then, $\ell = rm + s$ and $i + 1 = rq + s$ for some non-negative integers m and q . Now, the distances of v_i and v_j from v_ℓ are given by

$$d(v_\ell, v_i) = m - q + 1, \quad d(v_\ell, v_j) = \left\lceil \frac{\ell - j}{r} \right\rceil \leq \left\lceil \frac{\ell - i - 1}{r} \right\rceil = \left\lceil \frac{r(m - q)}{r} \right\rceil = m - q.$$

Thus, v_ℓ resolves the pair (v_i, v_j) when $\ell \neq i + 1$. If $\ell = i + 1$, then v_ℓ cannot resolve the pair (v_i, v_j) as $d(v_\ell, v_i) = 1 = d(v_\ell, v_j)$. Then, by similar argument, it can be shown that the vertex $v_{\ell'}$ resolves the pairs (v_i, v_j) , where $v_{\ell'}$ is the vertex that resolves the pair of consecutive vertices (v_{i-1}, v_i) .

Case 2. $j - i > r$. If W consists of a vertex v_ℓ such that either $\ell \geq j$ or $\ell \leq i$, then $d(v_\ell, v_i) \neq d(v_\ell, v_j)$ because

$$d(v_\ell, v_j) = \left\lceil \frac{j - \ell}{r} \right\rceil \geq \left\lceil \frac{i + r - \ell}{r} \right\rceil = d(v_\ell, v_i) + 1 \quad \text{when } \ell \leq i,$$

and

$$d(v_\ell, v_i) = \left\lceil \frac{\ell - i}{r} \right\rceil \geq \left\lceil \frac{\ell - j + r}{r} \right\rceil = d(v_\ell, v_j) + 1 \quad \text{when } \ell \geq j.$$

When all the vertices are between v_i and v_j , then we can prove the result by using a similar argument as shown in **Case 1**. $\square$

Lemma 4. A set $F \subset V(P_n^r)$ is an FTRS of P_n^r if and only if for each of the consecutive vertices u and v of P_n^r , there are at least two elements of F that resolve u and v .

Proof. Using Lemmas 1 and 3, we obtain the result. $\square$

Lemma 5. If P_{n+1}^r has an FTRS having the cardinality p , then P_n^r has also an FTRS with the cardinality p . In particular, $\beta'(P_n^r) \leq \beta'(P_{n+1}^r)$.

Proof. Let F be an FTRS of P_{n+1}^r . Note that P_n^r is a distance-hereditary graph. If $v_n \notin F$, then F will also be an FTRS of P_n^r . Hence the result is true in this case. So we assume $v_n \in F$. Let ℓ be the largest index such that $\ell \equiv n \pmod{r}$. Let us define a set F_1 by

$$F_1 = \begin{cases} F \cup \{v_\ell\} \setminus \{v_n\} & \text{if } v_\ell \notin F, \\ F \cup \{v_{n \pmod{r}-1}\} \setminus \{v_n\} & \text{if } v_\ell \in F \text{ and } n \not\equiv 0 \pmod{r}, \\ F \cup \{v_{r-1}\} \setminus \{v_n\} & \text{if } v_\ell \in F \text{ and } n \equiv 0 \pmod{r}. \end{cases}$$

Our claim F_1 is an FTRS of P_n^r . Due to Lemma 4, it is sufficient to prove that every pair of consecutive vertices are resolved by at least two elements of F_1 . Let (v_{i-1}, v_i) be a pair consecutive vertices. If $i \not\equiv n \pmod{r}$, then v_n can not resolve this pair due to Lemma 2. Therefore, there exists at least two vertices v_x and v_y in $F \setminus \{v_n\}$ that resolve the pair (v_{i-1}, v_i) . Hence, this pair is resolved by two elements of F_1 when $i \not\equiv n \pmod{r}$. So, we assume $i \equiv n \pmod{r}$. Since F is a fault-tolerant resolving set of P_n^r , so there exists a vertex v_x ($x \neq n$) that resolve (v_{i-1}, v_i) . Then due to Lemma 2, either v_x is a left side vertex of v_{i-1} with $x \equiv i-1 \pmod{r}$ or v_x is a right side vertex of v_i with $x \equiv i \pmod{r}$. In any case the pair (v_{i-1}, v_i) are resolved by at least two elements of F when $i \equiv n \pmod{r}$. This completes the proof. $\square$

Algorithm to Construct an FTRS of P_m^r from an FTRS of P_n^r , Where $m < n$

In this section, we have presented an algorithm to construct an FTRS of P_m^r from an FTRS of P_n^r , where $m < n$. In this algorithm, the cost of Step III is $O(r)$ and the cost of Step V is $O(|F|)$. These are the most expensive cases. So our proposed algorithm has linear time complexity.

Remark 2. Using Lemma 5 we can prove the correctness of the proposed algorithm (Algorithm 1).

Algorithm 1: Construction of an FTRS of P_m^r from an FTRS of P_n^r , where $m < n$.

Input: A k -size FTRS $F = \{v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$ for P_n^r , where $i_1 < i_2 < \dots < i_k$.

Initialization : The set $V(P_n^r) = \{v_0, v_1, \dots, v_{n-1}\}$ and $V(P_m^r) = \{v_0, v_1, \dots, v_{m-1}\}$, where $m < n$.

Step I : If $F \setminus V(P_m^r) = \emptyset$, then go to Step-VII.

Step II : For a vertex $u \in F \setminus V(P_m^r)$, calculate the class of u , say it is a -class element.

Step III : Find the largest a -class element in P_m^r , say it is v_ℓ .

Step IV : Update F by

$$F = \begin{cases} F \cup \{v_\ell\} \setminus \{u\} & \text{if } v_\ell \notin F, \\ F \cup \{v_{a-1}\} \setminus \{u\} & \text{if } v_\ell \in F \text{ and } a \neq 0, \\ F \cup \{v_{r-1}\} \setminus \{u\} & \text{if } v_\ell \in F \text{ and } a = 0. \end{cases}$$

Step V : Remove duplicate elements from F .

Step VI : Repeat Step I to Step IV with the new F .

Step VII : F is an FTRS.

Example 1. In this example, we explain the intermediate stages of our presented algorithm by taking the graphs P_{21}^5 and P_{41}^5 . We start with the FTRS of P_{41}^5 as $F = \{v_4, v_8, v_{12}, v_{16}, v_{20}, v_{24}, v_{28}, v_{32}, v_{36}\}$, $V(P_{41}^5) = \{v_0, v_1, \dots, v_{40}\}$ and $V(P_{21}^5) = \{v_0, v_1, \dots, v_{20}\}$.

Step I: Here, $v_{24} \in F \setminus V(P_{21}^5)$ and it is a 4-class element of $V(P_{41}^5)$.

Step II: The vertex v_{19} is the largest 4-class element of $V(P_{21}^5)$ and $v_{19} \notin F$.

Step III: So, we update F by replacing v_{24} by v_{19} (according to Step IV of Algorithm 1).

