Optimal Investment of Merton Model for Multiple Investors with Frictions

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Abstract: We investigate the classical optimal investment problem of the Merton model in a discrete time with market friction due to loss of wealth in trading. We consider the case of a finite number of investors, with the friction for each investor represented by a convex penalty function. This model cover the transaction costs and liquidity models studied previously in the literature. We suppose that each investor maximizes their utility function over all controls that keep the value of the portfolio after liquidation non-negative. In the main results of this paper, we prove the existence of an optimal strategy of investment by using a new approach based on the formulation of an equivalent general equilibrium economy model via constructing a truncated economy, and the optimal strategy is obtained using a classical argument of limits.

Keywords: Merton model; multiple investors; penalty functions; general equilibrium; truncated economy; optimal strategy

MSC: 91G99; 91G10

1. Introduction

The optimal investment of the Merton model introduced in [1,2] has been investigated by researchers and extended in different contexts since its appearance. One important extension in continuous time is that developed by Magill and Constantinides [3], where a linear transaction cost function is used in the context of the Merton problem. In discrete time, the study of the Merton model with linear transaction costs was developed by Jouini and Kallal in [4]. We can also cite the papers of Shreve and Soner [5], which extended the Merton problem by including viscosity theory, and Cetin, Jarrow, and Protter [6], who studied the Merton model for illiquid markets. In continuous time, only the problem of the super-replication of a contingent claim was analysed by Cetin and Rogers [7]. Later, this analysis was conducted in discrete time by Gokey and Soner [8].

In another context of the Merton model in continuous time, Swishchuk in [9] solved the optimal investment stochastic control problem in finance and insurance. They considered the wealth portfolio consisting of a bond and a stock price described by general compound Hawkes process (GCHP), and for a capital of an insurance company with the amount of claims described by risk model based on GCHP (see also [10,11]).

In discrete time, recently, Chebbi and Soner, in [12], extended the Merton model in a finite horizon to the case of a market with frictions represented by a convex penalty function defined for one investor. They proved the existence of an optimal strategy by solving a dynamic optimization problem. Later, Ounaies, Bonnisseau, Chebbi and Soner in [13] extended this model to the infinite horizon and proved the existence of an optimal strategy by using an argument of fixed points.
In the literature, we can find several sources of market frictions. However, the first one that received the most attention is transaction cost, defined as a consequence of the bid and ask spread. The transaction model was first studied in the context of the Merton problem by Magill and Constantinides [3] and later by Constantinides [14]. The mathematical modelling of this problem in continuous time was developed by Davis and Norman [15], and the maximal growth rate problem was studied by Dumas and Luciano [16]. Another concept of friction is defined by Cetin, Jarrow, and Protter [6] for illiquid markets and a notion of the supply curve is used in modelling frictions, which gives the price of stock as a function of the trade size.

The advantage of studying the Merton optimal problem of investment in discrete time is that we can model market friction through general penalty functions that are supposed to be convex. In continuous time, only the structure of the penalty function near the origin is relevant and one has to discuss the differentiability at the origin. In this case, the corresponding techniques depend on this property. In contrast to continuous time, a unified approach is possible in discrete time by assuming the penalty function convex, covering the model of transaction costs, and the model of an illiquid market.

In this paper, we will take this direction of extension in order to prove the existence of an optimal strategy for the Merton model for market frictions in an infinite horizon when there are finite number of investors. Our approach is very different and is based on constructing an equivalent general equilibrium model with multiple agents. The idea to use the general equilibrium theory is inspired by the paper of Le Van and Dana [17].

The sections of this paper are organized as follows. In Section 2, we give a description of the Merton model of the investment problem in an infinite horizon and with market frictions modelled by convex penalty functions defined for each investor, and, consequently, define constraint conditions for the liquidation value. In Section 3, we construct a general equilibrium economy model equivalent to the Merton model of investment. In Section 4, we prove the the existence of an equilibrium for the model of general equilibrium economy and prove that the optimal strategy of the Merton problem of investment will be this obtained equilibrium.

2. The Model

Let \((\Omega, \mathcal{F}, P)\) be the probability space where \(\Omega = \mathbb{R}^N\) is the space of events \((\omega_t)_{t \geq 1}\). For \(t \in \mathbb{N}^*\), let \(\mathcal{F}_t = \sigma(B_{j} ; s \in \{1, 2, \ldots, t\})\) be the \(\sigma\)-field generated by the canonical mapping process \(B_t(\omega) = \omega_t, t \geq 1, \omega \in \Omega\). We denote by \(\mathcal{F}_\infty = \sigma(\bigcup_{t \in \mathbb{N}} \mathcal{F}_t)\), where \(\mathcal{F}_0 = \{\emptyset, \Omega\}\) is the trivial \(\sigma\)-algebra and by \(P : \mathcal{F} \to [0, 1]\), the probability measure.

