

Article

Representation and Stability of General Nonic Functional Equation

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Abstract: In this paper, we introduce a way of representing a given mapping as the sum of odd and even mappings. Then, using this representation, we investigate the stability of various forms of the following general nonic functional equation: $\sum_{i=0}^{10} {}_{10}C_i (-1)^{10-i} f(x + iy) = 0$.

Keywords: stability; hyperstability; general nonic functional equation

MSC: 39B82; 39B52



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1. Introduction

The various types of stability for functional equations are a very interesting subject in the field of mathematical analysis. The stability problems of functional equations have been studied by several authors (see [1–4]). In particular, Gilányi [5] investigated the stability of a monomial functional equation in real normed spaces. Subsequent studies improved upon the results of Gilányi (for example, [6–8]). Moreover, hyperstability results for monomial functional equations can be found in [8–10]. The hyperstability of a functional equation requires that any mapping satisfying the equation approximately (in some sense) must be a real solution to it (refer to [11]).

Let X be a real normed space and Y a real Banach space. We also assume that $f : X \rightarrow Y$ is a mapping.

Now, we consider the following functional equation:

$$\sum_{i=0}^{n+1} {}_{n+1}C_i (-1)^{n+1-i} f(x + iy) = 0 \quad (1)$$

and the n -monomial functional equation

$$\sum_{i=0}^n {}_nC_i (-1)^{n-i} f(x + iy) - n!f(y) = 0.$$

The n -monomial functional equation is called a Cauchy, quadratic, cubic, quartic, quintic, sextic, septic, octic, or nonic functional equation for $n = 1, 2, 3, 4, 5, 6, 7, 8, 9$, respectively. In this paper, we discuss the general nonic functional equation:

$$\sum_{i=0}^{10} {}_{10}C_i (-1)^{10-i} f(x + iy) = 0. \quad (2)$$

If we let

$$C_n f(x, y) := \sum_{i=0}^n {}_n C_i (-1)^{n-i} f(x + iy) - n! f(y)$$

and

$$D_n f(x, y) := \sum_{i=0}^n {}_n C_i (-1)^{n-i} f(x + iy),$$

then $D_n f(x, y) = C_{n-1} f(x + y, y) - C_{n-1} f(x, y)$ and $D_n f(x, y) = D_{n-1} f(x + y, x) - D_{n-1} f(x, y)$ hold. Moreover, a mapping f satisfying one of the functional equations $C_1 f(x, y) = 0$, $C_2 f(x, y) = 0$, $\dots$, $C_9 f(x, y) = 0$, $D_2 f(x, y) = 0$, $D_3 f(x, y) = 0$, $\dots$, $D_8 f(x, y) = 0$, or $D_9 f(x, y) = 0$ satisfies the general nonic function equation $D_{10} f(x, y) = 0$. Hence, we refer to the functional equation described in Equation (2) as the general nonic functional equation.

In other words, the function $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) := ax^n$ satisfies the n -monomial functional equation for $n \leq 9$ (Lemma 1 in [12]), and the function f given by $f(x) := ax^n$ for $n \leq 9$ satisfies the general nonic functional equation. Therefore, the function $g: \mathbb{R} \rightarrow \mathbb{R}$ given by $g(x) := \sum_{i=0}^9 a_i x^i$ is a particular solution of the functional equation $D_{10} f(x, y) = 0$. More information on the functional equation $D_n f(x, y) = 0$ can be found in Baker's paper [13], especially in Theorem A.

The stability results for n -monomial functional equations with $n < 9$ can be found in [6,14–24]. In the papers [16,25–33], one can find hyperstability results for various types of functional equation. Recent results regarding the stability of the functional equation described in Equation (1) with $n < 9$ can be found in [34–39], and hyperstability results for this functional equation with $n < 9$ can be found in [38,40]. Note that the superstability of a functional equation requires that any mapping satisfying the equation approximately (in some sense) must be either a real solution to it or a bounded mapping, while hyperstability requires that any mapping satisfying the equation approximately (in some sense) must be a real solution to it. Superstability results for several functional equations can be found in [28,41,42].

However, no study has yet been conducted on the stability of the general nonic functional equation. The significant advantage of this paper is the uniqueness of the solution for the stability of the general nonic functional equation. The uniqueness of the solution for the stability of the monomial functional equation has been discussed in many research papers. However, the uniqueness of the solution for the stability of general nonic functional equations is a more complicated problem. Using a special representation of a given mapping, we solved this problem.

The rest of this paper is organized as follows. In Section 2, we study a way of representing a given mapping as the sum of odd and even mappings. In Section 3, we present hyperstability results for the general nonic functional equation. Precisely, let $p < 0$ be a real number. If $f: X \rightarrow Y$ satisfies the inequality

$$\frac{\|\sum_{i=0}^{10} {}_{10} C_i (-1)^{10-i} f(x + iy)\|}{\|x\|^p + \|y\|^p} \leq \theta \text{ for all } x, y \in X \setminus \{0\},$$

then we prove that f must be a solution mapping of the general nonic functional equation. Finally, in Section 4, we discuss the stability problem of the general nonic functional equation.

The functional equation described in Equation (1) is a particular case of a general linear functional equation, which was labeled as Equation (4) in [25]. Stability results for general linear functional equations can be found in [43–46]. Moreover, hyperstability results for general linear functional equations can be found in [25,47,48].

Lastly, readers should recall that X is a normed space and Y is a Banach space throughout this paper.

2. Representation of a Given Mapping

In this section, we will introduce a way of representing a given mapping as the sum of odd and even mappings.

For a given mapping $f: X \rightarrow Y$, we denote

$$f_o(x) := \frac{f(x) - f(-x)}{2}, \quad f_e(x) := \frac{f(x) + f(-x)}{2}.$$

Let us consider the following system of nonhomogeneous linear equations:

$$\begin{cases} f_1(x) + f_3(x) + f_5(x) + f_7(x) + f_9(x) = f_o(x), \\ 2^1 f_1(x) + 8^1 f_3(x) + 32^1 f_5(x) + 128^1 f_7(x) + 512^1 f_9(x) = f_o(2x), \\ 2^2 f_1(x) + 8^2 f_3(x) + 32^2 f_5(x) + 128^2 f_7(x) + 512^2 f_9(x) = f_o(4x), \\ 2^3 f_1(x) + 8^3 f_3(x) + 32^3 f_5(x) + 128^3 f_7(x) + 512^3 f_9(x) = f_o(8x), \\ 2^4 f_1(x) + 8^4 f_3(x) + 32^4 f_5(x) + 128^4 f_7(x) + 512^4 f_9(x) = f_o(16x) \end{cases}$$

and

$$\begin{cases} f_2(x) + f_4(x) + f_6(x) + f_8(x) = f_e(x), \\ 4^1 f_2(x) + 16^1 f_4(x) + 64^1 f_6(x) + 256^1 f_8(x) = f_e(2x), \\ 4^2 f_2(x) + 16^2 f_4(x) + 64^2 f_6(x) + 256^2 f_8(x) = f_e(4x), \\ 4^3 f_2(x) + 16^3 f_4(x) + 64^3 f_6(x) + 256^3 f_8(x) = f_e(8x) \end{cases}$$

for all $x \in X$. Then, we obtain the following lemmas by the uniqueness of the solution as stated in Cramer's rule.

Lemma 1. Let $f: X \rightarrow Y$ be a given mapping and

$$M := \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 8 & 32 & 128 & 512 \\ 2^2 & 8^2 & 32^2 & 128^2 & 512^2 \\ 2^3 & 8^3 & 32^3 & 128^3 & 512^3 \\ 2^4 & 8^4 & 32^4 & 128^4 & 512^4 \end{vmatrix}.$$

Then, we have the mappings $f_1, f_3, f_5, f_7, f_9: X \rightarrow Y$ defined by the formulas

$$\begin{aligned} f_1(x) &= \frac{\begin{vmatrix} f_o(x) & 1 & 1 & 1 & 1 \\ f_o(2x) & 8 & 32 & 128 & 512 \\ f_o(4x) & 8^2 & 32^2 & 128^2 & 512^2 \\ f_o(8x) & 8^3 & 32^3 & 128^3 & 512^3 \\ f_o(16x) & 8^4 & 32^4 & 128^4 & 512^4 \end{vmatrix}}{M}, & f_3(x) &= \frac{\begin{vmatrix} 1 & f_o(x) & 1 & 1 & 1 \\ 2 & f_o(2x) & 32 & 128 & 512 \\ 2^2 & f_o(4x) & 32^2 & 128^2 & 512^2 \\ 2^3 & f_o(8x) & 32^3 & 128^3 & 512^3 \\ 2^4 & f_o(16x) & 32^4 & 128^4 & 512^4 \end{vmatrix}}{M}, \\ f_5(x) &= \frac{\begin{vmatrix} 1 & 1 & f_o(x) & 1 & 1 \\ 2 & 8 & f_o(2x) & 128 & 512 \\ 2^2 & 8^2 & f_o(4x) & 128^2 & 512^2 \\ 2^3 & 8^3 & f_o(8x) & 128^3 & 512^3 \\ 2^4 & 8^4 & f_o(16x) & 128^4 & 512^4 \end{vmatrix}}{M}, & f_7(x) &= \frac{\begin{vmatrix} 1 & 1 & 1 & f_o(x) & 1 \\ 2 & 8 & 32 & f_o(2x) & 512 \\ 2^2 & 8^2 & 32^2 & f_o(4x) & 512^2 \\ 2^3 & 8^3 & 32^3 & f_o(8x) & 512^3 \\ 2^4 & 8^4 & 32^4 & f_o(16x) & 512^4 \end{vmatrix}}{M}, \\ f_9(x) &= \frac{\begin{vmatrix} 1 & 1 & 1 & 1 & f_o(x) \\ 2 & 8 & 32 & 128 & f_o(2x) \\ 2^2 & 8^2 & 32^2 & 128^2 & f_o(4x) \\ 2^3 & 8^3 & 32^3 & 128^3 & f_o(8x) \\ 2^4 & 8^4 & 32^4 & 128^4 & f_o(16x) \end{vmatrix}}{M} \end{aligned}$$

for all $x \in X$. Furthermore, $f_o(x) = f_1(x) + f_3(x) + f_5(x) + f_7(x) + f_9(x)$ for all $x \in X$.

