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Euler–Lagrange-Type Equations for Functionals Involving Fractional Operators and Antiderivatives

Ricardo Almeida

Center for Research and Development in Mathematics and Applications, Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal; ricardo.almeida@ua.pt

Abstract: The goal of this paper is to present the necessary and sufficient conditions that every extremizer of a given class of functionals, defined on the set $C^1[a,b]$, must satisfy. The Lagrange function depends on a generalized fractional derivative, on a generalized fractional integral, and on an antiderivative involving the previous fractional operators. We begin by obtaining the fractional Euler–Lagrange equation, which is a necessary condition to optimize a given functional. By imposing convexity conditions over the Lagrange function, we prove that it is also a sufficient condition for optimization. After this, we consider variational problems with additional constraints on the set of admissible functions, such as the isoperimetric and the holonomic problems. We end by considering a generalization of the fundamental problem, where the fractional order is not restricted to real values between 0 and 1, but may take any positive real value. We also present some examples to illustrate our results.

Keywords: fractional calculus; calculus of variations; generalized fractional derivative

MSC: 26A33; 49K05

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1. Introduction

Solving an optimization problem means, as the name implies, looking for the best result according to some pre-established requests. In mathematics optimization problems, we are interested in maximum and minimum problems such as maximum profit, minimum cost, minimum time, or shortest path. An area of mathematics that is very useful in solving optimization problems is calculus of variations (see, e.g., [1–3]), which generalizes the theory of maxima and minima of differential calculus for functions whose domain consists of a set of admissible curves. Although the history of the calculus of variations dates back to Ancient Greece, it was not until the 17th century in Western Europe that substantial progress was made. In 1696, Isaac Newton used variational principles to determine the shape of a body moving in the air, so that resistance is minimal. The brothers Johann and Jacob Bernoulli are often considered the inventors of the calculus of variations for having proposed the brachistochrone problem. Lagrange and Euler developed a new theory using variational techniques to determine a differential equation to solve such optimization problems. Such methods are still used nowadays.

Fractional calculus deals with integrals and derivatives of non-integer order [4–6]. Unlike integer calculus (where derivatives and integrals are of order $n = 1$ (first-order derivative and single integral), $n = 2$ (second-order derivative and double integral), etc.), fractional calculus (with arbitrary fractional order $\alpha \in \mathbb{R}^+$) has a specific emergence date. The beginnings of fractional calculus refer to an exchange of correspondence between Leibniz and l'Hôpital in 1695, in which Leibniz questioned the generalization from a usual derivative to a derivative with fractional order. Leibniz replied that the possibility of the concept of such a derivative "will lead to a paradox, from which one day useful consequences will be drawn". Later, numerous important mathematicians tried to develop these concepts, like Laplace, Lacroix, Fourier, Letnikov, etc. It was in the works of Riemann and Liouville that the important definitions of fractional derivatives and integrals appeared, and are still used nowadays. Starting with Cauchy's formula to evaluate an n-tuple integral, a generalization of integrals for a real order α is given, and then with this new concept, a definition for fractional derivatives was introduced. Later, several definitions were presented, most of them motivated by the works of Riemann and Liouville. More recently, with the concepts of fractional operators with respect to another function, we can generalize some of those operators into a single one and present results valid for a wide class of fractional operators [5,7–10]. We remark that these fractional operators depend on an arbitrary function g, and for particular choices of such function, we can recover some well-known fractional operators like the Riemann–Liouvile, the Caputo, the Hadamard, or the Erdelyi–Kober fractional operators. With this new approach, we generalize the results presented in [11,12], where the calculus of variation problems was addressed for (classical) fractional operators. In [13], the authors considered functionals defined on a set of non-differentiable functions, but instead of considering the dynamics modeled by a fractional derivative, a quantum derivative was considered. In [14], functionals involving these generalized operators are considered, but only depending on the fractional derivative. Here, we add a fractional integral and an antiderivative of these operators. Applications have been found in different fields. For example, in [4], a number of works are collected with various applications in physics. To mention a few, applications in biophysics, anomalous diffusion, Markovian chains, polymer dynamics, rheological equations, Hamiltonian chaotic systems, and fractal time series were detailed. In [15], a study of a forced mass-spring-damper in a vertical position was carried out, with the derivation of the fractional Euler–Lagrange equation. For further reading on this topic, we suggest [16–19].

One area where fractional calculus has been applied with success is in the calculus of variations. Typically, the goal is to minimize/maximize a functional, depending on time, the state function, and its first order derivative [20]. With the pioneering work of Riewe [21,22], where the integer-order derivative was replaced by a derivative of order $1/2$, these new models proved to be more realistic in modeling real-world phenomena. Since then, this field has attracted the attention of several researchers and, consequently, a lot of work has been carried out (see, e.g., [15,23–28]). One of the topics that can be studied is the case where the Lagrange function depends not only on a state function and its fractional derivative, but also on an antiderivative involving all the variables of the Lagrange function [29–31]. In this present paper, we extended the work performed with fractional operators by considering a generalized form of fractional integrals and fractional derivatives.

Our objective is to consider different calculus of variation problems, involving the generalized fractional operators and with the presence of an antiderivative. Applying a variational technique and fractional integration by parts formulae, we are able to deduce the necessary and sufficient optimization conditions for the problems to be studied.

The work is organized in the following way: In Section 2, we present a brief description of fractional calculus, needed in for the following work. In Section 3, we obtain the necessary and sufficient conditions for finding the extremum of a given functional. Instead of being described by an ordinary differential equation, it is given by a fractional differential equation, known as the fractional Euler–Lagrange equation. The sufficiency condition is obtained by assuming some convexity condition over the Lagrange function. The case where the lower bound of the integral is greater than the lower bound of the fractional operators is also considered. In Section 4, we consider variational problems subject to additional constraints. Specifically, isoperimetric and holonomic constraints are imposed on the set of admissible functions. We end with Section 5, where the variational problem is defined for arbitrary fractional orders, in contrast to the previous sections, where the order of the derivative belongs to the interval $(0, 1)$.

2. Preliminary Results

First, we introduce some definitions and preliminary facts of fractional calculus theory. Let $\alpha, \beta \in \mathbb{R}^+ \setminus \mathbb{N}$ and n be an integer number with $\alpha \in (n-1, n)$. Furthermore, consider two functions $x, g : [a,b] \to \mathbb{R}$, with g differentiable and $g'(t) > 0$, for all $t \in [a,b]$.

Definition 1 ([5,7])**.** *The (left) Riemann–Liouville fractional integral of x, order β, with respect to function g, is defined as*

$$\mathbb{I}_{a+}^{\beta,g} x(t) = \frac{1}{\Gamma(\beta)} \int_a^t g'(\tau)(g(t)-g(\tau))^{\beta-1} x(\tau)\, d\tau.$$

If x, g are of C^n, the (left) Caputo fractional derivative of the function x is given by the formula

$${}^{C}\mathbb{D}_{a+}^{\alpha,g} x(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t g'(\tau)(g(t)-g(\tau))^{n-\alpha-1} \left(\frac{1}{g'(\tau)} \frac{d}{d\tau} \right)^n x(\tau)\, d\tau.$$

For the following, we also will need the two following notions: the right Riemann–Liouville fractional integral of x:

$$\mathbb{I}_{b-}^{\beta,g} x(t) = \frac{1}{\Gamma(\beta)} \int_t^b g'(\tau)(g(\tau)-g(t))^{\beta-1} x(\tau)\, d\tau,$$

and the right Riemann–Liouville fractional derivative of x:

$$\mathbb{D}_{b-}^{\alpha,g} x(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{1}{g'(t)} \frac{d}{dt} \right)^n \int_t^b g'(\tau)(g(\tau)-g(t))^{n-\alpha-1} x(\tau)\, d\tau.$$

Fractional integration by parts formulae are an important tool to obtain the variational necessary conditions for optimizing functionals. Here, we recall two of them for fractional integrals and for fractional derivatives.

