

Article

Properties of Anti-Invariant Submersions and Some Applications to Number Theory

Ali H. Hakami and Mohd. Danish Siddiqi * 

Department of Mathematics, College of Science, Jazan University, P.O. Box 277, Jazan 4512, Saudi Arabia; aalhakami@jazanu.edu.sa

* Correspondence: msiddiqi@jazanu.edu.sa or anallintegral@gmail.com

Abstract: In this article, we investigate anti-invariant Riemannian and Lagrangian submersions onto Riemannian manifolds from the Lorentzian para-Sasakian manifold. We demonstrate that, for these submersions, horizontal distributions are not integrable and their fibers are not totally geodesic. As a result, they are not totally geodesic maps. The harmonicity of such submersions is also examined. We specifically prove that they are not harmonic when the Reeb vector field is horizontal. Finally, we provide an illustration of our findings and mention some number-theoretic applications for the same submersions.

Keywords: anti-invariant submersion; Lagrangian submersion; Lorentzian para-Sasakian manifold; homotopy groups

MSC: 53C15; 53B20; 11F23



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1. Introduction

The hypothesis of Riemannian submersions developed by O'Neill [1] and Gray [2] is now the most fascinating area of differential geometry research. Riemannian submersions among nearly-Hermitian manifolds were regarded as almost-Hermitian submersions by Watson [3]. After that, various subclasses of almost-Hermitian manifolds have been the subject of intense research with regard to almost-Hermitian submersions. Additionally, under the term of contact-Riemannian submersions, Riemannian submersions were expanded to include a number of subclasses of almost-contact manifolds. The book [4] contains vast investigations of Riemannian, nearly-Hermitian, and contact-Riemannian submersions.

Since Şahin [5] introduced such submersions along nearly-Hermitian manifolds onto Riemannian manifolds, the topic of Lagrangian submersions and anti-invariant submersions has become a particularly active research area. Anti-invariant and Lagrangian submersions have been researched in a variety of structures, including nearly-Kähler [6], Kähler [5,7], local-product Riemannian [8], almost-product [9], Sasakian [10–13], Kenmotsu [13,14], and cosymplectic structure [15].

It should be emphasized that the theory of anti-invariant Riemannian submersion has been enlarged into the concept of conformal anti-invariant submersion [16]. The bulk of the research on anti-invariant and Lagrangian submersions is reported in [17].

The following is how this document is structured. Section 2 introduces the background of the Lorentzian para-Sasakian manifold. Section 3 provides some context for Riemannian submersions. The authors review the definitions of anti-invariant and Lagrangian submersions in Section 4. Section 5 investigates anti-invariant submersions from Lorentzian para-Sasakian manifolds onto Riemannian manifolds conceding vertical Reeb vector fields, provides an example, and discusses some of its characteristics. Section 6 discusses the scenario where the Reeb vector field is horizontal. Section 7 investigates the geometry of vertical and horizontal distributions for Lagrangian submersions accommodating a vertical Reeb vector field. We also provide a necessary and sufficient condition for such harmonic

submersions. Comparable investigations for Lagrangian submersions conceding horizontal Reeb vector fields are given at the end.

Lorentzian almost-para-contact manifolds were introduced by Matsumoto [18]. Following that, several geometers investigated various structures on these manifolds. After the work in [19–21], the study of Lorentzian para-Sasakian manifolds has become a topic of increasing scholarly interest.

2. Preliminaries

Let $(\mathcal{L}, \delta)$ be a $(2r + 1)$ -dimensional Lorentzian manifold. Then, $\mathcal{L}$ is called a *Lorentzian almost-para-contact metric manifold* [18] with the Riemannian metric δ if there exists a tensor ψ of type $(1, 1)$ and a global vector field ς , which is called the *Reeb vector field* [22] or the *characteristic vector field*, such that, if θ is the dual 1-form of ς , then we have

$$\begin{aligned} \psi\varsigma &= 0, \theta(\varsigma) = -1, \psi^2 = I + \theta \otimes \varsigma, \\ \delta(\psi p, \psi q) &= \delta(p, q) + \theta(p)\theta(q) \end{aligned} \quad (1)$$

where $p, q \in \chi(\mathcal{L})$. Furthermore, it is clear from the preceding arguments that

$$\theta \circ \psi = 0 \quad \text{and} \quad \theta(p) = \delta(p, \varsigma), \quad \text{rank}(\psi) = r - 1. \quad (2)$$

This structure $(\psi, \varsigma, \theta, \delta)$ is referred to as the *Lorentzian para-contact metric structure* $\mathcal{L}$.

A Lorentzian para-contact metric structure $(\psi, \varsigma, \theta, \delta)$ on a connected manifold [23] $\mathcal{L}$ is known as a *Lorentzian para-Sasakian manifold* if [18,20]

$$(\nabla_p \psi)q = \delta(p, q)\varsigma + \theta(q)p + 2\theta(p)\theta(q)\varsigma, \quad (3)$$

$$\nabla_p \varsigma = \psi p. \quad (4)$$

for all p and q tangent to $\mathcal{L}$, where ∇ indicates the Levi–Civita connection with respect to metric δ .

3. Riemannian Submersions

This section provides the necessary context for Riemannian submersions. Let $(\mathcal{L}^{2r+1}, \delta)$ and $(\aleph^s, \delta_\aleph)$ be Riemannian manifolds with $\dim(\mathcal{L}) > \dim(\aleph)$ and a surjective map $\Phi : (\mathcal{L}^{2r+1}, \delta) \rightarrow (\aleph^s, \delta_\aleph)$ is referred to as a *Riemannian submersion* [1] if the following conditions are met.

(C1) $\text{Rank}(\Phi) = \dim(\aleph)$.

In this scenario, $\Phi^{-1}(x) = \Phi_x^{-1}$ for all points $x \in \aleph$ is a submanifold of $\mathcal{L}$ with dimension t and is referred to as a *fiber*, where

$$\dim(\mathcal{L}) = t + \dim(\aleph).$$

If a vector field on $\mathcal{L}$ is always tangent (resp. orthogonal) to fibers, it is claimed to be *vertical* (resp. *horizontal*). Confirmed all should be retained. If a horizontal vector field α on $\mathcal{L}$ is Φ -related to a vector field α_* on $\aleph$, then

$$\Phi_*(\alpha_l) = \alpha_{*\Phi(l)}$$

for all $l \in \mathcal{L}$, where Φ_* is the differential mapping of Φ . The projections on the vertical distribution $\text{Ker}\Phi_*$ and the horizontal distribution $\text{Ker}\Phi_*^\perp$ are denoted by $\mathbb{V}$ and $\mathbb{H}$, respectively.

The manifold $(\mathcal{L}, \delta)$ is termed the *total manifold*, and the manifold $(\aleph, \delta_\aleph)$ is called the *base manifold* of the submersion $\Phi : (\mathcal{L}, \delta) \rightarrow (\aleph, \delta_\aleph)$.

(C2) The lengths of the horizontal vectors are preserved by Φ_* .

This criterion implies that the derivative map Φ_* of Φ , constrained to $\mathcal{Ker}\Phi_*^\perp$, is a linear isometry.

O'Neill's [1] tensors T and A describe the geometry of Riemannian submersions, which are defined as follows:

$$T_p q = V\nabla_{Vp} Hq + H\nabla_{Vp} Vq, \quad (5)$$

$$A_p q = V\nabla_{Hp} Hq + H\nabla_{Hp} Vq \quad (6)$$

for any vector fields p and q on $\mathcal{L}$, wherein ∇ is the Levi-Civita connection of δ . It is clear that the T_p and A_p operators are skew-symmetric on the tangent bundle of $\mathcal{L}$ and reverse the horizontal and vertical distributions. The properties of the tensor field T and A are next outlined. Let p, q be vertical vector fields and α, β be horizontal vector fields on $\mathcal{L}$; then, we gain

$$T_p q = T_q p, \quad (7)$$

$$A_\alpha \beta = -A_\beta \alpha = \frac{1}{2} V[\alpha, \beta]. \quad (8)$$

In addition, from Equations (5) and (6), we obtain

$$\nabla_p q = T_p q + \hat{\nabla}_p q, \quad (9)$$

$$\nabla_p \alpha = T_p \alpha + H\nabla_p \alpha, \quad (10)$$

$$\nabla_\alpha p = A_\alpha p + V\nabla_\alpha p, \quad (11)$$

$$\nabla_\alpha \beta = H\nabla_\alpha \beta + A_\alpha \beta, \quad (12)$$

where $\hat{\nabla}_p q = V\nabla_p q$. Furthermore, if α is basic, then we get $H\nabla_p \alpha = A_\alpha p$. It is easy to see that T works as the second fundamental form, but A operates on the horizontal distribution and measures the obstacle to its integrability. For further information on Riemannian submersions, see O'Neill's work [1] and the book [4].

