

Borel Chain Conditions of Borel Posets

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Abstract: We study the coarse classification of partial orderings using chain conditions in the context of descriptive combinatorics. We show that (unlike the Borel counterpart of many other combinatorial notions), we have a strict hierarchy of different chain conditions, similar to the classical case.

Keywords: mathematical logic; set theory; descriptive set theory; descriptive combinatorics

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1. Introduction

Let X be a Polish space (i.e., a completely metrizable separable topological space). A partial order $<$ over X is said to be a Borel partial order if it is a Borel subset of X^2 . This class of partial orders has been found to play a central role in the theory of forcing, particularly in the theory of cardinal characteristics of the continuum. The first systematical study of these posets was by Harrington, Marker, and Shelah in [1], where they observed a typical dichotomy:

Theorem 1 ([1]). *If $(X, <)$ is a Borel partial order, then either:*

1. *It is a union of countably many Borel chains; or*
2. *It includes a perfect pair-wise incomparable subset.*

If we let (X, E) be the incomparability graph of $\leq$ (i.e., $E = X^2 \setminus (< \cup \geq)$), the above theorem can be restated as follows: either (X, E) has a countable Borel chromatic number or it includes a complete subgraph, which is also topologically perfect (this subject is then further developed in [2]). This statement is in the same spirit as the G_0 -dichotomy of Kechris, Solecki, and Todorcevic on the theory of the Borel chromatic number:

Theorem 2 ([3]). *There is a Borel graph G_0 on 2^ω , such that for every analytic graph G on a Polish space X , exactly one of the following holds:*

1. *X is a union of countably many Borel anti-cliques (in other words, it has a countable Borel chromatic number); or*
2. *There is a continuous map from 2^ω into X preserving edges (i.e., its square sends G_0 into G).*

In fact, Theorem 1 can be proved as a corollary of G_0 -dichotomy (see, e.g., [4]).

The incomparability graph is not the only combinatorial notion that draws our attention. The main focus of this paper is the incompatibility graph, which, at first sight, seems similar to the incomparability graph. However, the phenomenon we are going to observe only belong to the incompatibility graph.

Our subject is based on the following notions:

Definition 1. *Let P be a poset and $A \subset P$.*

1. *Let $n > 1$ be an integer. A is n -linked if for every subset $A' \subset A$ of size n , there is $z \in P$ so that $z \leq x$ for all $x \in A'$.*
2. *A is linked if it is 2-linked.*



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3. A is centered if it is n -linked for all $n > 1$.
4. $x, y \in P$ are compatible if the set $\{x, y\}$ is linked.
5. $x, y \in P$ are incompatible if they are not compatible.
6. A is an antichain if it is pairwise incompatible.

The chain condition method is a way of classifying partial orders by looking at certain combinatorial properties of compatibilities and incompatibilities. The importance of the chain conditions was first noticed in the characterization of topologies on linearly ordered sets and was quickly applied to the measure theory and in the theory of forcing.

Among the many chain conditions that have been studied, here is a list of the most:

Definition 2. Let P be a poset.

1. P satisfies the σ -finite chain condition if there is a countable partition $P = \bigcup_n P_n$, so that each P_n includes no infinite antichain.
2. P satisfies the σ -bounded chain condition if there is a countable partition $P = \bigcup_n P_n$ so that each P_n includes no antichain of size $\geq n$.
3. P is σ - n -linked if there is a countable partition $P = \bigcup_k P_k$ so that each P_k is n -linked.
4. P is σ -centered if there is a countable partition $P = \bigcup_n P_n$ so that each P_n is centered.

While these conditions are obviously listed from weaker to stronger, the fact that their strength is strictly increasing is non-trivial, especially for the σ -finite chain condition and σ -bounded chain condition, which were first studied and conjectured to be different in [5], and whose strength was just differentiated during the last decade in [6] (also see [7] for a Borel solution).