Theorem 1 ([32])**.** *Given $x, y \in C[a,b]$, we have that, for all $\beta > 0$,*

$$\int_a^b x(t) \cdot \mathbb{I}_{a+}^{\beta,g} y(t)\, dt = \int_a^b \mathbb{I}_{b-}^{\beta,g} \left(\frac{x(t)}{g'(t)} \right) \cdot y(t) g'(t)\, dt.$$

Theorem 2 ([7])**.** *Given $x, y \in C^n[a,b]$, we have that, for all $\alpha > 0$,*

$$\begin{aligned} \int_a^b x(t) \cdot {}^{C}\mathbb{D}_{a+}^{\alpha,g} y(t)\, dt &= \int_a^b \mathbb{D}_{b-}^{\alpha,g} \left(\frac{x(t)}{g'(t)} \right) \cdot y(t) g'(t)\, dt \\ &+ \left[\sum_{k=0}^{n-1} \left(-\frac{1}{g'(t)} \frac{d}{dt} \right)^k \mathbb{I}_{b-}^{n-\alpha,g} \left(\frac{x(t)}{g'(t)} \right) \cdot \left(\frac{1}{g'(t)} \frac{d}{dt} \right)^{n-k-1} y(t) \right]_{t=a}^{t=b}. \end{aligned}$$

For the particular case when the fractional order α is between 0 and 1, the integration by parts formula given in Theorem 2 reads as

$$\int_a^b x(t) \cdot {}^{C}\mathbb{D}_{a+}^{\alpha,g} y(t)\, dt = \int_a^b \mathbb{D}_{b-}^{\alpha,g} \left(\frac{x(t)}{g'(t)} \right) \cdot y(t) g'(t)\, dt + \left[\mathbb{I}_{b-}^{1-\alpha,g} \left(\frac{x(t)}{g'(t)} \right) \cdot y(t) \right]_{t=a}^{t=b}.$$

3. Fractional Calculus of Variations

Consider the following functional

$$F(x) = \int_a^b \mathcal{L}(t, x(t), {}^{C}\mathbb{D}_{a+}^{\alpha,g} x(t), \mathbb{I}_{a+}^{\beta,g} x(t), z(t))\, dt, \quad x \in C^1[a,b], \tag{1}$$

with the antiderivative

$$z(t) = \int_a^t \mathcal{Z}(\tau, x(\tau), {}^C\mathbb{D}_{a+}^{\alpha,g} x(\tau), \mathbb{I}_{a+}^{\beta,g} x(\tau))\, d\tau,$$

where $\alpha \in (0,1)$, $\beta > 0$, $\mathcal{L} : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ and $\mathcal{Z} : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$ are two continuously differentiable functions. To simplify the notation, by $[x]_{\mathcal{L}}$ and $[x]_{\mathcal{Z}}$ we mean

$$[x]_{\mathcal{L}}(t) = (t, x(t), {}^C\mathbb{D}_{a+}^{\alpha,g} x(t), \mathbb{I}_{a+}^{\beta,g} x(t), z(t)) \quad \text{and} \quad [x]_{\mathcal{Z}}(t) = (t, x(t), {}^C\mathbb{D}_{a+}^{\alpha,g} x(t), \mathbb{I}_{a+}^{\beta,g} x(t)),$$

for $t \in [a,b]$. The goal is to find the curves $\overline{x}$ for which functional (1) attains an extremum value, possibly under some boundary conditions. For that purpose, consider an arbitrary curve $v \in C^1[a,b]$, ϵ an arbitrary real number, and consider the admissible variation $[a,b] \ni t \mapsto \overline{x}(t) + \epsilon v(t)$. We remark that, in the problem's formulation, if boundary conditions are imposed at $t = a$ or $t = b$, then function v must satisfy the conditions $v(a) = 0$ or $v(b) = 0$, so that the curve $\overline{x} + \epsilon v$ is an admissible variation of the problem. Since functional F attains an extremum value at $\overline{x}$, its first variation must vanish when evaluated on that curve. Starting with

$$F(\overline{x} + \epsilon v) = \int_a^b \mathcal{L}(t, \overline{x}(t) + \epsilon v(t), {}^C\mathbb{D}_{a+}^{\alpha,g} \overline{x}(t) + \epsilon {}^C\mathbb{D}_{a+}^{\alpha,g} v(t), \mathbb{I}_{a+}^{\beta,g} \overline{x}(t) + \epsilon \mathbb{I}_{a+}^{\beta,g} v(t), z(t))\, dt,$$

with

$$z(t) = \int_a^t \mathcal{Z}(\tau, \overline{x}(\tau) + \epsilon v(\tau), {}^C\mathbb{D}_{a+}^{\alpha,g} \overline{x}(\tau) + \epsilon {}^C\mathbb{D}_{a+}^{\alpha,g} v(\tau), \mathbb{I}_{a+}^{\beta,g} \overline{x}(\tau) + \epsilon \mathbb{I}_{a+}^{\beta,g} v(\tau))\, d\tau,$$

differentiating with respect to ϵ, we obtain that

$$\begin{aligned}
&\int_a^b \frac{\partial \mathcal{L}}{\partial x}[\overline{x}]_{\mathcal{L}}(t) v(t) + \frac{\partial \mathcal{L}}{\partial {}^C\mathbb{D}_{a+}^{\alpha,g} x}[\overline{x}]_{\mathcal{L}}(t) {}^C\mathbb{D}_{a+}^{\alpha,g} v(t) + \frac{\partial \mathcal{L}}{\partial \mathbb{I}_{a+}^{\beta,g} x}[\overline{x}]_{\mathcal{L}}(t) \mathbb{I}_{a+}^{\beta,g} v(t) + \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(t) \\
&\times \int_a^t \left[\frac{\partial \mathcal{Z}}{\partial x}[\overline{x}]_{\mathcal{Z}}(\tau) v(\tau) + \frac{\partial \mathcal{Z}}{\partial {}^C\mathbb{D}_{a+}^{\alpha,g} x}[\overline{x}]_{\mathcal{Z}}(\tau) {}^C\mathbb{D}_{a+}^{\alpha,g} v(\tau) + \frac{\partial \mathcal{Z}}{\partial \mathbb{I}_{a+}^{\beta,g} x}[\overline{x}]_{\mathcal{Z}}(\tau) \mathbb{I}_{a+}^{\beta,g} v(\tau) \right] d\tau\, dt = 0.
\end{aligned}$$

Using fractional integration by parts (cf. Theorems 1 and 2), we get

$$\begin{aligned}
\int_a^b \frac{\partial \mathcal{L}}{\partial {}^C\mathbb{D}_{a+}^{\alpha,g} x}[\overline{x}]_{\mathcal{L}}(t) {}^C\mathbb{D}_{a+}^{\alpha,g} v(t)\, dt &= \int_a^b \mathbb{D}_{b-}^{\alpha,g} \left(\frac{\partial \mathcal{L}}{\partial {}^C\mathbb{D}_{a+}^{\alpha,g} x}[\overline{x}]_{\mathcal{L}}(t) \frac{1}{g'(t)} \right) \cdot v(t) g'(t)\, dt \\
&\quad + \left[\mathbb{I}_{b-}^{1-\alpha,g} \left(\frac{\partial \mathcal{L}}{\partial {}^C\mathbb{D}_{a+}^{\alpha,g} x}[\overline{x}]_{\mathcal{L}}(t) \frac{1}{g'(t)} \right) \cdot v(t) \right]_{t=a}^{t=b}
\end{aligned}$$

and

$$\int_a^b \frac{\partial \mathcal{L}}{\partial \mathbb{I}_{a+}^{\beta,g} x}[\overline{x}]_{\mathcal{L}}(t) \mathbb{I}_{a+}^{\beta,g} v(t)\, dt = \int_a^b \mathbb{I}_{b-}^{\beta,g} \left(\frac{\partial \mathcal{L}}{\partial \mathbb{I}_{a+}^{\beta,g} x}[\overline{x}]_{\mathcal{L}}(t) \frac{1}{g'(t)} \right) \cdot v(t) g'(t)\, dt.$$

Now, using standard integration by parts, we obtain

$$\begin{aligned}
&\int_a^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(t) \left(\int_a^t \frac{\partial \mathcal{Z}}{\partial x}[\overline{x}]_{\mathcal{Z}}(\tau) v(\tau)\, d\tau \right) dt \\
&\qquad = \int_a^b \left(-\frac{d}{dt} \int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\, d\tau \right) \cdot \left(\int_a^t \frac{\partial \mathcal{Z}}{\partial x}[\overline{x}]_{\mathcal{Z}}(\tau) v(\tau)\, d\tau \right) dt \\
&\qquad = \int_a^b \left(\int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\, d\tau \right) \cdot \frac{\partial \mathcal{Z}}{\partial x}[\overline{x}]_{\mathcal{Z}}(t) v(t)\, dt.
\end{aligned}$$

Finally, combining standard and fractional integration by parts, we get

$$\begin{aligned}
&\int_a^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(t)\left(\int_a^t \frac{\partial \mathcal{Z}}{\partial {}^C\mathbb{D}_{a+}^{\alpha,g}x}[\overline{x}]_{\mathcal{Z}}(\tau)\,{}^C\mathbb{D}_{a+}^{\alpha,g}v(\tau)\,d\tau\right)dt \\
&\quad = \int_a^b \mathbb{D}_{b-}^{\alpha,g}\left(\int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\,d\tau \cdot \frac{\partial \mathcal{Z}}{\partial {}^C\mathbb{D}_{a+}^{\alpha,g}x}[\overline{x}]_{\mathcal{Z}}(t)\frac{1}{g'(t)}\right)\cdot v(t)g'(t)\,dt \\
&\qquad + \left[\mathbb{I}_{b-}^{1-\alpha,g}\left(\int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\,d\tau \cdot \frac{\partial \mathcal{Z}}{\partial {}^C\mathbb{D}_{a+}^{\alpha,g}x}[\overline{x}]_{\mathcal{Z}}(t)\frac{1}{g'(t)}\right)\cdot v(t)\right]_{t=a}^{t=b}
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(t)\left(\int_a^t \frac{\partial \mathcal{Z}}{\partial \mathbb{I}_{a+}^{\beta,g}x}[\overline{x}]_{\mathcal{Z}}(\tau)\mathbb{I}_{a+}^{\beta,g}v(\tau)\,d\tau\right)dt \\
&\quad = \int_a^b \mathbb{I}_{b-}^{\beta,g}\left(\int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\,d\tau \cdot \frac{\partial \mathcal{Z}}{\partial \mathbb{I}_{a+}^{\beta,g}x}[\overline{x}]_{\mathcal{Z}}(t)\frac{1}{g'(t)}\right)\cdot v(t)g'(t)\,dt.
\end{aligned}$$