Let $(\mathcal{L}, \delta)$ and $(\mathfrak{N}, \delta_{\mathfrak{N}})$ be Riemannian manifolds and $\Phi : (\mathcal{L}, \delta) \rightarrow (\mathfrak{N}, \delta_{\mathfrak{N}})$ a smooth map. Then, the second fundamental form of Ψ is provided by

$$(\nabla \Phi_*)(p, q) = \nabla_p^\Phi \Phi_* p - \Phi_*(\nabla_p q) \quad (13)$$

for $p, q \in \Gamma(T\mathcal{L})$, where ∇^Φ is the pull back connection, and we denote for convenience the Riemannian connections of the metrics δ and $\delta_{\mathfrak{N}}$ with ∇ . It is well known that the second fundamental form is symmetric, wherein ∇^Φ denotes the pull back connection and ∇ indicates the Riemannian connections of the metrics δ and $\delta_{\mathfrak{N}}$ for convenience. In addition, Ψ is said to be *totally geodesic* [24] if

$$(\nabla \Phi_*)(p, q) = 0,$$

for each $p, q \in \Gamma(T\mathcal{L})$, and Φ is said to be a *harmonic map* [24] if

$$\text{Trace}(\nabla \Phi_*) = 0.$$

Example 1. Let $\mathcal{L}^6 = \{(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6) | \sigma_6 \neq 0\}$ be a six-dimensional differentiable manifold where (σ_i) signifies the standard coordinates of a point in $\mathbb{R}^6$, and $i = 1, 2, 3, 4, 5, 6$.

$$\delta_1 = \partial\sigma_1, \quad \delta_2 = \partial\sigma_2, \quad \delta_3 = \partial\sigma_3,$$

$$\delta_4 = \partial\sigma_4, \quad \delta_5 = \partial\sigma_5, \quad \delta_6 = \partial\sigma_6$$

is the basis for the tangent space $T(\mathcal{L}^6)$ since it consists of a set of linearly independent vector fields at each point of the manifold $\mathcal{L}^6$. A definite positive metric g on $\mathcal{L}^6$ is defined as follows. With $i, j = 1, 2, 3, 4, 5, 6$, it is defined as

$$g = \sum_{i,j=1}^6 d\sigma_i \otimes d\sigma_j.$$

Let θ be a 1-form such that $\theta(U) = g(U, \varsigma)$, where $\delta_6^\sharp = \varsigma$.

Thus, $(\mathcal{L}^6, g)$ is a total manifold. In addition, let $\bar{\nabla}$ be the Levi-Civita connection with respect to g . Then, we have

$$[\delta_1, \delta_2] = 0, \quad [\delta_1, \delta_6] = \delta_1, \quad [\delta_2, \delta_6] = \delta_2, \quad [\delta_3, \delta_6] = \delta_3,$$

$$[\delta_4, \delta_6] = \delta_4, \quad [\delta_5, \delta_6] = \delta_6, \quad [\delta_i, \delta_j] = 0,$$

where $1 \leq i \neq j \leq 5$.

Next, let us take a map $\Phi : (\mathbb{R}^6, g) \rightarrow (\mathbb{R}^3, g_3)$ as a submersion defined by

$$\Phi(x_1, x_2, \dots, x_6) = (y_1, y_2, y_3),$$

where

$$y_1 = \frac{x_1 + x_2}{\sqrt{2}}, \quad y_2 = \frac{x_3 + x_4}{\sqrt{2}} \text{ and } y_3 = \frac{x_5 + x_6}{\sqrt{2}}.$$

Then, the Jacobian matrix of Φ is given as

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

The Jacobi matrix of Φ has rank three at that point. This indicates that Φ is a submersion. Simple calculations produce

$$\begin{aligned} (\mathcal{Ker}\Phi_*) &= \text{Span}\{V_1 = \frac{1}{\sqrt{2}}(-\partial x_1 + \partial x_2), V_2 = \frac{1}{\sqrt{2}}(-\partial x_3 + \partial x_4), \\ &V_3 = \frac{1}{\sqrt{2}}(-\partial x_5 + \partial x_6)\}, \end{aligned}$$

and

$$\begin{aligned} (\mathcal{Ker}\Phi_*)^\perp &= \text{Span}\{H_1 = \frac{1}{\sqrt{2}}(\partial x_1 + \partial x_2), H_2 = \frac{1}{\sqrt{2}}(\partial x_3 + \partial x_4), \\ &H_3 = \frac{1}{\sqrt{2}}(\partial x_5 + \partial x_6)\}. \end{aligned}$$

Also, direct computations yields

$$\Phi_*(H_1) = \partial y_1, \quad \Phi_*(H_2) = \partial y_2 \text{ and } \Phi_*(H_3) = \partial y_3.$$

It is easy to observe that

$$g_{\mathbb{R}^6}(H_i, H_i) = g_{\mathbb{R}^3}(\Phi_*(H_i), \Phi_*(H_i)), \quad i = 1, 2, 3.$$

Hence, we can easily observe that Φ meets the requirement (C2). As a result, Φ is a Θ .

4. Submersions from Lorentzian Para-Sasakian Manifolds

We begin by recalling the essence of an anti-invariant Riemannian submersion with a Lorentzian para-contact manifold as its total manifold.

Definition 1 ([10,11]). Let $\mathcal{L}$ be a $(2r + 1)$ -dimensional Lorentzian para-contact manifold with Lorentzian para-contact metric structure $(\psi, \zeta, \theta, \delta)$ and $\mathcal{N}$ be a Riemannian manifold with Riemannian metric $\delta_{\mathcal{N}}$. Let there exist a Riemannian submersion $\Phi : \mathcal{L} \rightarrow \mathcal{N}$ with a vertical distribution $\text{Ker}\Phi_*$ that is anti-invariant such that we have ψ_* ; i.e., $\psi(\text{Ker}\Phi_*) \subseteq \text{Ker}\Phi_*^\perp$. The Riemannian submersion Φ is then known as an anti-invariant Riemannian submersion. Such submersions will be denoted for short by Θ .

A horizontal distribution $\text{Ker}\Phi_*^\perp$ is decomposed in this situation as

$$\text{Ker}\Phi_*^\perp = \psi(\text{Ker}\Phi_*) \oplus \nu, \quad (14)$$

wherein ν is the orthogonal complementary distribution of $\psi\text{Ker}\Phi_*$ in $\text{Ker}\Phi_*^\perp$ and is invariant with regard to ψ .

We have an anti-invariant submersion $\Phi : \mathcal{L} \rightarrow \mathcal{N}$ with a vertical Reeb vector field (briefly, $v^\sharp$ -RVF) if the Reeb vector field ζ is tangent to $\text{Ker}\Phi_*$ and a horizontal Reeb vector field (briefly, $\hbar$ -RVF) if the Reeb vector field ζ is normal to $\text{Ker}\Phi_*$. It is clear that ν contains the Reeb vector field (RVF) ζ in the case that $\Phi : \mathcal{L} \rightarrow \mathcal{N}$ with the $\hbar$ -RVF ζ . For more information on anti-invariant submersions from various structures, see [10,11,14,15].

Remark 1. In this article, we treat a Lorentzian para-Sasakian manifold [25] $(\mathcal{L}, \psi, \zeta, \theta, \delta)$ as a total manifold of a Θ .

Lagrangian submersion is a subcategory of Θ . We next review the description of a Lagrangian submersion from a Lorentzian para-contact manifold onto a Riemannian manifold.

