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New Stability Results for Abstract Fractional Differential Equations with Delay and Non-Instantaneous Impulses

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Abstract: This research delves into the field of fractional differential equations with both non-instantaneous impulses and delay within the framework of Banach spaces. Our objective is to establish adequate conditions that ensure the existence, uniqueness, and Ulam–Hyers–Rassias stability results for our problems. The studied problems encompass abstract impulsive fractional differential problems with finite delay, infinite delay, state-dependent finite delay, and state-dependent infinite delay. To provide clarity and depth, we augment our theoretical results with illustrative examples, illustrating the practical implications of our work.

Keywords: Caputo fractional-order derivative; mild solution; impulse; delay; Ulam stability

MSC: 26A33; 34A08; 34A37; 34G20

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1. Introduction

Fractional calculus is a highly effective tool in applied mathematics, offering a means to investigate a wide range of problems in various scientific and engineering fields. In recent years, there has been significant progress in both ordinary and partial fractional differential equations. For more details on the applications of fractional calculus, the reader is directed to the books of Abbas et al. [1–3], Herrmann [4], Hilfer [5], Kilbas et al. [6], Samko et al. [7], and Zhou [8] and papers [9–15]. In [16,17], Benchohra et al. demonstrated the existence, uniqueness, and stability results for various classes of problems with different conditions and some form of extension of the well-known Hilfer fractional derivative, which unifies the Riemann–Liouville and Caputo fractional derivatives.

Ulam initially introduced the topic of stability in functional equations during a talk at Wisconsin University in 1940. The problem he presented was as follows: Under what conditions does the existence of an additive mapping near an approximately additive mapping hold? (for more details, refer to [18]). Hyers provided the first solution to Ulam's question in 1941, specifically for the case of Banach spaces [19]. Subsequently, this type of stability became known as Ulam–Hyers stability. In 1978, Rassias introduced a notable extension of the Ulam–Hyers stability by taking into account variables [20]. The concept of stability in functional equations arises when the original equation is replaced by an inequality, serving as a perturbation. Hence, the issue of stability in functional equations revolves around the disparity between the solutions of the inequality and those of the given functional equation. Considerable attention has been devoted to investigating Ulam–Hyers and Ulam–Hyers–Rassias stability in various forms of functional equations, as discussed in the monographs by [21,22]. Ulam–Hyers stability in operatorial equations and inclusions has been examined by Bota-Boriceanu and Petrusel [23], Petru et al. [24], and Rus [25,26]. Castro and Ramos [27] explored Hyers–Ulam–Rassias stability for a specific class of Volterra integral equations. Wang et al. [28,29] proposed Ulam stability for fractional differential equations involving the Caputo derivative. For further historical insights and recent developments with respect to these stabilities, consult monographs [21,22,30] and papers by [25,28–31].

The study of differential equations with impulses was initially explored by Milman and Myshkis [32]. In several fields, such as physics, chemical technology, population dynamics, and natural sciences, numerous phenomena and evolutionary processes can undergo sudden changes or short-term disturbances [33] (and references therein). These brief disturbances can be interpreted as impulses. Impulsive problems also arise in various practical applications, including communications, chemical technology, mechanics (involving jump discontinuities in velocity), electrical engineering, medicine, and biology. These perturbations can be perceived as impulses. For instance, in the periodic treatment of certain diseases, impulses correspond to the administration of drug treatment. In environmental sciences, impulses represent seasonal changes in water levels in artificial reservoirs. Mathematical models involving impulsive differential equations and inclusions are used to describe these situations. Several mathematical results, such as the existence of solutions and their asymptotic behavior, have been obtained thus far [33–37] (and references therein).

In [38], the authors discussed the following second-order integrodifferential equations with state-dependent delay described in the following form:

$$\begin{cases} x''(\vartheta) = A(\vartheta)x(\vartheta) + \mathcal{K}\Big(\vartheta, x_{\rho(\vartheta,x_\vartheta)}, (\Psi x)(\vartheta)\Big) + \int_0^\vartheta \Upsilon(\vartheta,s)x(s)ds + \mathcal{P}u(\vartheta), \text{ if } \vartheta \in J, \\ x'(0) = \zeta_0 \in E, \;\; x(\vartheta) = \Phi(\vartheta), \text{ if } \vartheta \in \mathbb{R}_-, \end{cases}$$

where $J = [0,T]$, $A(\vartheta) : D(A(\vartheta)) \subset E \to E$, $\Upsilon(\vartheta,s)$ are closed linear operators on E, with a dense domain ($D(A(\vartheta))$), which is independent of ϑ, and $D(A(s)) \subset D(\Upsilon(\vartheta,s))$, the operator ($\Psi$), is defined by

$$(\Psi x)(\vartheta) = \int_0^T \Xi(\vartheta, s, x(s))ds,$$

The nonlinear terms $\Xi : J \times J \times E \to E$, $\mathcal{K} : J \times \mathbb{B} \times E \to E$, $\Phi : \mathbb{R}_- \to E$, $\rho : J \times \mathbb{B} \to (-\infty,\infty)$ are expressed by functions. The control function (u) is expressed by the function $L^2(J,U)$. The Banach space of admissible controls with U is expressed as a Banach space. $\mathcal{P}$ is a bounded linear operator from U into E, and $(E, \|\cdot\|)$ is a Banach space.

In [39–41], the authors studied some new classes of differential equations with non-instantaneous impulses. For more recent results we refer, for instance, to [42] and papers [43–46]. Motivated by the mentioned works, by using the Banach fixed-point theorem, we investigate the existence, uniqueness, and Ulam–Hyers–Rassias stability of the following abstract impulsive fractional differential equations with finite delay in the following form:

$$\begin{cases} {}^cD^\alpha_{\delta_\jmath} x(\vartheta) = Ax(\vartheta) + \Psi(\vartheta, x_\vartheta); \text{ if } \vartheta \in J_\jmath, \ \jmath = 0, \dots, \omega, \\ x(\vartheta) = \widehat{\Psi}_\jmath(\vartheta, x(\vartheta)); \text{ if } \vartheta \in \widehat{J}_\jmath, \ \jmath = 1, \dots, \omega, \\ x(\vartheta) = \phi(\vartheta); \text{ if } \vartheta \in [-\kappa_2, 0], \end{cases} \tag{1}$$

where $J_0 := [0, \vartheta_1]$, $\widehat{J}_\jmath := (\vartheta_\jmath, \delta_\jmath]$, $J_\jmath := (\delta_\jmath, \vartheta_{\jmath+1}]$; $\jmath = 1, \dots, \omega$, ${}^cD^\alpha_{\delta_\jmath}$ is the fractional Caputo derivative of order $\alpha \in (0,1]$, $0 = \delta_0 < \vartheta_1 \le \delta_1 \le \vartheta_2 < \cdots < \delta_{\omega-1} \le \vartheta_\omega \le \delta_\omega \le \vartheta_{\omega+1} = \kappa_1$, $\kappa_2, \kappa_1 > 0$, $\Psi : J_\jmath \times \mathcal{C} \to E$; $\jmath = 0, \dots, \omega$, $\widehat{\Psi}_\jmath : \widehat{J}_\jmath \times E \to E$; $\jmath = 1, \dots, \omega$, $\phi : [-\kappa_2, 0] \to E$ are given piecewise continuous functions, E is a Banach space, A is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $\{B(\vartheta);\ \vartheta > 0\}$ in E, and $\mathcal{C}$ is the Banach space defined by

$$\mathcal{C} = C_{\kappa_2} = \{x : [-\kappa_2, 0] \to E : \text{ continuous and there exist } \varepsilon_\jmath \in (-\kappa_2, 0);\ \jmath = 1, \ldots, \omega, \text{ such that } x(\varepsilon_\jmath^-) \text{ and } x(\varepsilon_\jmath^+) \text{ exist with } x(\varepsilon_\jmath^-) = x(\varepsilon_\jmath)\},$$

with the norm

$$\|x\|_{\mathcal{C}} = \sup_{\vartheta \in [-\kappa_2, 0]} \|x(\vartheta)\|_E.$$

x_ϑ denotes the element of $\mathcal{C}$ defined by

$$x_\vartheta(\varepsilon) = x(\vartheta + \varepsilon);\ \varepsilon \in [-\kappa_2, 0],$$

where $x_\vartheta(\cdot)$ represents the history of the state from time $\vartheta - \kappa_2$ to the present time (ϑ).

Next, as a continuation of [36,46], we consider the following abstract impulsive fractional differential equations with infinite delay in the following form:

$$\begin{cases} {}^cD^\alpha_{\delta_\jmath} x(\vartheta) = Ax(\vartheta) + \Psi(\vartheta, x_\vartheta);\ \text{if } \vartheta \in J_\jmath,\ \jmath = 0, \ldots, \omega, \\ x(\vartheta) = \widehat{\Psi}_\jmath(\vartheta, x(\vartheta));\ \text{if } \vartheta \in \widehat{J}_\jmath,\ \jmath = 1, \ldots, \omega, \\ x(\vartheta) = \phi(\vartheta);\ \text{if } \vartheta \in \mathbb{R}_- := (-\infty, 0], \end{cases} \tag{2}$$

where A and $\widehat{\Psi}_\jmath;\ \jmath = 1, \ldots, \omega$ are as in problem (1); $\Psi : J_\jmath \times \Bbbk \to E;\ \jmath = 0, \ldots, \omega,\ \phi : \mathbb{R}_- \to E$ are expressed as piecewise continuous functions; and $\Bbbk$ is a phase space specified in Section 4. This particular problem has more requirements and involves the incorporation of new concepts, specifically the inclusion of the phase space and its associated characteristics. Through the utilization of the Banach fixed point theorem and by employing the properties of the phase space, we thoroughly explore and establish results pertaining to existence, uniqueness, and Ulam–Hyers–Rassias stability.

The third problem is the abstract impulsive fractional differential equations with state-dependent delay of the following form:

$$\begin{cases} {}^cD^\alpha_{\delta_\jmath} x(\vartheta) = Ax(\vartheta) + \Psi(\vartheta, x_{\rho(\vartheta, x_\vartheta)});\ \text{if } \vartheta \in J_\jmath,\ \jmath = 0, \ldots, \omega, \\ x(\vartheta) = \widehat{\Psi}_\jmath(\vartheta, x(\vartheta));\ \text{if } \vartheta \in \widehat{J}_\jmath,\ \jmath = 1, \ldots, \omega, \\ x(\vartheta) = \phi(\vartheta);\ \text{if } \vartheta \in [-\kappa_2, 0], \end{cases} \tag{3}$$

where $A,\ \Psi,\ \phi$ and $\widehat{\Psi}_\jmath;\ \jmath = 1, \ldots, \omega$ are as in problem (1), and $\rho : J_\jmath \times \mathcal{C} \to \mathbb{R};\ \jmath = 0, \ldots, \omega$, is expressed as a piecewise continuous function.

Finally, we consider the abstract impulsive fractional differential equations with state-dependent delay in the following form:

$$\begin{cases} {}^cD^\alpha_{\delta_\jmath} x(\vartheta) = Ax(\vartheta) + \Psi(\vartheta, x_{\rho(\vartheta, x_\vartheta)});\ \text{if } \vartheta \in J_\jmath,\ \jmath = 0, \ldots, \omega, \\ x(\vartheta) = \widehat{\Psi}_\jmath(\vartheta, x(\vartheta));\ \text{if } \vartheta \in \widehat{J}_\jmath,\ \jmath = 1, \ldots, \omega, \\ x(\vartheta) = \phi(\vartheta);\ \text{if } \vartheta \in \mathbb{R}_-, \end{cases} \tag{4}$$

where A, Ψ, ϕ and $\widehat{\Psi}_\jmath;\ \jmath = 1, \ldots, \omega$ are as in problem (2), and $\rho : J_\jmath \times \Bbbk \to \mathbb{R};\ \jmath = 0, \ldots, \omega$, is expressed as a piecewise continuous function.