Step IV: Repeat Step I to Step III for the vertices v_{28} and v_{32} and update F by replacing v_{28} by v_{18} and replacing v_{32} by v_{17} . However, for the vertex v_{36} we obtain v_{16} as the largest class element of $V(P_{21}^5)$ and $v_{16} \in F$. So, here, we update F by replacing v_{36} by v_0 (according to STEP-IV of Algorithm 1).

OUTPUT: $F = \{v_4, v_8, v_{12}, v_{16}, v_{20}, v_{19}, v_{18}, v_{17}, v_0\}$ is an FTRS for P_{21}^5 .

Example 2. In this example, we explain the intermediate stages of our presented algorithm by taking the graphs P_{26}^5 and P_{41}^5 . We start with the FTRS $F = \{v_4, v_8, v_{12}, v_{16}, v_{20}, v_{24}, v_{28}, v_{32}, v_{36}\}$ of P_{41}^5 , $V(P_{41}^5) = \{v_0, v_1, \dots, v_{40}\}$ and $V(P_{26}^5) = \{v_0, v_1, \dots, v_{25}\}$.

Step I: Here, $v_{28} \in F \setminus V(P_{26}^5)$ and it is a 3-class element of $V(P_{41}^5)$.

Step II: The vertex v_{23} is the largest 3-class element of $V(P_{26}^5)$ and $v_{23} \notin F$.

Step III: So, we update F by replacing v_{28} by v_{23} (according to Step IV of Algorithm 1).

Step IV: Repeat Step I to Step III for the other two elements v_{32} and v_{36} and update F by replacing v_{32} by v_{22} and by replacing v_{36} by v_{21} .

OUTPUT: $F = \{v_4, v_8, v_{12}, v_{16}, v_{20}, v_{24}, v_{23}, v_{22}, v_{21}\}$ is an FTRS for P_{26}^5 .

4. FTRS of P_n^r

In this section, we determine fault-tolerant resolving sets of P_n^r and jotted down different examples to justify our result. In the next section, we show that these are optimal FTRS values.

Theorem 1. The set $F = \{v_0, v_1, \dots, v_{2r-1}\}$ is a fault-tolerant resolving set of P_n^r .

Proof. To prove the result, we have to show that for every pair of distinct vertices u and v of P_n^r , there exist at least two elements in F that resolved u and v . Let v_i and v_j be two vertices of P_n^r . Without loss of generality, we assume $i < j$. If v_i and v_j both are in F , then we have nothing to prove because in that case, one resolves the another. We consider the following two cases according to v_i and v_j are in F or not.

Case 1. Both v_i and v_j are outside of F . Let $i \equiv s \pmod{r}$, where $s \in \{0, 1, \dots, r-1\}$, i.e., $i = mr + s$ for some positive integer m . Define $U = \{v_s, v_{r+s}\} \subset F$. The distances of v_i and v_j from every vertex $v_\ell \in U$ are given below.

$$d(v_i, v_\ell) = \left\lceil \frac{i-\ell}{r} \right\rceil = \begin{cases} m & \text{if } \ell = s, \\ m-1 & \text{if } \ell = r+s. \end{cases}$$

and

$$d(v_j, v_\ell) = \left\lceil \frac{j-\ell}{r} \right\rceil \geq \begin{cases} m+1 & \text{if } \ell = s, \\ m & \text{if } \ell = r+s. \end{cases} \quad [\text{as } j > i \text{ and } i-\ell \text{ is multiple of } r]$$

Therefore, $d(v_i, v_\ell) \neq d(v_j, v_\ell)$ for all $v_\ell \in U$, i.e., v_i and v_j are resolved by both v_s and v_{r+s} whenever $i \equiv s \pmod{r}$.

Case 2. Exactly one of v_i and v_j is in F . As $i < j$ and F consists of consecutive vertices starting from the lower index vertex, in this case, $v_i \in F$, and $v_j \notin F$; otherwise, $v_j \in F$

gives $v_i \in F$. Here, we show there exists an element $v_\ell \in F$ ($\ell \neq i$) that resolved v_i and v_j . Since $v_i \in F$ and $v_j \notin F$, so $0 \leq i \leq 2r - 1$ and $j \geq 2r$. We consider the following sub-cases.

Case 2.1. $0 \leq i \leq r - 1$. In this case, for each $\ell \neq i$ with $0 \leq \ell \leq r - 1$, we obtain

$$\begin{aligned} d(v_i, v_\ell) &= 1, \\ d(v_j, v_\ell) &= \left\lceil \frac{j - \ell}{r} \right\rceil \geq \left\lceil \frac{2r - (r - 1)}{r} \right\rceil \geq 2. \end{aligned}$$

Thus, if $0 \leq i \leq r - 1$, then v_i and v_j are resolved by v_ℓ , where $0 \leq \ell \leq r - 1$. For better understanding, the reader may follow Example 3. In this example, we see that $d(v_\ell, v_j) = 2$ or 1, and $d(v_\ell, v_j) \geq 2$ for all $j \geq 10$ and $0 \leq i, \ell \leq 5$.

Case 2.2. $r \leq i \leq 2r - 1$. In this case, $v_{i-r} \in F$ and

$$\begin{aligned} d(v_i, v_{i-r}) &= 1, \\ d(v_j, v_{i-r}) &= \left\lceil \frac{j + r - i}{r} \right\rceil \geq \left\lceil \frac{3r - (2r - 1)}{r} \right\rceil \geq 2. \end{aligned}$$

Thus, in this case, v_i and v_j are resolved by v_{i-r} . $\square$

Example 3. In the following, we construct the code matrix $C(P_{24}^5)$ of P_{24}^5 with respect to $F = \{v_0, v_1, \dots, v_9\}$. In this matrix, the i -th column, $0 \leq i \leq 23$, represents the code of the vertex v_i with respect to the set F .

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}	v_{18}	v_{19}	v_{20}	v_{21}	v_{22}	v_{23}
v_0	0	1	1	1	1	1	2	2	2	2	2	3	3	3	3	3	4	4	4	4	4	5	5	5
v_1	1	0	1	1	1	1	1	2	2	2	2	2	3	3	3	3	3	4	4	4	4	4	5	5
v_2	1	1	0	1	1	1	1	1	2	2	2	2	2	3	3	3	3	4	4	4	4	4	4	5
v_3	1	1	1	0	1	1	1	1	1	2	2	2	2	2	3	3	3	3	4	4	4	4	4	4
v_4	1	1	1	1	0	1	1	1	1	1	2	2	2	2	2	3	3	3	3	4	4	4	4	4
v_5	1	1	1	1	1	0	1	1	1	1	1	2	2	2	2	2	3	3	3	3	4	4	4	4
v_6	2	1	1	1	1	1	0	1	1	1	1	1	2	2	2	2	2	3	3	3	3	4	4	4
v_7	2	2	1	1	1	1	1	0	1	1	1	1	1	2	2	2	2	2	3	3	3	3	4	4
v_8	2	2	2	1	1	1	1	1	0	1	1	1	1	1	2	2	2	2	2	3	3	3	3	3
v_9	2	2	2	2	1	1	1	1	1	0	1	1	1	1	1	2	2	2	2	2	3	3	3	3

From this code matrix, we see that every pair of vertices is resolved by at least two elements of F . For example, the vertices v_{10} and v_{11} are resolved by v_0 and v_5 . More generally, any pair (v_i, v_{i+1}) of two consecutive vertices which are not in F is resolved by exactly two elements of F , namely, $v_{i \pmod{5}}$ and $v_{5+i \pmod{5}}$. The pair (v_8, v_{19}) is resolved by every vertex of F . More generally, the pair (v_i, v_j) with $|i - j| \geq 11$ is resolved by every vertex of F .