Definition 2 ([12]). Let π be a Θ from a Lorentzian para-Sasakian manifold $(\mathcal{L}, \psi, \zeta, \theta, \delta)$ onto a Riemannian manifold $(\mathcal{N}, \delta_{\mathcal{N}})$. If it holds that

$$(i) \quad \nu = \{0\} \quad \text{or} \quad \nu = \text{span}\{\zeta\}, \quad \text{i.e.,} \quad \text{Ker}\Phi_*^\perp = \psi(\text{Ker}\Phi_*^\perp) \quad \text{or} \\ (ii) \quad \text{Ker}\Phi_*^\perp = \psi(\text{Ker}\Phi_*) \oplus \langle \zeta \rangle,$$

respectively, then we state that Φ is a Lagrangian submersion and it is denoted by Ξ .

Remark 2. This scenario has been investigated partially as a specific example of a Θ ; for more information, see [10–12,14,15].

5. Θ with Vertical Reeb Vector Field

We investigate Θ conceding a $v^\sharp$ -RVF from a Lorentzian para-Sasakian manifold $(\mathcal{L}, \psi, \zeta, \theta, \delta)$ in this section.

Let Φ be a Θ from a Lorentzian para-Sasakian manifold $(\mathcal{L}, \psi, \zeta, \theta, \delta)$ onto a Riemannian manifold $(\mathcal{N}, g_{\mathcal{N}})$. For any $\alpha \in \text{Ker}\Phi_*^\perp$, we compose

$$\psi\alpha = \mathcal{P}\alpha + \mathcal{Q}\alpha, \quad (15)$$

wherein $\mathcal{P}\alpha \in \Gamma(\text{Ker}\Phi_*)$ and $\mathcal{Q}\alpha \in \Gamma(\text{Ker}\Phi_*^\perp)$. First, we investigate how the Lorentzian para-Sasakian structure affects T and A of the submersion Φ .

Lemma 1. Let Φ be a Θ from a Lorentzian para-Sasakian manifold $(\mathcal{L}, \psi, \zeta, \theta, \delta)$ onto a Riemannian manifold $(\mathcal{N}, \delta_{\mathcal{N}})$ admitting a $v^\sharp$ -RVF. Then, we gain

$$\mathcal{T}_p\psi q - \delta(p, q)\zeta - 2\theta(p)\theta(q)\zeta = \mathcal{P}\mathcal{T}_p q + \theta(q)p, \quad (16)$$

$$\mathbb{H}\nabla_p\psi q = \mathcal{Q}\mathbb{T}_p q + \psi\hat{\nabla}_p q \quad (17)$$

$$\hat{\nabla}_q \mathcal{P}\alpha + \mathbb{T}_q \mathcal{Q}\alpha = \mathcal{P}\mathbb{H}\nabla_q \alpha \quad (18)$$

$$\mathbb{T}_q \mathcal{P}\alpha + \mathbb{H}\nabla_q \mathcal{Q}\alpha = \mathcal{Q}\mathbb{H}\nabla_q \alpha + \psi\mathbb{T}_q \alpha \quad (19)$$

$$\mathbb{A}_\alpha \psi q = \mathcal{P}\mathbb{A}_\alpha q \quad (20)$$

$$\mathbb{H}\nabla_\alpha \psi q + \theta(q)\alpha = \psi(\mathbb{V}\nabla_\alpha q) + \mathcal{Q}\mathbb{A}_\alpha q \quad (21)$$

$$\mathbb{V}\nabla_\alpha \mathcal{P}\beta + \mathbb{A}_\alpha \mathcal{Q}\beta = \mathcal{P}\mathbb{H}\nabla_\alpha \beta + \delta(\alpha, \beta)\zeta + 2\theta(\alpha)\theta(\beta)\zeta \quad (22)$$

$$\mathbb{A}_\alpha \mathcal{P}\beta + \mathbb{H}\nabla_\alpha \mathcal{Q}\beta = \mathcal{Q}\mathbb{H}\nabla_\alpha \beta + \psi\mathbb{A}_\alpha \beta, \quad (23)$$

wherein $p, q \in \Gamma(\mathcal{Ker}\Phi_*)$ and $\alpha, \beta \in \Gamma(\mathcal{Ker}\Phi_*^\perp)$.

Proof. For any $p, q \in \Gamma(\mathcal{Ker}\Phi_*)$, using Equation (3), we find

$$\nabla_p \psi q = \varphi \nabla_p q + [\delta(p, q)\zeta + \theta(q)p + 2\theta(p)\theta(q)\zeta].$$

As a result, using Equations (9), (10), and (33), we obtain

$$\begin{aligned} \mathbb{H}\nabla_p \psi q + \mathbb{T}_p \psi q = & \mathcal{P}\mathbb{T}_p q + \mathcal{Q}\mathbb{T}_p q + \psi\hat{\nabla}_p q \\ & + [\delta(q, p)\zeta + \theta(p)q + 2\theta(q)\theta(p)\zeta]. \end{aligned} \quad (24)$$

Given that ζ is vertical, we may take the horizontal and vertical components of Equation (24) to gain Equations (16) and (17), respectively.

Let α and β be any horizontal vector fields. We have Equation (3) once more.

$$\nabla_\alpha \psi \beta = \psi \nabla_\alpha \beta + [\delta(\alpha, \beta)\zeta + \theta(\beta)\alpha + 2\theta(\alpha)\theta(\beta)\zeta]$$

Adopting Equations (11), (12), and (33), we get

$$\begin{aligned} & \mathbb{A}_\alpha \mathcal{P}\beta + \mathbb{V}\nabla_\alpha \mathcal{P}\beta + \mathbb{H}\nabla_\alpha \mathcal{Q}\beta + \mathbb{A}_\alpha \mathcal{Q}\beta \\ & = \mathcal{P}\mathbb{H}\nabla_\alpha \beta + \mathcal{Q}\mathbb{H}\nabla_\alpha \beta + \psi\mathbb{A}_\alpha \beta + [\delta(\alpha, \beta)\zeta + \theta(\beta)\alpha + 2\theta(\alpha)\theta(\beta)\zeta]. \end{aligned} \quad (25)$$

We can simply generate Equations (22) and (23) by taking the horizontal and vertical portions of Equation (25) and employing the idea that ζ is vertical.

The other assertions can be acquired in the same manner. $\square$

In this section, we explore a Θ from a Lorentzian para-Sasakian manifold onto a Riemannian manifold with a $v^\sharp$ -RVF ζ .

Let Φ be a Θ admitting a $v^\sharp$ -RVF from a Lorentzian para-Sasakian manifold $(\mathcal{L}, \psi, \zeta, \theta, \delta)$ onto a Riemannian manifold $(\mathfrak{N}, \delta_\mathfrak{N})$. Then, adopting Equation (33) and the condition (C2), we deduce

$$\delta(\Phi_* \psi q, \Phi_* \mathcal{Q}\alpha) = 0,$$

for every $\alpha \in \Gamma(\mathcal{Ker}\Phi_*^\perp)$ and $q \in \Gamma(\mathcal{Ker}\Phi_*)$, which entails that

$$\mathbb{T}\mathfrak{N} = \Phi_*(\psi(\mathcal{Ker}\Phi_*)) \oplus \Phi_*(v). \quad (26)$$

Theorem 1. Let $(\mathcal{L}, \psi, \zeta, \theta, \delta)$ be a Lorentzian para-Sasakian manifold of dimension $(2r + 1)$ and $(\mathfrak{N}, \delta_{\mathfrak{N}})$ a Riemannian manifold of dimension s . Let $\Phi : (\mathcal{L}, \psi, \zeta, \theta, \delta) \rightarrow (\mathfrak{N}, \delta_{\mathfrak{N}})$ be a Θ such that $\psi(\mathcal{Ker}\Phi_*) = \mathcal{Ker}\Phi_*^\perp$. Then, the RVF ζ is vertical and $r = s$.

Proof. With the hypothesis $\psi(\mathcal{Ker}\Phi_*) = \mathcal{Ker}\Phi_*^\perp$, for any $p \in \mathcal{Ker}\Phi_*$ we gain

$$\delta(\zeta, \psi p) = -\delta(\psi \zeta, p) = 0,$$

This demonstrates that the RVF is vertical.