Therefore, we conclude from the previous relations that

$$\begin{aligned}
&\int_a^b \left[\frac{\partial \mathcal{L}}{\partial x}[\overline{x}]_{\mathcal{L}}(t) + \int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\,d\tau \cdot \frac{\partial \mathcal{Z}}{\partial x}[\overline{x}]_{\mathcal{Z}}(t)\right. \\
&\quad + g'(t)\cdot \mathbb{D}_{b-}^{\alpha,g}\left(\frac{\partial \mathcal{L}}{\partial {}^C\mathbb{D}_{a+}^{\alpha,g}x}[\overline{x}]_{\mathcal{L}}(t)\frac{1}{g'(t)} + \int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\,d\tau \cdot \frac{\partial \mathcal{Z}}{\partial {}^C\mathbb{D}_{a+}^{\alpha,g}x}[\overline{x}]_{\mathcal{Z}}(t)\frac{1}{g'(t)}\right) \\
&\quad \left. + g'(t)\cdot \mathbb{I}_{b-}^{\beta,g}\left(\frac{\partial \mathcal{L}}{\partial \mathbb{I}_{a+}^{\beta,g}x}[\overline{x}]_{\mathcal{L}}(t)\frac{1}{g'(t)} + \int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\,d\tau \cdot \frac{\partial \mathcal{Z}}{\partial \mathbb{I}_{a+}^{\beta,g}x}[\overline{x}]_{\mathcal{Z}}(t)\frac{1}{g'(t)}\right)\right]\cdot v(t)\,dt \\
&\quad + \left[\mathbb{I}_{b-}^{1-\alpha,g}\left(\frac{\partial \mathcal{L}}{\partial {}^C\mathbb{D}_{a+}^{\alpha,g}x}[\overline{x}]_{\mathcal{L}}(t)\frac{1}{g'(t)} + \int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\,d\tau \cdot \frac{\partial \mathcal{Z}}{\partial {}^C\mathbb{D}_{a+}^{\alpha,g}x}[\overline{x}]_{\mathcal{Z}}(t)\frac{1}{g'(t)}\right)\cdot v(t)\right]_{t=a}^{t=b} = 0.
\end{aligned}$$

From the last relation, and from the arbitrariness of function v, we conclude the following:

Theorem 3. *Assume that $\overline{x}$ minimizes or maximizes functional F given by (1). Then, for all $t \in [a,b]$, the equation*

$$\begin{aligned}
&\frac{\partial \mathcal{L}}{\partial x}[\overline{x}]_{\mathcal{L}}(t) + \int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\,d\tau \cdot \frac{\partial \mathcal{Z}}{\partial x}[\overline{x}]_{\mathcal{Z}}(t) \\
&\quad + g'(t)\cdot \mathbb{D}_{b-}^{\alpha,g}\left(\frac{\partial \mathcal{L}}{\partial {}^C\mathbb{D}_{a+}^{\alpha,g}x}[\overline{x}]_{\mathcal{L}}(t)\frac{1}{g'(t)} + \int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\,d\tau \cdot \frac{\partial \mathcal{Z}}{\partial {}^C\mathbb{D}_{a+}^{\alpha,g}x}[\overline{x}]_{\mathcal{Z}}(t)\frac{1}{g'(t)}\right) \\
&\quad + g'(t)\cdot \mathbb{I}_{b-}^{\beta,g}\left(\frac{\partial \mathcal{L}}{\partial \mathbb{I}_{a+}^{\beta,g}x}[\overline{x}]_{\mathcal{L}}(t)\frac{1}{g'(t)} + \int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\,d\tau \cdot \frac{\partial \mathcal{Z}}{\partial \mathbb{I}_{a+}^{\beta,g}x}[\overline{x}]_{\mathcal{Z}}(t)\frac{1}{g'(t)}\right) = 0 \qquad (2)
\end{aligned}$$

holds. Moreover, if $x(a)$ or $x(b)$ is arbitrary, then the natural boundary condition

$$\mathbb{I}_{b-}^{1-\alpha,g}\left(\frac{\partial \mathcal{L}}{\partial {}^C\mathbb{D}_{a+}^{\alpha,g}x}[\overline{x}]_{\mathcal{L}}(t)\frac{1}{g'(t)} + \int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\,d\tau \cdot \frac{\partial \mathcal{Z}}{\partial {}^C\mathbb{D}_{a+}^{\alpha,g}x}[\overline{x}]_{\mathcal{Z}}(t)\frac{1}{g'(t)}\right) = 0$$

holds at $t = a$ or $t = b$, respectively.

The fractional differential Equation (2) is usually referred to in the literature as the fractional Euler–Lagrange equation associated with the variational problem. We remark that, when $g(t) = t$, we obtain the result proven in [29].

Remark 1. *If $\overline{x} = (\overline{x}_1, \dots, \overline{x}_n)$ is an extremizer for functional*

$$F(x) = \int_a^b \mathcal{L}(t, x(t), {}^C\mathbb{D}_{a+}^{\alpha,g} x(t), \mathbb{I}_{a+}^{\beta,g} x(t), z(t))\, dt,$$

with the antiderivative

$$z(t) = \int_a^t \mathcal{Z}(\tau, x(\tau), {}^C\mathbb{D}_{a+}^{\alpha,g} x(\tau), \mathbb{I}_{a+}^{\beta,g} x(\tau))\, d\tau,$$

where $x \in C^1[a,b] \times \dots \times C^1[a,b]$, ${}^C\mathbb{D}_{a+}^{\alpha,g} x(t) = \left({}^C\mathbb{D}_{a+}^{\alpha,g} x_1(t), \dots, {}^C\mathbb{D}_{a+}^{\alpha,g} x_n(t)\right)$, $\mathbb{I}_{a+}^{\beta,g} x(t) = \left(\mathbb{I}_{a+}^{\beta,g} x_1(t), \dots, \mathbb{I}_{a+}^{\beta,g} x_n(t)\right)$, $\mathcal{L} : [a,b] \times \mathbb{R}^{3n+1} \to \mathbb{R}$, and $\mathcal{Z} : [a,b] \times \mathbb{R}^{3n} \to \mathbb{R}$, then the associated fractional Euler–Lagrange equations are obtained as in (2), replacing x with x_i, for $i \in \{1, \dots, n\}$, obtaining in this way the n fractional differential equations:

$$\begin{aligned}
&\frac{\partial \mathcal{L}}{\partial x_i}[\overline{x}]_{\mathcal{L}}(t) + \int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\, d\tau \cdot \frac{\partial \mathcal{Z}}{\partial x_i}[\overline{x}]_{\mathcal{Z}}(t) \\
&\quad + g'(t) \cdot \mathbb{D}_{b-}^{\alpha,g}\left(\frac{\partial \mathcal{L}}{\partial {}^C\mathbb{D}_{a+}^{\alpha,g} x_i}[\overline{x}]_{\mathcal{L}}(t)\frac{1}{g'(t)} + \int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\, d\tau \cdot \frac{\partial \mathcal{Z}}{\partial {}^C\mathbb{D}_{a+}^{\alpha,g} x_i}[\overline{x}]_{\mathcal{Z}}(t)\frac{1}{g'(t)}\right) \\
&\quad + g'(t) \cdot \mathbb{I}_{b-}^{\beta,g}\left(\frac{\partial \mathcal{L}}{\partial \mathbb{I}_{a+}^{\beta,g} x_i}[\overline{x}]_{\mathcal{L}}(t)\frac{1}{g'(t)} + \int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\, d\tau \cdot \frac{\partial \mathcal{Z}}{\partial \mathbb{I}_{a+}^{\beta,g} x_i}[\overline{x}]_{\mathcal{Z}}(t)\frac{1}{g'(t)}\right) = 0. \qquad (3)
\end{aligned}$$

If some convexity condition over the Lagrange function is assumed, then we can obtain a sufficient condition for optimization. First, we recall the concepts of convex and concave functions.

Definition 2. *Let $k \in \{1, \dots, m\}$ and $\Phi : D \subseteq \mathbb{R}^m \to \mathbb{R}$ be a function such that $\frac{\partial \Phi}{\partial t_i}$ exists and is continuous for all $i \in \{k, \dots, m\}$. We say that Φ is convex in $(t_k, \dots, t_m)$ if*

$$\Phi(t_1 + v_1, \dots, t_m + v_m) - \Phi(t_1, \dots, t_m) \geq \frac{\partial \Phi}{\partial t_k}(t_1, \dots, t_m)v_k + \dots + \frac{\partial \Phi}{\partial t_m}(t_1, \dots, t_m)v_m, \qquad (4)$$

for all $(t_1 + v_1, \dots, t_m + v_m), (t_1, \dots, t_m) \in D$. We say that Φ is concave in $(t_k, \dots, t_m)$ if Equation (4) holds, replacing "$\geq$" with "$\leq$".