In the discrete time model of this paper, we suppose that the market has a money market account paying a return \(r > 0\) and \(N\) risky assets that provide a random return of \(R = (R_t)_{t \geq 1}\) with values in \([-1, \infty)^N\) that are supposed to be identically and independently distributed over time. We denote by \((p^{j})_{1 \leq j \leq N}\) the strictly positive asset price process that is supposed to satisfy the following condition

\[
p_t^j = p_0^j \prod_{k=1}^{t} [1 + R_k^j] \iff R_t^j = \frac{p_t^j - p_{t-1}^j}{p_{t-1}^j}, \quad j = 1, \ldots, N. \tag{1}
\]

where \(p_0^j\) is the initial stock value. The return vector at time \(t\) is given by

\[
R_t(\omega) = B_t(\omega) = \omega_t, t \in \mathbb{N}^*, j = 1, \ldots, N.
\]

then, \(R_t\)s are \(\mathcal{F}_t\)-measurable and, consequently, \(R = (R_t)_{t \geq 1}\) is an \((\mathbb{R})^n\)-valued, \(\mathcal{F}\)-adapted process. The process \(p\) is an \((\mathbb{R}^+)^N\)-valued \(\mathcal{F}\)-adapted process.

In our multiple investors model, we suppose that there are a finite number \(m\) of investors, labelled \(i\), \((i = 1, 2, \ldots, m)\). Each investor has to choose a portfolio of assets \(j, (j = 0, 1, 2, \ldots, N)\). We denote by \(y = (y_{ij})_{t \geq 1}\) the individual \(i\) process of money invested in the \(j\)-th stock at any time \(t\) prior to the portfolio adjustment. The riskless asset
$x = (x_{jt})_{t \geq 1}$ will be the process of money invested in the money market account at any time $t$. Shares are traded at the determined price vector $p_t = (p^{1}_{t}, \cdots, p^{N}_{t})$. For $t \geq 1$, the process $z_{jt}$ will denote the number of shares held by the $i$-th investor at time $t$ with values in $\mathbb{R}^{N}$, and we have

$$y_{jt} = z_{jt}^{j} p^{j}_{t}, \quad j = 1, \cdots, N, \quad i = 1, \cdots, m, \quad t \geq 1.$$  

In our model of markets with frictions, we assume that there is a penalty function $g_{i}: \mathbb{R}^{N} \rightarrow \mathbb{R}^{N}$ for each investor $i$ due to transaction costs. The dynamics of the riskless asset will be as follows

$$x_{jt+1} = (x_{jt} - a_{jt} \cdot 1 - p_{t} g_{i}((z_{jt+1} - z_{jt}) \cdot 1 - c_{jt}) (1 + r), \quad t \geq 1, \quad (2)$$

where the $\mathcal{F}$-adapted process $c_{j}$ denotes the consumption of the $i$-th investor, and $a_{i}$ is the portfolio adjustment process given by:

$$a_{jt} := p_{t} \Delta z_{jt} = p_{t} ((z_{jt+1}^{j} - z_{jt}^{j}) \cdot 1, \quad j = 1, \cdots, N, \quad t \geq 1. \quad (3)$$

Note that the rebalancing of the portfolio will occur between time $t$ and time $t+1$, and it is easy to see that

$$y_{jt+1} = (y_{jt} + a_{jt}^{j}) (1 + R_{jt}^{j}), \quad (4)$$

and the mark-to-market value is given by:

$$\omega_{jt} := x_{jt} + y_{jt} \cdot 1 = x_{jt} + \sum_{j=1}^{N} y_{jt}^{j} \quad \quad$$