Lemma 2. Let $f: X \rightarrow Y$ be a given mapping and

$$M' := \begin{vmatrix} 1 & 1 & 1 & 1 \\ 4 & 16 & 64 & 256 \\ 4^2 & 16^2 & 64^2 & 256^2 \\ 4^3 & 16^3 & 64^3 & 256^3 \end{vmatrix}.$$

Then, we have the mappings $f_2, f_4, f_6, f_8: X \rightarrow Y$ defined by the formulas

$$\begin{aligned} f_2(x) &= \frac{\begin{vmatrix} f_e(x) & 1 & 1 & 1 \\ f_e(2x) & 16 & 64 & 256 \\ f_e(4x) & 16^2 & 64^2 & 256^2 \\ f_e(8x) & 16^3 & 64^3 & 256^3 \end{vmatrix}}{M'}, & f_4(x) &= \frac{\begin{vmatrix} 1 & f_e(x) & 1 & 1 \\ 4 & f_e(2x) & 64 & 256 \\ 4^2 & f_e(4x) & 64^2 & 256^2 \\ 4^3 & f_e(8x) & 64^3 & 256^3 \end{vmatrix}}{M'}, \\ f_6(x) &= \frac{\begin{vmatrix} 1 & 1 & f_e(x) & 1 \\ 4 & 16 & f_e(2x) & 256 \\ 4^2 & 16^2 & f_e(4x) & 256^2 \\ 4^3 & 16^3 & f_e(8x) & 256^3 \end{vmatrix}}{M'}, & f_8(x) &= \frac{\begin{vmatrix} 1 & 1 & 1 & f_e(x) \\ 4 & 16 & 64 & f_e(2x) \\ 4^2 & 16^2 & 64^2 & f_e(4x) \\ 4^3 & 16^3 & 64^3 & f_e(8x) \end{vmatrix}}{M'}, \end{aligned}$$

for all $x \in X$ and $f_e(x) = f_2(x) + f_4(x) + f_6(x) + f_8(x)$ for all $x \in X$.

Remark 1. By Lemmas 1 and 2, we have the following results. For all $x \in X$,

$$\begin{aligned} f_1(x) &:= \frac{f_o(16x) - 680f_o(8x) + 91392f_o(4x) - 2785280f_o(2x) + 16777216f_o(x)}{722925 \cdot 16}, \\ f_2(x) &:= \frac{f_e(8x) - 336f_e(4x) - 21504f_e(2x) - 262144f_e(x)}{2835 \cdot 64}, \\ f_3(x) &:= -\frac{5440(f_o(16x) - 674f_o(8x) + 87360f_o(4x) - 2269184f_o(2x) + 4194304f_o(x))}{722925 \cdot 65536}, \\ f_4(x) &:= -\frac{f_e(8x) - 324f_e(4x) - 17664f_e(2x) - 65536f_e(x)}{135 \cdot 1024}, \\ f_5(x) &:= \frac{1428(f_o(16x) - 650f_o(8x) + 71952f_o(4x) - 665600f_o(2x) + 1048576f_o(x))}{722925 \cdot 65536}, \\ f_6(x) &:= \frac{f_e(8x) - 276f_e(4x) - 5184f_e(2x) - 16384f_e(x)}{135 \cdot 4096}, \\ f_7(x) &:= -\frac{85(f_o(16x) - 554f_o(8x) + 21840f_o(4x) - 172544f_o(2x) + 262144f_o(x))}{722925 \cdot 65536}, \\ f_8(x) &:= -\frac{f_e(8x) - 84f_e(4x) - 1344f_e(2x) - 4096f_e(x)}{2835 \cdot 4096}, \\ f_9(x) &:= \frac{f_o(16x) - 170f_o(8x) + 5712f_o(4x) - 43520f_o(2x) + 65536f_o(x)}{722925 \cdot 65536}. \end{aligned}$$

Moreover,

$$f(x) = \sum_{i=1}^9 f_i(x) \text{ for all } x \in X.$$

Below, we define the mappings needed to prove the main theorems.

Definition 1. For a given mapping $f: X \rightarrow Y$, we define

$$\begin{aligned}\tilde{f}(x) &:= f(x) - f(0), \\ Df(x, y) &:= \sum_{i=0}^{10} 10C_i(-1)^{10-i} f(x + iy), \\ \Gamma f(x) &:= Df_o(12x, 4x) + 10Df_o(8x, 4x) + 55Df_o(4x, 4x) + 220Df_o(10x, 2x) \\ &\quad + 2200Df_o(8x, 2x) + 9988Df_o(6x, 2x) + 27280Df_o(4x, 2x) \\ &\quad + 45352Df_o(2x, 2x) + 17920Df_o(5x, x) + 179200Df_o(4x, x) \\ &\quad + 760320Df_o(3x, x) + 1689600Df_o(2x, x) + 1790976Df_o(x, x), \\ \Delta f(x) &:= Df_e(6x, 2x) + 10Df_e(4x, 2x) + 55Df_e(2x, 2x) + 110Df_e(0, 2x) \\ &\quad + 320Df_e(3x, x) + 3200Df_e(2x, x) + 12992Df_e(x, x) + 12160Df_e(0, x)\end{aligned}$$

for all $x, y \in X$.

As the results of tedious calculations, we obtain the following lemmas.

Lemma 3. Let $f: X \rightarrow Y$ be an arbitrary mapping. Then, the equalities

$$\begin{aligned}\tilde{f}(x) &= \sum_{i=1}^9 \tilde{f}_i(x), \\ D\tilde{f}(x, y) &= Df(x, y), \\ \Gamma \tilde{f}(x) &= f_o(32x) - 682f_o(16x) + 92752f_o(8x) - 2968064f_o(4x), \\ &\quad + 22347776f_o(2x) - 33554432f_o(x), \\ \Delta \tilde{f}(x) &= f_e(16x) - 340f_e(8x) + 22848f_e(4x) - 348160f_e(2x) + 1048576f_e(x)\end{aligned}$$

hold for all $x, y \in X$.

Lemma 4. Let $f: X \rightarrow Y$ be an arbitrary mapping. Then,

$$\begin{aligned}\tilde{f}_1(x) - \frac{\tilde{f}_1(2x)}{2} &= \frac{\Gamma \tilde{f}_o(x)}{722925 \cdot 32}, & \tilde{f}_2(x) - \frac{\tilde{f}_2(2x)}{4} &= \frac{\Delta \tilde{f}_e(x)}{2835 \cdot 256}, \\ \tilde{f}_3(x) - \frac{\tilde{f}_3(2x)}{8} &= -\frac{5440\Gamma \tilde{f}_o(x)}{722925 \cdot 65536 \cdot 8}, & \tilde{f}_4(x) - \frac{\tilde{f}_4(2x)}{16} &= -\frac{21\Delta \tilde{f}_e(x)}{2835 \cdot 256 \cdot 64}, \\ \tilde{f}_5(x) - \frac{\tilde{f}_5(2x)}{32} &= \frac{1428\Gamma \tilde{f}_o(x)}{722925 \cdot 32 \cdot 65536}, & \tilde{f}_6(x) - \frac{\tilde{f}_6(2x)}{64} &= \frac{21\Delta \tilde{f}_e(x)}{2835 \cdot 256 \cdot 1024}, \\ \tilde{f}_7(x) - \frac{\tilde{f}_7(2x)}{128} &= -\frac{85\Gamma \tilde{f}_o(x)}{722925 \cdot 65536 \cdot 128}, & \tilde{f}_8(x) - \frac{\tilde{f}_8(2x)}{256} &= -\frac{\Delta \tilde{f}_e(x)}{2835 \cdot 256 \cdot 4096}, \\ \tilde{f}_9(x) - \frac{\tilde{f}_9(2x)}{512} &= \frac{\Gamma \tilde{f}_o(x)}{722925 \cdot 65536 \cdot 512}\end{aligned}$$

are fulfilled for all $x, y \in X$.