Theorem 2. There exists a fault-tolerant resolving set of P_n^r with a cardinality of $2r - 1$ if $r(r - 1) + 2 \leq n \leq 2r(r - 1) + 1$.

Proof. Since $\beta'(P_m^r) \leq \beta'(P_n^r)$ for $m \leq n$ due to Lemma 5, it is sufficient to show that there is an FTRS of $P_{2r(r-1)+1}^r$ with the cardinality $2r - 1$. Our claim is to show that the set $F = \{v_{\ell(r-1)} : 1 \leq \ell \leq 2r - 1\}$ is an FTRS of $P_{2r(r-1)+1}^r$. To determine the classes of the elements of F , we may write $\ell(r - 1)$, where $1 \leq \ell \leq 2r - 1$, as in below

$$\ell(r - 1) = \begin{cases} (\ell - 1)r + r - \ell & \text{for } 1 \leq \ell \leq r, \\ (\ell - 2)r + 2r - \ell & \text{if } r + 1 \leq \ell \leq 2r - 1. \end{cases}$$

Thus, $|F \cap V^s(P_n^r)| = 2$ for each $s \in \{1, 2, \dots, r - 1\}$ and $|F \cap V^0(P_n^r)| = 1$. In addition, the first and the last elements in F are $r - 1$ class and 1 class, respectively. First, we show that any two consecutive vertices of $P_{2r(r-1)+1}^r$ are resolved by at least two elements of F . Let v_i

and v_{i+1} be two consecutive vertices of $P_{2r(r-1)+1}^r$. Then, $0 \leq i \leq 2r(r-1) - 1$. We consider the following cases.

Case 1. $0 \leq i \leq r-1$. The pair (v_{r-1}, v_r) is resolved by the vertices v_{r-1} and $v_{r(r-1)}$. Thus, we assume $0 \leq i \leq r-2$. Then, both the vertices $v_{(r-(i+1))(r-1)}$ and $v_{(2r-(i+1))(r-1)}$ are in $F \cap V^{i+1}(P_{2r(r-1)+1}^r)$. In addition, note that these two vertices are on the right side of the pair (v_i, v_{i+1}) . Thus, by using Lemma 2, the pair (v_i, v_{i+1}) are resolved by both the vertices $v_{(r-(i+1))(r-1)}$ and $v_{(2r-(i+1))(r-1)}$.

Case 2. $(2r-1)(r-1) \leq i \leq 2r(r-1) - 1$. The pair $(v_{(2r-1)(r-1)}, v_{(2r-1)(r-1)+1})$, if taken, is resolved by both the vertices $v_{(r-1)(r-1)}$ and $v_{(2r-1)(r-1)}$ of F . So, we assume $(2r-1)(r-1) + 1 \leq i \leq 2r(r-1) - 1$. Then, $v_i \in V^s(P_{2r(r-1)+1}^r)$ for some integer s with $1 \leq s \leq r-1$. Thus, we may assume $i \equiv s \pmod{r}$, where $1 \leq s \leq r-1$. Then, the vertices $v_{(r-s)(r-1)}$ and $v_{(2r-s)(r-1)}$ are in $F \cap V^s(P_{2r(r-1)+1}^r)$. In addition, both the elements are on the left side of the pair (v_i, v_{i+1}) . By using Lemma 2, in this case, the pair (v_i, v_{i+1}) is resolved by both the vertices $v_{(r-s)(r-1)}$ and $v_{(2r-s)(r-1)}$ of F .

Case 3. $r \leq i \leq (2r-1)(r-1) - 1$. Let $i \equiv s \pmod{r}$, where $0 \leq s \leq r-1$. We consider the following sub-cases.

Case 3.1. $s \in \{1, 2, \dots, r-2\}$. Recall that $|F \cap V^s(P_n^r)| = 2$. If the two elements of $F \cap V^s(P_n^r)$ are in the left side of the pair (v_i, v_{i+1}) or the two elements of $F \cap V^{s+1}(P_n^r)$ are in the right side of (v_i, v_{i+1}) , then Lemma 2 gives that the pair (v_i, v_{i+1}) is resolved by two elements of F . Again, if at least one element of $F \cap V^s(P_n^r)$ is in the left side of the pair (v_i, v_{i+1}) and at least one element of $F \cap V^{s+1}(P_n^r)$ is in the right side of (v_i, v_{i+1}) , then by the same lemma, this pair is resolved by two elements of F . The largest $s+1$ -class element of F is $v_{(2r-(s+1))(r-1)}$ and the smallest s -class element of F is $v_{(r-s)(r-1)}$. Now, we show that the following statement does not hold for F : "All elements of $F \cap V^{s+1}(P_n^r)$ are in the left side of the pair (v_i, v_{i+1}) and all the elements of $F \cap V^s(P_n^r)$ are in the right of (v_i, v_{i+1}) ". If all the elements of $F \cap V^{s+1}(P_n^r)$ are in the left side of the pair (v_i, v_{i+1}) and all the elements of $F \cap V^s(P_n^r)$ are in the right of (v_i, v_{i+1}) , then $(2r-(s+1))(r-1) \leq i \leq (r-s)(r-1)$, which is not possible because $r \geq 2$. Thus, in this case, the pair (v_i, v_{i+1}) is resolved by at least two vertices of F .

Case 3.2. $s \in \{0, r-1\}$. First, we assume $s = 0$. If $i \leq (r-1)(r-1) - 1$, then $v_{(r-1)(r-1)}$ and $v_{(2r-1)(r-1)}$ resolved the pair (v_i, v_{i+1}) due to Lemma 2. Since $(r-1)(r-1) - 1 \equiv 0 \pmod{r} \equiv i \pmod{r}$, so we take $i \geq r(r-1)$. By using same lemma, this pair is resolved by $v_{r(r-1)}$ and $v_{(2r-1)(r-1)}$, provided $r(r-1) \leq i \leq (2r-1)(r-1)$. By similar argument, we can prove the result when $s = r-1$. $\square$

Example 4. In this example, we construct an optimal FTRS of P_{13}^3 as presented in Theorem 2. Here, $r = 3$ and $n = 13$, so $n = 6 \cdot 2 + 1 = 2r(r-1) + 1$. According to Theorem 2, the set F is given by

$$F = \{v_{2i} : 1 \leq i \leq 5\} = \{v_2, v_4, v_6, v_8, v_{10}\}.$$

Note that $F \cap V^0(P_{13}^3) = \{v_6\}$, $F \cap V^1(P_{13}^3) = \{v_4, v_{10}\}$ and $F \cap V^2(P_{13}^3) = \{v_2, v_8\}$. According to **Case 1** of Theorem 2, each element of $\{v_2, v_6\}$ resolves the vertices v_2 and v_3 , and each vertex of $\{v_4, v_{10}\}$ resolves vertices v_0 and v_1 (see the code matrix $C(P_{13}^3)$). As for **Case 2** of Theorem 2, the pair (v_{10}, v_{11}) is resolved by the vertices v_4 and v_{10} . From the code matrix $C(P_{13}^3)$ with respect to