3. **General Equilibrium Model of the Merton Investment Problem**

Given a portfolio position $(x, y) \in \mathbb{R} \times (\mathbb{R}^{+})^{N}$, the after-liquidation value will be defined as follows

$$L(x_{jt}, y_{jt}) = a_{jt} + b_{jt} \cdot 1 - p_{t} g_{i}((z_{jt+1} - z_{jt}) \cdot 1 - c_{jt}) (1 + r) = x_{jt} + p_{t} z_{jt} \cdot 1 - p_{t} g_{i}((z_{jt+1} - z_{jt}) \cdot 1 \quad (5)$$

and the solvency condition is given by the requirement that $L(x_{jt}, y_{jt}) \geq 0$ for all $t \geq 1, P_{t}$ almost surely. Hence, our optimal investment problem will be formulated by the following optimization problem

$$Q_{t}(x, y) := \sup_{(c_{jt}, z_{jt})} E \left[ \sum_{j=0}^{\infty} p_{j} u_{j}(c_{jt}) \right]$$

subject to : $x_{jt} + p_{t} z_{jt} \cdot 1 - p_{t} g_{i}((z_{jt+1} - z_{jt}) \cdot 1 \geq 0 \quad a.e.$

where for each investor $i$, $u_{i}$ is the utility function and $p_{j}^{t}$ is the impatience parameter.

The infinite-horizon sequence of prices and quantities is given by

$$(p_{t}, (c_{jt}, z_{jt})_{j=1}^{m})$$

where, for each $i = 1, \cdots, m$,

$$(p_{t}, c_{jt}, z_{jt}) = (((p_{t})_{t=0}^{\infty}, (c_{jt})_{t=0}^{\infty}, (z_{jt})_{t=0}^{\infty}) \in (\mathbb{R}^{+})^{N} \times \mathbb{R}^{+\infty} \times (\mathbb{R}^{+\infty})^{N} ,$$

Now, let $\mathcal{E}$ be the economy, characterized by

$$\mathcal{E} = (\mathbb{R}^{N}, (u_{i}, p_{t}, z_{i-1})_{i=1}^{m}).$$
The equilibrium of this economy is determined by the set of consumption policies and price processes for which each agent maximizes their expected utility. More precisely:

**Definition 1.** The process \( \{ \bar{p}_t, (c_{i,t}, z_{i,t})\}_{t=0}^\infty \) is an equilibrium of the economy \( E \) if the following conditions are satisfied:

1. **Price positivity:** \( \bar{p}_t > 0 \) for \( t \geq 0 \).
2. **Market clearing:** At each \( t \geq 0 \),
   \[
   \sum_{i=1}^m \bar{c}_{i,t} + p_i \bar{g}_i((z_{i,t+1} - z_{i,t}) \cdot 1 = \omega_t, \quad a.e.
   \]
   \[
   \sum_{i=1}^N \bar{z}_{i,t}^j = 1 \quad a.e., \quad \forall i \in \{1, \ldots, m\},
   \]
   \[
   \sum_{i=1}^m \bar{z}_{i,t}^0 = 0 \quad a.e.
   \]
3. **Optimal consumption plans:** For each \( i \), \( \{(\bar{c}_{i,t}, \bar{z}_{i,t})\}_{t=0}^\infty \) is a solution of the problem \( Q_i(x, y) \).

**4. Existence of Equilibrium**

We will use the following standard assumptions in order to prove the existence of equilibrium:

- **Assumption (H1):** For each \( i = 1, \ldots, m \), \( u_i \) is a continuously differentiable, strictly increasing and concave function satisfying \( u_i(0) = 0, u_i'(0) = \infty \).
- **Assumption (H2):** At the initial period 0, \( z_{i,-1} \geq 0 \), and \( z_{i-1} \neq 0 \) for \( i = 1, \ldots, m \) with \( \sum_{i=1}^m z_{i,-1} = 1 \).
- **Assumption (H3):** \( \bar{g}_i : \mathbb{R}^N \to \mathbb{R}_+^N \) is convex with \( \bar{g}_i(0) = 0 \) and \( \bar{g}_i \geq 0 \) for \( i = 1, \ldots, m \).
- **Assumption (H4):** The utility of each agent \( i \) is finite:
   \[
   \sum_{t=0}^\infty \rho^t_i u_i(\bar{c}_{i,t}) < \infty.
   \]

We now construct the \( T \)-truncated economy \( E^T \) as \( E \) in which we suppose that there are no activities from period \( T + 1 \) to infinity, and by using a classical argument, wecompact this economy by using the bounded economy \( E^T_b \) as \( E^T \), in which all random variables are bounded. Consider a finite-horizon bounded economy which goes on for \( T + 1 \) periods \( t = 0, \ldots, T \) with \( B_c, B_z \) defined by:

\[
C_i := \{ (c_{i,0}, \ldots, c_{i,T}) : 0 \leq c_{i,t} \leq B_c, \quad \forall t \in \{1, \ldots, T\} \} = [0, B_c]^{T+1};
\]
\[
Z_i := \{ (z_{i,0}^j, \ldots, z_{i,T}^j) : 0 \leq z_{i,j}^j \leq B_z, \quad \forall t \in \{1, \ldots, T\} \} = [0, B_z]^T.
\]