Using the same notation as used before, the result reads as follows:

Theorem 4. *Assume that $\overline{x}$ satisfies Equation (2). If $\mathcal{L}$ is convex in $(x, {}^C\mathbb{D}_{a+}^{\alpha,g} x, \mathbb{I}_{a+}^{\beta,g} x, z)$ and one of the two following conditions hold:*

1. *$\mathcal{Z}$ is convex in $(x, {}^C\mathbb{D}_{a+}^{\alpha,g} x, \mathbb{I}_{a+}^{\beta,g} x)$ and $\frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(t) \geq 0$ for all $t \in [a,b]$;*
2. *$\mathcal{Z}$ is concave in $(x, {}^C\mathbb{D}_{a+}^{\alpha,g} x, \mathbb{I}_{a+}^{\beta,g} x)$ and $\frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(t) \leq 0$ for all $t \in [a,b]$,*

then $\overline{x}$ is a solution of the following variational problem: minimize F given by (1), subject to the boundary conditions $x(a) = x_a$ and $x(b) = x_b$, with $x_a, x_b \in \mathbb{R}$.

Proof. Let $v \in C^1[a,b]$, with $v(a)=0$ and $v(b)=0$. Then, since

$$
\begin{aligned}
F(\overline{x}+v)-F(\overline{x}) = \int_a^b \mathcal{L}\Bigg(& t, \overline{x}(t)+v(t), {}^C\mathbb{D}^{\alpha,g}_{a+}\overline{x}(t)+{}^C\mathbb{D}^{\alpha,g}_{a+}v(t), \mathbb{I}^{\beta,g}_{a+}\overline{x}(t)+\mathbb{I}^{\beta,g}_{a+}v(t), \\
& \int_a^t \mathcal{Z}(\tau,\overline{x}(\tau)+v(\tau), {}^C\mathbb{D}^{\alpha,g}_{a+}\overline{x}(\tau)+{}^C\mathbb{D}^{\alpha,g}_{a+}v(\tau), \mathbb{I}^{\beta,g}_{a+}\overline{x}(\tau)+\mathbb{I}^{\beta,g}_{a+}v(\tau))\,d\tau\Bigg) \\
& -\mathcal{L}\left(t,\overline{x}(t), {}^C\mathbb{D}^{\alpha,g}_{a+}\overline{x}(t), \mathbb{I}^{\beta,g}_{a+}\overline{x}(t), \int_a^t \mathcal{Z}(\tau,\overline{x}(\tau), {}^C\mathbb{D}^{\alpha,g}_{a+}\overline{x}(\tau), \mathbb{I}^{\beta,g}_{a+}\overline{x}(\tau))\,d\tau\right)dt,
\end{aligned}
$$

using the assumptions of the theorem, we conclude that

$$
\begin{aligned}
F(\overline{x}+v)-F(\overline{x}) \geq & \int_a^b \Bigg[\frac{\partial\mathcal{L}}{\partial x}[\overline{x}]_{\mathcal{L}}(t)v(t)+\frac{\partial\mathcal{L}}{\partial {}^C\mathbb{D}^{\alpha,g}_{a+}x}[\overline{x}]_{\mathcal{L}}(t){}^C\mathbb{D}^{\alpha,g}_{a+}v(t)+\frac{\partial\mathcal{L}}{\partial \mathbb{I}^{\beta,g}_{a+}x}[\overline{x}]_{\mathcal{L}}(t)\mathbb{I}^{\beta,g}_{a+}v(t) \\
& +\frac{\partial\mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(t)\int_a^t \Big[\mathcal{Z}(\tau,\overline{x}(\tau)+v(\tau), {}^C\mathbb{D}^{\alpha,g}_{a+}\overline{x}(\tau)+{}^C\mathbb{D}^{\alpha,g}_{a+}v(\tau), \mathbb{I}^{\beta,g}_{a+}\overline{x}(\tau)+\mathbb{I}^{\beta,g}_{a+}v(\tau))\,d\tau\Big) \\
& -\mathcal{Z}(\tau,\overline{x}(\tau), {}^C\mathbb{D}^{\alpha,g}_{a+}\overline{x}(\tau), \mathbb{I}^{\beta,g}_{a+}\overline{x}(\tau))\,d\tau\Big)\Big]\,d\tau\Bigg]dt \\
\geq & \int_a^b \Bigg[\frac{\partial\mathcal{L}}{\partial x}[\overline{x}]_{\mathcal{L}}(t)v(t)+\frac{\partial\mathcal{L}}{\partial {}^C\mathbb{D}^{\alpha,g}_{a+}x}[\overline{x}]_{\mathcal{L}}(t){}^C\mathbb{D}^{\alpha,g}_{a+}v(t)+\frac{\partial\mathcal{L}}{\partial \mathbb{I}^{\beta,g}_{a+}x}[\overline{x}]_{\mathcal{L}}(t)\mathbb{I}^{\beta,g}_{a+}v(t) \\
& +\frac{\partial\mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(t)\int_a^t \Big[\frac{\partial\mathcal{Z}}{\partial x}[\overline{x}]_{\mathcal{Z}}(\tau)v(\tau)+\frac{\partial\mathcal{Z}}{\partial {}^C\mathbb{D}^{\alpha,g}_{a+}x}[\overline{x}]_{\mathcal{Z}}(\tau){}^C\mathbb{D}^{\alpha,g}_{a+}v(\tau)+\frac{\partial\mathcal{Z}}{\partial \mathbb{I}^{\beta,g}_{a+}x}[\overline{x}]_{\mathcal{Z}}(\tau)\mathbb{I}^{\beta,g}_{a+}v(\tau)\Big]\,d\tau\Bigg]dt.
\end{aligned}
$$

Applying standard and fractional integration by parts, we get

$$
\begin{aligned}
F(\overline{x}+v)-F(\overline{x}) \geq \int_a^b \Bigg[& \frac{\partial\mathcal{L}}{\partial x}[\overline{x}]_{\mathcal{L}}(t)+\int_t^b \frac{\partial\mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\,d\tau\cdot\frac{\partial\mathcal{Z}}{\partial x}[\overline{x}]_{\mathcal{Z}}(t) \\
& +g'(t)\cdot\mathbb{D}^{\alpha,g}_{b-}\left(\frac{\partial\mathcal{L}}{\partial {}^C\mathbb{D}^{\alpha,g}_{a+}x}[\overline{x}]_{\mathcal{L}}(t)\frac{1}{g'(t)}+\int_t^b \frac{\partial\mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\,d\tau\cdot\frac{\partial\mathcal{Z}}{\partial {}^C\mathbb{D}^{\alpha,g}_{a+}x}[\overline{x}]_{\mathcal{Z}}(t)\frac{1}{g'(t)}\right) \\
& +g'(t)\cdot\mathbb{I}^{\beta,g}_{b-}\left(\frac{\partial\mathcal{L}}{\partial \mathbb{I}^{\beta,g}_{a+}x}[\overline{x}]_{\mathcal{L}}(t)\frac{1}{g'(t)}+\int_t^b \frac{\partial\mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\,d\tau\cdot\frac{\partial\mathcal{Z}}{\partial \mathbb{I}^{\beta,g}_{a+}x}[\overline{x}]_{\mathcal{Z}}(t)\frac{1}{g'(t)}\right)\Bigg]\cdot v(t)\,dt.
\end{aligned}
$$

Since $\overline{x}$ satisfies (2), we conclude that $F(\overline{x}+v)-F(\overline{x}) \geq 0$, and by the arbitrariness of v, we prove that $\overline{x}$ is in fact a solution of the variational problem. □

Remark 2. *If, in Theorem 4, we replace "convex" by "concave", and vice versa, we prove that $\overline{x}$ maximizes F.*

For example, let $\overline{x}(t)=(g(t)-g(0))^2$, where $g \in C^1[0,1]$ with $g'(t)>0$ for all $t \in [0,1]$. Then, given $\alpha \in (0,1)$ and $\beta > 0$, we have the following:

$$
{}^C\mathbb{D}^{\alpha,g}_{0+}\overline{x}(t)=\frac{2}{\Gamma(3-\alpha)}(g(t)-g(0))^{2-\alpha} \quad \text{and} \quad \mathbb{I}^{\beta,g}_{0+}\overline{x}(t)=\frac{2}{\Gamma(3+\beta)}(g(t)-g(0))^{2+\beta}. \tag{5}
$$