$F = \{v_2, v_4, v_6, v_8, v_{10}\}$, we see that every pair of vertices is resolved by at least two elements of F , where

$$C(P_{13}^3) = \begin{matrix} & v_0 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} & v_{11} & v_{12} \\ \begin{matrix} v_2 \\ v_4 \\ v_6 \\ v_8 \\ v_{10} \end{matrix} & \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 \\ 2 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ 2 & 2 & 2 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 3 & 3 & 2 & 2 & 2 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 2 \\ 4 & 3 & 3 & 3 & 2 & 2 & 2 & 1 & 1 & 1 & 0 & 1 & 1 \end{pmatrix} \end{matrix}.$$

5. Lower Bounds of $\beta'(P_n^r)$

Theorems 1 and 2 give two fault-tolerant resolving sets for P_n^r . However, the question that can be posed is: are these optimal? In this section, first, we determine lower bounds of $\beta'(P_n^r)$ for all n and r , and then, we show that the FTRS presented in Theorems 1 and 2 are optimal for all most all values of n . To present lower bounds, we need the following notations, lemmas and definitions.

Lemma 6. Let F be fault-tolerant resolving set for P_n^r and K_{r+1} be a clique in P_n^r . If $F \cap I(K_{r+1}) = \emptyset$, then $|F| \geq 2r$, where $I(K_{r+1})$ denotes the set of all intermediate vertices of K_{r+1} .

Proof. Let $V(K_{r+1}) = \{v_i, v_{i+1}, \dots, v_{i+r-1}, v_{i+r}\}$. So, $I(K_{r+1}) = \{v_{i+1}, \dots, v_{i+r-1}\}$. Since $F \cap I(K_{r+1}) = \emptyset$, applying Corollary 1, we have $(F \cap W_{a,a+1}) \cap (F \cap W_{a+1,a+2}) = \emptyset$ for each $a \in \{i, i+1, \dots, i+r-2\}$. Again, from Remark 1, we have $|F \cap W_{a,a+1}| \geq 2$ for every $a \in \{i, i+1, \dots, i+r-1\}$. Hence, we obtain the result as $(F \cap W_{a,a+1}) \cap (F \cap W_{a+1,a+2}) = \emptyset$ for each $a \in \{i, i+1, \dots, i+r-2\}$. $\square$

Lemma 7. For any fault-tolerant resolving set F and a clique K_{r+1} , $|F| \geq 2r - |F \cap I(K_{r+1})|$, where $I(K_{r+1})$ denotes the set of all intermediate vertices of K_{r+1} .

Proof. Let $V(K_{r+1}) = \{v_i, v_{i+1}, \dots, v_{i+r-1}, v_{i+r}\}$. For a fault-tolerant resolving set F with $F \cap V(K_{r+1}) = \emptyset$, $|F| \geq 2r$ due to Lemma 6. Now, our claim is that if $v_\ell \in I(K_{r+1}) \cap F$, then the cardinality of F is reduced by at most one. Let $F \cap I(K_{r+1}) = \{v_\ell\}$. Let $F_1 = F \setminus \{v_\ell\}$. Then, $|W_{a,a+1} \cap F_1| \geq 2$ or $|W_{a,a+1} \cap F_1| \geq 1$ according to $a \in \{i, i+1, \dots, i+r-1\} \setminus \{\ell-1, \ell\}$ or $a \in \{\ell-1, \ell\}$. Thus, $|F_1| \geq 2(r-2) + 2$ and consequently, $|F| \geq 2r - 1$. Hence, the result is true for $|F \cap I(K_{r+1})| = 1$. $\square$

Corollary 2. Let F be the FTRS of P_n^r having $2r - a$ elements. Then, for every clique K_{m+1} , $|F \cap I(K_{m+1})| \geq 2m - 2r + a$.

From here onward, we denote K_{p+1}^ℓ as the clique with the vertex set $\{v_\ell, v_{\ell+1}, \dots, v_{\ell+p}\}$.

Lemma 8. Let F be a fault-tolerant resolving set for P_n^r consisting of a gap r or more; then, $|F| \geq 2r$.

Proof. Since F has a gap r or more, there exists vertices $v_{i+1}, \dots, v_{i+r-1}$ which are not in F . Now, consider the clique K_{r+1} with $V(K_{r+1}) = \{v_i, v_{i+1}, \dots, v_{i+r}\}$. Then, $F \cap I(K_{r+1}) = \emptyset$ and so applying Lemma 6, we have $|F| \geq 2r$. $\square$

Lemma 9. Let $n \geq 2r(r-1) + 2$. Then, every fault-tolerant resolving set for P_n^r has a cardinality of at least $2r$.

Proof. Let F be a fault-tolerant resolving set of P_n^r with cardinality m . Without loss of generality, we may assume $F = \{v_{i_1}, v_{i_2}, \dots, v_{i_m}\}$, where $i_1 < i_2 < \dots < i_m$. We consider the following two cases.

Case 1. F consists of an internal gap with a size r or more. In this case, $i_{\ell+1} - i_\ell \geq r$ for some $\ell \in \{1, 2, \dots, m-1\}$; then, $F \cap I(K_{r+1}) = \emptyset$, where $V(K_{r+1}) = \{v_{i_\ell}, v_{i_\ell+1}, \dots, v_{i_\ell+r-1}, v_{i_\ell+r}\}$. Then, applying Lemma 6 to the clique K_{r+1} , we obtain $|F| \geq 2r$.

Case 2. F does not have any internal gap with size r or more. In this case, $i_{\ell+1} - i_\ell \leq r-1$ for every $\ell \in \{1, 2, \dots, m-1\}$. Our claim is that an external gap exists with size r or more. Since $i_{\ell+1} - i_\ell \leq r-1$ for each $\ell \in \{1, 2, \dots, m-1\}$, we obtain

$$i_m - i_1 \leq (m-1)(r-1). \quad (1)$$

We take the following two sub-cases.

Case 2.1. $i_1 \geq r$. In this case, F does not contain any vertex from the segment $v_0 v_1 \dots v_{r-1}$. Let K'_{r+1} be the clique on $(r+1)$ vertices $v_0, v_1, \dots, v_r$. Then, $F \cap I(K'_{r+1}) = \emptyset$ and so applying Lemma 6 to this clique, we obtain $|F| \geq 2r$.