The solvency set is given by:

\[
\bar{U}_i^T(x, y) := \{ (c_{i,t}, z_t) \in C_i \times Z_i : x_{i,t} + p_i z_{i,t} \cdot 1 - p_i \bar{g}_i(z_{i,t+1} - z_{i,t}) \cdot 1 \geq 0, \quad P - a.s. \}.
\]

Now, we define the economy \( E^T_{b,\varepsilon} \) for each \( \varepsilon > 0 \) such that \( \varepsilon < 1 \), by adding \( \varepsilon \) units for each agent at date 0. This condition assures the non-emptiness of the solvency set. Thus, the feasible set of each agent \( i \) will be:

\[
\bar{U}_i^{T,\varepsilon}(x, y) := \{ (c_{i,t}, z_t) \in \mathbb{R}_+^{T+1} \times (\mathbb{R}_+^{T+1})^N : (x_{i,0} + p_0(z_{i,0} + \varepsilon) \cdot 1 - p_0 \bar{g}_0((-z_{i,0} + \varepsilon) \cdot 1 - c_{i,0})(1 + r) \geq 0, \quad \text{for each } 1 \leq t \leq T : x_{i,t} + p_z z_{i,t} \cdot 1 - p_z \bar{g}_z(z_{i,t+1} - z_{i,t}) \cdot 1 \geq 0, \quad P - a.s. \}
\]
\[ L_i^{T,\epsilon}(x, y) := \left\{ (c_i, z_i) \in \mathbb{R}^{T+1}_+ \times (\mathbb{R}^{T+1}_+)^N : \\
(x_{i,0} + p_0(z_{i,0} + \epsilon) \cdot 1 - p_0g_i(-(z_{i,0} + \epsilon)) \cdot 1 - c_{i,0})(1 + r) > 0, \\
\text{for each } 1 \leq t \leq T : x_{i,t} + p_tz_{i,t} \cdot 1 - p_tg_i(z_{i,t+1} - z_{i,t}) \cdot 1 > 0, \ P - a.s. \right\} \]

**Lemma 1.** The set \( L_i^{T,\epsilon}(x, y) \) is non-empty for \( t = 0, \cdots, T \).

**Proof.** Indeed,

\[
L(x_{i,0}, y_{i,1}) = L \left( (x_{i,0} + p_0(z_{i,0} + \epsilon) \cdot 1 - p_0g_i(-(z_{i,0} + \epsilon)) \cdot 1 - c_{i,0})(1 + r), (y_{i,0}^{\epsilon} + \alpha_{i,0}^{\epsilon}) (1 + R_i^\epsilon) \right) \\
= L \left( (x_{i,0} + p_0(z_{i,0} + \epsilon) \cdot 1 - p_0g_i(-(z_{i,0} + \epsilon)) \cdot 1 - c_{i,0})(1 + r), 0 \right) \\
= (x_{i,0} + p_0(z_{i,0} + \epsilon) \cdot 1 - p_0g_i(-(z_{i,0} + \epsilon)) \cdot 1 - c_{i,0})(1 + r) \geq 0
\]

Now, since \( \epsilon, (z_{i,0} + \epsilon) > 0 \), we can select \( c_{i,0} \in (0, B_c) \) and \( z_{i,0} \in (0, B_z) \) such that

\[
(x_{i,0} + p_0(z_{i,0} + \epsilon) \cdot 1 - p_0g_i(-(z_{i,0} + \epsilon)) \cdot 1 - c_{i,0})(1 + r) > 0
\]

\( \square \)

**Lemma 2.** The set \( U_i^T(x, y) \) has convex values.

**Proof.** Now, we want to show that \( U(x, y) \) is convex. Take \( (c_{i,t}^k, \alpha_{i,t}^k) \in U(x^k, y^k) \), for \( k = 1, 2 \) and \( t \geq 1 \). For \( \lambda \in [0, 1] \), we note by \( c_{i,t} = \lambda c_{i,t}^1 + (1 - \lambda)c_{i,t}^2 \) and similarly \( x_{i,t} \), \( z_{i,t} \). We have