Consider the following functional:

$$F(x)=\int_0^1\left(\left({}^C\mathbb{D}_{0+}^{\alpha,g}x(t)-\frac{2}{\Gamma(3-\alpha)}(g(t)-g(0))^{2-\alpha}\right)^2\right.$$
$$\left.+\left(\mathbb{I}_{0+}^{\beta,g}x(t)-\frac{2}{\Gamma(3+\beta)}(g(t)-g(0))^{2+\beta}\right)^2+z(t)\right)dt,$$

defined on $C^1[0,1]$, where the antiderivative is defined as

$$z(t)=\int_0^t\left(x(\tau)-(g(\tau)-g(0))^2\right)^2d\tau.$$

Furthermore, the boundary conditions

$$x(0)=0\quad\text{and}\quad x(1)=(g(1)-g(0))^2$$

are imposed in the formulation of the problem. The fractional Euler–Lagrange Equation (2) reads as

$$(1-t)\left(x(t)-(g(t)-g(0))^2\right)$$
$$+g'(t)\mathbb{D}_{1-}^{\alpha,g}\left(\left({}^C\mathbb{D}_{0+}^{\alpha,g}x(t)-\frac{2}{\Gamma(3-\alpha)}(g(t)-g(0))^{2-\alpha}\right)\frac{1}{g'(t)}\right)$$
$$+g'(t)\mathbb{I}_{1-}^{\beta,g}\left(\left(\mathbb{I}_{0+}^{\beta,g}x(t)-\frac{2}{\Gamma(3+\beta)}(g(t)-g(0))^{2+\beta}\right)\frac{1}{g'(t)}\right)=0,$$

for $t\in[0,1]$. Taking into consideration the formulae presented in (5), we conclude that $\overline{x}$ satisfies the Euler–Lagrange equation. Furthermore, since the Lagrange function is convex and condition *1.* of Theorem 4 is satisfied, then we conclude that $\overline{x}$ is a minimizer of the given functional. Observe that for all $x\in C^1[0,1]$, $F(x)\geq 0$ and $F(\overline{x})=0$.

In the previous theorems, the lower bounds of the fractional operators are the same as the lower bound of the integral of the Lagrange function ($t=a$). We can extend this result by considering a different lower bound for the integral. Let $A\in(a,b)$ and consider the functional

$$F_A(x)=\int_A^b\mathcal{L}[\overline{x}]_{\mathcal{L}}(t)\,dt,\quad\text{where}\quad z(t)=\int_A^t\mathcal{Z}[\overline{x}]_{\mathcal{Z}}(\tau)\,d\tau.$$

In this case, the necessary conditions are given in the next theorem.

Theorem 5. *Assume that $\overline{x}$ minimizes or maximizes functional F_A. Then, the following equations hold:*

$$g'(t)\cdot\mathbb{D}_{b-}^{\alpha,g}\left(\frac{\partial\mathcal{L}}{\partial^C\mathbb{D}_{a+}^{\alpha,g}x}[\overline{x}]_{\mathcal{L}}(t)\frac{1}{g'(t)}+\int_t^b\frac{\partial\mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\,d\tau\cdot\frac{\partial\mathcal{Z}}{\partial^C\mathbb{D}_{a+}^{\alpha,g}x}[\overline{x}]_{\mathcal{Z}}(t)\frac{1}{g'(t)}\right)$$
$$-g'(t)\cdot\mathbb{D}_{A-}^{\alpha,g}\left(\frac{\partial\mathcal{L}}{\partial^C\mathbb{D}_{a+}^{\alpha,g}x}[\overline{x}]_{\mathcal{L}}(t)\frac{1}{g'(t)}+\int_t^b\frac{\partial\mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\,d\tau\cdot\frac{\partial\mathcal{Z}}{\partial^C\mathbb{D}_{a+}^{\alpha,g}x}[\overline{x}]_{\mathcal{Z}}(t)\frac{1}{g'(t)}\right)$$
$$+g'(t)\cdot\mathbb{I}_{b-}^{\beta,g}\left(\frac{\partial\mathcal{L}}{\partial\mathbb{I}_{a+}^{\beta,g}x}[\overline{x}]_{\mathcal{L}}(t)\frac{1}{g'(t)}+\int_t^b\frac{\partial\mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\,d\tau\cdot\frac{\partial\mathcal{Z}}{\partial\mathbb{I}_{a+}^{\beta,g}x}[\overline{x}]_{\mathcal{Z}}(t)\frac{1}{g'(t)}\right)$$
$$-g'(t)\cdot\mathbb{I}_{A-}^{\beta,g}\left(\frac{\partial\mathcal{L}}{\partial\mathbb{I}_{a+}^{\beta,g}x}[\overline{x}]_{\mathcal{L}}(t)\frac{1}{g'(t)}+\int_t^b\frac{\partial\mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\,d\tau\cdot\frac{\partial\mathcal{Z}}{\partial\mathbb{I}_{a+}^{\beta,g}x}[\overline{x}]_{\mathcal{Z}}(t)\frac{1}{g'(t)}\right)=0,$$

and

$$\begin{aligned}
&\frac{\partial \mathcal{L}}{\partial x}[\overline{x}]_{\mathcal{L}}(t) + \int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\, d\tau \cdot \frac{\partial \mathcal{Z}}{\partial x}[\overline{x}]_{\mathcal{Z}}(t) \\
&\quad + g'(t) \cdot \mathbb{D}^{\alpha,g}_{b-}\left(\frac{\partial \mathcal{L}}{\partial {}^C\mathbb{D}^{\alpha,g}_{a+}x}[\overline{x}]_{\mathcal{L}}(t)\frac{1}{g'(t)} + \int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\, d\tau \cdot \frac{\partial \mathcal{Z}}{\partial {}^C\mathbb{D}^{\alpha,g}_{a+}x}[\overline{x}]_{\mathcal{Z}}(t)\frac{1}{g'(t)} \right) \\
&\qquad + g'(t) \cdot \mathbb{I}^{\beta,g}_{b-}\left(\frac{\partial \mathcal{L}}{\partial \mathbb{I}^{\beta,g}_{a+}x}[\overline{x}]_{\mathcal{L}}(t)\frac{1}{g'(t)} + \int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\, d\tau \cdot \frac{\partial \mathcal{Z}}{\partial \mathbb{I}^{\beta,g}_{a+}x}[\overline{x}]_{\mathcal{Z}}(t)\frac{1}{g'(t)} \right) = 0.
\end{aligned}$$

for all $t \in [a, A]$ and for all $t \in [A, b]$, respectively.

Furthermore, we can obtain the following natural boundary conditions:

1. *If $x(b)$ is free, then*

$$\mathbb{I}^{1-\alpha,g}_{b-}\left(\frac{\partial \mathcal{L}}{\partial {}^C\mathbb{D}^{\alpha,g}_{a+}x}[\overline{x}]_{\mathcal{L}}(t)\frac{1}{g'(t)} + \int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\, d\tau \cdot \frac{\partial \mathcal{Z}}{\partial {}^C\mathbb{D}^{\alpha,g}_{a+}x}[\overline{x}]_{\mathcal{Z}}(t)\frac{1}{g'(t)} \right) = 0,$$

at $t = b$.

2. *If $x(A)$ is free, then*

$$\mathbb{I}^{1-\alpha,g}_{A-}\left(\frac{\partial \mathcal{L}}{\partial {}^C\mathbb{D}^{\alpha,g}_{a+}x}[\overline{x}]_{\mathcal{L}}(t)\frac{1}{g'(t)} + \int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\, d\tau \cdot \frac{\partial \mathcal{Z}}{\partial {}^C\mathbb{D}^{\alpha,g}_{a+}x}[\overline{x}]_{\mathcal{Z}}(t)\frac{1}{g'(t)} \right) = 0,$$

at $t = A$.

3. *If $x(a)$ is also free, then*

$$\begin{aligned}
&\mathbb{I}^{1-\alpha,g}_{b-}\left(\frac{\partial \mathcal{L}}{\partial {}^C\mathbb{D}^{\alpha,g}_{a+}x}[\overline{x}]_{\mathcal{L}}(t)\frac{1}{g'(t)} + \int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\, d\tau \cdot \frac{\partial \mathcal{Z}}{\partial {}^C\mathbb{D}^{\alpha,g}_{a+}x}[\overline{x}]_{\mathcal{Z}}(t)\frac{1}{g'(t)} \right) \\
&\quad = \mathbb{I}^{1-\alpha,g}_{A-}\left(\frac{\partial \mathcal{L}}{\partial {}^C\mathbb{D}^{\alpha,g}_{a+}x}[\overline{x}]_{\mathcal{L}}(t)\frac{1}{g'(t)} + \int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\, d\tau \cdot \frac{\partial \mathcal{Z}}{\partial {}^C\mathbb{D}^{\alpha,g}_{a+}x}[\overline{x}]_{\mathcal{Z}}(t)\frac{1}{g'(t)} \right),
\end{aligned}$$

at $t = a$.

Proof. The proof is similar to the one presented in Theorem 3, but before applying the fractional integration by parts formulae, the integral defined on the interval $[A, b]$ is first split into two in the following way

$$\int_A^b \ldots\, dt = \int_a^b \ldots\, dt - \int_a^A \ldots\, dt,$$

and only after applying the fractional integration by parts formulae to each integral. □

4. Optimization Problems Subject to Constraints

In the previous section, we already considered variational problems under some additional constraint (boundary conditions). In this section, we impose two type of constraints on the set of admissible functions to the problem. In the first one, an integral constraint is imposed (knows as an isoperimetric problem) and in the second one, a relation between the position variables and time (known as a holonomic constraint) is established.