Lemma 5. If $f: X \rightarrow Y$ is a mapping such that $Df(x, y) = 0$ for all $x, y \in X$ with $f(0) = 0$, then for each $i \in \{1, 2, \dots, 9\}$, the mappings $f_i(x)$ in Remark 1 satisfy the equalities $Df_i(x, y) = 0$ for all $x \in X$ and $f_i(2x) = 2^i f_i(x)$ for all $x, y \in X$.

Proof. Since $Df(x, y) = 0$ for all $x, y \in X$, by the definition of $f_i, \Gamma f$ and Δf , we have

$$Df_i(x, y) = 0, \Delta f(x) = 0, \Gamma f(x) = 0.$$

Applying Lemma 4, we arrive at $f_i(2x) = 2^i f_i(x)$. $\square$

3. Hyperstability of the General Nonic Functional Equation

In this section, we will prove the hyperstability of the general nonic functional equation. To prove the main theorem, we will use the functions introduced in the previous section.

Theorem 1. Let $p < 0$ be a real number. Suppose that $f: X \rightarrow Y$ is a mapping such that

$$\frac{\|Df(x, y)\|}{\|x\|^p + \|y\|^p} \leq \theta \text{ for all } x, y \in X \setminus \{0\}. \quad (3)$$

Then, $Df(x, y) = 0$ is fulfilled for all $x, y \in X$.

Proof. STEP 1: Using the definition of $\tilde{f}$ together with

$$\sum_{i=0}^{10} {}_{10}C_i (-1)^{10-i} = 0,$$

we have

$$D\tilde{f}(x, y) = Df(x, y), \quad \tilde{f}(0) = 0.$$

From the expression (3) and the definitions of Γf and Δf , we obtain

$$\|\Gamma \tilde{f}(x)\| \leq 47377612800K'\theta\|x\|^p, \quad \|\Delta \tilde{f}(x)\| \leq 11612160K\theta\|x\|^p$$

for all $x \in X \setminus \{0\}$, where

$$K' := \frac{220 \cdot 20^p + 55 \cdot 16^p + 10 \cdot 12^p + 17920 \cdot 10^p + 45353 \cdot 8^p + 27280 \cdot 6^p}{47377612800} + \frac{1801030 \cdot 4^p + 1689600 \cdot 3^p + 847560 \cdot 2^p + 4617216}{47377612800},$$

$$K := \frac{110 \cdot 10^p + 55 \cdot 8^p + 10 \cdot 6^p + 12160 \cdot 5^p + 12993 \cdot 4^p + 3200 \cdot 3^p + 496 \cdot 2^p + 28672}{11612160}.$$

Then, Lemma 3 and the definitions of f_k ($k \in \{1, 2, \dots, 9\}$) ensure the inequalities

$$\begin{aligned} \left\| \frac{\tilde{f}_1(2^i x)}{2^i} - \frac{\tilde{f}_1(2^{i+1} x)}{2^{i+1}} \right\| &= \left\| -\frac{4096\Gamma \tilde{f}_o(x)}{47377612800 \cdot 2^{i+1}} \right\| \leq \frac{4096K'\theta\|x\|^p}{2^{i+1}}, \\ \left\| \frac{\tilde{f}_2(2^i x)}{4^i} - \frac{\tilde{f}_2(2^{i+1} x)}{4^{i+1}} \right\| &= \left\| -\frac{64\Delta \tilde{f}_e(x)}{11612160 \cdot 4^{i+1}} \right\| \leq \frac{64K\theta\|x\|^p}{4^{i+1}}, \\ \left\| \frac{\tilde{f}_3(2^i x)}{8^i} - \frac{\tilde{f}_3(2^{i+1} x)}{8^{i+1}} \right\| &= \left\| \frac{5440\Gamma \tilde{f}_o(x)}{47377612800 \cdot 8^{i+1}} \right\| \leq \frac{5440K'\theta\|x\|^p}{8^{i+1}}, \\ \left\| \frac{\tilde{f}_4(2^i x)}{16^i} - \frac{\tilde{f}_4(2^{i+1} x)}{16^{i+1}} \right\| &= \left\| \frac{84\Delta \tilde{f}_e(x)}{11612160 \cdot 16^{i+1}} \right\| \leq \frac{84K\theta\|x\|^p}{16^{i+1}}, \\ \left\| \frac{\tilde{f}_5(2^i x)}{32^i} - \frac{\tilde{f}_5(2^{i+1} x)}{32^{i+1}} \right\| &= \left\| -\frac{1428\Gamma \tilde{f}_o(x)}{47377612800 \cdot 32^{i+1}} \right\| \leq \frac{1428K'\theta\|x\|^p}{32^{i+1}}, \\ \left\| \frac{\tilde{f}_6(2^i x)}{64^i} - \frac{\tilde{f}_6(2^{i+1} x)}{64^{i+1}} \right\| &= \left\| -\frac{21\Delta \tilde{f}_e(x)}{11612160 \cdot 64^{i+1}} \right\| \leq \frac{21K\theta\|x\|^p}{64^{i+1}}, \\ \left\| \frac{\tilde{f}_7(2^i x)}{128^i} - \frac{\tilde{f}_7(2^{i+1} x)}{128^{i+1}} \right\| &= \left\| \frac{85\Gamma \tilde{f}_o(x)}{47377612800 \cdot 128^{i+1}} \right\| \leq \frac{85K'\theta\|x\|^p}{128^{i+1}}, \\ \left\| \frac{\tilde{f}_8(2^i x)}{256^i} - \frac{\tilde{f}_8(2^{i+1} x)}{256^{i+1}} \right\| &= \left\| \frac{\Delta \tilde{f}_e(x)}{11612160 \cdot 256^{i+1}} \right\| \leq \frac{K\theta\|x\|^p}{256^{i+1}}, \\ \left\| \frac{\tilde{f}_9(2^i x)}{512^i} - \frac{\tilde{f}_9(2^{i+1} x)}{512^{i+1}} \right\| &= \left\| \frac{\Gamma \tilde{f}_o(x)}{47377612800 \cdot 512^{i+1}} \right\| \leq \frac{K'\theta\|x\|^p}{512^{i+1}} \end{aligned}$$

for all $x \in X \setminus \{0\}$. Note that

$$\sum_{k=1}^9 \frac{\tilde{f}_k(2^n x)}{2^{kn}} - \sum_{k=1}^9 \frac{\tilde{f}_k(2^{n+m} x)}{2^{k(n+m)}} = \sum_{i=n}^{n+m-1} \left(\sum_{k=1}^9 \frac{\tilde{f}_k(2^i x)}{2^{ki}} - \sum_{k=1}^9 \frac{\tilde{f}_k(2^{i+1} x)}{2^{k(i+1)}} \right).$$

This implies that

$$\left\| \sum_{k=1}^9 \frac{\tilde{f}_k(2^n x)}{2^{kn}} - \sum_{k=1}^9 \frac{\tilde{f}_k(2^{n+m} x)}{2^{k(n+m)}} \right\| \leq \sum_{i=n}^{n+m-1} \left(\frac{4096K'}{2^{i+1}} + \frac{64K}{4^{i+1}} + \frac{5440K'}{8^{i+1}} + \frac{84K}{16^{i+1}} + \frac{1428K'}{32^{i+1}} + \frac{21K}{64^{i+1}} + \frac{85K'}{128^{i+1}} + \frac{K}{256^{i+1}} + \frac{K'}{512^{i+1}} \right) \cdot \theta \|x\|^p \quad (4)$$

for all $x \in X \setminus \{0\}$ and $n, m \in \mathbb{N} \cup \{0\}$. If $p < 0$, the sequences $\{\frac{\tilde{f}_1(2^n x)}{2^n}\}, \dots, \{\frac{\tilde{f}_9(2^n x)}{2^{9n}}\}$ and $\{\sum_{k=1}^9 \frac{\tilde{f}_k(2^n x)}{2^{kn}}\}$ are Cauchy for all $x \in X \setminus \{0\}$. Since Y is complete and $\tilde{f}(0) = 0$, the sequences $\{\frac{\tilde{f}_1(2^n x)}{2^n}\}, \dots, \{\frac{\tilde{f}_9(2^n x)}{2^{9n}}\}$ and $\{\sum_{k=1}^9 \frac{\tilde{f}_k(2^n x)}{2^{kn}}\}$ converge for all $x \in X$. Hence, for each $k \in \{1, 2, \dots, 9\}$, we can define mappings $F_k, F: X \rightarrow Y$ by

$$F_k(x) := \lim_{n \rightarrow \infty} \frac{\tilde{f}_k(2^n x)}{2^{kn}}, \quad F(x) := \lim_{n \rightarrow \infty} \sum_{k=1}^9 \frac{\tilde{f}_k(2^n x)}{2^{kn}} \quad (x \in X).$$