In this work, we study these chain conditions on Borel partial orders defined on Polish spaces and restrict ourselves to only Borel witnesses. Namely, we study the following list of properties:

Definition 3. Let $(P, \leq)$ be a Borel poset (i.e., P is a Polish space or a standard Borel space, and the partial order $\leq$ is a Borel subset of P^2).

1. P satisfies the Borel σ -finite chain condition if there is a countable partition $P = \bigcup_n P_n$ so that each P_n is Borel and includes no infinite antichain.
2. P satisfies the Borel σ -bounded chain condition if there is a countable partition $P = \bigcup_n P_n$ so that each P_n is Borel and includes no antichain of size $\geq n$.
3. P is Borel σ - n -linked if there is a countable partition $P = \bigcup_n P_n$ so that each P_n is Borel and n -linked.
4. P is Borel σ -centered if there is a countable partition $P = \bigcup_n P_n$ so that each P_n is Borel and centered.

Our main theorem states that this hierarchy is indeed a non-trivial one:

Theorem 3. All properties listed above are distinct.

As mentioned above, such a non-trivial hierarchical structure would not occur in the theory of incomparabilities of a Borel poset. On the other hand, these Borel chain conditions are also significantly different from the classical ones. As we will see, all examples that differentiate this hierarchy can be taken to be σ -centered.

In Section 5, we will include a case that should belong to the above list as well, but actually coincides with ones that are already included.

Theorem 4. Let P be a Borel poset with Borel incompatibility (i.e., the incompatibility induced by this partial order is a Borel relation). If there is an integer k and a countable partition $P = \bigcup P_n$ into Borel subsets, so that for every n , every antichain $A \subset P_n$ has size $< k$, then P is Borel σ -linked.

The notions used will be defined in that section before we provide proof of this theorem.

We follow standard notations in the descriptive set theory. See, e.g., [8]. In particular, we will frequently use two similar symbols that have completely different meanings. For a tree, T , $[T]$ is the space of all its infinite branches, equipped with product topology. For a set, X , $[X]^n$ is the set of all its size- n subsets and $[X]^{<\omega}$ is the set of all its finite subsets, both equipped with product topology. The reader needs to be careful when encountering these notations.

2. Preparation

For a set X , a (symmetric) hypergraph over X is a pair (X, H) , where H (called the set of edges) is a subset of $[X]^{<\omega} \setminus X$. If all edges are of the same finite size d , we say X is a d -dimensional hypergraph. We will write the pair (X, H) as X when there is no confusion as to which hypergraph structure we are talking about. For a hypergraph (X, H) , a subset $A \subset X$ is called an anti-clique if there is no subset $A' \subset A$ that satisfies $A' \in H$. (X, H) is called a Borel hypergraph if X is a Polish space and H is a Borel subset of $[X]^{<\omega}$ equipped with the usual product topology. The Borel chromatic number $\chi_B(X, H)$ is the smallest cardinality of a Polish space to which there is a Borel map being non-constant on every edge.

We are going to heavily use the concepts related to trees. In this work, a tree T (order theoretical) is always a subset of $\omega^{<\omega}$. Given a tree T , we denote by $[T]$ the set of all its infinite branches.

Definition 4. Let X be a hypergraph. Denote as $P(X)$ the poset of all finite anti-cliques of X , ordered by reverse inclusion.

Note that when (X, H) is a Borel hypergraph, $P(X)$ is a Borel poset.

The hypergraphs we are going to use are defined on the set of branches of several trees. Let T_n be the tree $n^{<\omega}$ for each n and $T_\infty = \bigcup_{n < \omega} n^n$. For each T_n and T_∞ , fix a subset $D_n \subset T_n$, $D_\infty \subset T_\infty$, respectively, so that each of them is dense and intersects each level with exactly one node.