For the first case, we assume that we intend to extremize functional (1) subject to the constraint

$$G(x) = \int_a^b \mathcal{L}_G(t, x(t), {}^C\mathbb{D}^{\alpha,g}_{a+}x(t), \mathbb{I}^{\beta,g}_{a+}x(t), z(t))\, dt = K, \quad K \in \mathbb{R}, \tag{6}$$

with the antiderivative

$$z(t) = \int_a^t \mathcal{Z}_G(\tau, x(\tau), {}^C\mathbb{D}_{a+}^{\alpha,g}x(\tau), \mathbb{I}_{a+}^{\beta,g}x(\tau))\, d\tau,$$

with $\alpha, \beta, \mathcal{L}_G$ and $\mathcal{Z}_G$ as before. Furthermore, we say that a curve x is an extremal of functional (6) if is satisfies the Euler–Lagrange Equation (2); that is, if we replace $\mathcal{L}$ by $\mathcal{L}_G$ and $\mathcal{Z}$ by $\mathcal{Z}_G$ in (2), then x is a solution of such equation.

Theorem 6. *Let $\overline{x}$ be a solution of the variational problem:*

extremize F in (1), subject to the constraint $G(x) = K$, ($K \in \mathbb{R}$) and the boundary conditions $x(a) = x_a$, $x(b) = x_b$ ($x_a, x_b \in \mathbb{R}$).

If $\overline{x}$ is not an extremal for functional G, then there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that, if we define functions $H : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$, $\overline{H} : [a,b] \times \mathbb{R}^4 \to \mathbb{R}^2$, and $\overline{h} : [a,b] \times \mathbb{R}^3 \to \mathbb{R}^2$ by $H = \mathcal{L} + \lambda\mathcal{L}_G$, $\overline{H} = (\mathcal{L}, \lambda\mathcal{L}_G)$, and $\overline{h} = (\mathcal{Z}, \mathcal{Z}_G)$, then for all $t \in [a,b]$,

$$\begin{aligned}
&\frac{\partial H}{\partial x}[\overline{x}]_{\mathcal{L}}(t) + \int_t^b \left\langle \frac{\partial \overline{H}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau), \frac{\partial \overline{h}}{\partial x}[\overline{x}]_{\mathcal{Z}}(t) \right\rangle d\tau \\
&\quad + g'(t)\cdot \mathbb{D}_{b-}^{\alpha,g}\left(\frac{\partial H}{\partial {}^C\mathbb{D}_{a+}^{\alpha,g}x}[\overline{x}]_{\mathcal{L}}(t)\frac{1}{g'(t)} + \int_t^b \left\langle \frac{\partial \overline{H}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau), \frac{\partial \overline{h}}{\partial {}^C\mathbb{D}_{a+}^{\alpha,g}x}[\overline{x}]_{\mathcal{Z}}(t) \right\rangle d\tau \frac{1}{g'(t)} \right) \\
&\quad\quad + g'(t)\cdot \mathbb{I}_{b-}^{\beta,g}\left(\frac{\partial H}{\partial \mathbb{I}_{a+}^{\beta,g}x}[\overline{x}]_{\mathcal{L}}(t)\frac{1}{g'(t)} + \int_t^b \left\langle \frac{\partial \overline{H}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau), \frac{\partial \overline{h}}{\partial \mathbb{I}_{a+}^{\beta,g}x}[\overline{x}]_{\mathcal{Z}}(t) \right\rangle d\tau \frac{1}{g'(t)} \right) = 0. \quad (7)
\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product of $\mathbb{R}^2$.

Proof. Consider a variation of $\overline{x}$ of the form $[a,b] \ni t \mapsto \overline{x}(t) + \epsilon_1 v_1(t) + \epsilon_2 v_2(t)$, where $v_1, v_2 \in C^1[a,b]$ with $v_i(a) = v_i(b) = 0$, for $i = 1,2$, and $\epsilon_1, \epsilon_2 \in \mathbb{R}$. Next, define the functions f, g, in a neighborhood of $(0,0)$, as

$$f(\epsilon_1, \epsilon_2) = F(\overline{x} + \epsilon_1 v_1 + \epsilon_2 v_2) \quad \text{and} \quad g(\epsilon_1, \epsilon_2) = G(\overline{x} + \epsilon_1 v_1 + \epsilon_2 v_2) - K.$$

Since

$$\begin{aligned}
\frac{\partial g}{\partial \epsilon_2}(0,0) = \int_a^b &\left[\frac{\partial \mathcal{L}_G}{\partial x}[\overline{x}]_{\mathcal{L}}(t) + \int_t^b \frac{\partial \mathcal{L}_G}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\, d\tau \cdot \frac{\partial \mathcal{Z}_G}{\partial x}[\overline{x}]_{\mathcal{Z}}(t) \right. \\
&+ g'(t)\cdot \mathbb{D}_{b-}^{\alpha,g}\left(\frac{\partial \mathcal{L}_G}{\partial {}^C\mathbb{D}_{a+}^{\alpha,g}x}[\overline{x}]_{\mathcal{L}}(t)\frac{1}{g'(t)} + \int_t^b \frac{\partial \mathcal{L}_G}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\, d\tau \cdot \frac{\partial \mathcal{Z}_G}{\partial {}^C\mathbb{D}_{a+}^{\alpha,g}x}[\overline{x}]_{\mathcal{Z}}(t)\frac{1}{g'(t)} \right) \\
&\left. + g'(t)\cdot \mathbb{I}_{b-}^{\beta,g}\left(\frac{\partial \mathcal{L}_G}{\partial \mathbb{I}_{a+}^{\beta,g}x}[\overline{x}]_{\mathcal{L}}(t)\frac{1}{g'(t)} + \int_t^b \frac{\partial \mathcal{L}_G}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\, d\tau \cdot \frac{\partial \mathcal{Z}_G}{\partial \mathbb{I}_{a+}^{\beta,g}x}[\overline{x}]_{\mathcal{Z}}(t)\frac{1}{g'(t)} \right) \right] \cdot v_2(t)\, dt,
\end{aligned}$$

and $\overline{x}$ is not an extremal for functional G, then there exists a function v_2 for which $\frac{\partial g}{\partial \epsilon_2}(0,0) \neq 0$. Furthermore, $g(0,0) = 0$ and so, by the Implicit Function Theorem, there exists a neighborhood of 0 and a differentiable function ϕ, defined on that neighborhood, such that $g(\epsilon_1, \phi(\epsilon_1)) = 0$.

Now, we prove the desired necessary condition. For this, observe that the variational problem is equivalent to the following finite dimensional one:

$$\text{extremize } f, \text{ subject to the constraint } g = 0,$$

and since $(0,0)$ is a solution of this problem, and we proved that $\nabla g(0,0) \neq (0,0)$, by the Lagrange multiplier rule, there exists a real λ such that $\nabla(f + \lambda g)(0,0) = (0,0)$. Computing $\frac{\partial(f+\lambda g)}{\partial \epsilon_1}(0,0) = 0$, we prove the result. $\square$

Remark 3. *In Theorem 6, we can drop the assumption that "$\overline{x}$ is not an extremal for functional G". In such a case, we can prove that there exist Lagrange multipliers λ_0, λ (both not zero) such that, if we define functions by $H = \lambda_0\mathcal{L} + \lambda\mathcal{L}_G$ and $\overline{H} = (\lambda_0\mathcal{L}, \lambda\mathcal{L}_G)$ (and $\overline{h}$ as before), the result is still valid. In fact, if $\overline{x}$ is not an extremal for functional G, then we consider $\lambda_0 = 1$ and, in the other case, we take $\lambda_0 = 0$ and $\lambda = 1$.*

In the next problem that we will study, a holonomic constraint is imposed in the problem's formulation. Furthermore, we will consider a two-dimensional state function, that is, $x = (x_1, x_2)$, where its fractional derivative and integral are given by ${}^C\mathbb{D}^{\alpha,g}_{a+}x = \left({}^C\mathbb{D}^{\alpha,g}_{a+}x_1, {}^C\mathbb{D}^{\alpha,g}_{a+}x_2\right)$ and $\mathbb{I}^{\beta,g}_{a+}x = \left(\mathbb{I}^{\beta,g}_{a+}x_1, \mathbb{I}^{\beta,g}_{a+}x_2\right)$, respectively. The functional to be considered is the following:

$$F(x) = \int_a^b \mathcal{L}(t, x(t), {}^C\mathbb{D}^{\alpha,g}_{a+}x(t), \mathbb{I}^{\beta,g}_{a+}x(t), z(t))\, dt, \tag{8}$$

with

$$z(t) = \int_a^t \mathcal{Z}(\tau, x(\tau), {}^C\mathbb{D}^{\alpha,g}_{a+}x(\tau), \mathbb{I}^{\beta,g}_{a+}x(\tau))\, d\tau,$$

where $\mathcal{L} : [a,b] \times \mathbb{R}^7 \to \mathbb{R}$ and $\mathcal{Z} : [a,b] \times \mathbb{R}^6 \to \mathbb{R}$ are two continuously differentiable functions, and $x = (x_1, x_2) \in C^1[a,b] \times C^1[a,b]$. Furthermore, we will assume some boundary conditions $x(a) = X_a$ and $x(b) = X_b$ for some fixed $X_a, X_b \in \mathbb{R}^2$. Again, we will use the notation

$$[x]_{\mathcal{L}}(t) = (t, x(t), {}^C\mathbb{D}^{\alpha,g}_{a+}x(t), \mathbb{I}^{\beta,g}_{a+}x(t), z(t)) \quad \text{and} \quad [x]_{\mathcal{Z}}(t) = (t, x(t), {}^C\mathbb{D}^{\alpha,g}_{a+}x(t), \mathbb{I}^{\beta,g}_{a+}x(t)),$$

for $t \in [a,b]$, where x is a two-dimensional vector.