The paper is organized as follows. In Section 2, we commence by introducing essential notations and offering a review of preliminary concepts concerning fractional calculus, Ulam stability, and various auxiliary findings. Section 3 is dedicated to establishing the existence and uniqueness of mild solutions, utilizing the Banach fixed-point theorem. Additionally, we explore Ulam stability for the problem (1). Section 4 offers an in-depth analysis of the phase space, presenting crucial properties and associated observations. Within Section 5, our focus shifts to the existence, uniqueness, and stability results for problem (2). In Section 6, we present the uniqueness and Ulam stability results for problems (3) and (4). Finally, the concluding section is devoted to presenting a collection of examples that illustrate the concepts discussed throughout the paper.

2. Preliminaries

Let $J = [0, \kappa_1]$; $\kappa_1 > 0$, denote $L^1(J)$ the space of Bochner-integrable functions ($x : J \to E$) with the norm

$$\|x\|_{L^1} = \int_0^{\kappa_1} \|x(\vartheta)\|_E d\vartheta,$$

where $\|\cdot\|_E$ denotes a norm on E.

As usual, $AC(J)$ denotes the space of absolutely continuous functions from J to E, and $\mathcal{C} := C(J)$ is the Banach space of all continuous functions from J to E, with the norm $\|.\|_\infty$ defined by

$$\|x\|_\infty = \sup_{\vartheta \in J} \|x(\vartheta)\|_E.$$

Consider that Banach space

$$\begin{aligned} PC = \{ & x : [-\kappa_2, \kappa_1] \to E : x|_{[-\kappa_2,0]} = \phi,\ x|_{\widehat{J}_J} = \widehat{\Psi}_J;\ J = 1, \ldots, \omega,\ x|_{J_J};\ J = 1, \ldots, \omega \\ & \text{is continuous and there exist } x(\delta_J^-),\ x(\delta_J^+),\ x(\vartheta_J^-) \text{ and } x(\vartheta_J^+) \\ & \text{with } x(\delta_J^+) = \widehat{\Psi}_J(\delta_J, x(\delta_J)) \text{ and } x(\vartheta_J^-) = \widehat{\Psi}_J(\vartheta_J, x(\vartheta_J)) \}, \end{aligned}$$

with the norm

$$\|x\|_{PC} = \sup_{\vartheta \in [-\kappa_2, \kappa_1]} \|x(\vartheta)\|_E.$$

Let $\alpha > 0$, for $x \in L^1(J)$. The expression

$$(I_0^\alpha x)(\vartheta) = \frac{1}{\Gamma(\alpha)} \int_0^\vartheta (\vartheta - \varepsilon)^{\alpha - 1} x(\varepsilon) d\varepsilon,$$

is called the left-sided mixed Riemann–Liouville integral of order α, where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by $\Gamma(\varsigma) = \int_0^\infty \vartheta^{\varsigma - 1} e^{-\vartheta} d\vartheta$; $\varsigma > 0$.

In particular,

$$(I_0^0 x)(\vartheta) = x(\vartheta),\ (I_0^1 x)(\vartheta) = \int_0^\vartheta x(\varepsilon) d\varepsilon;\ \ \vartheta \in J.$$

For instance, $I_0^\alpha x$ exists for all $\alpha \in (0, \infty)$, when $x \in L^1(J)$. Note also that when $x \in C(J)$; then, $(I_0^\alpha x) \in C(J)$.

Definition 1 ([2,7])**.** *Let $\alpha \in (0,1]$ and $x \in L^1(J)$. The Caputo fractional-order derivative of order α of x is expressed by*

$${}^c D_0^\alpha x(\vartheta) = (I_0^{1-\alpha} \frac{d}{d\vartheta} x)(\vartheta) = \frac{1}{\Gamma(1-\alpha)} \int_0^\vartheta (\vartheta - \varepsilon)^{-\alpha} \frac{d}{d\varepsilon} x(\varepsilon) d\varepsilon.$$

Example 1. *Let $\varpi \in (-1, 0) \cup (0, \infty)$ and $\alpha \in (0, 1]$; then,*

$${}^c D_0^\alpha \frac{\vartheta^\varpi}{\Gamma(1 + \varpi)} = \frac{\vartheta^{\varpi - \alpha}}{\Gamma(1 + \varpi - \alpha)};\ \text{for almost all } \vartheta \in J.$$

Let $a_1 \in [0, \kappa_1]$, $\widehat{J}_1 = (a_1, \kappa_1]$, $\alpha > 0$. For $x \in L^1(\widehat{J}_1)$, the expression

$$(I_{\kappa_1^+}^\alpha x)(\vartheta) = \frac{1}{\Gamma(\alpha)} \int_{a_1^+}^\vartheta (\vartheta - \varepsilon)^{\alpha - 1} x(\varepsilon) d\varepsilon,$$

is called the left-sided mixed Riemann–Liouville integral of order α of x. See [2,7] for more details.

Definition 2 ([2,7])**.** *For $x \in L^1(\widehat{J}_1)$, where $\frac{d}{d\vartheta}x$ is Bochner-integrable on $\widehat{J}_1$, the Caputo fractional-order derivative of order α of x is defined by the following expression:*

$$({}^cD^{\alpha}_{\kappa_1+}x)(\vartheta) = (I^{1-\alpha}_{\kappa_1+}\frac{d}{d\vartheta}x)(\vartheta).$$

Definition 3 ([47])**.** *A function ($x : [-\kappa_2, \kappa_1] \to E$) is said to be a mild solution of (1) if x satisfies*

$$\begin{cases} x(\vartheta) = \mathfrak{F}_\alpha(\vartheta)\phi(0) + \int_0^{\vartheta}(\vartheta-\varepsilon)^{\alpha-1}B_\alpha(\vartheta-\varepsilon)\Psi(\varepsilon, x_\varepsilon)d\varepsilon; \ \text{if } \vartheta \in [0, \vartheta_1], \\ x(\vartheta) = \mathfrak{F}_\alpha(\vartheta-\delta_\jmath)\widehat{\Psi}_\jmath(\delta_\jmath, x(\delta_\jmath)) \\ + \int_{\delta_\jmath}^{\vartheta}(\vartheta-\varepsilon)^{\alpha-1}B_\alpha(\vartheta-\varepsilon)\Psi(\varepsilon, x_\varepsilon)d\varepsilon; \ \text{if } \vartheta \in J_\jmath, \ \jmath = 1, \dots, \omega, \\ x(\vartheta) = \widehat{\Psi}_\jmath(\vartheta, x(\vartheta)); \ \text{if } \vartheta \in \widehat{J}_\jmath, \ \jmath = 1, \dots, \omega, \\ x(\vartheta) = \phi(\vartheta); \ \text{if } \vartheta \in [-\kappa_2, 0], \end{cases}$$

where

$$\mathfrak{F}_\alpha(\vartheta) = \int_0^\infty \mu_\alpha(\eta)B(\vartheta^\alpha\eta)d\eta, \ B_\alpha(\vartheta) = \alpha\int_0^\infty \eta\mu_\alpha(\eta)B(\vartheta^\alpha\eta)d\eta, \ \mu_\alpha(\eta) = \frac{1}{\alpha}\eta^{-1-\frac{1}{\alpha}}\overline{\tau}_\alpha(\eta^{-\frac{1}{\alpha}}) \geq 0,$$

and

$$\overline{\tau}_\alpha(\eta) = \frac{1}{\pi}\sum_{\imath=0}^{\infty}(-1)^{\imath-1}\eta^{-\imath\alpha-1}\frac{\Gamma(\imath\alpha+1)}{\imath!}\sin(\imath\alpha\pi); \ \eta \in (0, \infty).$$

μ_α is a probability density function on $(0, \infty)$, that is $\int_0^\infty \mu_\alpha(\eta)d\eta = 1$.

Remark 1. *We can deduce that for $\varkappa \in [0, 1]$, we have*

$$\int_0^\infty \eta^{\varkappa}\mu_\alpha(\eta)d\eta = \int_0^\infty \eta^{-\alpha\varkappa}\overline{\tau}_\alpha(\eta)d\eta = \frac{\Gamma(1+\varkappa)}{\Gamma(1+\alpha\varkappa)}.$$

Lemma 1 ([47])**.** *For any $\vartheta \geq 0$, the operators $\mathfrak{F}_\alpha(\vartheta)$ and $B_\alpha(\vartheta)$ have the following properties:*

(*a*) *For $\vartheta \geq 0$, $\mathfrak{F}_\alpha$ and B_α are linear and bounded operators, ie., for any $x \in E$,*

$$\|\mathfrak{F}_\alpha(\vartheta)x\|_E \leq \Delta\|x\|_E, \ \|B_\alpha(\vartheta)x\|_E \leq \frac{\Delta}{\Gamma(\alpha)}\|x\|_E;$$

(*b*) *$\{\mathfrak{F}_\alpha(\vartheta); \ \vartheta \geq 0\}$ and $\{B_\alpha(\vartheta); \ \vartheta \geq 0\}$ are strongly continuous;*

(*c*) *For every $\vartheta \geq 0$, $\mathfrak{F}_\alpha(\vartheta)$ and $B_\alpha(\vartheta)$ are also compact operators.*

Now, we consider the Ulam stability for (1). Let $\upsilon > 0$, $\mathcal{Y} \geq 0$ and $\mathcal{Z} : J \to [0, \infty)$ be a continuous function. Let

$$\begin{cases} \|x(\vartheta) - \mathfrak{F}_\alpha(\vartheta)\phi(0) - \int_0^{\vartheta}(\vartheta-\varepsilon)^{\alpha-1}B_\alpha(\vartheta-\varepsilon)\Psi(\varepsilon, x_\varepsilon)d\varepsilon\|_E \leq \upsilon; \ \text{if } \vartheta \in [0, \vartheta_1], \\ \|x(\vartheta) - \mathfrak{F}_\alpha(\vartheta-\delta_\jmath)\widehat{\Psi}_\jmath(\delta_\jmath, x(\delta_\jmath)) \\ - \int_{\delta_\jmath}^{\vartheta}(\vartheta-\varepsilon)^{\alpha-1}B_\alpha(\vartheta-\varepsilon)\Psi(\varepsilon, x_\varepsilon)d\varepsilon\|_E \leq \upsilon; \ \text{if } \vartheta \in J_\jmath, \ \jmath = 1, \dots, \omega, \\ \|x(\vartheta) - \widehat{\Psi}_\jmath(\vartheta, x(\vartheta))\|_E \leq \upsilon; \ \text{if } \vartheta \in \widehat{J}_\jmath, \ \jmath = 1, \dots, \omega. \end{cases} \tag{5}$$

$$\begin{cases} \|x(\vartheta) - \mathfrak{F}_\alpha(\vartheta)\phi(0) - \int_0^\vartheta (\vartheta-\varepsilon)^{\alpha-1} B_\alpha(\vartheta-\varepsilon)\Psi(\varepsilon, x_\varepsilon)d\varepsilon\|_E \le \mathcal{Z}(\vartheta); \text{ if } \vartheta \in [0, \vartheta_1], \\ \|x(\vartheta) - \mathfrak{F}_\alpha(\vartheta - \delta_\jmath)\widehat{\Psi}_\jmath(\delta_\jmath, x(\delta_\jmath)) \\ - \int_{\delta_\jmath}^\vartheta (\vartheta-\varepsilon)^{\alpha-1} B_\alpha(\vartheta-\varepsilon)\Psi(\varepsilon, x_\varepsilon)d\varepsilon\|_E \le \mathcal{Z}(\vartheta); \text{ if } \vartheta \in J_\jmath,\ \jmath = 1, \dots, \omega, \\ \|x(\vartheta) - \widehat{\Psi}_\jmath(\vartheta, x(\vartheta))\|_E \le \mathcal{Y}; \text{ if } \vartheta \in \widehat{J}_\jmath,\ \jmath = 1, \dots, \omega. \end{cases} \tag{6}$$

$$\begin{cases} \|x(\vartheta) - \mathfrak{F}_\alpha(\vartheta)\phi(0) - \int_0^\vartheta (\vartheta-\varepsilon)^{\alpha-1} B_\alpha(\vartheta-\varepsilon)\Psi(\varepsilon, x_\varepsilon)d\varepsilon\|_E \le \upsilon\mathcal{Z}(\vartheta); \text{ if } \vartheta \in [0, \vartheta_1], \\ \|x(\vartheta) - \mathfrak{F}_\alpha(\vartheta - \delta_\jmath)\widehat{\Psi}_\jmath(\delta_\jmath, x(\delta_\jmath)) \\ - \int_{\delta_\jmath}^\vartheta (\vartheta-\varepsilon)^{\alpha-1} B_\alpha(\vartheta-\varepsilon)\Psi(\varepsilon, x_\varepsilon)d\varepsilon\|_E \le \upsilon\mathcal{Z}(\vartheta); \text{ if } \vartheta \in J_\jmath,\ \jmath = 1, \dots, \omega, \\ \|x(\vartheta) - \widehat{\Psi}_\jmath(\vartheta, x(\vartheta))\|_E \le \upsilon\mathcal{Y}; \text{ if } \vartheta \in \widehat{J}_\jmath,\ \jmath = 1, \dots, \omega. \end{cases} \tag{7}$$