For each n , define $H_n = \{ \{d \frown i \frown x : 0 \leq i < n\} : d \in D_n, x \in [T_n] \}$ to make each $[T_n]$ an n -dimensional hypergraph. In the same spirit, let $H_\infty^0 = \{ \{d \frown i \frown x : 0 \leq i < |d|\} : d \in D_\infty, x \in [\bigcup_{n > |d|+1} n^n] \}$ and $H_\infty^1 = \{ \{d \frown i \frown x, d \frown j \frown x\} : 0 \leq i \neq j < |d|, d \in D_\infty, x \in [\bigcup_{n > |d|+1} n^n] \}$.

These hypergraphs naturally generalize graphs G_0 defined in [3] and are well-studied in descriptive combinatorics (see, for example, [9]). They have uncountable Borel chromatic numbers:

Fact 1. The hypergraphs $([T_n], H_n)$, $([T_\infty], H_\infty^0)$, and $([T_\infty], H_\infty^1)$ all have uncountable Borel chromatic numbers. Moreover, for every countable partition of $([T_\infty], H_\infty^1)$ into Borel subsets, one fragment includes complete subgraphs of arbitrarily large sizes.

Here, we go through a standard argument using the property of Baire for case $[T_n]$; other cases follow the same method.

Proof. Suppose not. Then, there is a Borel map $f : [T_n] \rightarrow \mathbb{N}$ that is non-constant on every edge. As $[T_n]$ is a Polish space, there must be an integer k and a node $t \in D_n$, so that $f^{-1}(k)$ is comeager in the basic open set $\{t \frown x : x \in [T_n]\}$. Therefore, for each $0 \leq i < k$, $f^{-1}(n)$ is comeager in $\{t \frown i \frown x : x \in [T_n]\}$. Let $U_i = \{x : t \frown i \frown x \in f^{-1}(k)\}$. Each U_i is comeager in $[T_n]$; thus, they have a non-empty intersection. Take an x from this intersection, $\{t \frown i \frown x : 0 \leq i < n\}$ forms an edge on which f is constant, a contradiction. $\square$

3. Proof of the Theorem

The purpose of this section is to show the following facts:

Theorem 5.

1. $P([T_\infty], H_\infty^1)$ is the Borel σ -finite chain condition, but not the Borel σ -bounded chain condition.
2. $P([T_2], H_2)$ is the Borel σ -bounded chain condition, but not the Borel σ -linked.
3. For every $n > 2$, $P([T_n], H_n)$ is Borel σ -($n - 1$)-linked, but not Borel σ - n -linked.
4. $P([T_\infty], H_\infty^0)$ is Borel σ - n -linked for every $n > 2$, but not Borel σ -centered.

Which together imply Theorem 3.

First, we show the “not” part:

Proof. For each hypergraph $[T]$ mentioned, we can naturally identify $[T]$ with the subset of $P([T])$ consisting of all singletons. For T being 2-dimensional ($[T_2]$ and $([T_\infty], H_\infty^1)$), every complete subgraph is also an antichain in $P([T])$. In $[T_n]$, a subset is centered if and only if it is an anti-clique (to understand this, note that a finite subset of $[T]$, regarded as a subset of $P([T])$, has no common extension if and only if there is no element of $P([T])$ that includes them as subsets simultaneously. This only happens if their union contains a hyperedge of $[T]$, and if and only if—when regarded as a sub-hypergraph—they do not form an anti-clique). In $([T_\infty], H_\infty^0)$, a subset is n -linked if and only if it does not include any edge of size $\leq n$. Then the “not” part follows from Fact 1. $\square$

For the rest, we first describe the construction of Borel partitions for $P([T_n], H_n)$ witnessing the Borel σ -($n - 1$)-link for $n > 2$. For the three posets, the partition would be the same but we need to reason in a slightly different manner to see why they work.