Now, let $p_1, \dots, p_{k-1}, \zeta = p_k$ be an orthonormal set of $\mathcal{Ker}\Phi_*$, wherein $k = 2r - s + 1$. Since

$$\psi(\mathcal{Ker}\Phi_*) = (\mathcal{Ker}\Phi_*^\perp), \quad p_1, \dots, p_{k-1}, \zeta = p_k$$

with an orthonormal set of $\Gamma(\mathcal{Ker}\Phi_*^\perp)$. So, in light of Equation (33), we get $s + 1 = k$, which entails that $r = s$. $\square$

Theorem 2. Let $\Phi : (\mathcal{L}, \delta) \rightarrow (\mathfrak{N}, \delta_{\mathfrak{N}})$ be a Θ admitting a $v^\sharp$ -RVF; then, the fibers are not totally umbilical.

Proof. Adopting Equations (9) and (3), we get

$$T_p \zeta = \psi p \tag{27}$$

for any $p \in \Gamma(\mathcal{Ker}\Phi_*)$, and with totally umbilical fibers, one can get

$$T_p q = \delta(p, q)\mathcal{H},$$

wherein $\mathcal{H}$ is the mean curvature vector field of any fiber. Since $T_\zeta \zeta = 0$, we get $\mathcal{H} = 0$, which demonstrates that fibers are minimal. Consequently, the fibers are totally geodesic, which is inappropriate to the condition $T_p \zeta = \psi p \neq 0$. $\square$

We may derive the following Lemma from Equations (1) and (33).

Lemma 2. Let Φ be an be a Θ admitting a $v^\sharp$ -RVF from a Lorentzian para-Sasakian manifold $(\mathcal{L}, \psi, \zeta, \theta, \delta)$ to a Riemannian manifold $(\mathfrak{N}, \delta_{\mathfrak{N}})$. Then, we gain

$$\mathcal{P}\mathcal{Q}\alpha = 0, \quad \psi\mathcal{P}\alpha + \mathcal{Q}^2\alpha = \alpha$$

for any $\alpha \in \Gamma(\mathcal{Ker}\Phi_*^\perp)$.

Lemma 3. Let Φ be an be a Θ admitting a $v^\sharp$ -RVF from a Lorentzian para-Sasakian manifold $(\mathcal{L}, \psi, \zeta, \theta, \delta)$ to a Riemannian manifold $(\mathfrak{N}, \delta_{\mathfrak{N}})$. Then, we gain

$$\mathcal{Q}\alpha = \mathbf{A}_\alpha \zeta, \tag{28}$$

$$\delta(\mathbf{A}_\alpha \zeta, \psi p) = 0, \tag{29}$$

$$\delta(\nabla_\beta \mathbf{A}_\alpha \zeta, \psi p) = -\delta(\mathbf{A}_\alpha \zeta, \psi \mathbf{A}_\beta p) + \theta(p)\delta(\mathbf{A}_\alpha \zeta, \beta) \tag{30}$$

$$\delta(\alpha, \mathbf{A}_\beta \zeta) = -\delta(\beta, \mathbf{A}_\alpha \zeta) \tag{31}$$

for $\alpha, \beta \in \Gamma(\mathcal{Ker}\Phi_*^\perp)$ and $p \in \Gamma(\mathcal{Ker}\Phi_*)$.

Proof. We have Equation (28) as a result of Equations (4) and (11). For $\alpha \in \Gamma(\text{Ker}\Phi_*^\perp)$ and $p \in \Gamma(\text{Ker}\Phi_*)$, in light of Equations (6), (28) and (33), get

$$\delta(A_\alpha \zeta, \psi p) = -\delta(\psi \alpha - \mathcal{P}\alpha, \psi p) \quad (32)$$

$$= -\delta(\alpha, p) - \theta(\alpha)\theta(p) - \delta(\psi \mathcal{P}\alpha, p).$$

Since $\psi \mathcal{P}\alpha \in \Gamma(\text{Ker}\Phi_*^\perp)$ and $\zeta \in \Gamma(\text{Ker}\Phi_*)$, Equation (32) implies Equation (29). Using Equation (29) we find

$$\delta(\nabla_\beta A_\alpha \zeta, \psi p) = -\delta(A_\alpha \zeta, \nabla_\beta \psi p)$$

for $\alpha, \beta \in \Gamma(\text{Ker}\Phi_*^\perp)$ and $p \in \Gamma(\text{Ker}\Phi_*)$. Then, applying the geodesic condition and Equation (3), we obtain

$$\delta(\nabla_\beta A_\alpha \zeta, \psi p) = -\delta(A_\alpha \zeta, \psi A_\beta p) - \delta(A_\alpha \zeta, \psi(\nabla_\beta p))$$

$$+ \theta(p)\delta(A_\alpha \zeta, \beta).$$

Since $\psi(\nabla_\beta p) \in \Gamma(\psi \text{Ker}\Phi_*) = \Gamma(\text{Ker}\Phi_*^\perp)$, we get Equation (30). Since $\mathcal{A}$ is skew-symmetric and using Equation (8), we directly gain Equation (31). $\square$

6. Anti-Invariant Submersions with $\hbar$ -RVF

Example 2. Let $\mathbb{R}^9$ be a nine-dimensional Riemannian space given by $\mathbb{R}^9 = \{(\sigma^1, \dots, \sigma^n, \mu^1, \dots, \mu^n, \omega) | u^j, v^j, \omega \in \mathbb{R}, j = 1, \dots, 9\}$.

Then, we choose a Lorentzian para-contact structure $(\psi, \zeta, \theta, \delta)$ on $\mathbb{R}^9$, such as

$$\delta = -\theta \otimes \theta + \frac{1}{9} \sum_j d\sigma^j \otimes d\sigma^j \oplus d\mu^j \otimes d\mu^j, \quad \zeta = 3 \frac{\partial}{\partial \omega}, \quad \theta = \frac{1}{3}(-d\omega + \sum_j \mu^j d\sigma^j),$$

$$\psi(\partial\sigma^1) = \partial\mu^1, \psi(\partial\sigma^2) = \partial\mu^2, \psi(\partial\sigma^3) = \partial\sigma^3, \psi(\partial\sigma^4) = \partial\mu^4, \psi(\partial\mu^1) = \partial\sigma^1,$$

$$\psi(\partial\mu^2) = \partial\sigma^1, \psi(\partial\mu^3) = -\partial\sigma^3, \psi(\partial\mu^4) = -\partial\mu^4, \psi(\partial\omega) = 0,$$

wherein $\partial\sigma^j, \partial\mu^j = B_j \in \mathfrak{T}(\mathbb{R}^9)$, $1 \leq j \leq 4$ are vector fields. Thus, $(\mathbb{R}^9, \psi, \zeta, \theta, \delta)$ is a Lorentzian para-Sasakian manifold.

Next, consider a mapping $\Phi : (\mathbb{R}^9, \psi, \zeta, \theta, \delta) \longrightarrow (\mathbb{R}^5, \delta_5)$ defined by

$$\Phi(\sigma^1, \sigma^2, \sigma^3, \sigma^4, \mu^1, \mu^2, \mu^3, \mu^4, \omega) \longmapsto \left(\sigma^1 + \sigma^2, \mu^1 + \mu^2, \frac{\sigma^3 - \mu^3}{\sqrt{3}}, \frac{\sigma^4 - \mu^4}{\sqrt{3}}, 3\omega \right),$$

where δ_5 is the Riemannian metric of $\mathbb{R}^5$. Then, the Jacobian matrix of Φ is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & -\frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & -\frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

As the rank of the Jacobian matrix is 5, the map Φ is a submersion. However, we can easily show that Φ meets the requirement (C2). As a result, Φ is a Lorentzian submersion. Now, after some calculations, we arrive at

$$(\mathcal{Ker}\Phi_*) = \text{Span}\{v_1^\sharp = B_5 + B_6, v_2^\sharp = B_1 + B_2, v_3^\sharp = \frac{1}{\sqrt{3}}(B_3 + B_7), \\ v_4^\sharp = \frac{1}{\sqrt{3}}(B_4 + B_8)\},$$

and

$$(\mathcal{Ker}\Phi_*)^\perp = \text{Span}\{\hbar_1 = B_1 + B_2, \hbar_2 = B_5 + B_6, \hbar_3 = \frac{1}{\sqrt{3}}(B_3 - B_7), \\ \hbar_4 = \frac{1}{\sqrt{3}}(B_4 - B_8), \hbar_5 = \varsigma\}.$$

Furthermore, we can observe that $\psi(v_i^\sharp) = \hbar_i$ for $1 \leq j \leq 4$, which indicates

$$\psi(\mathcal{Ker}\Phi_*) \subset (\mathcal{Ker}\Phi_*)^\perp.$$

Hence, Φ is a Θ and ς is a $\hbar$ -RVF.