The holonomic constraint that all admissible functions to the problem must satisfy is given by

$$\Lambda(t, x_1(t), x_2(t)) = 0, \quad t \in [a,b], \tag{9}$$

where $\Lambda : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ is a C^1 function. The next result provides a necessary condition for optimizing functional (8).

Theorem 7. *Assume that $\overline{x} = (\overline{x}_1, \overline{x}_2)$ minimizes or maximizes functional (8), and that*

$$\frac{\partial \Lambda}{\partial x_2}(t, \overline{x}_1(t), \overline{x}_2(t)) \neq 0, \quad \forall t \in [a,b].$$

Then, there exists a continuous function $\lambda : [a,b] \to \mathbb{R}$ such that, for all $t \in [a,b]$, equations

$$\begin{aligned}
&\frac{\partial \mathcal{L}}{\partial x_1}[\overline{x}]_{\mathcal{L}}(t) + \int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\, d\tau \cdot \frac{\partial \mathcal{Z}}{\partial x_1}[\overline{x}]_{\mathcal{Z}}(t) \\
&\quad + g'(t) \cdot \mathbb{D}^{\alpha,g}_{b-}\left(\frac{\partial \mathcal{L}}{\partial {}^C\mathbb{D}^{\alpha,g}_{a+}x_1}[\overline{x}]_{\mathcal{L}}(t)\frac{1}{g'(t)} + \int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\, d\tau \cdot \frac{\partial \mathcal{Z}}{\partial {}^C\mathbb{D}^{\alpha,g}_{a+}x_1}[\overline{x}]_{\mathcal{Z}}(t)\frac{1}{g'(t)}\right) \\
&\quad + g'(t) \cdot \mathbb{I}^{\beta,g}_{b-}\left(\frac{\partial \mathcal{L}}{\partial \mathbb{I}^{\beta,g}_{a+}x_1}[\overline{x}]_{\mathcal{L}}(t)\frac{1}{g'(t)} + \int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\, d\tau \cdot \frac{\partial \mathcal{Z}}{\partial \mathbb{I}^{\beta,g}_{a+}x_1}[\overline{x}]_{\mathcal{Z}}(t)\frac{1}{g'(t)}\right) \\
&\quad + \lambda(t)\frac{\partial \Lambda}{\partial x_1}(t, \overline{x}_1(t), \overline{x}_2(t)) = 0
\end{aligned} \tag{10}$$

and

$$\frac{\partial \mathcal{L}}{\partial x_2}[\overline{x}]_{\mathcal{L}}(t) + \int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\,d\tau \cdot \frac{\partial \mathcal{Z}}{\partial x_2}[\overline{x}]_{\mathcal{Z}}(t)$$
$$+ g'(t)\cdot \mathbb{D}^{\alpha,g}_{b-}\left(\frac{\partial \mathcal{L}}{\partial {}^C\mathbb{D}^{\alpha,g}_{a+}x_2}[\overline{x}]_{\mathcal{L}}(t)\frac{1}{g'(t)} + \int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\,d\tau \cdot \frac{\partial \mathcal{Z}}{\partial {}^C\mathbb{D}^{\alpha,g}_{a+}x_2}[\overline{x}]_{\mathcal{Z}}(t)\frac{1}{g'(t)}\right)$$
$$+ g'(t)\cdot \mathbb{I}^{\beta,g}_{b-}\left(\frac{\partial \mathcal{L}}{\partial \mathbb{I}^{\beta,g}_{a+}x_2}[\overline{x}]_{\mathcal{L}}(t)\frac{1}{g'(t)} + \int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\,d\tau \cdot \frac{\partial \mathcal{Z}}{\partial \mathbb{I}^{\beta,g}_{a+}x_2}[\overline{x}]_{\mathcal{Z}}(t)\frac{1}{g'(t)}\right)$$
$$+ \lambda(t)\frac{\partial \Lambda}{\partial x_2}(t,\overline{x}_1(t),\overline{x}_2(t)) = 0 \quad (11)$$

hold.

Proof. To simplify the notation, by ELE_1 and ELE_2 we mean the Euler–Lagrange Equation (3) with respect to the functions x_1 and x_2, respectively. If we define function λ as

$$\lambda(t) = -\frac{ELE_2}{\frac{\partial \Lambda}{\partial x_2}(t,\overline{x}_1(t),\overline{x}_2(t))},$$

then (11) is proved immediately. To prove (10), a variational technique is used. Consider a variation of the optimal curve, given by $[a,b] \ni t \mapsto \overline{x}(t) + \epsilon v(t)$, where $v = (v_1, v_2) \in C^1[a,b] \times C^1[a,b]$ and $\epsilon \in \mathbb{R}$. Since we are assuming boundary conditions at $t = a$ and $t = b$, we need to assume that $v(a) = (0,0)$ and $v(b) = (0,0)$. Furthermore, since the variation curve must satisfy the holonomic constraint (9), that is,

$$\Lambda(t,\overline{x}_1(t)+\epsilon v_1(t),\overline{x}_2(t)+\epsilon v_2(t)) = 0, \quad t \in [a,b],$$

we conclude that

$$\frac{\partial \Lambda}{\partial x_1}(t,\overline{x}_1(t),\overline{x}_2(t))v_1(t) + \frac{\partial \Lambda}{\partial x_2}(t,\overline{x}_1(t),\overline{x}_2(t))v_2(t) = 0, \quad t \in [a,b]. \tag{12}$$

On the other hand, the first variation of functional (8) is zero when evaluated at the optimal curve, and so after performing integration by parts, as explained in the previous section, we arrive at

$$\int_a^b [ELE_1]v_1(t) + [ELE_2]v_2(t)\,dt = 0. \tag{13}$$

By (12) and the definition of function λ, we conclude that

$$[ELE_2]v_2(t) = -\lambda(t)\frac{\partial \Lambda}{\partial x_2}(t,\overline{x}_1(t),\overline{x}_2(t))v_2(t) = \lambda(t)\frac{\partial \Lambda}{\partial x_1}(t,\overline{x}_1(t),\overline{x}_2(t))v_1(t),$$

and so (13) becomes

$$\int_a^b \left([ELE_1] + \lambda(t)\frac{\partial \Lambda}{\partial x_1}(t,\overline{x}_1(t),\overline{x}_2(t))\right)v_1(t)\,dt = 0,$$

proving formula (10) from the arbitrariness of function v_1. $\square$

As an example, again consider the curve $\overline{x}(t) = (g(t)-g(0))^2$, where $g \in C^1[0,1]$ with $g'(t) > 0$ for all $t \in [0,1]$. Consider the following isoperimetric problem: extremize the functional

$$F(x) = \int_0^1 \left(\left({}^C\mathbb{D}^{\alpha,g}_{0+}x(t)\right)^2 + \left(\frac{2}{\Gamma(3-\alpha)}(g(t)-g(0))^{2-\alpha}\right)^2\right.$$
$$\left. + \left(\mathbb{I}^{\beta,g}_{0+}x(t) - \frac{2}{\Gamma(3+\beta)}(g(t)-g(0))^{2+\beta}\right)^2 + z(t)\right) dt,$$

defined on $C^1[0,1]$, where

$$z(t) = \int_0^t \left(x(\tau) - (g(\tau) - g(0))^2 \right)^2 d\tau,$$

subject to the integral constraint

$$\int_0^1 {}^C\mathbb{D}_{0+}^{\alpha,g} x(t) \cdot \frac{2}{\Gamma(3-\alpha)} (g(t) - g(0))^{2-\alpha} \, dt = K$$

and the boundary conditions

$$x(0) = 0 \quad \text{and} \quad x(1) = (g(1) - g(0))^2.$$

Here,

$$K = \int_0^1 \left(\frac{2}{\Gamma(3-\alpha)} (g(t) - g(0))^{2-\alpha} \right)^2 dt.$$

If we take $\lambda = -2$, we can show that $\overline{x}$ is a solution of the fractional differential equation given in Theorem 6.