Proof. For each hypergraph X , let $P_k(X) = \{p \in P(X) : |p| = k + 1\}$. Each $P_k(X)$ is Borel if X is (since $p \in P_k$ are finite, all quantifiers involved in the definition can be taken as first-order quantifiers), and $P(X) = \bigcup P_n(X) \cup \{\emptyset\}$. $\square$

Claim 1. For each hypergraph $[T_n]$, for each $\{x_0, \dots, x_k\} \in P_k([T_n])$, there are $k + 1$ distinct nodes $\{t_0, \dots, t_k\} \subset T_n$ of the same height, such that $t_i \sqsubset x_i$, and for every tuple $\{y_0, \dots, y_k\} \subset [T_n]$ satisfying $t_i \sqsubset y_i$ for every $0 \leq i \leq k$, $\{y_0, \dots, y_k\}$ is an anti-clique.

Proof. Fix $[T_n]$. For every two branches x, y in a tree, denote by $\Delta(x, y)$ the longest initial segment of x and y . Fix $\{x_0, \dots, x_k\} \in P_k([T_n])$. For it to be an anti-clique, it must fall into one of three cases:

1. There are $i \neq j$, so that $\Delta(x_i, x_j) \notin D_n$; or
2. There are $i \neq j \neq k$, so that $\Delta(x_i, x_j) \neq \Delta(x_j, x_k)$; or
3. There is a $d \in D_n$, so that for every $i \neq j$, we have $\Delta(x_i, x_j) = d$, but there are $i \neq j$ and $l_{ij} > |d|$, so that $x_i(l_{ij}) \neq x_j(l_{ij})$.

In the first two cases, let $l = \sup\{|\Delta(x_i, x_j)| + 1\}_{0 \leq i \neq j \leq k}$. If the third case happens, pick $i \neq j$ and let $l = l_{ij} + 1$. Let $t_i = x_i|l$ (the initial segment of x_i of length l). Then each tuple in the open set $\{\{y_i\}_{0 \leq i \leq k} : t_i \sqsubset y_i\}$ realizes the same case as $\{x_0, \dots, x_k\}$ below level l and, thus, is an anti-clique. $\square$

Now, for each $p \in P_k([T_n])$, we pick $S_p = \{t_0, \dots, t_k\}$ and let U_p be the open neighbourhood of $P_k([T_n])$ defined by $U_p = \{\{y_i\}_{0 \leq i \leq k} : t_i \sqsubset y_i\}$. Clearly, U_p is Borel (in fact, it is open). We show that it is $n - 1$ -linked. Let $A = \{p_0, \dots, p_{n-2}\} \subset U_p$ be a subset of size $n - 1$. We show that it is centered (i.e., the union is still an anti-clique.)

If not, we take $(x_0, \dots, x_{n-1})$, being an edge in $\bigcup_{i < l} p_i$. By the above claim, we have $i \leq k$; moreover, x_{j_0} and x_{j_1} both extend t_i and, thus, $|\Delta(x_{j_0}, x_{j_1})| \geq t_i$. On the other hand, by the pigeonhole principle (and the fact that $n > n - 1$), there has to be $m < n_1$, $x_{l_0} \neq x_{l_1}$ both in p_m . By our definition of U_p , $|\Delta(x_{l_0}, x_{l_1})| < |t_i|$. Thus, we have $\Delta(x_{j_0}, x_{j_1}) \neq \Delta(x_{l_0}, x_{l_1})$.

However, by the definition of H_n , we have $\Delta(x_{i_0}, x_{i_1}) = \Delta(x_{j_0}, x_{j_1})$ for all pairs where $i_0 \neq i_1$ and $j_0 \neq j_1$. This contradiction shows that there cannot be any edge in $\bigcup_{0 \leq i < n-1} p_i$.

Lastly, notice that while there are uncountably many p ; there can only be countably many S_p since they are finite subsets of the countable set T_n . Also, it is clear that $p \in U_p$; thus, $P([T_n]) = \bigcup_{p \in P} U_p$ is actually a countable partition of $P[T_n]$ into countably many $n-1$ -linked Borel subsets, as required.