Case 2.2. $i_1 \leq r-1$. In this case, we prove the result by the contrary. Suppose that $|F| < 2r$. Since $i_1 \leq r-1$, by using (1), we have $i_m \leq m(r-1)$. Since $|F| \leq 2r-1$, $m \leq 2r-1$ and so $i_m \leq (2r-1)(r-1)$. Since $i_m \leq (2r-1)(r-1)$ and $n \geq 2r(r-1)+2$, $i_m + r - 1 \leq n - 2$. Then, $v_{i_m}, v_{i_m+1}, v_{i_m+2}, \dots, v_{i_m+r-1}, v_{i_m+r}$ forms a clique in P_n^r on $(r+1)$ vertices, and we name this clique as K_{r+1} . Then, applying Lemma 7 to the clique K_{r+1} , we have $|F \cap I(K_{r+1})| \geq 1$, i.e., $F \cap \{v_{i_m+1}, v_{i_m+2}, \dots, v_{i_m+r-1}\} \neq \emptyset$, which contradicts that i_m is the last index such that $v_{i_m} \in F$. Thus, $|F| \geq 2r$. In view of the above cases, we obtain the result. $\square$

Theorem 3. The fault-tolerant metric dimension of P_n^r is $2r$ provided $n \geq 2r(r-1)+2$.

Proof. The result follows immediately from Lemma 9 and Theorem 1. $\square$

Definition 5. The two vertices v_i and v_j in P_n^r are said to be the same class elements if $i \equiv j \pmod{r}$. For example, v_{33} and v_{41} are the same class elements in P_{47}^4 .

Theorem 3 determines the fault-tolerant metric dimension of P_n^r when $n \geq 2r(r-1)+2$. Now, we find the FTRS and fault-tolerant metric dimension of P_n^r for smaller values of n .

Lemma 10. Let $r(r-1)+2 \leq n \leq 2r(r-1)+1$ and F be a fault-tolerant resolving set of P_n^r . Then, $|F| \geq 2r-1$.

Proof. We prove the lemma by the contrary. Suppose that there is a fault-tolerant resolving set F with a cardinality of at most $2r-2$. Then, $|F \cap I(K_{r+1}^i)| \geq 2$ for every $i \in \{0, r-1, 2(r-1), \dots, (r-2)(r-1)\}$ and $|F \cap I(K_r^i)| \geq 1$ for $i = (r-1)(r-1)$, which implies that $|F| \geq 2r-1$. Thus, we obtain a contradiction. $\square$

Theorem 4. The fault-tolerant metric dimension of P_n^r is $2r-1$, provided $r(r-1)+2 \leq n \leq 2r(r-1)+1$.

Proof. The result follows immediately from Lemma 10 and Theorem 2. $\square$

On account of Theorems 3 and 4, we say that the exact value of the fault-tolerant metric dimension has been obtained when $n \geq r(r-1)+2$. Again, in virtue of Lemma 5 and Algorithm 1, $\beta'(P_n^r) \leq 2r-1$ for all $n \leq r(r-1)+1$. The lemmas and theorems given below give the range of $\beta'(P_n^r)$ for the values of $n \leq r(r-1)+1$.

Lemma 11. Let a be an integer such that $1 \leq a \leq \lfloor \frac{2r}{3} \rfloor$. If $n \geq \lceil \frac{2r-a}{a+1} \rceil (r-1) + 2$; then, $\beta'(P_n^r) \geq 2r - a$.

Proof. Let F be an arbitrary fault-tolerant resolving set of P_n^r . If there exists a clique K_{r+1} such that $|F \cap I(K_{r+1})| \leq a$, then using Lemma 7, F consists of at least $2r - a$ elements. Thus, we assume that $|F \cap I(K_{r+1})| \geq a + 1$ for every clique K_{r+1} on $r + 1$ vertices. The given condition $n \geq \lceil \frac{2r-a}{a+1} \rceil (r-1) + 2$ gives $(\lceil \frac{2r-a}{a+1} \rceil - 1)(r-1) + r \leq n - 1$. Let $p = \lceil \frac{2r-a}{a+1} \rceil$ and $S = \{0, r-1, 2(r-1), \dots, (p-1)(r-1)\}$. Then, for each $i \in S$, we obtain

$$|F \cap I(K_{r+1}^i)| \geq a + 1. \quad (2)$$

Note that $(F \cap I(K_{r+1}^i)) \cap (F \cap I(K_{r+1}^j)) = \emptyset$ for distinct $i, j \in S$, so using (2), we have

$$\begin{aligned} |F| &\geq \sum_{\ell \in S} |F \cap I(K_{r+1}^\ell)| \geq (a+1)|S| \\ &= (a+1)p \\ &= (a+1) \left\lceil \frac{2r-a}{a+1} \right\rceil \\ &\geq 2r - a. \end{aligned}$$

So finally, $|F| \geq 2r - a$ for every fault-tolerant resolving set F provided $n \geq \lceil \frac{2r-a}{a+1} \rceil (r-1) + 2$. This completes the proof. $\square$

Theorem 5. For P_n^r , $\beta'(P_n^r) \geq 2r$, provided $n \geq 2r(r-1) + 2$. Moreover, $\beta'(P_n^r) \geq 2r - a$ provided $\lceil \frac{2r-a}{a+1} \rceil (r-1) + 2 \leq n \leq \lceil \frac{2r-a+1}{a} \rceil (r-1) + 1$.

Proof. If $n \geq 2r(r-1) + 2$, then applying Lemma 7, we obtain that $|F| \geq 2r$ for every fault-tolerant resolving set F of P_n^r . Thus, $\beta'(P_n^r) \geq 2r$, provided $n \geq 2r(r-1) + 2$. Applying Lemma 11, we obtain that

$$\beta'(P_n^r) \geq \begin{cases} 2r - a & \text{if } n \geq \lceil \frac{2r-a}{a+1} \rceil (r-1) + 2, \\ 2r - a + 1 & \text{if } n \geq \lceil \frac{2r-a+1}{a} \rceil (r-1) + 2. \end{cases} \quad (3)$$

Inequality (3) gives the result. $\square$

Theorem 6. Let $n \equiv s \pmod{(r-1)}$ and a be a positive integer with $1 \leq a \leq \lfloor \frac{2r}{3} \rfloor$. If $\beta'(P_n^r) = 2r - a$, then the upper bound on n is as follows:

(a) For $s \notin \{0, 1\}$,

$$n \leq \begin{cases} \left\lfloor \frac{4r-2s-2a+2}{a} \right\rfloor (r-1) + s & \text{if } a > 2(r-s+1), \\ \left\lfloor \frac{2r-a}{a} \right\rfloor (r-1) + s & \text{otherwise.} \end{cases}$$

(b) For $s = 0$,

$$n \leq \begin{cases} \left\lfloor \frac{2r-a+4}{a} \right\rfloor (r-1) & \text{if } a > 4, \\ \left(\left\lfloor \frac{2r-a}{a} \right\rfloor + 1 \right) (r-1) & \text{otherwise.} \end{cases}$$

(c) For $s = 1$,

$$n \leq \begin{cases} \left\lfloor \frac{2r-a+2}{a} \right\rfloor (r-1) + 1 & \text{if } a > 2, \\ \left(\left\lfloor \frac{2r-a}{a} \right\rfloor + 1 \right) (r-1) + 1 & \text{otherwise.} \end{cases}$$

Proof. By division algorithm, we may write $n = m(r-1) + s$ for some integers m and s with $0 \leq s \leq r-2$. Let F be an FTRS such that $|F| = 2r - a$. Using Lemma 7, we obtain $|F \cap I(K_{r+1})| \geq a$ for every clique K_{r+1} . We consider the following two cases.