STEP 2: By (3) and the definitions of Df, F_1 , and f_1 , since $Df_1(x, y) = D\tilde{f}_1(x, y)$, we have

$$\begin{aligned} \|DF_1(x, y)\| &= \lim_{n \rightarrow \infty} \left\| \frac{D\tilde{f}_1(2^n x, 2^n y)}{2^n} \right\| = \lim_{n \rightarrow \infty} \left\| \frac{Df_1(2^n x, 2^n y)}{2^n} \right\| \\ &= \lim_{n \rightarrow \infty} \left\| \frac{16777216Df_0(2^n x, 2^n y)}{11566800 \cdot 2^n} - \frac{2785280Df_0(2^{n+1} x, 2^{n+1} y)}{11566800 \cdot 2^n} \right. \\ &\quad \left. + \frac{91392f_0(2^{n+2} x, 2^{n+2} y)}{11566800 \cdot 2^n} - \frac{680Df_0(2^{n+3} x, 2^{n+3} y)}{11566800 \cdot 2^n} + \frac{Df_0(2^{n+4} x, 2^{n+4} y)}{11566800 \cdot 2^n} \right\| \\ &\leq \lim_{n \rightarrow \infty} \left(\frac{16777216 + 2785280 \cdot 2^p + 91392 \cdot 2^{2p} + 680 \cdot 2^{3p} + 2^{4p}}{11566800} \right) \cdot \frac{2^{np}}{2^n} \theta (\|x\|^p + \|y\|^p) \\ &= 0 \end{aligned} \quad (5)$$

for all $x, y \in X \setminus \{0\}$. On the other hand, from the definition of DF_1 , we have

$$DF_1(x, 0) = \sum_{i=0}^{10} {}_{10}C_i (-1)^{10-i} F_1(x + i0) = F_1(x) \sum_{i=0}^{10} {}_{10}C_i (-1)^{10-i} = 0 \quad (6)$$

for all $x \in X$. And, in view of (5), we obtain

$$DF_1(0, y) = DF_1(-10y, y) = 0 \quad \text{for all } y \in X \setminus \{0\}. \quad (7)$$

Therefore, the relations (5)–(7) yield that $DF_1(x, y) = 0$ for all $x, y \in X$. Similarly, we can show that $DF_k(x, y) = 0$ for each $k \in \{2, 3, \dots, 9\}$ and all $x, y \in X$. Since $DF(x, y) = \sum_{k=1}^9 DF_k(x, y)$ for all $x, y \in X$, we obtain $DF(x, y) = 0$ for all $x, y \in X$.

STEP 3: Observe that for all $x \in X$,

$$\begin{aligned} D\tilde{f}(x, y) - DF(x, y) &= \sum_{i=0}^{10} {}_{10}C_i (-1)^{10-i} (\tilde{f}(x + iy) - F(x + iy)), \\ DF((1-n)x, nx) &= 0. \end{aligned}$$

Then, we see that

$$D\tilde{f}((1-n)x, nx) = \tilde{f}((1-n)x) - F((1-n)x) - 10\tilde{f}(x) + 10F(x) + \sum_{i=2}^{10} {}_{10}C_i(-1)^{10-i}(\tilde{f}((1-n)x + inx) - F((1-n)x + inx)) \quad (8)$$

for any $n \in N$ and $x \in X \setminus \{0\}$. Moreover, by letting $n = 0$ and taking the limit $m \rightarrow \infty$ in (4), we obtain the inequalities

$$\|\tilde{f}(x) - F(x)\| \leq \left(\frac{4096K'}{2-2^p} + \frac{64K}{4-2^p} + \frac{5440K'}{8-2^p} + \frac{84K}{16-2^p} + \frac{1428K'}{32-2^p} + \frac{21K}{64-2^p} + \frac{85K'}{128-2^p} + \frac{K}{256-2^p} + \frac{K'}{512-2^p} \right) \cdot \theta \|x\|^p \quad (9)$$

for all $x \in X$.

Since $p < 0$, by (3), (8), and (9), we are forced to conclude that

$$\begin{aligned} 10 \cdot \|\tilde{f}(x) - F(x)\| &\leq \lim_{n \rightarrow \infty} \|D\tilde{f}((1-n)x, nx)\| + \lim_{n \rightarrow \infty} \|\tilde{f}((1-n)x) - F((1-n)x)\| \\ &\quad + \sum_{i=2}^{10} \lim_{n \rightarrow \infty} \|{}_{10}C_i(\tilde{f}(((i-1)n+1)x) - F(((i-1)n+1)x))\| \\ &\leq \lim_{n \rightarrow \infty} ((n-1)^p + n^p) + M(n-1)^p + \sum_{i=2}^{10} {}_{10}C_i((i-1)n+1)^p \cdot \theta \|x\|^p \\ &= 0 \end{aligned}$$

for all $x \in X \setminus \{0\}$, where

$$M := \frac{4096K'}{2-2^p} + \frac{64K}{4-2^p} + \frac{5440K'}{8-2^p} + \frac{84K}{16-2^p} + \frac{1428K'}{32-2^p} + \frac{21K}{64-2^p} + \frac{85K'}{128-2^p} + \frac{K}{256-2^p} + \frac{K'}{512-2^p}.$$

Thus, we have $\|\tilde{f}(x) - F(x)\| = 0$ for all $x \in X \setminus \{0\}$. Since $\tilde{f}(0) = 0 = F(0)$, we have $\tilde{f}(x) = F(x)$ for all $x \in X$. Therefore, $D\tilde{f}(x, y) = DF(x, y) = 0$ for all $x, y \in X$. From the fact that $D\tilde{f}(x, y) = Df(x, y)$, we finally have

$$Df(x, y) = D\tilde{f}(x, y) = DF(x, y) = 0,$$

which completes the proof. $\square$

4. Stability of the General Nonic Functional Equation

In this section, we will consider the stability of the general nonic functional equation

$$\sum_{i=0}^{10} {}_{10}C_i(-1)^{10-i}f(x + iy) = 0.$$

Theorem 2. Let $p \neq 1, 2, 3, 4, 5, 6, 7, 8, 9$ be a non-negative real number. Suppose that $f : X \rightarrow Y$ is a mapping such that for all $x, y \in X$,

$$\|Df(x, y)\| \leq \theta(\|x\|^p + \|y\|^p). \quad (10)$$

Then, there exists a unique mapping F satisfying $DF(x, y) = 0$ and

$$\begin{aligned} \|\tilde{f}(x) - F(x)\| &\leq \left(\frac{4096}{|2-2^p|} + \frac{5440}{|8-2^p|} + \frac{1428}{|32-2^p|} + \frac{85}{|128-2^p|} + \frac{1}{|512-2^p|} \right) \cdot \frac{K'\theta\|x\|^p}{4096} \\ &\quad + \left(\frac{64}{|4-2^p|} + \frac{84}{|16-2^p|} + \frac{21}{|64-2^p|} + \frac{1}{|256-2^p|} \right) \cdot \frac{K\theta\|x\|^p}{64}, \end{aligned} \quad (11)$$

for all $x \in X$, where $K := 6^p + 10 \cdot 4^p + 320 \cdot 3^p + 3431 \cdot 2^p + 41664$ and

$$K' := 12^p + 220 \cdot 10^p + 2210 \cdot 8^p + 9988 \cdot 6^p + 17920 \cdot 5^p + 206601 \cdot 4^p + 760320 \cdot 3^p + 1819992 \cdot 2^p + 6228992.$$

Proof. From the definition of $\tilde{f}$, we obtain $D\tilde{f}(x, y) = Df(x, y)$ and $\tilde{f}(0) = 0$. By (10) and the definition of Γf and Δf , we have

$$\begin{aligned} \|\Gamma \tilde{f}(x)\| &= \|Df_o(12x, 4x) + 10Df_o(8x, 4x) + 55Df_o(4x, 4x) + 220Df_o(10x, 2x) \\ &\quad + 2200Df_o(8x, 2x) + 9988Df_o(6x, 2x) + 27280Df_o(4x, 2x) \\ &\quad + 45352Df_o(2x, 2x) + 17920Df_o(5x, x) + 179200Df_o(4x, x) \\ &\quad + 760320Df_o(3x, x) + 1689600Df_o(2x, x) + 1790976Df_o(x, x)\| \\ &\leq (12^p + 4^p + 10 \cdot 8^p + 10 \cdot 4^p + 110 \cdot 4^p + 220 \cdot 10^p + 220 \cdot 2^p \\ &\quad + 2200 \cdot 8^p + 2200 \cdot 2^p + 9988 \cdot 6^p + 9988 \cdot 2^p + 27280 \cdot 4^p + 27280 \cdot 2^p \\ &\quad + 90704 \cdot 2^p + 17920 \cdot 5^p + 17920 + 179200 \cdot 4^p + 179200 \\ &\quad + 760320 \cdot 3^p + 760320 + 1689600 \cdot 2^p + 1689600 + 3581952) \cdot \theta \|x\|^p \\ &\leq 722925 \cdot 16K'\theta \|x\|^p, \end{aligned} \quad (12)$$

$$\begin{aligned} \|\Delta \tilde{f}(x)\| &= \|Df_e(6x, 2x) + 10Df_e(4x, 2x) + 55Df_e(2x, 2x) + 110Df_e(0, 2x) \\ &\quad + 320Df_e(3x, x) + 3200Df_e(2x, x) + 12992Df_e(x, x) + 12160Df_e(0, x)\| \\ &\leq (6^p + 2^p + 10 \cdot 4^p + 10 \cdot 2^p + 110 \cdot 2^p + 110 \cdot 2^p \\ &\quad + 320 \cdot 3^p + 320 + 3200 \cdot 2^p + 3200 + 25984 + 12160) \cdot \theta \|x\|^p \\ &\leq 2835 \cdot 64K'\theta \|x\|^p \end{aligned} \quad (13)$$

for all $x \in X$.