Let Φ be a Θ from a Lorentzian para-Sasakian manifold $(\mathcal{L}, \psi, \varsigma, \theta, \delta)$ onto a Riemannian manifold $(\mathbb{N}, \delta_{\mathbb{N}})$. For any $\alpha \in \mathcal{Ker}\Phi_*^\perp$, we compose

$$\psi\alpha = \mathcal{P}\alpha + \mathcal{Q}\alpha, \quad (33)$$

wherein $\mathcal{P}\alpha \in \Gamma(\mathcal{Ker}\Phi_*)$ and $\mathcal{Q}\alpha \in \Gamma(\mathcal{Ker}\Phi_*^\perp)$. We begin by investigating how the Lorentzian para-Sasakian manifold $(\mathcal{L}, \psi, \varsigma, \theta, \delta)$ affects the tensor fields T and A of the submersion Φ .

Lemma 4. Let Φ be a Θ from a Lorentzian para-Sasakian manifold $(\mathcal{L}, \psi, \varsigma, \theta, \delta)$ onto a Riemannian manifold $(\mathbb{N}, \delta_{\mathbb{N}})$ admitting a $\hbar$ -RVF. Then, we gain

$$T_p\psi q = \mathcal{P}T_pq, \quad (34)$$

$$H\nabla_p\psi q - \delta(p, q)\varsigma - 2\theta(p)\theta(q)\varsigma = \mathcal{Q}T_pq + \psi\hat{\nabla}_pq, \quad (35)$$

$$\hat{\nabla}_p\mathcal{P}\alpha + T_p\mathcal{Q}\alpha = \mathcal{P}H\nabla_p\alpha - \theta(\alpha)q, \quad (36)$$

$$T_q\mathcal{P}\alpha + H\nabla_q\mathcal{Q}\alpha = \mathcal{Q}H\nabla_q\alpha + \psi T_q\alpha,, \quad (37)$$

$$A_\alpha\psi q = \mathcal{P}A_\alpha q, \quad (38)$$

$$H\nabla_\alpha\psi V = \psi(V\nabla_\alpha q) + \mathcal{Q}A_\alpha q,, \quad (39)$$

$$V\nabla_\alpha\mathcal{P}\beta + A_\alpha\mathcal{Q}\beta = \mathcal{P}H\nabla_\alpha\beta,, \quad (40)$$

$$A_\alpha\mathcal{P}\beta + H\nabla_\alpha\mathcal{Q}\beta = \mathcal{Q}H\nabla_\alpha\beta + \psi A_\alpha\beta + \delta(\alpha, \beta)\varsigma \\ + \theta(\beta)\alpha + 2\theta(\alpha)\theta(\beta)\varsigma \quad (41)$$

wherein $p, q \in \Gamma(\mathcal{Ker}\Phi_*)$ and $\alpha, \beta \in \Gamma(\mathcal{Ker}\Phi_*^\perp)$.

Proof. The proof is pretty similar to the Lemma 1 proof. As a result, we leave it out. $\square$

In this part, we examine a Θ from a Lorentzian para-Sasakian manifold onto a Riemannian manifold with a $\hbar$ -RFV ζ .

Using Equation (33), we have

$$\nu = \psi\nu \oplus \{\zeta\}.$$

Assume that q is a vertical vector field and α is a horizontal vector field. Applying the aforementioned relationship and Equation (3), we get

$$\delta(\psi q, \mathcal{Q}\alpha) = 0.$$

We have this last relationship $\delta(\Phi_*\psi q, \Phi_*\mathcal{Q}\alpha) = 0$, which implies that

$$T\mathbb{N} = \Phi_*(\psi\mathcal{Ker}\Phi_*) \oplus \Phi_*(\nu). \quad (42)$$

Equations (3) and (33) entail the following lemma.

Lemma 5. Let Φ be a Θ admitting a $\hbar$ -RVF from a Lorentzian para-Sasakian manifold $(\mathcal{L}, \psi, \zeta, \theta, \delta)$ onto a Riemannian manifold $(\mathbb{N}, \delta_{\mathbb{N}})$. Then, we gain

$$\mathcal{P}\mathcal{Q}\alpha = 0, \quad \psi^2 X = \psi\mathcal{P}\alpha + \mathcal{Q}^2\alpha$$

for any $\alpha \in \Gamma(\mathcal{Ker}\Phi_*)$.

Lemma 6. Let Φ be a Θ with a $\hbar$ -RVF from a Lorentzian para-Sasakian manifold $(\mathcal{L}, \psi, \zeta, \theta, \delta)$ onto a Riemannian manifold $(\mathbb{N}, \delta_{\mathbb{N}})$. Then, we have

$$\mathcal{P}\alpha = A_\alpha\zeta, \quad (43)$$

$$T_p\zeta = 0, \quad (44)$$

$$\delta(A_\alpha\zeta, \psi p) = 0, \quad (45)$$

$$\delta(\nabla_Y A_\alpha\zeta, \psi p) = -\delta(A_\alpha\zeta, \psi A_\beta p), \quad (46)$$

$$\delta(\nabla_\alpha \mathcal{Q}\beta, \psi p) = -\delta(\mathcal{Q}\beta, \psi A_\alpha p) \quad (47)$$

for $\alpha, \beta \in \Gamma(\mathcal{Ker}\Phi_*^\perp)$ and $p \in \Gamma(\mathcal{Ker}\Phi_*)$.

Proof. In light of Equations (4), (12), and (33), we get Equation (43). Adopting Equations (4) and (10), we get Equation (44). Since ψp is horizontal and $A_\alpha\zeta$ is vertical, for $\alpha \in \Gamma(\mathcal{Ker}\Phi_*^\perp)$ and $p \in \Gamma(\mathcal{Ker}\Phi_*)$, we get Equation (45). Then, employing Equation (45), we get

$$\delta(\nabla_\beta A_\alpha\zeta, \psi p) = -\delta(A_\alpha\zeta, \nabla_\beta \psi p)$$

for $\alpha, \beta \in \Gamma(\mathcal{Ker}\Phi_*^\perp)$ and $p \in \Gamma(\mathcal{Ker}\Phi_*)$. Then, adopting Equations (3) and (11), we have

$$\delta(\nabla_\beta A_\alpha\zeta, \psi p) = -\delta(A_\alpha\zeta, \psi A_\beta p) - \delta(A_\alpha\zeta, \psi(q\nabla_\beta p))$$

Since $\psi(q\nabla_\beta p) \in \Gamma(\text{Ker}\Phi_*^\perp)$, we get Equation (46).
From Equation (14), we get

$$\delta(\mathcal{Q}q, \psi p) = 0$$

$$0 = \delta(\nabla_\alpha \mathcal{Q}\beta, \psi p) + \delta(\mathcal{Q}\beta, \nabla_\alpha \psi p)$$

$$= \delta(\nabla_\alpha \mathcal{Q}\beta, \psi p) + \delta(\mathcal{Q}\beta, \psi \nabla_\alpha p)$$

$$\delta(\nabla_\alpha \mathcal{Q}\beta, \psi p) = \delta(\mathcal{Q}\beta, \psi(\mathbb{A}_\alpha p)).$$

Thus, we gain Equation (47). $\square$

7. Lagrangian Submersions with $v^\sharp$ -RVFs from Lorentzian Para-Sasakian Manifolds

In this section, we examine the integrability and total geodesicity of the horizontal distribution of Ξ conceding a $v^\sharp$ -RVF from a Lorentzian para-Sasakian manifold. Initially, we estimate the behavior of the tensor T of Ξ . In light of Lemma 4, one can obtain the following outcomes.