Definition 4 ([9,25]). *Problem (1) is Ulam–Hyers stable if there exists a real number ($c_{\Psi,\widehat{\Psi}_\jmath} > 0$) such that for each $\upsilon > 0$ and for each solution($x \in PC$) of the inequalities (5), there exists a mild solution ($\varkappa \in PC$) of problem (1) with*

$$\|x(\vartheta) - \varkappa(\vartheta)\|_E \le \upsilon c_{\Psi,\widehat{\Psi}_\jmath};\ \vartheta \in J.$$

Definition 5 ([9,25]). *Problem (1) is generalized Ulam–Hyers stable if there exists $\eta_{\Psi,\widehat{\Psi}_\jmath} : C([0,\infty),[0,\infty))$ with $\eta_{\Psi,\widehat{\Psi}_\jmath}(0) = 0$ such that for each $\upsilon > 0$ and for each solution ($x \in PC$) of the inequalities (5), there exists a mild solution ($\varkappa \in PC$) of problem (1) with*

$$\|x(\vartheta) - \varkappa(\vartheta)\|_E \le \eta_{\Psi,\widehat{\Psi}_\jmath}(\upsilon);\ \vartheta \in J.$$

Definition 6 ([9,25]). *Problem (1) is Ulam–Hyers–Rassias stable with respect to $(\mathcal{Z}, \mathcal{Y})$ if there exists a real number ($c_{\Psi,\widehat{\Psi}_\jmath,\mathcal{Z}} > 0$) such that for each $\upsilon > 0$ and for each solution ($x \in PC$) of the inequalities (7), there exists a mild solution ($\varkappa \in PC$) of problem (1) with*

$$\|x(\vartheta) - \varkappa(\vartheta)\|_E \le \upsilon c_{\Psi,\widehat{\Psi}_\jmath,\mathcal{Z}}(\mathcal{Y} + \mathcal{Z}(\vartheta));\ \vartheta \in J.$$

Definition 7 ([9,25]). *Problem (1) is generally Ulam–Hyers–Rassias stable with respect to $(\mathcal{Z}, \mathcal{Y})$ if there exists a real number ($c_{\Psi,\widehat{\Psi}_\jmath,\mathcal{Z}} > 0$) such that for each solution ($x \in PC$) of the inequalities (6), there exists a mild solution $\varkappa \in PC$ of problem (1) with $\|x(\vartheta) - \varkappa(\vartheta)\|_E \le c_{\Psi,\widehat{\Psi}_\jmath,\mathcal{Z}}(\mathcal{Y} + \mathcal{Z}(\vartheta))$; $\vartheta \in J$.*

Remark 2. *It is clear that (i) Definition 4 $\Rightarrow$ Definition 5, (ii) Definition 6 $\Rightarrow$ Definition 7, (iii) Definition 6 for $\mathcal{Z}(\cdot) = \mathcal{Y} = 1 \ \Rightarrow$ Definition 4.*

Remark 3. *A function ($x \in PC$) is a solution of the inequalities (6) if and only if there exist a function ($G \in PC$) and a sequence ($G_\jmath$; $\jmath = 1, \dots, \omega$ in E) (which depend on x) such that:*
(i) $\|G(\vartheta)\|_E \le \mathcal{Z}(\vartheta)$ *and* $\|G_\jmath\|_E \le \mathcal{Y}$; $\jmath = 1, \dots, \omega$,

(*ii*) *The function* $x \in PC$ *satisfies*

$$\begin{cases} x(\vartheta) = G(\vartheta) + \mathfrak{F}_\alpha(\vartheta)\phi(0) + \int_0^\vartheta (\vartheta-\varepsilon)^{\alpha-1} B_\alpha(\vartheta-\varepsilon)\Psi(\varepsilon, x_\varepsilon)d\varepsilon; \;\; if \;\; \vartheta \in [0, \vartheta_1], \\ x(\vartheta) = G(\vartheta) + \mathfrak{F}_\alpha(\vartheta-\delta_\jmath)\widehat{\Psi}_\jmath(\delta_\jmath, x(\delta_\jmath)) \\ + \int_{\delta_\jmath}^\vartheta (\vartheta-\varepsilon)^{\alpha-1} B_\alpha(\vartheta-\varepsilon)\Psi(\varepsilon, x_\varepsilon)d\varepsilon; \;\; if \;\; \vartheta \in J_\jmath, \; \jmath = 1, \ldots, \omega, \\ x(\vartheta) = G_\jmath + \widehat{\Psi}_\jmath(\vartheta, x(\vartheta)); \;\; if \; \vartheta \in \widehat{J}_\jmath, \; \jmath = 1, \ldots, \omega. \end{cases}$$

Lemma 2 ([48]). *Suppose* $\beta > 0$, $a(\vartheta)$ *is a non-negative function locally integrable on* $0 \leq \vartheta < T$ *(some* $T \leq +\infty$*) and* $\widehat{\Psi}(\vartheta)$ *is a non-negative, non-decreasing continuous function defined as* $0 \leq \vartheta < T$, $\widehat{\Psi}(\vartheta) \leq \Delta$ *(constant), and suppose that* $x(\vartheta)$ *is non-negative and locally integrable on* $0 \leq \vartheta < T$ *with*

$$x(\vartheta) \leq a(\vartheta) + \widehat{\Psi}(\vartheta)\int_0^\vartheta (\vartheta-\delta)^{\beta-1} x(\delta)d\delta$$

on this interval. Then,

$$x(\vartheta) \leq a(\vartheta) + \int_0^\vartheta \left[\sum_{\imath=1}^{\infty} \frac{(\widehat{\Psi}(\vartheta)\Gamma(\beta))^\imath}{\Gamma(\imath\beta)}(\vartheta-\delta)^{\imath\beta-1}a(\delta)\right]d\delta, \; 0 \leq \vartheta < T.$$

3. Uniqueness and Ulam Stability Results with Finite Delay

Theorem 1. *Given that the following assumptions are satisfied:*

(H_1) *Semigroup* $B(\vartheta)$ *is compact for* $\vartheta > 0$;

(H_2) *For each* $\vartheta \in J_\jmath$; $\jmath = 0, \ldots, \omega$, *the function* $\Psi(\vartheta, \cdot) : E \to E$ *is continuous, and for each* $\varkappa \in \mathcal{C}$, *the function* $\Psi(\cdot, \varkappa) : J_\jmath \to E$ *is measurable;*

(H_3) *There exists a constant (*$l_\Psi > 0$*) such that*

$$\|\Psi(\vartheta, x) - \Psi(\vartheta, \overline{x})\|_E \leq l_\Psi\|x - \overline{x}\|_{\mathcal{C}}, \text{ for each } \vartheta \in J_\jmath;\ \jmath = 0, \ldots, \omega, \text{and each } x, \overline{x} \in \mathcal{C};$$

(H_4) *There exist constants (*$0 < l_{\widehat{\Psi}_\jmath} < 1$; $\jmath = 1, \ldots, \omega$,*) such that*

$$\|\widehat{\Psi}_\jmath(\vartheta, x) - \widehat{\Psi}_\jmath(\vartheta, \overline{x})\|_E \leq l_{\widehat{\Psi}_\jmath}\|x - \overline{x}\|_E,$$

for each $\vartheta \in \widehat{J}_\jmath$, *and each* $x, \overline{x} \in E$, $\jmath = 1, \ldots, \omega$.

If

$$\ell := \Delta l_{\widehat{\Psi}} + \frac{\Delta l_\Psi \kappa_1{}^\alpha}{\Gamma(\alpha)} < 1, \tag{8}$$

where $l_{\widehat{\Psi}} = \max_{\jmath=1,\ldots,\omega} l_{\widehat{\Psi}_\jmath}$, *then problem (1) has a unique mild solution on* $[-\kappa_2, \kappa_1]$.

Furthermore, if the following hypothesis holds:

(H_5) *There exists* $\varpi_{\mathcal{Z}} > 0$ *such that for each* $\vartheta \in J$, *we have*

$$\int_{\delta_\jmath}^\vartheta \left[\sum_{\imath=1}^{\infty} \frac{(\Delta l_\Psi)^\imath}{(1 - \Delta l_{\widehat{\Psi}})^\imath \Gamma(\imath\alpha)}(\vartheta-\varepsilon)^{\imath\alpha-1}\mathcal{Z}(\varepsilon)\right]d\varepsilon \leq \varpi_{\mathcal{Z}}\mathcal{Z}(\vartheta);\ \jmath = 0, \ldots, \omega,$$

then problem (1) is generalized Ulam–Hyers–Rassias stable.

Proof. Consider the operator ($\aleph : PC \to PC$) defined by

$$\begin{cases} (\aleph x)(\vartheta) = \mathfrak{F}_\alpha(\vartheta)\phi(0) + \int_0^\vartheta (\vartheta-\varepsilon)^{\alpha-1} B_\alpha(\vartheta-\varepsilon)\Psi(\varepsilon, x(\varepsilon))d\varepsilon; \text{ if } \vartheta \in [0,\vartheta_1], \\ (\aleph x)(\vartheta) = \mathfrak{F}_\alpha(\vartheta-\delta_\jmath)\widehat{\Psi}_\jmath(\delta_\jmath, x(\delta_\jmath)) \\ + \int_{\delta_\jmath}^\vartheta (\vartheta-\varepsilon)^{\alpha-1} B_\alpha(\vartheta-\varepsilon)\Psi(\varepsilon, x(\varepsilon))d\varepsilon; \text{ if } \vartheta \in J_\jmath,\ \jmath = 1,\ldots,\omega, \\ (\aleph x)(\vartheta) = \widehat{\Psi}_\jmath(\vartheta, x(\vartheta)); \text{ if } \vartheta \in \widehat{J}_\jmath,\ \jmath = 1,\ldots,\omega, \\ (\aleph x)(\vartheta) = \phi(\vartheta); \text{ if } \vartheta \in [-\kappa_2, 0], \end{cases}$$

Clearly, the fixed points of the operator ($\aleph$) are a solution to problem (1).