Next, for $i \in \{1, 2, \dots, 9\}$, we will find F_k to reach $F(x) = \sum_{k=1}^9 F_k(x)$. For each given $p \neq 1, 2, 3, 4, 5, 6, 7, 8, 9$, we will use a different approach to find the functions F_k .

- **Setting F_1 :** Let $0 \leq p < 1$. It follows from Lemma 4 and (12) that

$$\left\| \frac{\tilde{f}_1(2^i x)}{2^i} - \frac{\tilde{f}_1(2^{i+1} x)}{2^{i+1}} \right\| = \left\| \frac{\Gamma \tilde{f}(2^i x)}{722925 \cdot 32 \cdot 2^i} \right\| \leq \frac{K'\theta \|x\|^p}{2 \cdot 2^i},$$

for all $x \in X$. Notice that for all $x \in X$,

$$\frac{\tilde{f}_1(2^n x)}{2^n} - \frac{\tilde{f}_1(2^{n+m} x)}{2^{n+m}} = \sum_{i=n}^{n+m-1} \left(\frac{\tilde{f}_1(2^i x)}{2^i} - \frac{\tilde{f}_1(2^{i+1} x)}{2^{i+1}} \right),$$

which implies that

$$\left\| \frac{\tilde{f}_1(2^n x)}{2^n} - \frac{\tilde{f}_1(2^{n+m} x)}{2^{n+m}} \right\| \leq \sum_{i=n}^{n+m-1} \frac{K'2^{ip}\theta \|x\|^p}{2 \cdot 2^i} \quad (14)$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. By (14), the sequence $\{\frac{\tilde{f}_1(2^n x)}{2^n}\}$ is a Cauchy sequence for all $x \in X$, because of the fact that $0 \leq p < 1$. Since Y is complete, the sequence $\{\frac{\tilde{f}_1(2^n x)}{2^n}\}$ converges. Hence, we may define a mapping $F_1 : X \rightarrow Y$ by

$$F_1(x) := \lim_{n \rightarrow \infty} \frac{\tilde{f}_1(2^n x)}{2^n} \quad \text{for all } x \in X.$$

Moreover, letting $n = 0$ and taking the limit $m \rightarrow \infty$ in (14), we obtain the inequality

$$\|\tilde{f}_1(x) - F_1(x)\| \leq \frac{K'\theta \|x\|^p}{2 - 2^p} \quad \text{for all } x \in X.$$

By the definition of F_1 , we easily obtain $F_1(2x) = 2F_1(x)$ for all $x \in X$ and

$$\begin{aligned} \|DF_1(x, y)\| &= \lim_{n \rightarrow \infty} \left\| \frac{Df_1(2^n x, 2^n y)}{2^n} \right\| \\ &= \lim_{n \rightarrow \infty} \left\| \frac{16777216Df_o(2^n x, 2^n y)}{11566800 \cdot 2^n} - \frac{2785280Df_o(2^{n+1}x, 2^{n+1}y)}{11566800 \cdot 2^n} \right. \\ &\quad \left. + \frac{91392Df_o(2^{n+2}x, 2^{n+2}y)}{11566800 \cdot 2^n} - \frac{680Df_o(2^{n+3}x, 2^{n+3}y)}{11566800 \cdot 2^n} - \frac{Df_o(2^{n+4}x, 2^{n+4}y)}{11566800 \cdot 2^n} \right\| \\ &\leq \lim_{n \rightarrow \infty} \frac{16777216 \cdot 2^{np}\theta(\|x\|^p + \|y\|^p)}{11566800 \cdot 2^n} + \lim_{n \rightarrow \infty} \frac{2785280 \cdot 2^{(n+1)p}\theta(\|x\|^p + \|y\|^p)}{11566800 \cdot 2^n} \\ &\quad + \lim_{n \rightarrow \infty} \frac{91392 \cdot 2^{(n+2)p}\theta(\|x\|^p + \|y\|^p)}{11566800 \cdot 2^n} + \lim_{n \rightarrow \infty} \frac{680 \cdot 2^{(n+3)p}\theta(\|x\|^p + \|y\|^p)}{11566800 \cdot 2^n} \\ &\quad + \lim_{n \rightarrow \infty} \frac{2^{(n+4)p}\theta(\|x\|^p + \|y\|^p)}{11566800 \cdot 2^n} = 0 \end{aligned}$$

for all $x, y \in X$.

Let $p > 1$. It follows from Lemma 4 and (12) that

$$\left\| 2^i \tilde{f}_1(2^{-i}x) - 2^{i+1} \tilde{f}_1(2^{-i-1}x) \right\| \leq \left\| \frac{2^i \Gamma \tilde{f}(2^{-i-1}x)}{722925 \cdot 16} \right\| \leq \frac{K' 2^i \theta \|x\|^p}{2^{(i+1)p}}$$

for all $x \in X$. Because of the fact that

$$2^n \tilde{f}_1(2^{-n}x) - 2^{n+m} \tilde{f}_1(2^{-n-m}x) = \sum_{i=n}^{n+m-1} \left(2^i \tilde{f}_1(2^{-i}x) - (2^{i+1} \tilde{f}_1(2^{-i-1}x)) \right)$$

for all $x \in X$, we have

$$\|2^n \tilde{f}_1(2^{-n}x) - 2^{n+m} \tilde{f}_1(2^{-n-m}x)\| \leq \sum_{i=n}^{n+m-1} \frac{K' 2^i \theta \|x\|^p}{2^{(i+1)p}} \quad (15)$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Since $p > 1$, by (15), the sequence $\{2^n \tilde{f}_1(2^{-n}x)\}$ is a Cauchy sequence for all $x \in X$. By the completeness of Y , we know that the sequence $\{2^n \tilde{f}_1(2^{-n}x)\}$ converges. Hence, we can define a mapping $F_1: X \rightarrow Y$ by

$$F_1(x) := \lim_{n \rightarrow \infty} 2^n \tilde{f}_1(2^{-n}x) \text{ for all } x \in X.$$

However, letting $n = 0$ and passing the limit $m \rightarrow \infty$ in (15), we obtain the inequality

$$\|\tilde{f}_1(x) - F_1(x)\| \leq \frac{K' \theta \|x\|^p}{2^p - 2} \text{ for all } x \in X.$$

From the definition of F_1 , we easily obtain $F_1(2x) = 2F_1(x)$ for all $x \in X$ and $DF_1(x, y) = 0$ for all $x, y \in X$.

• **Setting F_2 :** Let $p < 2$. It follows from Lemma 4 and (13) that

$$\left\| \frac{\tilde{f}_2(2^n x)}{4^n} - \frac{\tilde{f}_2(2^{n+m}x)}{4^{n+m}} \right\| = \left\| \frac{\Delta \tilde{f}_e(2^i x)}{2835 \cdot 256 \cdot 4^i} \right\| \leq \sum_{i=n}^{n+m-1} \frac{K 2^{ip} \theta \|x\|^p}{4 \cdot 4^i} \quad (16)$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Since $p < 2$, we have from (16) that the sequence $\{\frac{\tilde{f}_2(2^n x)}{4^n}\}$ is a Cauchy sequence for all $x \in X$. By the completeness of Y , the sequence $\{\frac{\tilde{f}_2(2^n x)}{4^n}\}$ converges. Hence, we can define a mapping $F_2: X \rightarrow Y$ by

$$F_2(x) := \lim_{n \rightarrow \infty} \frac{\tilde{f}_2(2^n x)}{4^n} \text{ for all } x \in X.$$

Now, letting $n = 0$ and passing the limit $m \rightarrow \infty$ in (16), we obtain

$$\|\tilde{f}_2(x) - F_2(x)\| \leq \frac{K\theta\|x\|^p}{4 - 2^p} \quad \text{for all } x \in X. \quad (17)$$

From the definition of F_2 , we then have $F_2(2x) = 4F_2(x)$ for all $x \in X$ and $DF_2(x, y) = 0$ for all $x, y \in X$.