Now, we turn to $P([T_2], H_2)$. The claim from above still works for $n=2$, so we can still construct U_p . For this case, we want to show that each U_p only includes anti-chains of bounded size. Fix $p \in P([T_2], H_2)$, pick S_p as in the above proof, and let $A \subset U_p$ be an antichain. We order $S_p = \{t_0, \dots, t_k\}$. For each pair $q_0 \neq q_1 \in A$, as they are incompatible, there has to be an H_2 edge connecting $x \in q_0, y \in q_1$. By our claim, there has to be $0 \leq i \leq k$, so that x and y both end-extend t_i . We color this (unordered) pair q_0, q_1 with the smallest, such as i . A is an antichain, so $[A]^2$ is fully colored. By the Ramsey theorem, when $|A|$ is large enough (more precisely, when it is no less than the $|p|$ -colored Ramsey number $R(3, 3, \dots, 3)$), we have $t \in S_p, p_0 \neq p_1 \neq p_2 \in A$ and $x_i \in p_i$, so that $t \sqsubset x_i$ for $i=0, 1, 2$ and $x_0, x_1, x_2 \in [T_2]$ form a triangle (K_3). However, this is impossible. It is a well-known fact that $([T_2], H_2)$ (which is just G_0) is loop-free.

Thus, U_p does not include any antichain of a size larger than the $|p|$ -colored Ramsey number $R(3, 3, \dots, 3)$. This number clearly only depends on the size of p and is independent of our choice of S_p . Again, there are only countably many different possible S_p s, so $P([T_2], H_2) = \bigcup_{p \in P} U_p$ satisfies the σ -bounded chain condition.

For $P([T_\infty], H_\infty^0)$, we aim to construct a partition that proves Borel σ - n -linkedness for each n . For this purpose, we turn back to Claim 1. In addition to the requirements in the claim, we also require the $|t_i| > n$. This can be achieved simply by picking $l = n+1$ if the original $l \leq n$ (otherwise, we can just leave it unchanged). Once this is done, the same proof of Borel σ - n -linkedness works for $P([T_\infty], H_\infty^0)$.

Lastly, for $P([T_\infty], H_\infty^1)$, we show that for every p and any S_p , as in the claim, U_p does not include infinite anti-chains. The proof is exactly the same as in case $P([T_2], H_2)$, it only slightly differs with the use of the Ramsey theorem. Instead of K_3 , we use the Ramsey theorem to pick an infinite complete subgraph G from $([T_\infty], H_\infty^1)$. We now show that there cannot be any infinite complete subgraph. We pick $x \in G$. By the definition of H_∞^1 , for each $d \in D_\infty$, there are only finitely many $x' \in [T_\infty]$ satisfying $x' \in G$ and $\Delta(x, x') = d$. Therefore, there has to be $y \neq z$, so that $\Delta(x, y) = d_0 \sqsubset \Delta(x, z) = d_1$ for $d_0 \neq d_1 \in D_\infty$. In this case, we can see that $\Delta(y, z) = d_0$ as well. Since $y, z \in G$, there have to be integers $i \neq j$ and real r so that $y = d_0 \frown i \frown r$ and $z = d_0 \frown j \frown r$. However, this implies that $d_0 \frown j \sqsubset d_1 \sqsubset x$. By looking at the definition of H_∞^1 again, we notice that an edge from x to y makes $x = d_0 \frown j \frown r = z$, contradicting our choice of x, y, z being distinct.

4. Comparison with Classical Cases

It is worth noticing that every poset we mentioned above are all σ -centered if we do not require the fragmentation to be Borel.