Corollary 1. *Let Φ be a Ξ with a $v^\sharp$ -RVF from a Lorentzian para-Sasakian manifold $(\mathcal{L}, \psi, \varsigma, \theta, \delta)$ onto a Riemannian manifold $(\aleph, \delta_\aleph)$. Then, we gain*

$$T_p \psi q - \delta(p, q)\varsigma - 2\theta(p)\theta(q)\varsigma = \psi T_p q - \theta(q)p, \quad (48)$$

$$T_q \psi \alpha = \psi T_q \alpha, \quad (49)$$

$$T_q \varsigma = -\psi q, \quad (50)$$

$$T_\varsigma \alpha = -\psi \alpha, \quad (51)$$

for $p, q \in \Gamma(\text{Ker}\Phi_*)$ and $\alpha, \beta \in \Gamma(\text{Ker}\Phi_*^\perp)$.

Proof. For a Ξ , we have $\mathcal{Q}\alpha = 0$ for any $\alpha \in \Gamma(\text{Ker}\Phi_*^\perp)$. Henceforth, the claim from Equations (48) and (49) gives Equations (16) and (19), respectively. Equation (50) follows from Equations (4) and (9). The last expression is obtained from Equation (50). $\square$

Remark 3. Referring to [26], it can be noted that tensor T vanishes if the fibers of a Riemannian submersion are totally geodesic.

Corollary 1 entails that the tensor T cannot vanish. Hence, in light of Remark 3, we can articulate the following.

Theorem 3. *Let Φ be a Ξ with a $v^\sharp$ -RVF from a Lorentzian para-Sasakian manifold $(\mathcal{L}, \psi, \varsigma, \theta, \delta)$ onto a Riemannian manifold $(\aleph, \delta_\aleph)$. Then, the fibers of Φ are not totally geodesic.*

In addition, we provide a few findings about the characteristics of the tensor $\mathbb{A}$ of Ξ .

Corollary 2. *Let Φ be a Ξ with a $v^\sharp$ -RVF from a Lorentzian para-Sasakian manifold $(\mathcal{L}, \psi, \varsigma, \theta, \delta)$ onto a Riemannian manifold $(\aleph, \delta_\aleph)$. Then, we get*

$$\mathbb{A}_\alpha \psi q = \psi \mathbb{A}_\alpha q, \quad (52)$$

$$A_\alpha \psi \beta = \psi A_\alpha \beta, \quad (53)$$

$$A_\alpha \zeta = 0, \quad (54)$$

for $q \in \Gamma(\text{Ker} \Phi_*)$ and $\alpha, \beta \in \Gamma(\text{Ker} \Phi_*^\perp)$.

Proof. Equations (52) and (53) are obtained from Equations (20) and (24) and Equation (54) is obtained from Equations (4) and (11). $\square$

Remark 4. The total geodesicity and integrability of the horizontal distribution are equivalent to each other for a Riemannian submersion. The same condition can be observed from Equations (8) and (12). Therefore, the tensor A vanishes in this case.

We can observe that the tensor A cannot vanish for the submersion from Equation (54). Then, we get the following theorem.

Theorem 4. Let Φ be a Ξ with a $v^\sharp$ -RVF from a Lorentzian para-Sasakian manifold $(\mathcal{L}, \psi, \zeta, \theta, \delta)$ onto a Riemannian manifold $(\mathfrak{N}, \delta_\mathfrak{N})$. Then, the horizontal distribution of Φ is not integrable.

Remark 5. A mapping $\Phi : (\mathcal{L}, \psi, \zeta, \theta, \delta) \rightarrow (\mathfrak{N}, \delta_\mathfrak{N})$ between Riemannian manifolds is considered to be totally geodesic if Φ_* preserves parallel translation. According to Vilms [26], a Riemannian submersion Φ is totally geodesic if and only if both tensors T and A vanish.

Now, referring to Theorem 3 or Theorem 4 and Remark 5, we gain the following outcome.

Theorem 5. Let Φ be a Ξ with a $v^\sharp$ -RVF from a Lorentzian para-Sasakian manifold $(\mathcal{L}, \psi, \zeta, \theta, \delta)$ onto a Riemannian manifold $(\mathfrak{N}, \delta_\mathfrak{N})$. Then, the submersion Φ is not a totally geodesic map.

Finally, we provide a necessary and sufficient condition for Φ to be harmonic.

Theorem 6. Let Φ be a Ξ with a $v^\sharp$ -RVF from a Lorentzian para-Sasakian manifold $(\mathcal{L}, \psi, \zeta, \theta, \delta)$ onto a Riemannian manifold $(\mathfrak{N}, \delta_\mathfrak{N})$. Then, Φ is harmonic if and only if

$$\text{Trace}(\psi T_p)|_{\text{Ker} \Phi_*} = 0$$

for $p \in \Gamma(\text{Ker} \Phi_*)$, wherein $\psi T_p|_{\text{Ker} \Phi_*}$ is the restriction of ψT_p to $\text{Ker} \Phi_*$.

Proof. In view of [27], Φ is harmonic if and only if Φ has minimal fibers. Let $\{b_1, \dots, b_k, \zeta\}$ be an orthonormal set of $\text{Ker} \Phi_*$.

Thus, Φ is harmonic $\Leftrightarrow \sum_{j=1}^k T_{b_j} b_j + T_\zeta \zeta = 0$. Since $T_\zeta \zeta = 0$, it follows that

Φ is harmonic $\Leftrightarrow \sum_{j=1}^k T_{b_j} b_j = 0$. Now, we compute $\sum_{j=1}^k T_{b_i} b_j$. By orthonormal expansion, we can write

$$\sum_{j=1}^k T_{b_j} b_j = \sum_{j=1}^k \sum_{l=1}^k \delta(T_{b_i} b_l, \psi b_l) \psi b_j, \quad$$

wherein $\{\psi b_1, \dots, \psi b_k\}$ is an orthonormal frame of $\psi \text{Ker} \Phi_*$. Since T_{b_i} is skew-symmetric, we get

$$\sum_{j=1}^k T_{b_i} b_j = - \sum_{j,l=1}^k \delta(T_{b_j} \psi b_l, b_j) \psi b_l.$$

From Equation (48), we have

$$T_{b_j}\psi b_l = \psi T_{b_j}b_l + \delta(b_j, b_l)\zeta + \theta(b_l)b_j + 2\theta(b_j)\theta(b_l)\zeta.$$

Then, we get

$$\sum_{j=1}^k T_{b_j}b_j = - \sum_{j,l=1}^k \delta(\psi T_{b_j}b_l, b_j)\varphi e_j,$$

since $\theta(b_l) = 0$ and $\theta(b_j) = 0$. Adopting Equation (7), we get

$$\sum_{j=1}^k T_{b_j}b_j = - \sum_{j,l=1}^k \delta(\psi T_{b_l}b_j, b_j)\varphi e_j. \quad (55)$$

As $\psi b_1, \dots, \psi b_k$ are linearly independent, in view of Equation (55), we arrive at

$$\sum_{j=1}^k T_{b_j}b_j = 0 \Leftrightarrow \sum_{j,l=1}^k \delta(\psi T_{b_l}b_j, b_j) = 0. \quad (56)$$

It is not difficult to observe that

$$\sum_{j,l=1}^k \delta(\psi T_{b_l}b_j, b_j) = 0 \Leftrightarrow \sum_{j=1}^k \delta(\psi T_p b_j, b_j) = 0 \quad (57)$$

for any $p \in \Gamma(\text{Ker}\Phi_*)$. On the other side,

$$\text{Trace}\psi T_p|_{\text{Ker}\Phi_*} = \sum_{i=1}^k \delta(\psi T_p b_i, b_i) + \delta(T_p \zeta, \zeta)$$

and in light of Equations (1) and (50), we get

$$\text{Trace}\psi T_p|_{\text{Ker}\Phi_*} = \sum_{j=1}^k \delta(\psi T_p b_j, b_j). \quad (58)$$

Then, using Equations (56)–(58), the required result follows. $\square$

Example 3. Let $\mathbb{R}^5 = \{(\sigma^1, \sigma^2, \mu^1, \mu^2, \omega) | (\sigma^1, \sigma^2, \mu^1, \mu^2, \omega) \neq (0, 0, 0, 0, 0)\}$, wherein $(\sigma^1, \sigma^2, \mu^1, \mu^2, \omega)$ are the standard coordinates in $\mathbb{R}^5$, and $\mathbb{R}^2$ be a Lorentzian para-Sasakian manifold, as in the example in Equation (2).