Let $p > 2$. It follows from Lemma 4 and (13) that

$$\|4^n \tilde{f}_2(2^{-n}x) - 4^{n+m} \tilde{f}_2(2^{-n-m}x)\| \leq \sum_{i=n}^{n+m-1} \frac{K4^i \theta \|x\|^p}{2^{(i+1)p}} \quad (18)$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Since $p > 2$, we have by (18) that the sequence $4^n \tilde{f}_2(2^{-n}x)$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n \tilde{f}_2(2^{-n}x)\}$ converges. Then, we can define a mapping $F_2: X \rightarrow Y$ by

$$F_2(x) := \lim_{n \rightarrow \infty} 4^n \tilde{f}_2(2^{-n}x) \quad \text{for all } x \in X.$$

In (18), letting $n = 0$ and passing the limit $m \rightarrow \infty$, we obtain

$$\|\tilde{f}_2(x) - F_2(x)\| \leq \frac{K\theta\|x\|^p}{2^p - 4} \quad \text{for all } x \in X.$$

According to the definition of F_2 , we arrive at $F_2(2x) = 4F_2(x)$ for all $x \in X$ and $DF_2(x, y) = 0$ for all $x, y \in X$.

- **Setting F_3 :** Let $p < 3$. It follows by Lemma 4 and (12) that

$$\left\| \frac{\tilde{f}_3(2^n x)}{8^n} - \frac{\tilde{f}_3(2^{n+m} x)}{8^{n+m}} \right\| = \left\| \frac{5440\Gamma \tilde{f}(2^i x)}{722925 \cdot 524288 \cdot 8^i} \right\| \leq \sum_{i=n}^{n+m-1} \frac{5440K'2^{ip}\theta\|x\|^p}{32768 \cdot 8^i} \quad (19)$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Since $p < 3$, we see by (19) that the sequence $\{\frac{\tilde{f}_3(2^n x)}{8^n}\}$ is a Cauchy sequence for all $x \in X$. From the completeness of Y , the sequence $\{\frac{\tilde{f}_3(2^n x)}{8^n}\}$ converges. Thus, we may define a mapping $F_3: X \rightarrow Y$ by

$$F_3(x) := \lim_{n \rightarrow \infty} \frac{\tilde{f}_3(2^n x)}{8^n} \quad \text{for all } x \in X.$$

Putting $n = 0$ and passing to the limit as $m \rightarrow \infty$ in (19), we find that

$$\|\tilde{f}_3(x) - F_3(x)\| \leq \frac{5440K'\theta\|x\|^p}{4096(8 - 2^p)}$$

for all $x \in X$. Based on the definition of F_3 , we yield $F_3(2x) = 8F_3(x)$ for all $x \in X$ and $DF_3(x, y) = 0$ for all $x, y \in X$.

Let $p > 3$. Lemma 4 and (12) guarantee that

$$\|8^n \tilde{f}_3(2^{-n}x) - 8^{n+m} \tilde{f}_3(2^{-n-m}x)\| \leq \sum_{i=n}^{n+m-1} \frac{5440K'8^i\theta\|x\|^p}{4096 \cdot 2^{(i+1)p}} \quad (20)$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Since $p > 3$, we have from (20) that $\{8^n \tilde{f}_3(2^{-n}x)\}$ is a Cauchy sequence for all $x \in X$. Thus, because Y is complete, the sequence $\{8^n \tilde{f}_3(2^{-n}x)\}$ converges. Then, we may define a mapping $F_3: X \rightarrow Y$ by

$$F_3(x) := \lim_{n \rightarrow \infty} 8^n \tilde{f}_3(2^{-n}x) \quad \text{for all } x \in X.$$

Let $n = 0$ and take the limit $m \rightarrow \infty$ in (20); then,

$$\|\tilde{f}_3(x) - F_3(x)\| \leq \frac{5440K'\theta\|x\|^p}{4096(2^p - 8)} \quad \text{for all } x \in X.$$

The definition of F_3 shows that $F_3(2x) = 8F_3(x)$ for all $x \in X$ and $DF_3(x, y) = 0$ for all $x, y \in X$.

- **Setting F_4 :** Let $p < 4$. It follows from Lemma 4 and (13) that

$$\left\| \frac{\tilde{f}_4(2^n x)}{16^n} - \frac{\tilde{f}_4(2^{n+m} x)}{16^{n+m}} \right\| = \left\| \frac{21\Delta\tilde{f}_e(2^i x)}{2835 \cdot 16384 \cdot 16^i} \right\| \leq \sum_{i=n}^{n+m-1} \frac{21K2^i p \theta \|x\|^p}{256 \cdot 16^i} \quad (21)$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Considering $p < 4$ and (21), we know that the sequence $\{\frac{\tilde{f}_4(2^n x)}{16^n}\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{\tilde{f}_4(2^n x)}{16^n}\}$ converges. Thereby, we can define a mapping $F_4: X \rightarrow Y$ by

$$F_4(x) := \lim_{n \rightarrow \infty} \frac{\tilde{f}_4(2^n x)}{16^n} \quad \text{for all } x \in X.$$

Now, by letting $n = 0$ and passing the limit $m \rightarrow \infty$ in (21), we obtain the inequality

$$\|\tilde{f}_4(x) - F_4(x)\| \leq \frac{21K\theta\|x\|^p}{16(16 - 2^p)} \quad \text{for all } x \in X.$$

We know from the definition of F_4 that $F_4(2x) = 16F_4(x)$ for all $x \in X$ and $DF_4(x, y) = 0$ for all $x, y \in X$.

Let $p > 4$. Lemma 4 and (13) imply that

$$\|16^n \tilde{f}_4(2^{-n} x) - 16^{n+m} \tilde{f}_4(2^{-n-m} x)\| \leq \sum_{i=n}^{n+m-1} \frac{21K16^i \theta \|x\|^p}{16 \cdot 2^{(i+1)p}} \quad (22)$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Since $p > 4$, we have by (22) that $16^n \tilde{f}_4(2^{-n} x)$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{16^n \tilde{f}_4(2^{-n} x)\}$ converges. Thus, we can define a mapping $F_4: X \rightarrow Y$ by

$$F_4(x) := \lim_{n \rightarrow \infty} 16^n \tilde{f}_4(2^{-n} x) \quad \text{for all } x \in X.$$

Meanwhile, in (22), letting $n = 0$ and passing the limit $m \rightarrow \infty$, we then have the inequality

$$\|\tilde{f}_4(x) - F_4(x)\| \leq \frac{21K\theta\|x\|^p}{16(2^p - 16)} \quad \text{for all } x \in X.$$

From the definition of F_4 , we see that $F_4(2x) = 16F_4(x)$ for all $x \in X$ and $DF_4(x, y) = 0$ for all $x, y \in X$.

- **Setting F_5 :** Let $p < 5$. It follows from Lemma 4 and (12) that

$$\left\| \frac{\tilde{f}_5(2^n x)}{32^n} - \frac{\tilde{f}_5(2^{n+m} x)}{32^{n+m}} \right\| = \left\| \frac{1428\Gamma\tilde{f}(2^i x)}{722925 \cdot 2097152 \cdot 32^i} \right\| \leq \sum_{i=n}^{n+m-1} \frac{1428K'2^i p \theta \|x\|^p}{131072 \cdot 32^i} \quad (23)$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. We assume that $p < 5$. Thus, we have by (23) that $\{\frac{\tilde{f}_5(2^n x)}{32^n}\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{\tilde{f}_5(2^n x)}{32^n}\}$ converges. Hence, we can define a mapping $F_5 : X \rightarrow Y$ by

$$F_5(x) := \lim_{n \rightarrow \infty} \frac{\tilde{f}_5(2^n x)}{32^n} \quad \text{for all } x \in X.$$

Moreover, letting $n = 0$ and passing the limit $m \rightarrow \infty$ in (23), we obtain the inequality

$$\|\tilde{f}_5(x) - F_5(x)\| \leq \frac{1428K'\theta\|x\|^p}{4096(32 - 2^p)} \quad \text{for all } x \in X.$$

With the help of the definition of F_5 , we obtain $F_5(2x) = 32F_5(x)$ for all $x \in X$ and $DF_5(x, y) = 0$ for all $x, y \in X$.

Let $p > 5$. In view of Lemma 4 and (12), we have

$$\|32^n \tilde{f}_5(2^{-n}x) - 32^{n+m} \tilde{f}_5(2^{-n-m}x)\| \leq \sum_{i=n}^{n+m-1} \frac{1428 \cdot 32^i K' \theta \|x\|^p}{4096 \cdot 2^{(i+1)p}} \quad (24)$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. It follows from $p > 5$ and (24) that the sequence $\{32^n \tilde{f}_5(2^{-n}x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, we see that the sequence $\{32^n \tilde{f}_5(2^{-n}x)\}$ converges. Thus, one can define a mapping $F_5 : X \rightarrow Y$ by

$$F_5(x) := \lim_{n \rightarrow \infty} 32^n \tilde{f}_5(2^{-n}x) \quad \text{for all } x \in X.$$

Furthermore, setting $n = 0$ and passing the limit $m \rightarrow \infty$ in (24), we reach

$$\|\tilde{f}_5(x) - F_5(x)\| \leq \frac{1428K'\theta\|x\|^p}{4096(2^p - 32)} \quad \text{for all } x \in X.$$

By virtue of the definition of F_5 , we find that $F_5(2x) = 32F_5(x)$ for all $x \in X$ and $DF_5(x, y) = 0$ for all $x, y \in X$.