Theorem 6. *If H is a hypergraph with, at most, a continuum of connected components, and each connected component is countable, then $P(H)$ is σ -centered.*

Proof. Let $H = \bigcup_{\lambda < \mathfrak{c}} H_\lambda$, where each H_λ is a connected component of H . We equip it with the discrete topology and consider the topological space $X = \prod_{\lambda < \mathfrak{c}} P(H_\lambda)$ equipped with the usual product topology. Every $P(H_\lambda)$ is countable and, thus, separable. By the Hewitt–Marczewski–Pondiczery theorem, X is also separable. We take $D \subset X$ as a countable dense subset. For each $d_n \in D$, let $P_n = \{p : p \in P(H) \text{ and } p \cap H_\lambda = d_n(\lambda)\}$. Every P_n is centered since $\bigcup_{\lambda < \mathfrak{c}} d_n(\lambda)$ is an anti-clique in H . Also, for every $p \in P(H)$, the subset $\{x : x(\lambda) = p \cap H_\lambda \text{ or } p \cap H_\lambda = \emptyset \text{ for all } \lambda\}$ is open, so there is a d_n in it, and equivalently, $p \in P_n$. $\square$

Our hypergraphs $[T_n]$, $([T_\infty], H_\infty^0)$, and $([T_\infty], H_\infty^1)$ all satisfy the requirements of the above theorem, since any two vertices in an edge are eventually equal; thus, all posets we deal with are σ -centered.

For another interesting example that fails the Borel σ -finite chain condition and the usual σ -bounded chain condition, but satisfies the σ -finite chain condition, see [10].

5. Further Observations

When we look at the σ -bounded chain condition, a natural question that arises is whether replacing “bounded” with “uniformly bounded” would result in a new property that lies strictly in between the σ -bounded chain condition and σ -linkedness or not. More precisely, we consider the following property:

Definition 5. Let n be a positive integer. A poset P is said to satisfy the σ - n -chain condition if there is a countable partition $P = \bigcup P_i$, so that for every i , every antichain $A \subset P_i$ has size $< n$. When P is a Borel poset and P_i can be taken to be Borel simultaneously, we say that P satisfies the Borel σ - n -chain condition.

However, the following (unpublished, as far as the author knows) theorem by Galvin and Hajnal states that this property is actually just σ -linkedness:

Theorem 7 (Galvin, Hajnal). For any positive integer n , a poset P satisfies the σ - n -chain condition if and only if it is σ -linked.

Similarly, the same occurs for the Borel σ - n -chain condition, for a wide class of Borel posets:

Theorem 8. Let P be a Borel poset with Borel incompatibility (i.e., the incompatibility induced by this partial order is a Borel relation). If there is an integer k and a countable partition $P = \bigcup P_n$ into Borel subsets so that for every n , every antichain $A \subset P_n$ has size $< k$, then P is Borel σ -linked.

(In some articles, a Borel poset with Borel incompatibility is also called a “Souslin forcing”).

We provide proof of the later theorem, from which, if we omit all complexity checks, would result in the proof of the original theorem by Galvin and Hajnal.

Proof. First, let us look at the description of “there is no antichain of size $= n$ ” of a subset A : “for every sequence $x_0, \dots, x_{n-1} \in A$, there are $m_0 \neq m_1 \leq n-1$ so that x_{m_0} and x_{m_1} are compatible”. When “being compatible” is Borel. This condition is clearly Π_1^1 over Σ_1^1 ; therefore, the reflection lemma implies that we can relax the Borel partition in the definition of the σ - n -chain condition to the analytic (Σ_1^1) partition.

Now, suppose that P is not Borel σ -linked. Let $P = \bigcup_{i < \omega} P_i$ be a partition witnessing the Borel σ - n -chain condition for the smallest n possible. As the σ -2-bounded chain condition and σ -linkedness are the same properties, for the following, we assume that $n > 2$.