Now, let us assume the map $\Phi : (\mathbb{R}^5, \psi, \zeta, \theta, \delta) \longrightarrow (\mathbb{R}^2, \delta_2)$ defined by the following:

$$\Phi(\sigma^1, \sigma^2, \mu^1, \mu^2, \omega) \longmapsto \left(\frac{\sigma^1 - \mu^2}{\sqrt{3}}, \frac{\sigma^2 - \mu^1}{\sqrt{3}} \right),$$

where δ_2 is the Riemannian metric of $\mathbb{R}^2$. Thus, the Jacobian matrix of Φ is given as:

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & 0 \end{bmatrix}.$$

Since the rank of the matrix is equal to 2, the map Φ is a submersion. On the other hand, we can observe that Φ obeys the condition (C2). Then, by a direct estimation, we arrive at

$$(\text{Ker}\Phi_*) = \text{Span}\{v_1^\sharp = \frac{1}{\sqrt{3}}(B_1 + B_4), v_2^\sharp = \frac{1}{\sqrt{3}}(B_2 + B_3), v_3^\sharp = \zeta\},$$

and

$$(\mathcal{Ker}\Phi_*)^\perp = \text{Span}\{\hbar_1 = \frac{1}{\sqrt{3}}(B_1 - B_4), \hbar_2 = \frac{1}{\sqrt{3}}(B_2 - B_3)\}.$$

It is easy to identify that $\psi(v_1^\sharp) = \hbar_1$, $\psi(v_2^\sharp) = \hbar_2$, and $\varphi(v_3^\sharp) = 0$ entail that

$$\psi(\mathcal{Ker}\Phi_*) = (\mathcal{Ker}\Phi_*)^\perp.$$

Consequently, Φ is a Ξ such that ζ is a $v^\sharp$ -RVF.

8. Lagrangian Submersions with $\hbar$ -RVFs from Lorentzian Para-Sasakian Manifolds

We analyze Ξ admitting $\hbar$ -RVFs from Lorentzian para-Sasakian manifolds onto Riemannian manifolds in this section.

Theorem 7. Assume a Lorentzian para-Sasakian manifold $(\mathcal{L}, \psi, \zeta, \theta, \delta)$ of dimension $(2r + 1)$ and a Riemannian manifold $(\mathfrak{N}, \delta_\mathfrak{N})$ of dimension s . Let $\Phi : (\mathcal{L}, \psi, \zeta, \theta, \delta) \rightarrow (\mathfrak{N}, \delta_\mathfrak{N})$ be a Ξ admitting $\hbar$ -RVFs. Then, $s = r + 1$.

Proof. Let $I_1, \dots, I_k$ be an orthonormal set of $(\mathcal{Ker}\Phi_*)$, where $k = 2r - s + 1$. As such, $\psi(\mathcal{Ker}\Phi_*) = \mathcal{Ker}\Phi_*^\perp \oplus \{\zeta\}$, $\psi I_1, \dots, \psi I_k, \zeta$ generated from an orthonormal set of $\Gamma(\mathcal{Ker}\Phi_*^\perp)$. So, from Equation (26), we get $s - 1 = k$, which implies that $r + 1 = s$. $\square$

It should be emphasized that the proof of Theorem 7 is highly analogous to the proof of Theorem 8.1 in [10], but we include it for clarification.

From Lemma 1, we get the following corollary.

Corollary 3. Let Φ be a Ξ with a $\hbar$ -RVF from a Lorentzian para-Sasakian manifold $(\mathcal{L}, \psi, \zeta, \theta, \delta)$ onto a Riemannian manifold $(\mathfrak{N}, \delta_\mathfrak{N})$. Then, we have

$$T_p \psi q = \psi T_p q, \quad (59)$$

$$T_q \psi \alpha = \psi T_q \alpha, \quad (60)$$

$$T_q \zeta = 0. \quad (61)$$

for $p, q \in \Gamma(\mathcal{Ker}\Phi_*)$ and $\alpha \in \Gamma(\mathcal{Ker}\Phi_*)$.

Proof. Equations (59) and (60) are derived by using Equations (34) and (37). The final claim in Equation (61) follows from Equations (4) and (10) or straight from Equation (44). $\square$

We can see from Equation (61) that the tensor T cannot vanish; hence, we have the following result.

Theorem 8. Let Φ be a Ξ with a $\hbar$ -RVF from a Lorentzian para-Sasakian manifold $(\mathcal{L}, \psi, \zeta, \theta, \delta)$ onto a Riemannian manifold $(\mathfrak{N}, \delta_\mathfrak{N})$. Then, the fibers of Φ are not totally geodesic.

Corollary 4. Let Φ be a Ξ with a $\hbar$ -RVF from a Lorentzian para-Sasakian manifold $(\mathcal{L}, \psi, \zeta, \theta, \delta)$ onto a Riemannian manifold $(\mathfrak{N}, \delta_\mathfrak{N})$. Then, we get

$$A_\alpha \psi q = \psi A_\alpha q, \quad (62)$$

$$A_\alpha \mathcal{P}\beta = \psi A_\alpha \beta + \delta(p, q)H_\zeta + \theta(\beta)\alpha + 2\theta(\alpha)\theta(\beta)H_\zeta, \quad (63)$$

$$A_\zeta q = \psi q. \quad (64)$$

$$A_\zeta \alpha = \psi \alpha. \quad (65)$$

for $q \in \Gamma(\text{Ker}\Phi_*)$ and $\alpha, \beta \in \Gamma(\text{Ker}\Phi_*^\perp)$.

Proof. Equations (62) and (63) are derived using Equations (38) and (41). The third claim (Equation (64)) follows from Equations (4) and (11). The final claim is obtained through Equation (65). $\square$

It is clear from Equations (62) and (63) that the tensor A cannot vanish. As a result of Remark 4, we gain the desired result.

Theorem 9. Let Φ be a Ξ with a $\hbar$ -RVF from a Lorentzian para-Sasakian manifold $(\mathcal{L}, \psi, \zeta, \theta, \delta)$ onto a Riemannian manifold $(\mathfrak{N}, \delta_\mathfrak{N})$. The horizontal distribution of Φ is hence not integrable.

Based on Remark 5 from Theorem 8 or Theorem 9, we obtain the following outcome.

Theorem 10. Let Φ be a Ξ with a $\hbar$ -RVF from a Lorentzian para-Sasakian manifold $(\mathcal{L}, \psi, \zeta, \theta, \delta)$ onto a Riemannian manifold $(\mathfrak{N}, \delta_\mathfrak{N})$. Then, the submersion Φ is not a totally geodesic map.

Lastly, we can present a finding relating to the harmonicity of Ξ .

Theorem 11. Let Φ be a Ξ with a $\hbar$ -RVF from a Lorentzian para-Sasakian manifold $(\mathcal{L}, \psi, \zeta, \theta, \delta)$ onto a Riemannian manifold $(\mathfrak{N}, \delta_\mathfrak{N})$. Then, Φ is not harmonic.