\[
L(x_{i,t}, \alpha_{i,t}) = L(x_{i,t}^1 + p_tz_{i,t}^1 \cdot 1 - p_tg_i(z_{i,t+1} - z_{i,t}) \cdot 1 \\
= \lambda(x_{i,t}^1 + p_tz_{i,t}^2 \cdot 1) + (1 - \lambda)(x_{i,t}^2 + p_tz_{i,t}^2 \cdot 1) - p_tg_i(z_{i,t+1} - z_{i,t}) \cdot 1 \\
\geq p_t[\lambda g(z_{i,t+1}^1 - z_{i,t}^1) \cdot 1 + (1 - \lambda)g(z_{i,t+1}^2 - z_{i,t}^2) \cdot 1 - g(z_{i,t+1} - z_{i,t}) \cdot 1] \\
\geq 0
\]

since \( g \) is convex and \( (x_{i,t}^k, y_{i,t}^k) \in \mathbb{F} \) for \( k = 1, 2 \) and \( t \geq 1 \). \( \square \)

For simplicity, we denote \( U_i = C_i \times Z_i \).

**Lemma 3.** \( L_i^{T,\epsilon}(x, y) \) is lower semi-continuous correspondence on \( U_i \) and \( U_i^T(x, y) \) is upper semi-continuous with compact convex values.

**Proof.** Since \( L_i^{T,\epsilon}(x, y) \) is non-empty and has an open graph, then it has lower semi-continuous correspondence. Since \( U_i \) is compact and the correspondence \( U_i^T(x, y) \) has a closed graph, then \( U_i^T(x, y) \) is upper semi-continuous with compact values. \( \square \)

**Definition 2.** The stochastic process \( (p_t, (c_{i,t}, z_{i,t}))_{t=0}^m \) is an equilibrium of the economy \( \mathcal{E}^T \) if it satisfies the following conditions:

1. **Price positivity:** \( \bar{p}_t > 0 \) for \( t = 0, 1, \cdots, T \).
2. **Market clearing:**

\[
\sum_{i=1}^m c_{i,0} + p_0g_i(-(z_{i,0} + \epsilon)) = \sum_{i=1}^m x_{i,0} + p_0(z_{i,0}^t + \epsilon) \cdot \bar{1}, \ a.e. \\
\sum_{i=1}^m c_{i,t} + p_tg_i(z_{i,t+1} - z_{i,t}) = \sum_{i=1}^m x_{i,t} + p_tz_{i,t} \cdot \bar{1}, \ a.e.
\]
3. Optimal consumption plans: For each $i$, $(c_{i,t}, z_{i,t})_{t=1}^T$ is a solution of the maximization problem of agent $i$ with the feasible set $U_{i}^{T,x}(x, y)$ such that

$$Q_{i}^{T,x}(x, y) = \sup_{(c_{i,t}, z_{i,t})} \mathbb{E} \left[ \sum_{t=0}^{T} \rho^t u_i(c_{i,t}) \right].$$

For $i = 0, \ldots, m$, consider an element $h = (h_i)$ defined on $X := B \times \prod_{i=1}^{m} U_i$ by

$$h_i = \begin{cases} p & \text{for } i = 0 \\ (c_{i,t}, z_{i,t}) & \text{for } i = 1, \ldots, m \end{cases}$$

where $B = \{ p \in \mathbb{R}^N \parallel p \parallel \leq 1 \}.$

Now, let $\varphi_0$ be the correspondence defined by

$$\varphi_0 : \prod_{i=1}^{m} U_i \to 2^B$$

$$\varphi_0(h_i)_{i=0}^{m} := \arg \max_{p \in B} \left\{ \left( \sum_{i=1}^{m} c_{i,0} + p_0 g_i(- (z_{i,0} + \epsilon)) \cdot 1 - x_{i,0} - p_0 (z_{i,0} + \epsilon) \cdot 1 \right. \right.$$

$$+ \left. \sum_{i=1}^{T} \sum_{j=1}^{m} c_{i,j} + p_i g_i(z_{i,j+1} - z_{i,j}) \cdot 1 - x_{i,j} - p_i z_{i,j}^\epsilon \cdot 1 \right\}.$$  

and for each $i = 1, \ldots, m,$ consider

$$\varphi_i : B \to 2^{U_i}$$

$$\varphi_i(p) := \arg \max_{(c_{i,t}, z_{i,t}) \in U(x, y)} \mathbb{E} \left[ \sum_{t=0}^{T} \rho^t u_i(c_{i,t}) \right].$$

Lemma 4. The correspondence $\varphi_i$ is upper semi-continuous with non-empty, convex, compact values for each $i = 1, \ldots, m.$