Let $x, \varkappa \in PC$; then, for each $\vartheta \in J$, we have

$$\begin{cases} \|(\aleph x)(\vartheta) - (\aleph\varkappa)(\vartheta)\|_E \le \| \int_0^\vartheta (\vartheta-\varepsilon)^{\alpha-1} B_\alpha(\vartheta-\varepsilon) \\ \times[\Psi(\varepsilon, x_\varepsilon) - \Psi(\varepsilon, \varkappa_\varepsilon)]d\varepsilon\|_E; \text{ if } \vartheta \in [0,\vartheta_1], \\ \|(\aleph x)(\vartheta) - (\aleph\varkappa)(\vartheta)\|_E \le \|\mathfrak{F}_\alpha(\vartheta-\delta_\jmath)(\widehat{\Psi}_\jmath(\delta_\jmath, x(\delta_\jmath)) - \widehat{\Psi}_\jmath(\delta_\jmath, \varkappa(\delta_\jmath)))\|_E \\ +\| \int_{\delta_\jmath}^\vartheta (\vartheta-\varepsilon)^{r_1-1} B_\alpha(\vartheta-\delta_\jmath)[\Psi(\varepsilon, x_\varepsilon) - \Psi(\varepsilon, \varkappa_\varepsilon)]d\varepsilon\|_E; \text{ if } \vartheta \in J_\jmath,\ \jmath = 1,\ldots,\omega, \\ \|(\aleph x)(\vartheta) - (\aleph\varkappa)(\vartheta)\|_E = \|\widehat{\Psi}_\jmath(\vartheta, x(\vartheta)) - \widehat{\Psi}_\jmath(\vartheta, \varkappa(\vartheta))\|_E; \text{ if } \vartheta \in \widehat{J}_\jmath,\ \jmath = 1,\ldots,\omega. \end{cases}$$

Thus, we obtain

$$\begin{cases} \|(\aleph x)(\vartheta) - (\aleph\varkappa)(\vartheta)\|_E \le \int_0^\vartheta (\vartheta-\varepsilon)^{\alpha-1} l_\Psi \|B_\alpha(\vartheta-\varepsilon)(x_\varepsilon - \varkappa_\varepsilon)\|_C d\varepsilon; \\ \le \frac{\Delta l_\Psi \kappa_1{}^\alpha}{\Gamma(\alpha)} \|x-\varkappa\|_{PC}; \text{ if } \vartheta \in [0,\vartheta_1], \\ \|(\aleph x)\vartheta - (\aleph\varkappa)\vartheta\|_E \le l_{\widehat{\Psi}} \|\mathfrak{F}_\alpha(\vartheta-\delta_\jmath)(x(\vartheta) - \varkappa(\vartheta))\|_E \\ + \int_{\delta_\jmath}^\vartheta (\vartheta-\varepsilon)^{\alpha-1} l_\Psi \|B_\alpha(\vartheta-\varepsilon)(x_\varepsilon - \varkappa_\varepsilon)\|_C d\varepsilon \\ \le \left(\Delta l_{\widehat{\Psi}} + \frac{\Delta l_\Psi \kappa_1{}^\alpha}{\Gamma(\alpha)}\right) \|x-\varkappa\|_{PC}; \text{ if } \vartheta \in J_\jmath,\ \jmath = 1,\ldots,\omega, \\ \|(\aleph x)(\vartheta) - (\aleph\varkappa)(\vartheta)\|_E \le l_{\widehat{\Psi}} \|x-\varkappa\|_{PC}; \text{if } \vartheta \in \widehat{J}_\jmath,\ \jmath = 1,\ldots,\omega. \end{cases}$$

Hence,

$$\|\aleph(x) - \aleph(\varkappa)\|_{PC} \le \ell \|x-\varkappa\|_{PC}.$$

Based on (8), it can be deduced that $\aleph$ has contraction properties. Consequently, according to Banach's fixed-point theorem, it follows that $\aleph$ possesses a unique fixed point ($\varkappa$), which is a mild solution to (1). Then, we have

$$\begin{cases} \varkappa(\vartheta) = \mathfrak{F}_\alpha(\vartheta)\phi(0) + \int_0^\vartheta (\vartheta-\varepsilon)^{\alpha-1} B_\alpha(\vartheta-\varepsilon)\Psi(\varepsilon, \varkappa_\varepsilon)d\varepsilon; \text{ if } \vartheta \in [0,\vartheta_1], \\ \varkappa(\vartheta) = \mathfrak{F}_\alpha(\vartheta-\delta_\jmath)\widehat{\Psi}_\jmath(\delta_\jmath, \varkappa(\delta_\jmath)) \\ + \int_{\delta_\jmath}^\vartheta (\vartheta-\varepsilon)^{\alpha-1} B_\alpha(\vartheta-\varepsilon)\Psi(\varepsilon, \varkappa_\varepsilon)d\varepsilon; \text{ if } \vartheta \in J_\jmath,\ \jmath = 1,\ldots,\omega, \\ \varkappa(\vartheta) = \widehat{\Psi}_\jmath(\vartheta, \varkappa(\vartheta)); \text{ if } \vartheta \in \widehat{J}_\jmath,\ \jmath = 1,\ldots,\omega, \\ \varkappa(\vartheta) = \phi(\vartheta); \text{ if } \vartheta \in [-\kappa_2, 0]. \end{cases}$$

Let $x \in PC$ be a solution of inequality (6). According to Remark 3, (ii) and (H_5), for each $\vartheta \in J$, we obtain

$$\begin{cases} \|x(\vartheta) - \mathfrak{F}_\alpha(\vartheta)\phi(0) - \int_0^\vartheta (\vartheta-\varepsilon)^{\alpha-1} B_\alpha(\vartheta-\varepsilon)\Psi(\varepsilon, x_\varepsilon) d\varepsilon\|_E \\ \leq \mathcal{Z}(\vartheta); \text{ if } \vartheta \in [0, \vartheta_1], \\ \|x(\vartheta) - \mathfrak{F}_\alpha(\vartheta-\delta_\jmath)\widehat{\Psi}_\jmath(\delta_\jmath, x(\delta_\jmath)) - \int_{\delta_\jmath}^\vartheta (\vartheta-\varepsilon)^{\alpha-1} B_\alpha(\vartheta-\varepsilon)\Psi(\varepsilon, x_\varepsilon) d\varepsilon; \\ \leq \mathcal{Z}(\vartheta); \text{ if } \vartheta \in J_\jmath,\ \jmath = 1, \ldots, \omega, \\ \|x(\vartheta) - \widehat{\Psi}_\jmath(\vartheta, x(\vartheta))\|_E \leq \mathcal{Y}; \text{ if } \vartheta \in \widehat{J}_\jmath,\ \jmath = 1, \ldots, \omega. \end{cases}$$

Thus,

$$\begin{cases} \|x(\vartheta) - \varkappa(\vartheta)\|_E \leq \mathcal{Z}(\vartheta) + \| \int_0^\vartheta (\vartheta-\varepsilon)^{\alpha-1} B_\alpha(\vartheta-\varepsilon) \\ \times[\Psi(\varepsilon, x_\varepsilon) - \Psi(\varepsilon, \varkappa_\varepsilon)] d\varepsilon\|_E; \text{ if } \vartheta \in [0, \vartheta_1], \\ \|x(\vartheta) - \varkappa(\vartheta)\|_E \leq \mathcal{Z}(\vartheta) + \Delta\|\widehat{\Psi}_\jmath(\delta_\jmath, x(\delta_\jmath)) - \widehat{\Psi}_\jmath(\delta_\jmath, \varkappa(\delta_\jmath))\|_E \\ + \int_{\delta_\jmath}^\vartheta (\vartheta-\varepsilon)^{r_1-1}\|B_\alpha(\vartheta-\varepsilon)(\Psi(\varepsilon, x_\varepsilon) - \Psi(\varepsilon, \varkappa_\varepsilon))\|_E d\varepsilon; \\ \text{if } \vartheta \in J_\jmath,\ \jmath = 1, \ldots, \omega, \\ \|x(\vartheta) - \varkappa(\vartheta)\|_E \leq \mathcal{Y} + \|\widehat{\Psi}_\jmath(\vartheta, x(\vartheta)) - \widehat{\Psi}_\jmath(\vartheta, \varkappa(\vartheta))\|_E; \text{ if } \vartheta \in \widehat{J}_\jmath,\ \jmath = 1, \ldots, \omega. \end{cases}$$

Hence,

$$\begin{cases} \|x(\vartheta) - \varkappa(\vartheta)\|_E \leq \mathcal{Z}(\vartheta) + \int_0^\vartheta (\vartheta-\varepsilon)^{\alpha-1} l_\Psi \|B_\alpha(\vartheta-\varepsilon)(x_\varepsilon - \varkappa_\varepsilon)\|_C d\varepsilon \\ \leq \mathcal{Z}(\vartheta) + \frac{\Delta l_\Psi}{\Gamma(\alpha)} \int_0^\vartheta (\vartheta-\varepsilon)^{\alpha-1}\|x_\varepsilon - \varkappa_\varepsilon\|_C d\varepsilon; \text{ if } \vartheta \in [0, \vartheta_1] \times [0, b], \\ \|x(\vartheta) - \varkappa(\vartheta)\|_E \leq \mathcal{Z}(\vartheta) + \Delta l_{\widehat{\Psi}}\|x(\vartheta) - \varkappa(\vartheta)\|_E \\ + \frac{\Delta l_\Psi}{\Gamma(\alpha)} \int_{\delta_\jmath}^\vartheta (\vartheta-\varepsilon)^{r_1-1}\|x_\varepsilon - \varkappa_\varepsilon\|_C d\varepsilon; \text{ if } \vartheta \in J_\jmath,\ \jmath = 1, \ldots, \omega, \\ \|x(\vartheta) - \varkappa(\vartheta)\|_E \leq \mathcal{Y} + l_{\widehat{\Psi}}\|x(\vartheta) - \varkappa(\vartheta)\|_E; \text{ if } \vartheta \in \widehat{J}_\jmath,\ \jmath = 1, \ldots, \omega. \end{cases}$$

For each $\vartheta \in [0, \vartheta_1]$, we have

$$\|x(\vartheta) - \varkappa(\vartheta)\|_E \leq \mathcal{Z}(\vartheta) + \frac{\Delta l_\Psi}{\Gamma(\alpha)} \int_0^\vartheta (\vartheta-\varepsilon)^{\alpha-1}\|x_\varepsilon - \varkappa_\varepsilon\|_C d\varepsilon.$$

We consider the function (ϱ) defined by

$$\varrho(\vartheta) = \sup\{\|x(\varepsilon) - \varkappa(\varepsilon)\| : -\kappa_2 \leq \varepsilon \leq \vartheta\};\ \vartheta \in J.$$

Let $\vartheta^* \in [-\kappa_2, \vartheta]$ be such that $\varrho(\vartheta) = \|x(\vartheta^*) - \varkappa(\vartheta^*)\|_E$. If $\vartheta^* \in [-\kappa_2, 0]$, then $\varrho(\vartheta) = 0$. Now, if $\vartheta^* \in J$, then according to the previous inequality, for $\vartheta \in J$, we have

$$\varrho(\vartheta) \leq \mathcal{Z}(\vartheta) + \frac{\Delta l_\Psi}{\Gamma(\alpha)} \int_0^\vartheta (\vartheta-\varepsilon)^{\alpha-1}\varrho(\vartheta) d\varepsilon.$$

From Lemma 2, we have

$$\begin{aligned}\varrho(\vartheta) &\leq \mathcal{Z}(\vartheta)+\int_0^{\vartheta}\left[\sum_{\imath=1}^{\infty}\frac{(\Delta l_{\Psi})^{\imath}}{\Gamma(\imath\alpha)}(\vartheta-\varepsilon)^{\imath\alpha-1}\mathcal{Z}(\varepsilon)\right]d\varepsilon,\\ &\leq (1+\varpi_{\mathcal{Z}})\mathcal{Z}(\vartheta)\\ &:= c_{1,\Psi,\widehat{\Psi}_{\jmath},\mathcal{Z}}\mathcal{Z}(\vartheta).\end{aligned}$$