• **Setting F_6 :** Let $p < 6$. We combine Lemma 4 and (13) to find that

$$\left\| \frac{\tilde{f}_6(2^n x)}{64^n} - \frac{\tilde{f}_6(2^{n+m} x)}{64^{n+m}} \right\| = \left\| \frac{21\Delta \tilde{f}_e(2^i x)}{2835 \cdot 262144 \cdot 64^i} \right\| \leq \sum_{i=n}^{n+m-1} \frac{21K2^i \theta \|x\|^p}{4096 \cdot 64^i} \quad (25)$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Then, since $p < 6$, it follows by (25) that $\{\frac{\tilde{f}_6(2^n x)}{64^n}\}$ is a Cauchy sequence for all $x \in X$. The completeness of Y ensures that the sequence $\{\frac{\tilde{f}_6(2^n x)}{64^n}\}$ converges, so that we can define a mapping $F_6 : X \rightarrow Y$ by

$$F_6(x) := \lim_{n \rightarrow \infty} \frac{\tilde{f}_6(2^n x)}{64^n} \quad \text{for all } x \in X.$$

Then, we let $n = 0$ and take the limit as $m \rightarrow \infty$ in (25) to obtain

$$\|\tilde{f}_6(x) - F_6(x)\| \leq \frac{21K\theta\|x\|^p}{64(64 - 2^p)} \quad \text{for all } x \in X.$$

According to the definition of F_6 , we show that $F_6(2x) = 64F_6(x)$ for all $x \in X$ and $DF_6(x, y) = 0$ for all $x, y \in X$.

Let $p > 6$. It follows from Lemma 4 and (13) that

$$\|64^n \tilde{f}_6(2^{-n}x) - 64^{n+m} \tilde{f}_6(2^{-n-m}x)\| \leq \sum_{i=n}^{n+m-1} \frac{21 \cdot 64^i K \theta \|x\|^p}{64 \cdot 2^{(i+1)p}} \quad (26)$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Since $p > 6$, we see from (26) that the sequence $\{64^n \tilde{f}_6(2^{-n}x)\}$ is a Cauchy sequence for all $x \in X$. Thus, by the completeness of Y , we find that the sequence $\{64^n \tilde{f}_6(2^{-n}x)\}$ converges. Hence, we can define a mapping $F_6: X \rightarrow Y$ by

$$F_6(x) := \lim_{n \rightarrow \infty} 64^n \tilde{f}_6(2^{-n}x) \quad \text{for all } x \in X.$$

In addition, we let $n = 0$ and $m \rightarrow \infty$ in (26) to obtain

$$\|\tilde{f}_6(x) - F_6(x)\| \leq \frac{21K\theta\|x\|^p}{64(2^p - 64)} \quad \text{for all } x \in X.$$

From the definition of F_6 , we reach $F_6(2x) = 64F_6(x)$ for all $x \in X$ and $DF_6(x, y) = 0$ for all $x, y \in X$.

- **Setting F_7 :** Let $p < 7$. We know by Lemma 4 and (12) that

$$\left\| \frac{\tilde{f}_7(2^n x)}{128^n} - \frac{\tilde{f}_7(2^{n+m} x)}{128^{n+m}} \right\| = \left\| \frac{85\Gamma \tilde{f}(2^i x)}{722925 \cdot 8388608 \cdot 128^i} \right\| \leq \sum_{i=n}^{n+m-1} \frac{85K'2^{ip}\theta\|x\|^p}{524504 \cdot 128^i} \quad (27)$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Based on the fact that $p < 7$ and (27), the sequence $\{\frac{\tilde{f}_7(2^n x)}{128^n}\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{\tilde{f}_7(2^n x)}{128^n}\}$ converges. Thus, one can define a mapping $F_7: X \rightarrow Y$ by

$$F_7(x) := \lim_{n \rightarrow \infty} \frac{\tilde{f}_7(2^n x)}{128^n} \quad \text{for all } x \in X.$$

Moreover, letting $n = 0$ and passing the limit $m \rightarrow \infty$ in (27), we obtain

$$\|\tilde{f}_7(x) - F_7(x)\| \leq \frac{85K'\theta\|x\|^p}{4096(128 - 2^p)} \quad \text{for all } x \in X.$$

By the definition of F_7 , we have $F_7(2x) = 128F_7(x)$ for all $x \in X$ and $DF_7(x, y) = 0$ for all $x, y \in X$.

Let $p > 7$. Note that, by Lemma 4 and (12), we have

$$\|128^n \tilde{f}_7(2^{-n}x) - 128^{n+m} \tilde{f}_7(2^{-n-m}x)\| \leq \frac{85 \cdot 128^i K' \theta \|x\|^p}{4096 \cdot 2^{(i+1)p}} \quad (28)$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. On the basis of the assumption $p > 7$ and (28), we see that $\{128^n \tilde{f}_7(2^{-n}x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{128^n \tilde{f}_7(2^{-n}x)\}$ converges for all $x \in X$. Therefore, we can define a mapping $F_7: X \rightarrow Y$ by

$$F_7(x) := \lim_{n \rightarrow \infty} 128^n \tilde{f}_7(2^{-n}x) \quad \text{for all } x \in X.$$

In (28), we set $n = 0$ and then let $m \rightarrow \infty$ to find

$$\|\tilde{f}_7(x) - F_7(x)\| \leq \frac{85K'\theta\|x\|^p}{4096(2^p - 128)} \quad \text{for all } x \in X.$$

By the definition of F_7 , it is shown that $F_7(2x) = 128F_7(x)$ for all $x \in X$ and $DF_7(x, y) = 0$ for all $x, y \in X$.

- **Setting F_8 :** Let $p < 8$. It follows from Lemma 4 and (13) that

$$\left\| \frac{\tilde{f}_8(2^n x)}{256^n} - \frac{\tilde{f}_8(2^{n+m} x)}{256^{n+m}} \right\| = \left\| \frac{\Delta \tilde{f}_e(2^i x)}{2835 \cdot 256 \cdot 4096 \cdot 256^i} \right\| \leq \sum_{i=n}^{n+m-1} \frac{K2^{ip}\theta\|x\|^p}{16384 \cdot 256^i} \quad (29)$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. By the assumption $p < 8$ and (29), the sequence $\{\frac{\tilde{f}_8(2^n x)}{256^n}\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{\tilde{f}_8(2^n x)}{256^n}\}$ converges. Hence, we can define a mapping $F_8: X \rightarrow Y$ by

$$F_8(x) := \lim_{n \rightarrow \infty} \frac{\tilde{f}_8(2^n x)}{256^n} \text{ for all } x \in X.$$

Letting $n = 0$ and passing the limit $m \rightarrow \infty$ in (29), we then have the following inequality:

$$\|\tilde{f}_8(x) - F_8(x)\| \leq \frac{K\theta\|x\|^p}{64(256 - 2^p)} \text{ for all } x \in X.$$

From the definition of F_8 , we are forced to conclude that $F_8(2x) = 256F_8(x)$ for all $x \in X$ and $DF_8(x, y) = 0$ for all $x, y \in X$.

Let $p > 8$. Observe that, by Lemma 4 and (13), we obtain

$$\|256^n \tilde{f}_8(2^{-n}x) - 256^{n+m} \tilde{f}_8(2^{-n-m}x)\| \leq \sum_{i=n}^{n+m-1} \frac{256^i K\theta\|x\|^p}{64 \cdot 2^{(i+1)p}} \quad (30)$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. It follows from the assumption $p > 8$ and (30) that $\{256^n \tilde{f}_8(2^{-n}x)\}$ is a Cauchy sequence for all $x \in X$. The completeness of Y implies that the sequence $\{256^n \tilde{f}_8(2^{-n}x)\}$ converges, so that we can define a mapping $F_8: X \rightarrow Y$ by

$$F_8(x) := \lim_{n \rightarrow \infty} 256^n \tilde{f}_8(2^{-n}x) \text{ for all } x \in X.$$

We let $n = 0$ and then take $m \rightarrow \infty$ in (30) to obtain

$$\|\tilde{f}_8(x) - F_8(x)\| \leq \frac{K\theta\|x\|^p}{64(2^p - 256)} \text{ for all } x \in X.$$

According to the definition of F_8 , we find that $F_8(2x) = 256F_8(x)$ for all $x \in X$ and $DF_8(x, y) = 0$ for all $x, y \in X$.

- **Setting F_9 :** Let $p < 9$. It follows from Lemma 4 and (12) that

$$\left\| \frac{\tilde{f}_9(2^n x)}{512^n} - \frac{\tilde{f}_9(2^{n+m} x)}{512^{n+m}} \right\| = \left\| \frac{\Gamma \tilde{f}(2^i x)}{722925 \cdot 33554432 \cdot 512^i} \right\| \leq \sum_{i=n}^{n+m-1} \frac{K' 2^{ip} \theta \|x\|^p}{2097152 \cdot 512^i} \quad (31)$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. We then have by the assumption $p < 9$ and (31) that $\{\frac{\tilde{f}_9(2^n x)}{512^n}\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{\tilde{f}_9(2^n x)}{512^n}\}$ converges. Then, we can define a mapping $F_9: X \rightarrow Y$ by

$$F_9(x) := \lim_{n \rightarrow \infty} \frac{\tilde{f}_9(2^n x)}{512^n} \text{ for all } x \in X.$$

On the other hand, letting $n = 0$ and passing the limit $m \rightarrow \infty$ in (31), we deduce that

$$\|\tilde{f}_9(x) - F_9(x)\| \leq \frac{K'\theta\|x\|^p}{4096(512 - 2^p)} \text{ for all } x \in X.$$

We have from the definition of F_9 that $F_9(2x) = 512F_9(x)$ for all $x \in X$ and $DF_9(x, y) = 0$ for all $x, y \in X$.