Proof. This is a direct consequence of the maximum theorem. □

According to the Kakutani theorem, there exists $(\bar{p}, (\bar{c}_{i,t}, \bar{z}_{i,t}))$ such that

$$\bar{p} \in \varphi_0((c_{i,t}, z_{i,t})_{i=1}^{m})$$

$$(\bar{c}_{i,t}, \bar{z}_{i,t}) \in \varphi_i(\bar{p}).$$

For simplicity, we denote this using:

$$\bar{E}_t = \sum_{i=1}^{m} \bar{c}_{i,t} - x_{i,t}, \quad t \geq 0$$

$$\bar{F}_0 = \sum_{i=1}^{m} g_i(- (z_{i,0}^\epsilon + \epsilon)) - (z_{i,0} + \epsilon) \cdot 1$$

$$\bar{F}_t = \sum_{i=1}^{m} g_i(z_{i,t+1} - z_{i,t}) - z_{i,t}^\epsilon \cdot 1, \quad t \geq 1$$

Lemma 5. Under assumptions (H1), (H2), and (H3), there exists an equilibrium for the finite-horizon-bounded $\epsilon$-economy $\mathbb{E}_b^{T,x}.$
Proof. We start by proving that $\bar{E}_t + \bar{p}_t \bar{F}_t = 0$ and $\bar{p}_t > 0$ for $t = 0, \cdots, T$. Indeed, from (6), one can easily check that for every $p \in B$, we have:

$$\sum_{t=0}^{T} (p_t - \bar{p}_t) F_t \leq 0. \quad (8)$$

We recall the solvency constraint,

$$x_{i,t} + \bar{p}_t z_{i,t} \cdot 1 - \bar{p}_t g_i(z_{i,t+1} - z_{i,t}) \cdot 1 \geq 0$$

Moreover, the value of an agent’s consumption cannot exceed the value of their wealth, and the following inequality will be satisfied:

$$x_{i,t} + \bar{p}_t z_{i,t} \cdot 1 - \bar{p}_t g_i(z_{i,t+1} - z_{i,t}) \cdot 1 \geq \bar{c}_{i,t}$$

$$x_{i,t} - \bar{c}_{i,t} + \bar{p}_t z_{i,t} \cdot 1 - \bar{p}_t g_i(z_{i,t+1} - z_{i,t}) \cdot 1 \geq 0 \quad (9)$$

By summing inequality (9) over $i$, we obtain that, for each $t$:

$$\sum_{i=1}^{m} x_{i,t} - \bar{c}_{i,t} + \bar{p}_t \left[ \sum_{i=1}^{m} z_{i,t} \cdot 1 - g_i(z_{i,t+1} - z_{i,t}) \cdot 1 \right] \geq 0$$

$$\bar{E}_t + \bar{p}_t \bar{F}_t \leq 0 \quad (10)$$

If $\bar{p}_t = 0$, we deduce that $\bar{c}_{i,t} = B_c = \omega_{i,t}$. Therefore, for all $t$, $\sum_{i=1}^{m} c_{i,t} > \sum_{i=1}^{m} x_{i,t}$, which contradicts (10). Hence, we obtain $\bar{p}_t > 0$ as a result.

Since prices are strictly positive and the utility functions are strictly increasing, all budget constraints are binding. By summing over $i$ at date $t$, we obtain

$$\bar{E}_t + \bar{p}_t \bar{F}_t = 0.$$ 

Hence, the optimality of $(\bar{c}_i, \bar{z}_i)$ is from (7). □

Lemma 6. Supposing that assumptions (H1), (H2) and (H3) are satisfied, then there exists an equilibrium for the finite-horizon-bounded economy $\mathcal{E}_b^T$.

Proof. We have proved that for each $\epsilon = \frac{1}{n} > 0$, where $n$ is an integer and large enough, there exists an equilibrium denoted as follows:

$$\text{equi}(n) := (\bar{p}(n), (\bar{c}_{i,n}(n), z_{i,n}(n))_{i=1}^{m} )_{t=0}^{T};$$

for the economy, $\mathcal{E}_b^{T,\epsilon_n}$. Since prices and allocations are bounded, there exists a sub-sequence $(n_1, n_2, \cdots)$ such that equi$(n_i)$ converges. Without loss of generality, we can assume that

$$\left( \bar{p}(n), (\bar{c}_{i,n}(n), z_{i,n}(n))_{i=1}^{m} \right) \rightarrow (\bar{p}, (\bar{c}_i, \bar{z}_i)_{i=1}^{m})$$

when $n$ tends to infinity. Moreover, by taking the limit of market clearing conditions of the $\mathcal{E}_b^{T,\epsilon_n}$, we obtain the corresponding conditions of the bounded truncated economy $\mathcal{E}_b^{T}$. □

Remark 1. It should be noticed that at equilibrium, we have $\bar{p}_0 > 0$ according to (1).