Since for every $\vartheta\in[0,\vartheta_1]$, $\|x_{\vartheta}\|_{\mathcal{C}}\leq\varrho(\vartheta)$, we obtain

$$\|x(\vartheta)-\varkappa(\vartheta)\|_E\leq c_{1,\Psi,\widehat{\Psi}_{\jmath},\mathcal{Z}}(\mathcal{Y}+\mathcal{Z}(\vartheta)).$$

Now, for each $\vartheta\in J_{\jmath},\ \jmath=1,\ldots,\omega$, we have

$$\begin{aligned}&\|x(\vartheta)-\varkappa(\vartheta)\|_E\leq\mathcal{Z}(\vartheta)+\Delta l_{\widehat{\Psi}}\|x(\vartheta)-\varkappa(\vartheta)\|_E\\ &+\frac{\Delta l_{\Psi}}{\Gamma(\alpha)}\int_{\delta_{\jmath}}^{\vartheta}(\vartheta-\varepsilon)^{\alpha-1}\|x_{\varepsilon}-\varkappa_{\varepsilon}\|_{\mathcal{C}}d\varepsilon.\end{aligned}$$

Then, we obtain

$$\begin{aligned}&\|x(\vartheta)-\varkappa(\vartheta)\|_E\leq\frac{1}{1-\Delta l_{\widehat{\Psi}}}\mathcal{Z}(\vartheta)\\ &+\frac{\Delta l_{\Psi}}{(1-\Delta l_{\widehat{\Psi}})\Gamma(\alpha)}\int_{\delta_{\jmath}}^{\vartheta}(\vartheta-\varepsilon)^{\alpha-1}\|x_{\varepsilon}-\varkappa_{\varepsilon}\|_{\mathcal{C}}d\varepsilon.\end{aligned}$$

Again, from Lemma 2, we have

$$\begin{aligned}\|x(\vartheta)-\varkappa(\vartheta)\|_E &\leq \frac{1}{1-\Delta l_{\widehat{\Psi}}}\left(\mathcal{Z}(\vartheta)+\int_0^{\vartheta}\left[\sum_{\imath=1}^{\infty}\frac{(\Delta l_{\Psi})^{\imath}}{(1-\Delta l_{\widehat{\Psi}})^{\imath}\Gamma(\imath\alpha)}(\vartheta-\varepsilon)^{\imath\alpha-1}\mathcal{Z}(\varepsilon)\right]d\varepsilon\right)\\ &\leq \frac{1}{1-\Delta l_{\widehat{\Psi}}}(1+\varpi_{\mathcal{Z}})\mathcal{Z}(\vartheta)\\ &:= c_{2,\Psi,\widehat{\Psi}_{\jmath},\mathcal{Z}}\mathcal{Z}(\vartheta).\end{aligned}$$

Hence, for each $\vartheta\in J_{\jmath},\ \jmath=1,\ldots,\omega$, we obtain

$$\|x(\vartheta)-\varkappa(\vartheta)\|_E\leq c_{2,\Psi,\widehat{\Psi}_{\jmath},\mathcal{Z}}(\mathcal{Y}+\mathcal{Z}(\vartheta)).$$

Now, for each $\vartheta\in\widehat{J}_{\jmath},\ \jmath=1,\ldots,\omega$, we have

$$\|x(\vartheta)-\varkappa(\vartheta)\|_E\leq\mathcal{Y}+l_{\widehat{\Psi}}\|x(\vartheta)-\varkappa(\vartheta)\|_E.$$

This yields

$$\|x(\vartheta)-\varkappa(\vartheta)\|_E\leq\frac{\mathcal{Y}}{1-l_{\widehat{\Psi}}}:=c_{3,\Psi,\widehat{\Psi}_{\jmath},\mathcal{Z}}\mathcal{Y}.$$

Thus, for each $\vartheta\in\widehat{J}_{\jmath},\ \jmath=1,\ldots,\omega$, we obtain

$$\|x(\vartheta)-\varkappa(\vartheta)\|_E\leq c_{3,\Psi,\widehat{\Psi}_{\jmath},\mathcal{Z}}(\mathcal{Y}+\mathcal{Z}(\vartheta)).$$

Set $c_{\Psi,\widehat{\Psi}_{\jmath},\mathcal{Z}}:=\max\limits_{i\in\{1,2,3\}}c_{i,\Psi,\widehat{\Psi}_{\jmath},\mathcal{Z}}$.
Hence, for each $\vartheta\in J$, we obtain

$$\|x(\vartheta)-\varkappa(\vartheta)\|_{PC}\leq c_{\Psi,\widehat{\Psi}_{\jmath},\mathcal{Z}}(\mathcal{Y}+\mathcal{Z}(\vartheta)).$$

Consequently, problem (1) is generalized Ulam–Hyers–Rassias stable. □

4. The Phase Space

The notation for the phase space ($\mathbb{k}$) plays a significant role in the exploration of both qualitative and quantitative aspects within the field of functional differential equations. A common selection involves a seminormed space that adheres to specific axioms, a concept originally introduced by Hale and Kato [49]. To elaborate further, $\mathbb{k}$ denotes a vector space comprising functions defined from $\mathbb{R}_-$ to E accompanied by a seminorm designated as $\|\cdot\|_{\mathbb{k}}$. This seminorm must satisfy a set of predetermined axioms.

- (A_1) If $\xi : (-\infty, b) \to E$ is continuous on $[0, b]$ and $\xi_0 \in \mathbb{k}$, then for $\vartheta \in [0, b)$, the following conditions hold:
  (i) $\xi_\vartheta \in \mathbb{k}$;
  (ii) $\|\xi_\vartheta\|_{\mathbb{k}} \leq \widehat{\Delta}(\vartheta) \sup\{|\xi(\delta)| : 0 \leq \delta \leq \vartheta\} + \Delta(\vartheta)\|\xi_0\|_{\mathbb{k}}$;
  (iii) $|\xi(\vartheta)| \leq H\|\xi_\vartheta\|_{\mathbb{k}}$,
  where $H \geq 0$ is a constant, $\widehat{\Delta} : [0, b) \to [0, +\infty)$;
  $\Delta : [0, +\infty) \to [0, +\infty)$ with $\widehat{\Delta}$ continuous and Δ locally bounded; and H, $\widehat{\Delta}$, and Δ are independent of $\xi(.)$;
- (A_2) For the function ξ in (A_1), the function $\vartheta \to \xi_\vartheta$ is a $\mathbb{k}$-valued continuous function on $[0, b]$;
- (A_3) The space $\mathbb{k}$ is complete.

  Let $\widehat{\Delta}_b = \sup\{\widehat{\Delta}(\vartheta) : \vartheta \in [0, b]\}$ and $\Delta_b = \sup\{\Delta(\vartheta) : \vartheta \in [0, b]\}$.

Remark 4.
1. *[(iii)] is equivalent to $|\phi(0)| \leq H\|\phi\|_{\mathbb{k}}$ for every $\phi \in \mathbb{k}$;*
2. *Since $\|\cdot\|_{\mathbb{k}}$ is a seminorm, two (elements $\phi, \psi \in \mathbb{k}$) can verify $\|\phi - \psi\|_{\mathbb{k}} = 0$ without necessarily $\phi(\eta) = \psi(\eta)$ for all $\eta \leq 0$;*
3. *From the equivalence in the first remark, we can see that for all $\phi, \psi \in \mathbb{k}$ such that $\|\phi - \psi\|_{\mathbb{k}} = 0$; therefore, we necessarily have $\phi(0) = \psi(0)$.*

Example 2 ([50])**.** *Let:*
BC be the space of bounded continuous functions defined from $\mathbb{R}_-$ to E;
BUCbe the the space of bounded uniformly continuous functions defined from $\mathbb{R}_-$ to E;
$C^\infty := \left\{\phi \in BC : \lim_{\eta \to -\infty} \phi(\eta) \text{ exist in } E\right\}$;
$C^0 := \left\{\phi \in BC : \lim_{\eta \to -\infty} \phi(\eta) = 0\right\}$ *be endowed with the uniform norm*

$$\|\phi\| = \sup\{|\phi(\eta)| : \eta \leq 0\}.$$

Spaces BUC, C^∞, and C^0 satisfy conditions $(A_1) - (A_3)$. However, BC satisfies (A_1) and (A_3), but (A_2) is not satisfied.

Example 3 ([50])**.** *Consider spaces $C_{\widehat{\Psi}}$, $UC_{\widehat{\Psi}}$, $C^\infty_{\widehat{\Psi}}$, and $C^0_{\widehat{\Psi}}$.*
Let $\widehat{\Psi}$ be a positive continuous function on $(-\infty, 0]$.
$C_{\widehat{\Psi}} := \left\{\phi \in C(\mathbb{R}_-, E) : \frac{\phi(\eta)}{\widehat{\Psi}(\eta)} \text{ is bounded on } \mathbb{R}_-\right\}$;
$C^0_{\widehat{\Psi}} := \left\{\phi \in C_{\widehat{\Psi}} : \lim_{\eta \to -\infty} \frac{\phi(\eta)}{\widehat{\Psi}(\eta)} = 0\right\}$ *is endowed with the uniform norm*

$$\|\phi\| = \sup\left\{\frac{|\phi(\eta)|}{\widehat{\Psi}(\eta)} : \eta \leq 0\right\}.$$

Then, spaces $C_{\widehat{\Psi}}$ and $C^0_{\widehat{\Psi}}$ satisfy conditions $(A_1) - (A_3)$. We consider the following condition on the function $\widehat{\Psi}$.

(g_1) *For all $\kappa_1 > 0$, $\sup_{0 \leq \vartheta \leq \kappa_1} \sup\left\{\frac{\widehat{\Psi}(\vartheta+\eta)}{\widehat{\Psi}(\eta)} : -\infty < \eta \leq -\vartheta\right\} < \infty$.*

They satisfy conditions (A_1) and (A_2) if $(\widehat{\Psi}_1)$ holds.

Example 4 ([50]). *Consider space C_ϱ. For any real constant (ϱ), we define the functional space (C_ϱ) as*

$$C_\varrho := \left\{ \phi \in C(\mathbb{R}_-, E) : \lim_{\eta \to -\infty} e^{\varrho\eta}\phi(\eta) \text{ exists in } E \right\}$$

which is endowed with the following norm:

$$\|\phi\| = \sup\{e^{\varrho\eta}|\phi(\eta)| : \eta \leq 0\}.$$

Then, C_ϱ satisfies axioms $(A_1) - (A_3)$.

5. Uniqueness and Ulam Stability Results with Infinite Delay

In this section, we present conditions for the Ulam stability of problem (2). Consider the following space:

$$\Omega := \{x : (-\infty, \kappa_1] \to E : \ x_\vartheta \in \Bbbk \text{ for } \vartheta \in \mathbb{R}_- \ \text{ and } x|_J \in PC\}.$$

Theorem 2. *Assume that (H_1), (H_4), and the following hypotheses hold:*

(H_6) *For each $\vartheta \in J_\jmath$; $\jmath = 0, \ldots, \omega$, the function $\Psi(\vartheta, \cdot) : E \to E$ is continuous, and for each $\varkappa \in \Bbbk$, the function $\Psi(\cdot, \varkappa) : J_\jmath \to E$ is measurable;*

(H_7) *There exists a constant ($l'_\Psi > 0$) such that*

$$\|\Psi(\vartheta, x) - \Psi(\vartheta, \overline{x})\|_E \leq l'_\Psi \|x - \overline{x}\|_\Bbbk, \text{ for each } \vartheta \in J_\jmath;\ \jmath = 0, \ldots, \omega, \text{and each } x, \overline{x} \in \Bbbk.$$

If

$$\ell' := \Delta l_{\widehat{\Psi}} + \frac{\Delta\widehat{\Delta} l'_\Psi {\kappa_1}^\alpha}{\Gamma(\alpha)} < 1, \tag{9}$$

then problem (2) has a unique mild solution on $(-\infty, \kappa_1]$. Furthermore, if (H_5) holds, then problem (2) is generalized Ulam–Hyers–Rassias stable.