Let $p > 9$. Using Lemma 4 and (12), we have

$$\|512^n \tilde{f}_9(2^{-n}x) - 512^{n+m} \tilde{f}_9(2^{-n-m}x)\| \leq \frac{512^i K' \theta \|x\|^p}{4096 \cdot 2^{(i+1)p}} \quad (32)$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. From $p > 9$ and (32), it follows that the sequence $\{512^n \tilde{f}_9(2^{-n}x)\}$ is a Cauchy sequence for all $x \in X$. By the completeness of Y , the sequence $\{512^n \tilde{f}_9(2^{-n}x)\}$ converges. Thereby, we can define a mapping $F_9: X \rightarrow Y$ by

$$F_9(x) := \lim_{n \rightarrow \infty} 512^n \tilde{f}_9(2^{-n}x) \text{ for all } x \in X.$$

In particular, we let $n = 0$ and $m \rightarrow \infty$ in (32) to obtain

$$\|\tilde{f}_9(x) - F_9(x)\| \leq \frac{K' \theta \|x\|^p}{4096(2^p - 512)} \text{ for all } x \in X.$$

With the aid of the definition of F_9 , we obtain that $F_9(2x) = 512F_9(x)$ for all $x \in X$ and $DF_9(x, y) = 0$ for all $x, y \in X$.

Finally, we set a mapping F as

$$F(x) := \sum_{k=1}^9 F_k(x) \text{ for all } x \in X.$$

Since $DF_k(x, y) = 0$ for all $k \in \{1, 2, \dots, 9\}$, we have

$$DF(x, y) = \sum_{k=1}^9 DF_k(x, y) = 0 \text{ for all } x, y \in X.$$

Next, we are in the position to prove that the mapping F satisfies Inequality (11). Since $\tilde{f}(x) = \sum_{k=1}^9 \tilde{f}_k(x)$, we have

$$\|\tilde{f}(x) - F(x)\| \leq \sum_{k=1}^9 \|\tilde{f}_k(x) - F_k(x)\| \text{ for all } x \in X,$$

and so we obtain the desired result (11).

It remains to be proven that F is unique. Suppose that $F': V \rightarrow Y$ is another mapping with $F'(0) = 0$ satisfying the relation $DF'(x, y) = 0$ and the inequality (11). We know by Lemma 5 that for each $k \in \{1, 2, \dots, 9\}$, the mappings $F'_k: X \rightarrow Y$ satisfy

$$F'(x) = \sum_{k=1}^9 F'_k(x), \quad F'_k(2x) = 2^k F'_k(x) \quad (x \in X).$$

Next, it is sufficient to show the uniqueness of F only for $2 < p < 3$. This is because other cases of p can be shown in a similar fashion. Therefore, let us assume that $2 < p < 3$. Then, we see that for $2 < p < 3$,

$$\begin{aligned}
& \left\| 4^n \tilde{f}_2\left(\frac{x}{2^n}\right) - F'_2(x) \right\| = \left\| 4^n \tilde{f}_2\left(\frac{x}{2^n}\right) - 4^n F'_2\left(\frac{x}{2^n}\right) \right\| \\
&= \frac{4^n}{181440} \left\| 262144 \tilde{f}_e\left(\frac{x}{2^n}\right) - 21504 \tilde{f}_e\left(\frac{2x}{2^n}\right) + 336 \tilde{f}_e\left(\frac{4x}{2^n}\right) - \tilde{f}_e\left(\frac{8x}{2^n}\right) \right. \\
&\quad \left. - 262144 F'_e\left(\frac{x}{2^n}\right) + 21504 F'_e\left(\frac{2x}{2^n}\right) - 336 F'_e\left(\frac{4x}{2^n}\right) + F'_e\left(\frac{8x}{2^n}\right) \right\| \\
&\leq \frac{262144 \cdot 4^n}{181440} \left\| \tilde{f}_e\left(\frac{x}{2^n}\right) - F'_e\left(\frac{x}{2^n}\right) \right\| + \frac{21504 \cdot 4^n}{181440} \left\| \tilde{f}_e\left(\frac{2x}{2^n}\right) - F'_e\left(\frac{2x}{2^n}\right) \right\| \\
&\quad + \frac{336 \cdot 4^n}{181440} \left\| \tilde{f}_e\left(\frac{4x}{2^n}\right) - F'_e\left(\frac{4x}{2^n}\right) \right\| + \frac{4^n}{181440} \left\| \tilde{f}_e\left(\frac{8x}{2^n}\right) - F'_e\left(\frac{8x}{2^n}\right) \right\| \\
&\leq \frac{262144 + 21504 \cdot 2^p + 336 \cdot 4^p + 8^p}{181440} \cdot \frac{4^n}{2^{np}} M\theta \|x\|^p,
\end{aligned}$$

and

$$\begin{aligned}
& \left\| \frac{\tilde{f}_3(2^n x)}{8^n} - F'_3(x) \right\| = \left\| \frac{\tilde{f}_3(2^n x)}{8^n} - \frac{F'_3(2^n x)}{8^n} \right\| \\
&= \frac{5440}{722925 \cdot 65536} \left\| \frac{4194304((F'_0 - \tilde{f}_0)(2^n x)) - 2269184((F'_0 - \tilde{f}_0)(2^{n+1}x))}{8^n} \right. \\
&\quad \left. + \frac{87360((F'_0 - \tilde{f}_0)(2^{n+2}x)) - 674((F'_0 - \tilde{f}_0)(2^{n+3}x)) + ((F'_0 - \tilde{f}_0)(2^{n+4}x))}{8^n} \right\| \\
&\leq \left(\frac{4194304 + 2269184 \cdot 2^p + 87360 \cdot 2^{2p} + 674 \cdot 2^{3p} + 2^{4p}}{722925 \cdot 65536} \right) \cdot \frac{5440 \cdot 2^{np} M\theta \|x\|^p}{8^n}
\end{aligned}$$

for all $x \in X$ and all positive integers n , where

$$\begin{aligned}
M := & \left(\frac{4096}{|2 - 2^p|} + \frac{5440}{|8 - 2^p|} + \frac{1428}{|32 - 2^p|} + \frac{85}{|128 - 2^p|} + \frac{1}{|512 - 2^p|} \right) \cdot \frac{K'}{4096} \\
& + \left(\frac{64}{|4 - 2^p|} + \frac{84}{|16 - 2^p|} + \frac{21}{|64 - 2^p|} + \frac{1}{|256 - 2^p|} \right) \cdot \frac{K}{64}.
\end{aligned}$$

Taking the limit in the above relations as $n \rightarrow \infty$, we obtain the equality

$$F'_2(x) = \lim_{n \rightarrow \infty} 4^n \tilde{f}_2\left(\frac{x}{2^n}\right), \quad F'_3(x) = \lim_{n \rightarrow \infty} \frac{\tilde{f}_3(2^n x)}{8^n} \quad (x \in X),$$

which means that $F_2(x) = F'_2(x)$ and $F_3(x) = F'_3(x)$ for all $x \in X$. In a similar way, it is easily shown that for each $k \in \{1, 4, 5, 6, 7, 8, 9\}$, the equalities $F_k = F'_k$ hold. Note that

$$F(x) = \sum_{k=1}^9 F_k(x) = \sum_{k=1}^9 F'_k(x) = F'(x).$$

This completes the proof of the uniqueness of F for $2 < p < 3$.

For other cases of p , one can prove the uniqueness of F using the same reasoning as in the uniqueness proof for $2 < p < 3$. $\square$

5. Conclusions

Example 1 in [10] proved that a specially defined mapping $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the inequality $|\sum_{i=0}^n {}_n C_i (-1)^{n-i} f(x + iy) - n! f(y)| \leq 4(n+1)!(n+1)^{2n}(|x|^n + |y|^n)$, but there is no n -monomial mapping $F : \mathbb{R} \rightarrow \mathbb{R}$ and constant $d > 0$ that satisfy the inequality $|f(x) - F(x)| \leq d|x|^n$ (see also [15,49] and Theorem 2 in [5]). Since a mapping f satisfying any Cauchy functional equation, quadratic functional equation, ..., or nonic functional equation satisfies the general nonic functional equation, Theorem 2 fails to hold for $p = 1, 2, \dots, 9$.

For further study, this theory can be extended to the case of 11 instead of 10 in Equation (2). We will also investigate whether we can extend our theory to the functional equation described in Equation (1) as well.

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