Lemma 7. For each $i$, $(\bar{c}_i, \bar{z}_i)$ is optimal.

Proof. Since $\sum_{i=1}^{m} \bar{z}_{i,0} = 1$, for all $j \in \{1, \cdots, N\}$, there exists an agent $i$ such that $z_{i,-1} > 0$. According to Remark 1, we have $L_j^T(x,y) \neq \emptyset$. We now prove the optimality of $(\bar{c}_i, \bar{z}_i)$. Let $(c_i, z_i)$ be a feasible allocation of the maximization problem of agent $i$ with the feasible set $\mathcal{U}_i^T(x, y)$. We should prove that $\mathbb{E} \left[ \sum_{t=0}^{T-1} p_t u_i(c_{i,t}) \right] \leq \mathbb{E} \left[ \sum_{t=0}^{T-1} p_t \bar{u}_i(\bar{c}_{i,t}) \right].$
We deduce that when \( \bar{a} \) is the optimal solution. We now prove that \( \bar{a}_i > 0 \) for every \( i \). Indeed, if \( \bar{a}_i = 0 \), the optimality of \( \bar{a}_i, \bar{z}_i \) implies that \( \bar{a}_{i,t} = B_c > x_{i,t} \), which is a contradiction.

After proving the existence of the equilibrium when \( \varepsilon \) tends to 0, we deduce that this equilibrium holds for the truncated unbounded economy.

**Lemma 8.** An equilibrium for \( E_b^T \) is an equilibrium for \( E^T \).

**Proof.** Let \( \{t_i, \bar{a}_{i,t}, \bar{z}_{i,t}\}_{i=1}^{m} \) be an equilibrium of \( E_b^T \). Note that \( z_{i,T+1} = 0 \) for every \( i = 1, \cdots, T \). We can see that conditions (i) and (ii) in Definition (2) are satisfied. We will show that condition (iii) is also verified. Let \( a_i := (\bar{a}_{i,t}, \bar{z}_{i,t})_{t=0}^{T} \) be a feasible plan of agent \( i \). Suppose that \( \sum_{t=0}^{T} \rho_i^t u_i^t(\bar{a}_{i,t}) > \sum_{t=0}^{T} \rho_i^t u_i^t(\bar{a}_{i,t}) \). For each \( \gamma \in (0,1) \), we define \( a_i(\gamma) := \gamma a_i + (1 - \gamma) \bar{a}_i \). By definition of \( E_b^T \), we can choose \( \gamma \) sufficiently close to 0 such that \( a_i(\gamma) \in C_i \times Z_i \). It is clear that \( a_i(\gamma) \) is a feasible allocation. By the concavity of the utility function, we have

\[
\sum_{t=0}^{T} \rho_i^t u_i^t(a_i(\gamma)) \geq \gamma \sum_{t=0}^{T} \rho_i^t u_i^t(\bar{a}_{i,t}) + (1 - \gamma) \sum_{t=0}^{T} \rho_i^t u_i^t(\bar{a}_{i,t})
\]

\[
> \sum_{t=0}^{T} \rho_i^t u_i^t(\bar{a}_{i,t})
\]

We deduce that

\[
\mathbb{E} \left[ \sum_{t=0}^{T} \rho_i^t u_i^t(a_i(\gamma)) \right] > \mathbb{E} \left[ \sum_{t=0}^{T} \rho_i^t u_i^t(\bar{a}_{i,t}) \right]
\]

which contradicts the optimality of \( \bar{a}_i \).

We denote by \( (p^T, (c_i^T, z_i^T))_{i=1}^{m} \) an equilibrium of the T-truncated economy \( E^T \). Since \( \|p_t\| \leq 1 \), for every \( t \leq T, c_i^T \leq B_c \) and \( \sum_{t=1}^{m} z_i^T = 1 \). Thus, we can assume that

\[
(p^T, (c_i^T, z_i^T))_{i=1}^{m} \rightarrow (p, (c_i, z_i))_{i=1}^{m}
\]

when \( T \) goes to infinity.

One can easily check that all markets clear. \( \Box \)

Now, we can give the main results of this paper.
Theorem 1. If hypotheses \((H1), (H2), (H3), \) and \((H4)\) are satisfied, then there exists an equilibrium of the infinite horizon economy \(E\).