5. Optimization Problems with Arbitrary Fractional Orders

In this section, we allow the fractional order of the derivative to take any positive real value. More precisely, let $n \in \mathbb{N}$, $\alpha_k \in \mathbb{R}$ with $\alpha_k \in (k-1, k)$, for $k \in \{1, \ldots, n\}$, and define $\alpha = (\alpha_1, \ldots, \alpha_n)$ and

$$ {}^C\mathbb{D}_{a+}^{\alpha,g} x = \left({}^C\mathbb{D}_{a+}^{\alpha_1,g} x, \ldots, {}^C\mathbb{D}_{a+}^{\alpha_n,g} x \right). \tag{14}$$

Consider the following functional with dependence on higher-order fractional derivatives:

$$F(x) = \int_a^b \mathcal{L}(t, x(t), {}^C\mathbb{D}_{a+}^{\alpha,g} x(t), \mathbb{I}_{a+}^{\beta,g} x(t), z(t)) \, dt, \tag{15}$$

with the antiderivative

$$z(t) = \int_a^t \mathcal{Z}(\tau, x(\tau), {}^C\mathbb{D}_{a+}^{\alpha,g} x(\tau), \mathbb{I}_{a+}^{\beta,g} x(\tau)) \, d\tau,$$

where ${}^C\mathbb{D}_{a+}^{\alpha,g} x$ is understood as being the vector defined in (14), and functions $\mathcal{L} : [a,b] \times \mathbb{R}^{n+3} \to \mathbb{R}$ and $\mathcal{Z} : [a,b] \times \mathbb{R}^{n+2} \to \mathbb{R}$ are of class C^1. Furthermore, $[x]_{\mathcal{L}}$ and $[x]_{\mathcal{Z}}$ denote the same vectors as before, with ${}^C\mathbb{D}_{a+}^{\alpha,g} x$ as defined in (14). We will also assume that $x^{(k)}(a)$ and $x^{(k)}(b)$ are fixed for $k \in \{0, 1, \ldots, n-1\}$.

Theorem 8. *If $\overline{x}$ is a minimizer of maximizer curve of the functional defined in (15), then*

$$\begin{aligned}
&\frac{\partial \mathcal{L}}{\partial x}[\overline{x}]_{\mathcal{L}}(t) + \int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\, d\tau \cdot \frac{\partial \mathcal{Z}}{\partial x}[\overline{x}]_{\mathcal{Z}}(t) \\
&+ \sum_{k=1}^{n} g'(t) \cdot \mathbb{D}_{b-}^{\alpha_k,g} \left(\frac{\partial \mathcal{L}}{\partial {}^C\mathbb{D}_{a+}^{\alpha_k,g} x}[\overline{x}]_{\mathcal{L}}(t) \frac{1}{g'(t)} + \int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\, d\tau \cdot \frac{\partial \mathcal{Z}}{\partial {}^C\mathbb{D}_{a+}^{\alpha_k,g} x}[\overline{x}]_{\mathcal{Z}}(t) \frac{1}{g'(t)} \right) \\
&\qquad + g'(t) \cdot \mathbb{I}_{b-}^{\beta,g} \left(\frac{\partial \mathcal{L}}{\partial \mathbb{I}_{a+}^{\beta,g} x}[\overline{x}]_{\mathcal{L}}(t) \frac{1}{g'(t)} + \int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\, d\tau \cdot \frac{\partial \mathcal{Z}}{\partial \mathbb{I}_{a+}^{\beta,g} x}[\overline{x}]_{\mathcal{Z}}(t) \frac{1}{g'(t)} \right) = 0,
\end{aligned}$$

for all $t \in [a,b]$.

Proof. Let $v \in C^n[a,b]$ be a function with $v^{(k)}(a) = 0$ and $v^{(k)}(b) = 0$ for $k \in \{0,1,\ldots,n-1\}$, and consider the variation of the optimal curve as being $[a,b] \ni t \mapsto \overline{x}(t) + \epsilon v(t)$. The first variation of the functional, evaluated along this variation, is the following:

$$\int_a^b \frac{\partial \mathcal{L}}{\partial x}[\overline{x}]_{\mathcal{L}}(t)v(t) + \sum_{k=1}^{n} \frac{\partial \mathcal{L}}{\partial {}^C\mathbb{D}_{a+}^{\alpha_k,g}x}[\overline{x}]_{\mathcal{L}}(t)\,{}^C\mathbb{D}_{a+}^{\alpha_k,g}v(t) + \frac{\partial \mathcal{L}}{\partial \mathbb{I}_{a+}^{\beta,g}x}[\overline{x}]_{\mathcal{L}}(t)\mathbb{I}_{a+}^{\beta,g}v(t) + \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(t)$$
$$\times \int_a^t \left[\frac{\partial \mathcal{Z}}{\partial x}[\overline{x}]_{\mathcal{Z}}(\tau)v(\tau) + \sum_{k=1}^{n} \frac{\partial \mathcal{Z}}{\partial {}^C\mathbb{D}_{a+}^{\alpha_k,g}x}[\overline{x}]_{\mathcal{Z}}(\tau)\,{}^C\mathbb{D}_{a+}^{\alpha_k,g}v(\tau) + \frac{\partial \mathcal{Z}}{\partial \mathbb{I}_{a+}^{\beta,g}x}[\overline{x}]_{\mathcal{Z}}(\tau)\mathbb{I}_{a+}^{\beta,g}v(\tau)\right] d\tau\, dt = 0.$$

Since $v^{(k)}(a) = 0$ and $v^{(k)}(b) = 0$ for $k \in \{0,1,\ldots,n-1\}$, then

$$\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^k v(t) = 0, \quad \forall k \in \{0,1,\ldots,n-1\},$$

when we evaluate it at $t = a$ or at $t = b$, and so the fractional integration by parts reads as

$$\int_a^b \frac{\partial \mathcal{L}}{\partial {}^C\mathbb{D}_{a+}^{\alpha_k,g}x}[\overline{x}]_{\mathcal{L}}(t)\,{}^C\mathbb{D}_{a+}^{\alpha_k,g}v(t)\, dt = \int_a^b \mathbb{D}_{b-}^{\alpha_k,g}\left(\frac{\partial \mathcal{L}}{\partial {}^C\mathbb{D}_{a+}^{\alpha_k,g}x}[\overline{x}]_{\mathcal{L}}(t)\frac{1}{g'(t)}\right) \cdot v(t)g'(t)\, dt$$

and

$$\int_a^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(t)\left(\int_a^t \frac{\partial \mathcal{Z}}{\partial {}^C\mathbb{D}_{a+}^{\alpha_k,g}x}[\overline{x}]_{\mathcal{Z}}(\tau)\,{}^C\mathbb{D}_{a+}^{\alpha_k,g}v(\tau)\, d\tau\right) dt$$
$$= \int_a^b \mathbb{D}_{b-}^{\alpha_k,g}\left(\int_t^b \frac{\partial \mathcal{L}}{\partial z}[\overline{x}]_{\mathcal{L}}(\tau)\, d\tau \cdot \frac{\partial \mathcal{Z}}{\partial {}^C\mathbb{D}_{a+}^{\alpha_k,g}x}[\overline{x}]_{\mathcal{Z}}(t)\frac{1}{g'(t)}\right) \cdot v(t)g'(t)\, dt,$$

for all $k \in \{0,1,\ldots,n-1\}$. The rest of the proof is similar to the one given in Theorem 3. □

For example, let $n \in \mathbb{N}$, $\alpha \in (n-1,n)$ and $\beta > 0$. Consider the functional

$$F(x) = \int_0^1 \left({}^C\mathbb{D}_{0+}^{\alpha,g}x(t) - \Gamma(\alpha+2)(g(t)-g(0))\right)^2 + z(t)\, dt,$$

defined on $C^n[0,1]$, subject to the boundary conditions $x(0) = 0$ and $x(1) = (g(1) - g(0))^{\alpha+1}$, where

$$z(t) = \int_0^t \left(x(\tau) - (g(\tau)-g(0))^{\alpha+1}\right)^2 d\tau.$$

If we define function $\overline{x} \in C^n[0,1]$ as $\overline{x}(t) = (g(t)-g(0))^{\alpha+1}$, and since

$${}^C\mathbb{D}_{0+}^{\alpha,g}\overline{x}(t) = \Gamma(\alpha+2)(g(t)-g(0)),$$

we conclude that $\overline{x}$ is a solution for this arbitrary fractional order problem.

6. Conclusions and Future Work

In this paper, we studied some variational problems for the case where the Lagrange functions depend on some fractional operators and on an antiderivative. To make the work as general as possible, we included the fractional integral and fractional derivative of the state function with dependence on an arbitrary kernel. We proved the fractional Euler–Lagrange equation, and under convexity conditions, we showed that in fact it is also a sufficient condition. Then, we imposed some additional constraints on the set of state functions (isoperimetric and holonomic constraints) and deduced the respective necessary conditions for optimization. We ended by studying the case where fractional orders may take any positive real value. With the help of some examples, we show how the variational problem can be solved by computing the solutions of a certain fractional differential equation. For future work, it will be interesting to consider an optimal control approach to these variational problems, where the Lagrange function as well as the state equation depend on the same fractional operators. The goal will be to deduce the fractional Pontryagin's minimum principle for such a situation. Another path for research would be to develop numerical methods to determine an approximation of the solution to the variational problem. Although there are already numerous works dealing with different fractional operators, we have not yet found many studies dealing with these fractional integrals and derivatives with respect to another function, as presented in the present paper. Usually, such methods consist of discretizing the fractional operators and the cost integral, and thus converting the fractional problem into a (finite) system of difference equations.

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