Proof. Consider the operator $\aleph' : \Omega \to \Omega$ as defined by

$$\begin{cases} (\aleph' x)(\vartheta) = \mathfrak{F}_\alpha(\vartheta)\phi(0) + \int_0^\vartheta (\vartheta - \varepsilon)^{\alpha-1} B_\alpha(\vartheta - \varepsilon)\Psi(\varepsilon, x(\varepsilon))d\varepsilon; \ \text{if} \ \vartheta \in [0, \vartheta_1], \\ (\aleph' x)(\vartheta) = \mathfrak{F}_\alpha(\vartheta - \delta_\jmath)\widehat{\Psi}_\jmath(\delta_\jmath, x(\delta_\jmath)) \\ + \int_{\delta_\jmath}^\vartheta (\vartheta - \varepsilon)^{\alpha-1} B_\alpha(\vartheta - \varepsilon)\Psi(\varepsilon, x(\varepsilon))d\varepsilon; \ \text{if} \ \vartheta \in J_\jmath, \ \jmath = 1, \ldots, \omega, \\ (\aleph' x)(\vartheta) = \widehat{\Psi}_\jmath(\vartheta, x(\vartheta)); \text{ if } \vartheta \in \widehat{J}_\jmath, \ \jmath = 1, \ldots, \omega, \\ (\aleph' x)(\vartheta) = \phi(\vartheta); \text{ if } \vartheta \in \mathbb{R}_-, \end{cases}$$

Clearly, the fixed points of the operator ($\aleph'$) are mild solutions of problem (2). Consider the function $\varkappa(\cdot) : (-\infty, \kappa_1] \to E$ as defined by

$$\begin{cases} \varkappa(\vartheta) = 0; \ \text{if} \ \vartheta \in J, \\ \varkappa(\vartheta) = \phi(\vartheta); \ \text{if } \vartheta \in \mathbb{R}_-. \end{cases}$$

Then, $\varkappa_0 = \phi$. For each $\tau \in \mathcal{C}(J)$ with $\tau(0) = 0$, $\overline{\tau}$ denotes the function defined by

$$\begin{cases} \overline{\tau}(\vartheta) = \tau(\vartheta) \ \text{ if } \vartheta \in J, \\ \overline{\tau}(\vartheta) = 0, \text{ if } \vartheta \in \tilde{J}'. \end{cases}$$

If $x(\cdot)$ satisfies

$$\begin{cases} x(\vartheta) = \mathfrak{F}_\alpha(\vartheta)\phi(0) + \int_0^\vartheta (\vartheta-\varepsilon)^{\alpha-1} B_\alpha(\vartheta-\varepsilon)\Psi(\varepsilon, x(\varepsilon))d\varepsilon; \text{ if } \vartheta \in [0,\vartheta_1], \\ x(\vartheta) = \mathfrak{F}_\alpha(\vartheta-\delta_\jmath)\widehat{\Psi}_\jmath(\delta_\jmath, x(\delta_\jmath)) \\ + \int_{\delta_\jmath}^\vartheta (\vartheta-\varepsilon)^{\alpha-1} B_\alpha(\vartheta-\varepsilon)\Psi(\varepsilon, x(\varepsilon))d\varepsilon; \text{ if } \vartheta \in J_\jmath,\ \jmath = 1,\ldots,\omega, \\ x(\vartheta) = \widehat{\Psi}_\jmath(\vartheta, x(\vartheta)); \text{ if } \vartheta \in \widehat{J}_\jmath,\ \jmath = 1,\ldots,\omega, \\ x(\vartheta) = \phi(\vartheta); \text{ if } \vartheta \in \mathbb{R}_-, \end{cases}$$

we decompose $x(\vartheta)$ as $x(\vartheta) = \tau(\vartheta) + \varkappa(\vartheta);\ \vartheta \in J$, which implies $x_\vartheta = \tau_\vartheta + \varkappa_\vartheta;\ \vartheta \in J$ and the function τ satisfies $\tau_0 = 0$. Then, for $\vartheta \in J$, we obtain

$$\begin{cases} \tau(\vartheta) = \mathfrak{F}_\alpha(\vartheta)\phi(0) + \int_0^\vartheta (\vartheta-\varepsilon)^{\alpha-1} B_\alpha(\vartheta-\varepsilon)\Psi(\varepsilon, \overline{\tau}_\varepsilon + \varkappa_\varepsilon)d\varepsilon; \text{ if } \vartheta \in [0,\vartheta_1], \\ \tau(\vartheta) = \mathfrak{F}_\alpha(\vartheta-\delta_\jmath)\widehat{\Psi}_\jmath(\delta_\jmath, \overline{\tau}_{\delta_\jmath} + \varkappa_{\delta_\jmath}) \\ + \int_{\delta_\jmath}^\vartheta (\vartheta-\varepsilon)^{\alpha-1} B_\alpha(\vartheta-\varepsilon)\Psi(\varepsilon, \overline{\tau}_\varepsilon + \varkappa_\varepsilon)d\varepsilon; \text{ if } \vartheta \in J_\jmath,\ \jmath = 1,\ldots,\omega, \\ \tau(\vartheta) = \widehat{\Psi}_\jmath(\vartheta, \overline{\tau}_\vartheta + \varkappa_\vartheta); \text{ if } \vartheta \in \widehat{J}_\jmath,\ \jmath = 1,\ldots,\omega. \end{cases}$$

Set

$$C_0 = \{\tau \in PC:\ \tau(0) = 0\},$$

and let $\|\cdot\|_a$ be the seminorm in C_0 defined by

$$\|\tau\|_a = \|\tau_0\|_{\mathbb{k}} + \sup_{\vartheta\in J}\|\tau(\vartheta)\| = \sup_{\vartheta\in J}\|\tau(\vartheta)\|;\ \tau \in C_0.$$

Hence, C_0 is a Banach space with norm $\|\cdot\|_a$. Let the operator $P:\ C_0 \to C_0$ be defined by

$$\begin{cases} (Pw)(\vartheta) = \mathfrak{F}_\alpha(\vartheta)\phi(0) + \int_0^\vartheta (\vartheta-\varepsilon)^{\alpha-1} B_\alpha(\vartheta-\varepsilon)\Psi(\varepsilon, \overline{\tau}_\varepsilon + \varkappa_\varepsilon)d\varepsilon; \text{ if } \vartheta \in [0,\vartheta_1], \\ (Pw)(\vartheta) = \mathfrak{F}_\alpha(\vartheta-\delta_\jmath)\widehat{\Psi}_\jmath(\delta_\jmath, \overline{\tau}_{\delta_\jmath} + \varkappa_{\delta_\jmath}) \\ + \int_{\delta_\jmath}^\vartheta (\vartheta-\varepsilon)^{\alpha-1} B_\alpha(\vartheta-\varepsilon)\Psi(\varepsilon, \overline{\tau}_\varepsilon + \varkappa_\varepsilon)d\varepsilon; \text{ if } \vartheta \in J_\jmath,\ \jmath = 1,\ldots,\omega, \\ (Pw)(\vartheta) = \widehat{\Psi}_\jmath(\vartheta, \overline{\tau}_\vartheta + \varkappa_\vartheta); \text{ if } \vartheta \in \widehat{J}_\jmath,\ \jmath = 1,\ldots,\omega. \end{cases}$$

Obviously, the operator $\aleph'$ has a fixed point equivalent to P. We use the Banach contraction principle to prove that P has a fixed point. Consider $\tau, \tau^* \in C_0$. Then, for each $\vartheta \in J$, we obtain

$$\|P(\tau) - P(\tau^*)\|_a \leq \ell'\|\overline{\tau} - \overline{\tau}^*\|_a.$$

Based on condition (9), we conclude that P is a contraction. As a consequence of the Banach fixed-point theorem, we deduce that P has a unique fixed point (τ^*). Then, we have

$$\begin{cases} \tau^*(\vartheta) = \mathfrak{F}_\alpha(\vartheta)\phi(0) + \int_0^\vartheta (\vartheta-\varepsilon)^{\alpha-1} B_\alpha(\vartheta-\varepsilon)\Psi(\varepsilon, \overline{\tau}_\varepsilon + \varkappa_\varepsilon)d\varepsilon; \text{ if } \vartheta \in [0,\vartheta_1], \\ \tau^*(\vartheta) = \mathfrak{F}_\alpha(\vartheta-\delta_\jmath)\widehat{\Psi}_\jmath(\delta_\jmath, \overline{\tau}^*_{\delta_\jmath} + \varkappa_{\delta_\jmath}) \\ + \int_{\delta_\jmath}^\vartheta (\vartheta-\varepsilon)^{\alpha-1} B_\alpha(\vartheta-\varepsilon)\Psi(\varepsilon, \overline{\tau}^*_\varepsilon + \varkappa_\varepsilon)d\varepsilon; \text{ if } \vartheta \in J_\jmath,\ \jmath = 1,\ldots,\omega, \\ \tau^*(\vartheta) = \widehat{\Psi}_\jmath(\vartheta, \overline{\tau}^*_\vartheta + \varkappa_\vartheta); \text{ if } \vartheta \in \widehat{J}_\jmath,\ \jmath = 1,\ldots,\omega. \end{cases}$$

Let $\tau \in C_0$ be a solution of inequality (6). Thus, according to (H_5) and Lemma 2 and as in the proof of Theorem 1, we can show that for each $\vartheta \in J$,

$$\|\tau(\vartheta,\xi) - \tau^*(\vartheta,\xi)\|_E \leq c'_{\Psi,\widehat{\Psi}_J,\mathcal{Z}}(\mathcal{Y} + \mathcal{Z}(\vartheta,\xi)),$$

for some $c'_{\Psi,\widehat{\Psi}_J,\mathcal{Z}} > 0$, which shows that problem (2) is generalized Ulam–Hyers–Rassias stable. □

6. Uniqueness and Ulam Stability Results with State-Dependent Delay

In this section, we present (without proof) uniqueness and Ulam stability results for problems (3) and (4).

Set

$$\mathcal{R} := \{\rho(\delta,x) : (\delta,x) \in J_J \times \mathcal{D},\ \rho(\delta,x) \leq 0,\ J = 0,\ldots,\omega\},$$

where $\mathcal{D} \in \{\mathcal{C},\Bbbk\}$. We always assume that $\rho : J_J \times \mathcal{D} \to \mathbb{R}$; $J = 0,\ldots,\omega$ is continuous and that the function $\delta \longmapsto x_\delta$ is continuous from $\mathcal{R}$ into $\mathcal{D}$.

Theorem 3. *Assume that* (H_1), (H_2), (H_4), *and the following hypothesis hold:*

(H_8) *There exists a constant* ($l''_\Psi > 0$) *such that*

$$\|\Psi(\vartheta, x_{\rho(\vartheta,x_\vartheta)}) - \Psi(\vartheta,\overline{x}_{\rho(\vartheta,\overline{x}_\vartheta)})\|_E \leq l''_\Psi \|x_{\rho(\vartheta,x_\vartheta)} - \overline{x}_{\rho(\vartheta,\overline{x}_\vartheta)}\|_{\mathcal{C}};$$

for each $\vartheta \in J_J$; $J = 0,\ldots,\omega$, *and each* $x,\overline{x} \in \mathcal{C}$.