Case 1. $s \notin \{0, 1\}$, i.e., $2 \leq s \leq r-2$. Let $U = \{0, r-1, 2(r-1), \dots, (m-1)(r-1)\}$. Since $s \geq 2$, $(m-1)(r-1) + (r-1) = m(r-1) < n-1$ and so for each $i \in U$, we have the following inequality

$$|F \cap \{v_{i+1}, v_{i+2}, \dots, v_{i+r-1}\}| \geq a. \quad (4)$$

Note that $F \cap \{v_i, v_{i+1}, \dots, v_{i+r-1}\}$ and $F \cap \{v_j, v_{j+1}, \dots, v_{j+r-1}\}$ are disjoint for two distinct elements i and j of U . Thus, on account of the above inequalities for all $i \in U$, we obtain $|F \cap \{v_1, \dots, v_{m(r-1)}\}| \geq ma$. Again, if $a > 2(r-s+1)$, then applying Corollary 2 to the clique on the vertices $\{v_{m(r-1)}, \dots, v_{m(r-1)+s-1}\}$, we obtain

$$|F \cap \{v_{m(r-1)+1}, \dots, v_{m(r-1)+s-2}\}| \geq 2(s-1) - 2r + a = a - 2(r-s+1).$$

Since $|F| = 2r - a$, $ma \leq 2r - a$ or $ma + a - 2(r-s+1) \leq 2r - a$ according to $a \leq 2(r-s+1)$ or $a > 2(r-s+1)$. Thus, $m \leq \left\lfloor \frac{2r-a}{a} \right\rfloor$ or $m \leq \left\lfloor \frac{4r+2s-2a+2}{a} \right\rfloor$ according to $a \leq 2(r-s+1)$ or $a > 2(r-s+1)$. Using the maximum possible value of m in $n = m(r-1) + s$, we obtained the result.

Case 2. $s \in \{0, 1\}$. We consider the following two sub-cases.

Case 2.1. $s = 0$. In this case, $n = m(r-1)$. Let $U_1 = \{0, r-1, \dots, (m-2)(r-1)\}$. Then, by similar argument as described in Case 1, we obtain that the inequality

$$|F \cap \{v_i, v_{i+1}, \dots, v_{i+r-1}\}| \geq a$$

is true for each $i \in U_1$. Again, if $a > 4$, then using Corollary 2 to the clique with vertices $\{v_{(m-1)(r-1)}, \dots, v_{m(r-1)-1}\}$, we obtain

$$|F \cap \{v_{(m-1)(r-1)+1}, \dots, v_{m(r-1)-2}\}| \geq 2(r-2) - (2r-a) = a-4.$$

Thus, F consists of at least $(m-1)a + a - 4$ elements or at least $(m-1)a$ according to $a > 4$ or $a \leq 4$. Since $|F| = 2r - a$, so $(m-1)a + a - 4 \leq 2r - a$ or $(m-1)a \leq 2r - a$ according to $a > 4$ or $a \leq 4$. Thus, $m \leq \left\lfloor \frac{2r-a+4}{a} \right\rfloor$ or $m \leq \left\lfloor \frac{2r-a}{a} \right\rfloor + 1$ according to $a > 4$ or $a \leq 4$. By putting the maximum possible values of m in $n = m(r-1)$ for different values of a , we obtain the result.

Case 2.2. $s = 1$. In this case, $n = m(r-1) + 1$. Let $U_1 = \{0, r-1, \dots, (m-2)(r-1)\}$. By a similar argument as in Case 1, we have that the inequality

$$|F \cap \{v_i, v_{i+1}, \dots, v_{i+r-1}\}| \geq a$$

is true for each $i \in U_1$. Again, if $a > 2$, then using Corollary 2 to the clique with vertices $\{v_{(m-1)(r-1)}, \dots, v_{m(r-1)}\}$, we obtain

$$|F \cap \{v_{(m-1)(r-1)+1}, \dots, v_{m(r-1)-1}\}| \geq 2(r-1) - (2r-a) = a-2.$$

Then, by a similar argument as of **Case 2.1**, we obtain the result. $\square$

6. Applications of Bounds

In this section, we apply the results of lower and upper bounds that we have obtained in previous sections for P_n^r , where $r \in \{2, 3, 4, 5, 6, 8\}$. We have presented some examples to verify the sharpness of our bounds. Algorithm 1 is also used to construct an optimal fault-tolerant resolving set of P_n^r , when $r(r-1) + 2 \leq n \leq 2r(r-1) + 1$, from an optimal fault-tolerant resolving set of $P_{2r(r-1)+2}^r$, which can be found from the proof of Theorem 2.

Example 5. Theorem 5 gives that $\beta'(P_n^2) \geq 4$ for $n \geq 6$, whereas Theorem 1 shows that $\beta'(P_n^2) \leq 4$. Thus, we obtain the result that the fault-tolerant metric dimension of the square of path P_n^2 is 4 when $n \geq 6$. In the below, we calculate the codes of each vertex for P_{10}^2 with respect to the fault-tolerant resolving set $F = \{v_0, v_1, v_2, v_3\}$ (Figure 1). The distance matrix D for P_{10}^2 has been given below.

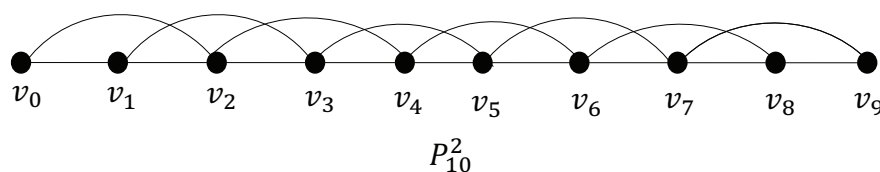


Figure 1. The figure shows a 10 vertex square path.

$$D = \begin{matrix} & v_0 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 \\ \begin{matrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \end{matrix} & \begin{pmatrix} 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 \\ 1 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \\ 1 & 1 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 \\ 2 & 1 & 1 & 0 & 1 & 1 & 2 & 2 & 3 & 3 \\ 2 & 2 & 1 & 1 & 0 & 1 & 1 & 2 & 2 & 3 \\ 3 & 2 & 2 & 1 & 1 & 0 & 1 & 1 & 2 & 2 \\ 3 & 3 & 2 & 2 & 1 & 1 & 0 & 1 & 1 & 2 \\ 4 & 3 & 3 & 2 & 2 & 1 & 1 & 0 & 1 & 1 \\ 4 & 4 & 3 & 3 & 2 & 2 & 1 & 1 & 0 & 1 \\ 5 & 4 & 4 & 3 & 3 & 2 & 2 & 1 & 1 & 0 \end{pmatrix} \end{matrix}.$$

In the matrix, D , the (i, j) -th entry represents the distance between v_i and v_j . Now, if we choose a 4×10 sub-matrix D_F consisting of the first four rows, we have the following matrix whose $(j+1)$ -th column represents the code of v_j with respect to the set $F = \{v_0, v_1, v_2, v_3\}$.