Proof. Let $\{b_1, \dots, b_k\}$ be an orthonormal set of $\text{Ker}\Phi_*$. Then, $\{\psi b_1, \dots, \psi b_k, \zeta\}$ is an orthonormal set of $\text{Ker}\Phi_*^\perp$. Thus, we arrive at

$$\sum_{j=1}^k T_{b_j} b_j = \sum_{j,l=1}^k \left\{ \delta(T_{b_j} b_j, \psi b_l) \psi b_l + \delta(T_{b_j} b_j, \zeta) \zeta \right\}.$$

Applying the skew-symmetry of T_{e_j} and Equation (59), we derive

$$\sum_{j=1}^k T_{b_j} b_j = \sum_{j,l=1}^k \left\{ \delta(\psi T_{b_j} b_l, b_j) \psi b_l - \delta(T_{b_j} \zeta, b_j) \zeta \right\}.$$

By combining Equations (7) and (61), we get

$$\sum_{j=1}^k T_{b_j} b_j = \sum_{j,l=1}^k \delta(\psi T_{b_l} b_j, b_j) \psi b_l. \quad (66)$$

Let us now consider that Φ is harmonic. Then,

$$\sum_{j=1}^k T_{b_j} b_j = 0.$$

We can conclude from Equation (66) that

$$\sum_{j,l=1}^k \delta(\psi T_{b_l} b_j, b_j) \psi b_l = 0.$$

This ensures that $\{\psi b_1, \dots, \psi b_k, \zeta\}$ are linearly independent. $\square$

Remark 6. Theorems 8, 9, and 10 also hold for a Θ allowing a $\hbar$ -RVF.

9. Some Number Theoretic Applications to Riemannian Submersions

The submersion $\Phi : \mathbb{R}^{r+m} \rightarrow \mathbb{R}^m$ is a smooth Hopf fibration [28]. Moreover, a large class of submersions are submersions between spheres of higher dimensions, such as

$$\Phi : \mathbb{S}^{r+m} \rightarrow \mathbb{S}^m$$

whose fibers have the dimension m . The Hopf fibration asserts that a fibration generalizes the idea of a fiber bundle and plays a significant role in algebraic topology [29].

Every fiber in a fibration is closely connected to the homotopy group and satisfies the homotopy property [30]. The homotopy group of spheres $\mathbb{S}^n$ essentially describes how several spheres of different dimensions may twist around one another. We can obtain the j -th homotopy group $\Phi_j(\mathbb{S}^r)$ in which the j -dimensional sphere $\mathbb{S}^j$ can be mapped continuously onto the r -dimensional sphere $\mathbb{S}^r$.

Now, we have the following remark:

Remark 7. We aim to figure out the homotopy groups for positive k using the formula $\Phi_{r+k}(\mathbb{S}^r)$. The homotopy groups $\Phi_{r+k}(\mathbb{S}^r)$ with $r > k + 1$ are known as stable homotopy groups of spheres and are denoted by $\Phi_k^{\mathbb{S}}$. They are finite abelian groups for $k \neq 0$. In view of Freudenthal's suspension Theorem [31], the groups are known as unstable homotopy groups of spheres for $r \leq k + 1$.

Now, in light of Theorem 1 and using Remark (7), we get the following outcomes.

Theorem 12. Let $(\mathcal{L}, \psi, \zeta, \theta, \delta)$ be a Lorentzian para-Sasakian manifold of dimension $(2r + 1)$, and $(\mathbb{N}, \delta_{\mathbb{N}})$ is a Riemannian manifold of dimension s . Let $\Phi : (\mathcal{L}, \psi, \zeta, \theta, \delta) \rightarrow (\mathbb{N}, \delta_{\mathbb{N}})$ be a Θ . Then, the homotopy group of Θ is $\Phi_{2r+1}(\mathbb{N}^s)$.

Example 4. Let us adopt the example from Equation (2). We have anti-invariant submersions with a $\hbar$ -RVF wherein $(\mathbb{R}^9, \psi, \zeta, \theta, \delta)$ is a Lorentzian para-Sasakian manifold.

Next, consider a mapping

$$\Phi : (\mathbb{R}^9, \psi, \zeta, \theta, \delta) \rightarrow (\mathbb{R}^5, \delta_5)$$

where δ_5 is the Riemannian metric of $\mathbb{R}^5$. Then, according to Hopf fibration (the fiber bundle), we have homotopy groups:

$$\Phi_9(\mathbb{R}^5, \psi, \zeta, \theta, \delta) = \Phi_9(\mathbb{R}^5) = \Phi_{5+4}(\mathbb{R}^5). \quad (67)$$

Therefore, the above remark entails that $r \geq k + 1$; i.e., $5 \geq 4 + 1$. Thus, the homotopy groups $\Phi_9(\mathbb{R}^5)$ are unstable homotopy groups.

Remark 8. For a prime number p , the homotopy p -exponent of a topological space $\mathcal{T}$, denoted by $\text{Exp}_p(\mathcal{T})$, is defined to be the largest $e \in \mathbb{N} = \{0, 1, 2, \dots\}$ such that some homotopy group $\Phi_j(\mathcal{T})$ has an element of order p^e . Cohen et al. [32] proved that

$$\text{Exp}_p(\mathcal{S}^{2n+1}) = n \quad \text{if } p \neq 2.$$

For a prime number p and an integer z , the p -adic order of z is given by $\text{Ord}_p(z) = \sup\{z \in \mathbb{N} : p^z | z\}$.

Example 5. In view of Equation (67) and Remark (8), the homotopy p -exponent of homotopy group $\Phi_9(\mathbb{R}^5)$ of the anti-invariant submersions Φ with $\hbar$ -RVFs is

$$\text{Exp}_9(\mathbb{R}^5) = 2, \quad p = 9, \quad p \neq 2. \quad (68)$$

Moreover, the homotopy group $\Phi_9(\mathbb{R}^5)$ of anti-invariant submersions Φ with $\hbar$ -RVFs has an element of the p -adic order 9^e , where $e \in \mathbb{N}$.

Through the above observation, in 2007, Davis and Sun proved an interesting inequality in terms of homotopy groups. For more details, see Theorem 1.1, page 2, in [33]. According to them, for any prime p and $z = 2, 3, \dots$, some homotopy group $\pi_i(SU(n))$ contains an element of order $p^{n-1+\text{Ord}_p(\lfloor n/p \rfloor!)}; i.e., the strong and elegant lower bound for the homotopy p -exponent of the homotopy group is then$

$$\text{Exp}_p(SU(n)) \geq n - 1 + \text{Ord}_p\left(\left\lfloor \frac{n}{p} \right\rfloor!\right), \quad (69)$$

where $S(U)(n)$ is a special unitary group of degree n .

Therefore, using Davis and Sun's result (Theorem 1.1 [33]) with Theorem 12, we gain an interesting inequality.

Theorem 13. For any prime number p and $s = 2, 3, \dots$, some homotopy group $\Phi_{2r+1}(\mathbb{N}^s)$ of Θ contains an element of order $p^{s-1+\text{Ord}_p(\lfloor s/p \rfloor!)}; and we get the inequality$

$$\text{Exp}_p(\Phi_{2r+1}(\mathbb{N}^s)) \geq s - 1 + \text{Ord}_p\left(\left\lfloor \frac{s}{p} \right\rfloor!\right). \quad (70)$$

Example 6. Again, considering the case of the example in Equation (4), the homotopy group of anti-invariant submersions Φ with $\hbar$ -RVFs is $\Phi_9(\mathbb{R}^5)$. Equation (13) also holds for the homotopy group $\Phi_9(\mathbb{R}^5)$ of submersion Φ such that

$$\text{Exp}_p(\Phi_9(\mathbb{R}^5)) \geq 4 + \text{Ord}_p\left(\left\lfloor \frac{5}{p} \right\rfloor!\right). \quad (71)$$

10. Conclusions

This study explored the possibility that, if an anti-invariant Riemannian submersion admits a vertical Reeb vector field, the fiber of the anti-invariant Riemannian submersion is not totally umbilical. Also, if a Lagrangian submersion admits a vertical Reeb vector field, the fiber of the Lagrangian submersion is not totally geodesic, and under the same condition, the horizontal distribution of the submersion is not integrable. In addition, this implies a necessary and sufficient condition for a Lagrangian submersion to be harmonic, and a Lagrangian submersion with a horizontal Reeb vector field is not totally geodesic. Furthermore, if a Lagrangian submersion admits a horizontal Reeb vector field, the horizontal distribution of the same submersion is not integrable. Finally, this Riemannian submersion from Lorentzian para-Sasakian manifolds can be used to construct a homotopy group and find the lower bound for the same homotopy group of the Riemannian submersion.

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Acronyms

Θ	anti-invariant Riemannian submersion
$\mathbb{E}J$	Lagrangian submersion
$v^\sharp$ -RVF	vertical Reeb vector field
h -RVF	horizontal Reeb vector field

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