If

$$\ell'' := \Delta l_{\widehat{\Psi}} + \frac{\Delta l''_\Psi {\kappa_1}^\alpha}{\Gamma(\alpha)} < 1, \tag{10}$$

then problem (3) has a unique mild solution on $[-\kappa_2,\kappa_1]$. *Furthermore, if* (H_5) *holds, then problem (3) is generalized Ulam–Hyers–Rassias stable.*

Proof. Following the same steps as for the proof of Theorem 1, we can deduce the uniqueness and Ulam stability results. □

Theorem 4. *Assume that* (H_1), (H_4), (H_6), *and the following hypothesis hold:*

(H_9) *There exists a constant* ($l'''_\Psi > 0$) *such that*

$$\|\Psi(\vartheta, x_{\rho(\vartheta,x_\vartheta)}) - \Psi(\vartheta,\overline{x}_{\rho(\vartheta,\overline{x}_\vartheta)})\|_E \leq l'_\Psi \|x_{\rho(\vartheta,x_\vartheta)} - \overline{x}_{\rho(\vartheta,\overline{x}_\vartheta)}\|_{\Bbbk};$$

for each $\vartheta \in J_J$; $J = 0,\ldots,\omega$, *and each* $x,\overline{x} \in \Bbbk$.

If

$$\ell''' := \Delta l_{\widehat{\Psi}} + \frac{\Delta\widehat{\Delta} l'''_\Psi {\kappa_1}^\alpha}{\Gamma(\alpha)} < 1, \tag{11}$$

then problem (4) has a unique mild solution on $(-\infty,\kappa_1]$. *Furthermore, if* (H_5) *holds, then problem (4) is generalized Ulam–Hyers–Rassias stable.*

Proof. Following the same steps as for the proof of Theorem 2, we can deduce the uniqueness and Ulam stability results. □

7. Examples

As applications of our results, we present two examples.

Example 5. *Consider the functional abstract fractional differential equations with non-instantaneous impulses of the following form:*

$$\begin{cases} D^{\alpha}_{0,\vartheta}\lambda(\vartheta,\xi) = \frac{\partial^2\lambda}{\partial\xi^2}(\vartheta,\xi) + \beth(\vartheta,\lambda(\vartheta-1,\xi)); & \vartheta\in[0,1]\cup(2,3], \quad \xi\in[0,\pi], \\ \lambda(\vartheta,\xi) = \widehat{\Psi}(\vartheta,\lambda(\vartheta,\xi)); & \vartheta\in(1,2], \quad \xi\in[0,\pi], \\ \lambda(\vartheta,0) = \lambda(\vartheta,\pi) = 0; & \vartheta\in[0,1]\cup(2,3], \\ \lambda(\vartheta,\xi) = \phi(\vartheta,\xi); & \vartheta\in[-1,0], \quad \xi\in[0,\pi], \end{cases} \tag{12}$$

where $D^{\alpha}_{0,\vartheta} := \frac{\partial^\alpha}{\partial\vartheta^\alpha}$ *is the Caputo fractional partial derivative of order* $\alpha\in(0,1]$ *with respect to* ϑ. *It is defined by the following expression:*

$${}^{c}D^{\vartheta}_{0,\vartheta}\lambda(\vartheta,\xi) = \frac{1}{\Gamma(1-\alpha)}\int_0^{\vartheta}(\vartheta-\varepsilon)^{-\alpha}\frac{\partial}{\partial\varepsilon}\lambda(\varepsilon,\xi)d\varepsilon,$$

$\mathcal{C} := C_1$, $\beth : ([0,1]\cup(2,3])\times\mathcal{C}\to\mathbb{R}$ *and* $\widehat{\Psi} : (1,2]\times\mathbb{R}\to\mathbb{R}$ *are expressed by*

$$\beth(\vartheta,\lambda(\vartheta-1,\xi)) = \frac{1}{(1+110e^{\vartheta})(1+|\lambda(\vartheta-1,\xi)|)};\ \vartheta\in[0,1]\cup(2,3],\ \xi\in[0,\pi],$$

$$\widehat{\Psi}(\vartheta,\lambda(\vartheta,\xi)) = \frac{1}{1+110e^{\vartheta+\xi}}\ln(1+\vartheta^2+|\lambda(\vartheta,\xi)|);\ \vartheta\in(1,2],\ \xi\in[0,\pi],$$

and $\phi : [-1,0]\times[0,\pi]\to\mathbb{R}$ *is a continuous function.*

Let $E = L^2([0,\pi],\mathbb{R})$ *and define* $A : D(A)\subset E\to E$ *as* $A\tau = \tau''$ *with the following domain*

$$D(A) = \{\tau\in E : \tau,\ \tau' \text{ are absolutely continuous},\ \tau''\in E,\ \tau(0)=\tau(\pi)=0\}.$$

It is well known that A *is the infinitesimal generator of an analytic semigroup on* E *(see [51]). Then,*

$$A\tau = -\sum_{\imath=1}^{\infty}\imath^2 < \tau, e_\imath > e_\imath;\tau\in D(A),$$

where

$$e_\imath(\xi) = \sqrt{\frac{2}{\pi}}\sin(\imath\xi);\ \xi\in[0,\pi],\ \imath=1,2,3,\ldots.$$

Semigroup $B(\vartheta);\ \vartheta\geq 0$ *is expressed by*

$$B(\vartheta)\tau = \sum_{\imath=1}^{\infty}e^{-\imath^2\vartheta} < \tau, e_\imath > e_\imath;\ \tau\in E.$$

Hence, the assumptions of (H_1) *and* (H_2) *are satisfied.*

For $\xi\in[0,\pi]$, *set* $x(\vartheta)(\xi) = \lambda(\vartheta,\xi);\quad \vartheta\in[0,3],\ \phi(\vartheta)(\xi) = \phi(\vartheta,\xi);\quad \vartheta\in[-1,0]$,

$$Ax(\vartheta)(\xi) = \frac{\partial^2\lambda}{\partial\xi^2}(\vartheta,\xi);\quad \vartheta\in[0,1]\cup(2,3],$$

$$\Psi(\vartheta,x(\vartheta))(\xi) = \beth(\vartheta,\lambda(\vartheta,\xi));\quad \vartheta\in[0,1]\cup(2,3],$$

and

$$\widehat{\Psi}(\vartheta,x(\vartheta))(\xi) = \widehat{\Psi}(\vartheta,\lambda(\vartheta,\xi));\quad \vartheta\in(1,2].$$

Consequently, employing the given definitions of ϕ, A, Ψ, *and* $\widehat{\Psi}$, *system (12) can be equivalently expressed as functional abstract problem (1).*

For each $\lambda, \overline{\lambda}, \in\mathcal{C}$, $\vartheta\in[0,1]\cup(2,3]$ *and* $\xi\in[0,\pi]$, *we have*

$$|\Psi(\vartheta,\lambda_\vartheta)(\xi) - \Psi(\vartheta,\overline{\lambda}_\vartheta)(\xi)| \leq \frac{1}{111}|\lambda(\vartheta,\xi) - \overline{\lambda}(\vartheta,\xi)|,$$

then, we obtain

$$\|\Psi(\vartheta,\lambda)-\Psi(\vartheta,\overline{\lambda})\|_E \leq \frac{1}{111}\|\lambda-\overline{\lambda}\|_{\mathcal{C}}.$$

Also, for each $\lambda,\overline{\lambda},\in E$, $\vartheta\in(1,2]$ *and* $\xi\in[0,\pi]$, *we can easily obtain*

$$\|\widehat{\Psi}(\vartheta,\lambda)-\widehat{\Psi}(\vartheta,\overline{\lambda})\|_E \leq \frac{1}{111}\|\lambda-\overline{\lambda}\|_E.$$

Thus, (H_3) *and* (H_4) *are verified with* $l_\Psi = l_{\widehat{\Psi}} = \frac{1}{111}$. *We show that condition (8) holds with* $\kappa_1 = 3$ *and* $\Delta = 1$. *Indeed, for each* $\alpha\in(0,1]$ *we obtain*

$$\begin{aligned}\ell &= \Delta l_{\widehat{\Psi}} + \frac{\Delta l_\Psi {\kappa_1}^\alpha}{\Gamma(\alpha)}\\ &= \frac{1}{111}+\frac{3^\alpha}{111\Gamma(\alpha)}\\ &< \frac{7}{111}\\ &< 1.\end{aligned}$$

Therefore, we guarantee the existence of a distinct mild solution defined on the interval $[-1,3]$ *for the given problem (12). In conclusion, condition* (H_5) *is fulfilled by* $\mathcal{Z}(\vartheta)=1$ *and*

$$\varpi_{\mathcal{Z}} = \sum_{\imath=1}^{\infty}\frac{1}{(110)^\imath\Gamma(1+\imath\alpha)}3^{\imath\alpha}.$$

Consequently, Theorem 1 implies that problem (12) is generalized Ulam–Hyers–Rassias stable.

Example 6. *Now consider the functional abstract fractional differential equations with state-dependent delay and non-instantaneous impulses of the following form*

$$\begin{cases} D^\alpha_{0,\vartheta}\lambda(\vartheta,\xi)=\frac{\partial^2\lambda}{\partial\xi^2}(\vartheta,\xi) & & \\ +\daleth(\vartheta,\lambda(\vartheta-\sigma(\lambda(\vartheta,\xi)),\xi)); & \vartheta\in[0,1]\cup(2,3], & \xi\in[0,\pi],\\ \lambda(\vartheta,\xi)=\widehat{\Psi}(\vartheta,\lambda(\vartheta,\xi)); & \vartheta\in(1,2], & \xi\in[0,\pi],\\ \lambda(\vartheta,0)=\lambda(\vartheta,\pi)=0; & \vartheta\in[0,1]\cup(2,3], & \\ \lambda(\vartheta,\xi)=\phi(\vartheta,\xi); & \vartheta\in(-\infty,0], & \xi\in[0,\pi],\end{cases} \tag{13}$$

where $D^\alpha_{0,\vartheta} := \frac{\partial^\alpha}{\partial\vartheta^\alpha}$ *is the Caputo fractional partial derivative of order* $\alpha\in(0,1]$ *with respect to* ϑ, $\sigma\in C(\mathbb{R},[0,\infty))$, $\daleth:([0,1]\cup(2,3])\times\Bbbk\to\mathbb{R}$ *and* $\widehat{\Psi}:(1,2]\times\mathbb{R}\to\mathbb{R}$ *are expressed by*

$$\daleth(\vartheta,\lambda(\vartheta-\sigma(\lambda(\vartheta,\xi)),\xi)) = \frac{1}{111(1+|\lambda(\vartheta-\sigma(\lambda(\vartheta,\xi)),\xi)|)};\ \vartheta\in[0,1]\cup(2,3],\ \xi\in[0,\pi],$$

$$\widehat{\Psi}(\vartheta,\lambda(\vartheta,\xi)) = \frac{\arctan(\vartheta^2+|\lambda(\vartheta,\xi)|)}{1+110e^{\vartheta+\xi}};\ \vartheta\in(1,2],\ \xi\in[0,\pi],$$

and $\phi:(-\infty,0]\times[0,\pi]\to\mathbb{R}$ *is a continuous function. We choose* $\Bbbk=\Bbbk_\varrho$ *as the phase space defined by*

$$\Bbbk_\varrho := \left\{\phi\in C((-\infty,0],E) : \lim_{\eta\to-\infty} e^{\varrho\eta}\phi(\eta)\ \textit{exists in}\ E\right\}$$

which is endowed with the norm

$$\|\phi\| = \sup\{e^{\varrho\eta}|\phi(\eta)| :\ \eta\leq 0\}.$$

Let $E = L^2([0,\pi],\mathbb{R})$ *and* A *be the operator defined in Example 1. For* $\xi \in [0,\pi]$, *set* $x(\vartheta)(\xi) = \lambda(\vartheta,\xi); \quad \vartheta \in [0,3], \ \phi(\vartheta)(\xi) = \phi(\vartheta,\xi); \quad \vartheta \in (-\infty,0],$

$$Ax(\vartheta)(\xi) = \frac{\partial^2 \lambda}{\partial \xi^2}(\vartheta,\xi); \quad \vartheta \in [0,1] \cup (2,3],$$

$$\Psi(\vartheta, x(\vartheta - \sigma(\lambda(\vartheta,\xi))))(\xi) = \beth(\vartheta, \lambda(\vartheta - \sigma(\lambda(\vartheta,\xi)),\xi)); \quad \vartheta \in [0,1] \cup (2,3],$$

and

$$\widehat{\Psi}(\vartheta, x(\vartheta))(\xi) = \widehat{\Psi}(\vartheta, \lambda(\vartheta,\xi)); \quad \vartheta \in (1,2].$$

Thus, under the above definitions of ϕ, A, Ψ, *and* $\widehat{\Psi}$, *system (13) can be represented by functional abstract problem (4). We can see that all hypotheses of Theorem 4 are fulfilled. Consequently, problem (13) has a unique mild solution defined on* $(-\infty,3]$. *Moreover, problem (13) is generalized Ulam–Hyers–Rassias stable.*

8. Conclusions

In this study, we undertook the task of establishing the existence, uniqueness, and Ulam–Hyers–Rassias stability of solutions for fractional differential equations with non-instantaneous impulses and delay. Operating within the framework of Banach spaces, our exploration extended to diverse problem cases, encompassing abstract impulsive fractional differential equations with finite, infinite, and state-dependent delay. Our approach to proving the results relied on the application of the principle of contraction of Banach combined with some properties of the phase space. The outcomes of our study present a novel contribution to the existing literature, enriching the ever-evolving and dynamic field of study in significant ways. Furthermore, we recognize the potential for further exploration along various avenues, such as coupled systems, problems incorporating anticipations, implicit problems, or those involving hybrid differential equations. We hope that this article will serve as a starting point for such an undertaking.