$$D_F = \begin{matrix} & c(v_0) & c(v_1) & c(v_2) & c(v_3) & c(v_4) & c(v_5) & c(v_6) & c(v_7) & c(v_8) & c(v_9) \\ \begin{matrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{matrix} & \begin{pmatrix} 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 \\ 1 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \\ 1 & 1 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 \\ 2 & 1 & 1 & 0 & 1 & 1 & 2 & 2 & 3 & 3 \end{pmatrix} \end{matrix}.$$

In the above matrix D_F , every pair of columns is different in two places, and hence, F is a fault-tolerant resolving set. For example, if we take the 6-th and 7-th columns, then these two columns differ in the 2nd and 4th places. Again, if we choose a 3×10 sub-matrix consisting of the first three

rows, we have the following matrix whose $(j + 1)$ -th column represents the code of v_j with respect to the set $F' = \{v_0, v_1, v_2\}$.

$$D_{F'} = \begin{matrix} & c(v_0) & c(v_1) & c(v_2) & c(v_3) & c(v_4) & c(v_5) & c(v_6) & c(v_7) & c(v_8) & c(v_9) \\ \begin{matrix} v_0 \\ v_1 \\ v_2 \end{matrix} & \begin{pmatrix} 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 \\ 1 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \\ 1 & 1 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 \end{pmatrix} \end{matrix}.$$

Here, the 6-th and 7-th columns differ in only one place. So, F' cannot be a fault-tolerant resolving set for P_n^2 .

Example 6. Applying the result of Theorem 5, we have

$$\beta'(P_n^3) \geq \begin{cases} 6 & \text{if } n \geq 14, \\ 5 & \text{if } 8 \leq n \leq 13. \end{cases} \quad (5)$$

Again, applying Theorem 1, the set $\{v_0, v_1, \dots, v_5\}$, say F , is a fault-tolerant resolving set of P_n^3 . Inequality (5) gives that the set F is optimal fault-tolerant when $n \geq 14$. Table 1 gives an optimal fault-tolerant resolving set of P_n^3 for $8 \leq n \leq 13$.

Table 1. Fault-tolerant metric basis of P_n^3 for $8 \leq n \leq 13$.

Values of n	Values of $\beta'(P_n^3)$	Elements of Fault-Tolerant Metric Basis
8	5	$\{v_0, v_1, v_2, v_3, v_5\}$
9	5	$\{v_0, v_1, v_2, v_4, v_6\}$
10, 11	5	$\{v_0, v_2, v_4, v_6, v_8\}$
12, 13	5	$\{v_1, v_3, v_5, v_7, v_9\}$

Example 7. Applying Theorem 5, we obtain a lower bound for the fault-tolerant metric dimension of P_n^4 as

$$\beta'(P_n^4) \geq \begin{cases} 8 & \text{if } n \geq 26, \\ 7 & \text{if } 14 \leq n \leq 25, \\ 6 & \text{if } 8 \leq n < 13. \end{cases} \quad (6)$$

Using inequality (6) and Theorem 1, the set $\{v_0, v_1, \dots, v_7\}$ is an optimal FTRS of P_n^4 provided $n \geq 26$. Again, using inequality (6) and Lemma 10, we obtain an optimal FTRS of P_n^4 , provided $14 \leq n \leq 25$. Using Sage-Math, we determine an optimal fault-tolerant resolving set for P_n^4 when $8 \leq n \leq 12$. In Table 2, we show an optimal fault-tolerant resolving set of P_n^4 for $8 \leq n \leq 25$.

Example 8. Theorem 5 generates a lower bound for the fault-tolerant metric dimension of P_n^5 as

$$\beta'(P_n^5) \geq \begin{cases} 10 & \text{if } n \geq 42, \\ 9 & \text{if } 22 \leq n \leq 41, \\ 8 & \text{if } 14 \leq n \leq 21, \\ 7 & \text{if } 10 \leq n \leq 13. \end{cases} \quad (7)$$

By similar argument as in the above example, $\{v_0, v_1, \dots, v_9\}$ is an optimal FTRS when $n \geq 42$. From the proof of Theorem 2 and inequality (7), we can say that $\{v_4, v_8, v_{12}, v_{16}, v_{20}, v_{24}, v_{28}, v_{32}, v_{36}\}$ forms an optimal FTRS of P_{41}^5 . In addition, using Algorithm 1, we can construct an FTRS of P_n^5 with a cardinality of 9. Thus, using (7), we obtain $\beta'(P_n^5) = 9$ for $22 \leq n \leq 41$. In Table 3, we

have provided an optimal FTRS of P_n^5 when $12 \leq n \leq 20$. For $n = 10$ and $n = 11$, the sets $F_1 = \{v_0, v_1, v_3, v_4, v_5, v_7, v_8\}$ and $F_2 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_8, v_9\}$ are the optimal FTRS of P_n^5 , respectively.

Example 9. In Table 4, we consider the graph P_n^6 and find the ranges of n for different values of $\beta'(P_n^6)$ and a from Theorem 5. In Table 5, we calculate the upper bounds on n for all possible values of a for the graph P_n^6 . In Table 6, we calculate the exact value of $\beta'(P_n^6)$ for different values of n using Sage-Math, where $V(P_n^6) = \{v_0, v_1, \dots, v_{n-1}\}$. Finally, we summarize the fault-tolerant metric basis for P_n^6 for different values of n .

$$\beta'(P_n^6) = \begin{cases} 7 & n \in [1, 7], \\ 8 & n \in [8, 12], \\ 9 & n \in [13, 17], \\ 10 & n \in [18, 30], \\ 11 & n \in [31, 61], \\ 12 & n \geq 62. \end{cases} \quad (8)$$

Table 2. Fault-tolerant metric basis of P_n^4 for $8 \leq n \leq 25$.

Values of n	Values of $\beta'(P_n^4)$	Elements of Fault-Tolerant Metric Basis
8	6	$\{v_0, v_1, v_2, v_3, v_4, v_6\}$
9, 10	6	$\{v_0, v_1, v_3, v_4, v_6, v_7\}$
11	6	$\{v_1, v_2, v_4, v_5, v_7, v_8\}$
12	6	$\{v_2, v_3, v_5, v_6, v_8, v_9\}$
13, 14	7	$\{v_0, v_1, v_2, v_3, v_4, v_7, v_{10}\}$
15	7	$\{v_0, v_1, v_2, v_3, v_5, v_8, v_{11}\}$
16	7	$\{v_0, v_1, v_2, v_3, v_6, v_9, v_{12}\}$
17, 18, 19	7	$\{v_0, v_1, v_3, v_6, v_9, v_{12}, v_{15}\}$
20, 21, 22	7	$\{v_0, v_3, v_6, v_9, v_{12}, v_{15}, v_{18}\}$
23	7	$\{v_1, v_4, v_7, v_{10}, v_{13}, v_{16}, v_{19}\}$
24	7	$\{v_2, v_5, v_8, v_{11}, v_{14}, v_{17}, v_{20}\}$
25	7	$\{v_3, v_6, v_9, v_{12}, v_{15}, v_{18}, v_{21}\}$

Table 3. Fault-tolerant metric basis of P_n^5 for $12 \leq n \leq 20$.