Proof. We have proved previously that for each \(T \geq 1\), there exists an equilibrium for the economy \(E^T\). Let \((c_{i,j}, z_{i,j})\) be a feasible allocation of the problem \(Q_i(\beta, z)\). We will prove that
\[
E \left[ \sum_{t=0}^{\infty} \rho_t u_i(c_{i,t}) \right] \leq E \left[ \sum_{t=0}^{\infty} \rho_t u_i(\tilde{c}_{i,t}) \right].
\]

We define \((c_{i,t}', z_{i,t}')\) as follows:
\[
\begin{align*}
z_{i,t}' &= z_{i,t} \quad \text{if} \quad t \leq T - 1, \\
c_{i,t}' &= c_{i,t} \quad \text{if} \quad t \leq T - 1, \\
c_{i,t}' &= 0 \quad \text{if} \quad t > T
\end{align*}
\]
\[
x_{i,T} + \bar{p}_T z_{i,T} - \bar{p}_T g_i(-z_{i,T}) = x_{i,T} + \bar{p}_T z_{i,T} - \bar{p}_T g_i(-z_{i,T})
\]
We can see that \((c_{i,t}', z_{i,t}')_{t=0}^{T} \in U_T^{T}(x, g)\).

Since \(\mathcal{L}_i^T(x, g) \neq \emptyset\), there exists a sequence \((\{c_{i,t}', z_{i,t}'\}_{t=0}^{\infty})_{n=0}^{\infty} \in \mathcal{L}_i^T(x, g)\) with \(z_{i,T+1} = 0\) and this sequence converges to \((c_{i,t}', z_{i,t}')_{t=0}^{\infty}\) when \(n\) tends to infinity. We have
\[
x_{i,t} + \bar{p}_t z_{i,t} - \bar{p}_t g_i(z_{i,t+1} - z_{i,t}) > 0.
\]
We can choose \(s_0\) high enough such that \(s_0 > T\) and for every \(s \geq s_0\), we have
\[
x_{i,t} + \bar{p}_t z_{i,t} - \bar{p}_t g_i(z_{i,t+1} - z_{i,t}) > 0.
\]
Consequently, \((c_{i,t}', z_{i,t}')_{t=0}^{T} \in U_T^{T}(x^{s}, g^{s})\). Therefore, we obtain
\[
\sum_{t=0}^{T} \rho_t u_i(c_{i,t}') \leq \sum_{t=0}^{s} \rho_t u_i(c_{i,t}).
\]

When \(s\) tends to infinity, we obtain \(\sum_{t=0}^{T} \rho_t u_i(c_{i,t}') \leq \sum_{t=0}^{\infty} \rho_t u_i(\tilde{c}_{i,t}).\) Now, if we let \(n\) tend to infinity, we obtain \(\sum_{t=0}^{T} \rho_t u_i(c_{i,t}') \leq \sum_{t=0}^{\infty} \rho_t u_i(\tilde{c}_{i,t})\) for every \(T\). Consequently,
\[
\sum_{t=0}^{T-1} \rho_t u_i(c_{i,t}) \leq \sum_{t=0}^{\infty} \rho_t u_i(\tilde{c}_{i,t}).
\]
Letting \(T\) tend to infinity, we obtain
\[
\sum_{t=0}^{\infty} \rho_t u_i(c_{i,t}) \leq \sum_{t=0}^{\infty} \rho_t u_i(\tilde{c}_{i,t}).
\]
Then,
\[
E \left[ \sum_{t=0}^{\infty} \rho_t u_i(c_{i,t}) \right] \leq E \left[ \sum_{t=0}^{\infty} \rho_t u_i(\tilde{c}_{i,t}) \right].
\]
Hence, we have proved the optimality of \((\tilde{c}_{i,t}, \tilde{z}_{i,t})\). Note that prices \(\bar{p}_t\) are strictly positive since the utility function of agent \(i\) is strictly increasing. \(\square\)

5. Conclusions

The obtained equilibrium is the expected optimal strategy of the Merton investment problem in the case of multiple investors and in markets with frictions formulated by penalty functions for every investor due to losses in trading. Our discrete time model and the main results extend the models and results obtained by Chebbi and Soner in [12] and by Ounaies, Bonnisseau, Chebbi and Soner in [13] in case of the Merton model with one investor. The approach used in this paper is very different and is based on projecting the Merton model in a general equilibrium model of economy and then exploiting
some classical methods on the existence of equilibrium to deduce the existence of an optimal strategy.

Motivated by recent studies of the Merton model of investment in [9–11], one question that arise and will be investigated is to extend the Merton model with frictions for multiple investors in a context of risk modelling in insurance and finance.

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