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References

1. Abbas, S.; Benchohra, M.; Lazreg, J.E.; Nieto, J.J.; Zhou, Y. *Fractional Differential Equations and Inclusions: Classical and Advanced Topics*; World Scientific Publishing Co. Pte. Ltd.: Hackensack, NJ, USA, 2023.
2. Abbas, S.; Benchohra, M.; N'Guérékata, G.M. *Topics in Fractional Differential Equations*: Springer: New York, NY, USA, 2012.
3. Abbas, S.; Benchohra, M.; N'Guérékata, G.M. *Advanced Fractional Differential and Integral Equations*; Nova Science Publishers: New York, NY, USA, 2015.
4. Herrmann, R. *Fractional Calculus: An Introduction for Physicists*; World Scientific Publishing Company: Singapore, 2011.
5. Hilfer, R. *Applications of Fractional Calculus in Physics*; World Scientific: Singapore, 2000.
6. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; North-Holland Mathematics Studies; Elsevier: Amsterdam, The Netherlands, 2006. Volume 204 .
7. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives. Theory and Applications*; Gordon and Breach: Yverdon, Switzerland, 1993.
8. Zhou, Y. *Basic Theory of Fractional Differential Equations*; World Scientific: Singapore, 2014.
9. Abbas, S.; Benchohra, M.; Nieto, J.J. Ulam stabilities for impulsive partial fractional differential equations. *Acta Univ. Palacki. Olomuc.* **2014**, *53*, 15–17.

10. Abbas, S.; Benchohra, M.; Zhou, Y. Fractional order partial hyperbolic functional differential equations with state-dependent delay. *Int. J. Dyn. Syst. Differ. Equ.* **2011**, *3*, 459–490. [CrossRef]
11. Salim, A.; Benchohra, M.; Graef, J.R.; Lazreg, J.E. Initial value problem for hybrid ψ-Hilfer fractional implicit differential equations. *J. Fixed Point Theory Appl.* **2022**, *24*, 14. [CrossRef]
12. Rayal, A.; Joshi, B.P.; Pandey, M.; Torres, D.F.M. Numerical investigation of the fractional oscillation equations under the context of variable order Caputo fractional derivative via fractional order Bernstein wavelets. *Mathematics* **2023**, *11*, 2503. [CrossRef]
13. Wanassi, O.K.; Bourguiba, R.; Torres, D.F.M. Existence and uniqueness of solution for fractional differential equations with integral boundary conditions and the Adomian decomposition method. *Math. Methods. Appl. Sci.* **2022** , 1–14. [CrossRef]
14. Shah, N.A.; Hamed, Y.S.; Abualnaja, K.M.; Chung, J.-D.; Shah, R.; Khan, A. A comparative analysis of fractional-order Kaup-Kupershmidt equation within different operators. *Symmetry* **2022**, *14*, 986. [CrossRef]
15. Shah, N.A.; Alyousef, H.A.; El-Tantawy, S.A.; Shah, R.; Chung, J.D. Analytical investigation of fractional-order Korteweg–De-Vries-type equations under Atangana–Baleanu–Caputo operator: Modeling nonlinear waves in a plasma and fluid. *Symmetry* **2022**, *14*, 739. [CrossRef]
16. Benchohra, M.; Karapınar, E.; Lazreg, J.E.; Salim, A. *Advanced Topics in Fractional Differential Equations: A Fixed Point Approach*; Springer: Cham, Switzerland, 2023. [CrossRef]
17. Benchohra, M.; Karapınar, E.; Lazreg, J.E.; Salim, A. *Fractional Differential Equations: New Advancements for Generalized Fractional Derivatives*; Springer: Cham, Switzerland, 2023.
18. Ulam, S.M. *A Collection of Mathematical Problems*; Interscience Publishers: New York, NY, USA, 1968.
19. Hyers, D.H. On the stability of the linear functional equation. *Proc. Natl. Acad. Sci. USA* **1941**, *27*, 222–224.
20. Rassias, T.M. On the stability of linear mappings in Banach spaces. *Proc. Am. Math. Soc.* **1978**, *72*, 297–300. [CrossRef]
21. Hyers, D.H.; Isac, G.; Rassias, T.M. *Stability of Functional Equations in Several Variables*; Birkhuser: Basel, Switzerland, 1998. [CrossRef]
22. Jung, S.-M. *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*; Hadronic Press: Palm Harbor, FL, USA, 2001.
23. Bota-Boriceanu, M.F.; Petrusel, A. Ulam-Hyers stability for operatorial equations and inclusions. *Analele Univ. I. Cuza Iasi* **2011**, *57*, 65–74.
24. Petru, T.P.; Bota, M.-F. Ulam-Hyers stabillity of operational inclusions in complete gauge spaces. *Fixed Point Theory* **2012**, *13*, 641–650.
25. Rus, I.A. Ulam stabilities of ordinary differential equations in a Banach space. *Carpathian J. Math.* **2010**, *26*, 103–107.
26. Rus, I.A. Remarks on Ulam stability of the operatorial equations. *Fixed Point Theory* **2009**, *10*, 305–320.
27. Castro, L.P.; Ramos, A. Hyers-Ulam-Rassias stability for a class of Volterra integral equations. *Banach J. Math. Anal.* **2009**, *3*, 36–43.
28. Wang, J.; Lv, L.; Zhou, Y. Ulam stability and data dependence for fractional differential equations with Caputo derivative. *Electron. J. Qual. Theory Diff. Equ.* **2011**, *63*, 1–10. [CrossRef]
29. Wang, J.; Lv, L.; Zhou, Y. New concepts and results in stability of fractional differential equations. *Commun. Nonlinear Sci. Numer. Simul.* **2012**, *17*, 2530–2538. [CrossRef]
30. Jung, S.-M. *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*; Springer: New York, NY, USA, 2011. [CrossRef]
31. Lazreg, J.E.; Benchohra, M.; Salim, A. Existence and Ulam stability of $\jmath$-generalized ψ-Hilfer fractional problem. *J. Innov. Appl. Math. Comput. Sci.* **2022**, *2*, 1–13 [CrossRef].
32. Milman, V.D.; Myshkis, A.A. On the stability of motion in the presence of impulses. *Sib. Math. J.* **1960**, *1*, 233–237. (In Russian) [CrossRef]
33. Lakshmikantham, V.; Bainov, D.D.; Simeonov, P.S. *Theory of Impulsive Differential Equations*; World Scientific: Singapore, 1989.
34. Benchohra, M.; Henderson, J.; Ntouyas, S.K. *Impulsive Differential Equations and Inclusions*; Hindawi Publishing Corporation: New York, NY, USA, 2006; Volume 2.
35. Samoilenko, A.M.; Perestyuk, N.A. *Impulsive Differential Equations*; World Scientific: Singapore, 1995.
36. Krim, S.; Salim, A.; Abbas, S.; Benchohra, M. On implicit impulsive conformable fractional differential equations with infinite delay in b-metric spaces. *Rend. Circ. Mat. Palermo* **2023**, *72*, 2579–2592. [CrossRef]
37. Hakkar, N.; Dhayal, R.; Debbouche, A.; Torres, D.F.M. Approximate controllability of delayed fractional stochastic differential systems with mixed noise and impulsive effects. *Fractal Fract.* **2023**, *7*, 104. [CrossRef]
38. Bensalem, A.; Salim, A.; Benchohra, M.; Nieto, J.J. Controllability results for second-order integro-differential equations with state-dependent delay. *Evol. Equ. Control Theory* **2023**, *12*, 1559–1576. [CrossRef]
39. Hernández, E.; O'Regan, D. On a new class of abstract impulsive differential equations. *Proc. Am. Math. Soc.* **2013**, *141*, 1641–1649. [CrossRef]
40. Pierri, M.; O'Regan, D.; Rolnik, V. Existence of solutions for semi-linear abstract differential equations with not instantaneous. *Appl. Math. Comput.* **2013**, *219*, 6743–6749. [CrossRef]
41. Wang, J.; Fečkan, M.; Zhou, Y. Ulam's type stability of impulsive ordinary differential equations. *J. Math. Anal. Appl.* **2012**, *395*, 258–264. [CrossRef]
42. Agarwal, R.P.; Hristova, S.; O'Regan, D. *Non-Instantaneous Impulses in Differential Equations*; Springer: New York, NY, USA, 2017. [CrossRef]

43. Agarwal, R.P.; Hristova, S.; O'Regan, D. Non-instantaneous impulses in Caputo fractional differential equations. *Fract. Calc. Appl. Anal.* **2017**, *20*, 595–622.
44. Agarwal, R.P.; Hristova, S.; O'Regan, D. Stability by Lyapunov like functions of nonlinear differential equations with non-instantaneous impulses. *J. Appl. Math. Comput.* **2017**, *53*, 147–168. [CrossRef]
45. Agarwal, R.P.; Hristova, S.; O'Regan, D. Monotone iterative technique for the initial value problem for differential equations with non-instantaneous impulses. *Appl. Math. Comput.* **2017**, *298*, 45–56. [CrossRef]
46. Benkhettou, N.; Aissani, K.; Salim, A.; Tunc, C. Controllability of fractional integro-differential equations with infinite delay and non-instantaneous impulses. *Appl. Anal. Optim.* **2022**, *6*, 79–94. [CrossRef]
47. Zhou, Y.; Jiao, F. Existence of mild solutions for fractional neutral evolution equations. *Comput. Math. Appl.* **2010**, *59*, 1063–1077.
48. Ye, H.; Gao, J.; Ding, Y. A generalized Gronwall inequality and its application to a fractional differential equation. *J. Math. Anal. Appl.* **2007**, *328*, 1075–1081. [CrossRef]
49. Hale, J.K.; Kato, J. Phase space for retarded equations with infinite delay. *Funk. Ekvacioj* **1978**, *21*, 11–41. [CrossRef]
50. Hino, Y.; Murakami, S.; Naito, T. *Functional Differential Equations with Unbounded Delay*; Springer: Berlin/Heidelberg, Germany, 1991.
51. Pazy, A. *Semigroups of Linear Operators and Applications to Partial Differential Equations*; Springer: New York, NY, USA, 1983.