Values of n	Values of $\beta'(P_n^5)$	Elements of Fault-Tolerant Metric Basis
12	8	$\{v_0, v_1, v_2, v_3, v_4, v_5, v_7, v_9\}$
13	8	$\{v_0, v_1, v_2, v_3, v_4, v_6, v_8, v_{10}\}$
14	8	$\{v_0, v_1, v_2, v_4, v_5, v_8, v_9, v_{12}\}$
15, 16, 17	8	$\{v_0, v_1, v_4, v_5, v_8, v_9, v_{12}, v_{13}\}$
18	8	$\{v_1, v_2, v_5, v_6, v_9, v_{10}, v_{13}, v_{14}\}$
19	8	$\{v_2, v_3, v_6, v_7, v_{10}, v_{11}, v_{14}, v_{15}\}$
20	8	$\{v_3, v_4, v_7, v_8, v_{11}, v_{12}, v_{15}, v_{16}\}$

Table 4. Lower bound on $\beta'(P_n^6)$ for different values of n .

Values of n	Values of a	Lower Bound on $\beta'(P_n^6)$ from Theorem 5
$n \geq 62$	0	12
$32 \leq n \leq 61$	1	11
$22 \leq n \leq 31$	2	10
$17 \leq n \leq 21$	3	9
$12 \leq n \leq 16$	4	8

Table 5. Upper bound on n from Theorem 6.

Values of $\beta'(P_n^6)$	Values of a	Upper Bound on n from Theorem 6
11	1	61
10	2	31
9	3	20
8	4	15
7	5	10

Table 6. Exact value of $\beta'(P_n^6)$ using Sage-Math.

Values of n	Values of $\beta'(P_n^6)$	Elements of Fault-Tolerant Metric Basis
7	7	$\{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$
8, 9	8	$\{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$
10	8	$\{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_8\}$
11, 12	8	$\{v_1, v_2, v_4, v_5, v_6, v_7, v_9, v_{10}\}$
13, 14	9	$\{v_0, v_1, v_2, v_3, v_5, v_6, v_8, v_{10}, v_{11}\}$
15, 16	9	$\{v_0, v_2, v_4, v_5, v_7, v_9, v_{10}, v_{12}, v_{14}\}$
17	9	$\{v_1, v_3, v_5, v_6, v_8, v_{10}, v_{11}, v_{13}, v_{15}\}$
18	10	$\{v_0, v_1, v_2, v_3, v_4, v_5, v_7, v_{10}, v_{12}\}$
30	10	$\{v_4, v_5, v_9, v_{10}, v_{14}, v_{15}, v_{19}, v_{20}, v_{24}, v_{25}\}$

Example 10. In this example, we take the graph P_n^8 . In Table 7, we find the lower bound on $\beta'(P_n^8)$ for different ranges of n and a .

In Table 8, we calculate the exact value of $\beta'(P_n^8)$ for different values of n using Sage-Math, where $V(P_n^8) = \{v_0, v_1, \dots, v_{n-1}\}$. In Table 9, we calculate the upper bounds on n for possible values of a for the graph P_n^8 . In Table 10, given below, we state the fault-tolerant metric dimension and an optimal FTRS of P_n^8 for different values of n .

Table 7. Lower bound on $\beta'(P_n^8)$ for different values of n .

Values of n	Values of a	Lower Bound on $\beta'(P_n^8)$ from Theorem 5.
$n \geq 114$	0	16
$58 \leq n \leq 113$	1	15
$37 \leq n \leq 57$	2	14
$30 \leq n \leq 36$	3	13
$23 \leq n \leq 29$	4	12
$16 \leq n \leq 22$	5	11

Table 8. Exact value of $\beta'(P_n^8)$ using Sage-Math.

Values of n	Values of $\beta'(P_n^8)$	Elements of Fault-Tolerant Metric Basis
16	11	$\{v_0, v_1, v_3, v_4, v_6, v_7, v_8, v_{10}, v_{11}, v_{13}, v_{14}\}$
17	11	$\{v_1, v_2, v_4, v_5, v_7, v_8, v_9, v_{11}, v_{12}, v_{14}, v_{15}\}$
24	12	$\{v_2, v_3, v_6, v_7, v_9, v_{10}, v_{13}, v_{14}, v_{16}, v_{17}, v_{20}, v_{21}\}$
25, 26, 27	13	$\{v_0, v_1, v_3, v_5, v_7, v_{10}, v_{12}, v_{14}, v_{17}, v_{19}, v_{21}, v_{23}, v_{24}\}$
28, 29, 30	13	$\{v_3, v_5, v_7, v_{10}, v_{12}, v_{14}, v_{17}, v_{19}, v_{21}, v_{23}, v_{24}, v_{25}, v_{26}\}$
31, 32	14	$\{v_0, v_1, v_2, v_6, v_7, v_{13}, v_{14}, v_{20}, v_{21}, v_{24}, v_{25}, v_{26}, v_{27}, v_{28}\}$
33	14	$\{v_0, v_1, v_2, v_6, v_7, v_{13}, v_{14}, v_{20}, v_{21}, v_{25}, v_{26}, v_{27}, v_{28}, v_{32}\}$
34	14	$\{v_0, v_1, v_2, v_6, v_7, v_{13}, v_{14}, v_{20}, v_{21}, v_{26}, v_{27}, v_{28}, v_{32}, v_{33}\}$
35	14	$\{v_0, v_1, v_2, v_6, v_7, v_{13}, v_{14}, v_{20}, v_{21}, v_{27}, v_{28}, v_{32}, v_{33}, v_{34}\}$

Table 9. Upper Bound on n from Theorem 6.

Values of $\beta'(P_n^8)$	Values of a	Upper bound on n from Theorem 6
15	1	113
14	2	57
13	3	36
12	4	28
11	5	21
10	6	15

Table 10. An optimal FTRS of P_n^8 and the values of $\beta'(P_n^8)$.

Values of n	Lower Bound of $\beta'(P_n^8)$	Exact Value of $\beta'(P_n^8)$	An Optimal FTRS
$n \geq 114$	16 (using Theorem 5)	16	$\{v_0, v_1, \dots, v_{15}\}$ by using Theorem 1
$58 \leq n \leq 113$	15 (using Lemma 10)	15	For $n = 113$, $\{v_7, v_{14}, \dots, v_{105}\}$ the set is an optimal FTRS (from the proof of Theorem 2). Algorithm 1 generates an optimal FTRS for each $n \in \{58, 59, \dots, 112\}$.

7. Concluding Remark

In this article, we study the fault-tolerant metric dimension of P_n^r for all n and r . We have presented an algorithm to construct a fault-tolerant resolving system of P_m^r from a fault-tolerant resolving system of P_n^r when $m < n$. We have found the exact value of $\beta'(P_n^r)$ and presented an optimal FTRS for $n \geq r(r-1) + 2$. For the other values of n , we presented

several bounds. These results have been applied to P_n^r for $r \in \{3, 4, 5, 6, 8\}$ and we obtained exact values or bounds for $\beta'(P_n^r)$. Although we have presented several bounds for $\beta'(P_n^r)$ when $n \leq r(r-1) + 1$, the problem below is still open.

Problem: Determine the fault-tolerant metric dimension of P_n^r when $n \leq r(r-1) + 1$.

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