Since P does not satisfy Borel σ - $n-1$ -chain condition, there must be $k < \omega$, so that for every partition of $P_k = \bigcup_{j < \omega} P_{k,j}$ into analytic sets, there is a fragment $P_{k,l}$ including an antichain of size no less than $n-1$ (and is thus equal to $n-1$). For each $p \in P_k$, let $L(p) = \{q \in P_k, \text{ so that } q \text{ is incompatible with } p\}$ and $R_i(p) = \{q \in P_k; r \in P_i \text{ extends both } p \text{ and } q\}$. Here, $L(p)$ is Borel and $R_i(p)$ is, at most, analytic. We can see that for every p , $L(p) \cup (\bigcup_i R_i(p)) = P_k$; thus, either $L(p)$ includes an antichain of size $n-1$ or there is an integer i so that $R_i(p)$ contains an antichain of size $n-1$. However, if we pick an antichain of size $n-1$ from $L(p)$, we can add p into it and make an antichain of size n , which contradicts the choice of P_k . Therefore, for every p , there has to be i . For each i , let $Q_i = \{p \in P_k: R_i(p) \text{ includes an antichain of size } n-1\}$. Q_i is again analytic since it is the i th fiber of the analytic set $\{(p, i)\}$. There is an antichain of size $n-1$ in $R_i(p)$. By our above argument, we have $\bigcup_i Q_i = P_k$; thus, there is l , so that Q_l contains an

antichain $p_1, \dots, p_{n-1}$ of size $n - 1$. Then, for each $i = 1, 2, \dots, n - 1$, we can find antichain $q_{i1}, \dots, q_{i(n-1)} \subset R_i$ of size $n - 1$. For each $i = 1, \dots, n - 1$ and $j = 1, 2, \dots, n - 1$, we fix r_{ij} in P_i , extending both p_i and q_{ij} . Since p_{i_0} and p_{i_1} are incompatible for different i_0 and i_1 , then $r_{i_0 j_0}$ and $r_{i_1 j_1}$ are also incompatible for different i_0, i_1 , disregarding what j_0 and j_1 are. Then, $\{r_{ij} : i, j = 1, 2, \dots, n - 1\}$ is an antichain and it is a subset of P_i . Since $(n - 1)^2 > n$ for $n > 2$, we have a contradiction with $n > 2$. $\square$

While Borel posets with non-Borel incompatibility do exist (for example, take two disjoint Polish spaces $X \cap Y = \emptyset$ and let $U \subset X \times Y$ be a closed set with non-Borel projection onto Y , considering U as a partial order on $X \cup Y$ results in such a poset), it is not known if the requirement of Borel incompatibility can be omitted.

Proof. In the previous proof, every $L(p)$ is Borel, $R_i(p)$, and Q_i were analytic. Also, notice that “every antichain has size $< n$ ” is Π_1^1 over Σ_1^1 , so by the reflection lemma, the above proof works for Borel posets with Borel incompatibility. $\square$

Also, due to the G_0 -dichotomy, the following fact is quickly followed:

Theorem 9. Let P be a Borel poset, such that the collection $\mathcal{C}$ of centered subsets is Borel and there is a Borel function $f : \mathcal{C} \rightarrow P$, such that for every $C \in \mathcal{C}$, $f(C) \leq p$ for every $p \in C$. Then exactly one of the following holds:

1. P is the Borel σ -linkedness; or
2. There is a $P' \subset P([T_2])$ failing Borel σ -linkedness, and there exists a Borel map $\phi : P' \rightarrow P$ that preserves incompatibility.

Proof. Let (P, G) be the incompatibility graph over P . Then P is Borel σ -linkedness if and only if the Borel chromatic number $\chi_B(P)$ is countable.

When P is not Borel σ -linkedness, there is a Borel map ψ that embeds $G_0([T_2])$ into (P, G) . Let $P' = \{p \in P([T_2]) : \{\psi(x) : x \in p\} \in \mathcal{C}\}$. Let $\phi(p) = f(\{\psi(x) : x \in p\})$. This P' and ϕ are then as required. $\square$

And similarly, we can replace Borel σ -linkedness and $[T_2]$ with other Borel chain conditions and corresponding posets. We finish with conjecturing the following strengthening of this theorem:

Question 1. Is it true that for every Borel poset P , exactly one of the following holds?

1. P is Borel σ -linkedness; or
2. There is a Borel map $\phi : P([T_2]) \rightarrow P$ that preserves incompatibility.

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