Fractal Divergences of Generalized Jacobi Polynomials

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Abstract: The notion of entropy (including macro state entropy and information entropy) is used, among others, to define the fractal dimension. Rényi entropy constitutes the basis for the generalized correlation dimension of multifractals. A motivation for the study of the information measures of orthogonal polynomials is because these polynomials appear in the densities of many quantum mechanical systems with shape-invariant potentials (e.g., the harmonic oscillator and the hydrogenic systems). With the help of a sequence of some generalized Jacobi polynomials, we define a sequence of discrete probability distributions. We introduce fractal Kullback–Leibler divergence, fractal Tsallis divergence, and fractal Rényi divergence between every element of the sequence of probability distributions introduced above and the element of the equiprobability distribution corresponding to the same index. Practically, we obtain three sequences of fractal divergences and show that the first two are convergent and the last is divergent.

Keywords: fractal Kullback–Leibler divergence; fractal Tsallis divergence; fractal Rényi divergence; orthogonal polynomials; generalized Jacobi polynomials

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1. Introduction

In order to define the fractals, two methods in the literature are used. The first method was introduced by the French mathematicians Fatou (see [1]) and Julia (see [2]) at the beginning of the 20th century, who used the complex function theory in the study of the iteration of rational functions on a Riemann sphere. Mandelbrot [3] put into real light these two papers (which remained practically not quite understood until the second half of the 20th century) and generated a vast array in the literature and domain (see [4,5]). The second method was introduced by the Australian mathematician Hutchinson (see [6]), who used general topology and functional analysis. Afterwards, Barnsley (see [7]), another Australian mathematician, continued this work and contributed in a decisive manner to its dissemination. The main idea of this method was to consider the so-called iterated function systems (IFS) and find their fixed points. For a given IFS, Hutchinson’s theory generates the fractal set and the fractal measure using the Banach–Caccioppoli–Picard contraction principle. So, the fractal measure is the fixed point of the Markov operator associated to the respective IFS. A short survey of the two aforementioned fractals can be found in [8]. For recent results concerning this topic, readers can consult [9–13].

The entropy and fractal dimension are measures of complexity (see [14–16]). For example, the entropy can be used to measure the multiscale complexity (see [17,18]). In the multifractal dimension set, the macro state entropy is used on the capacity dimension, and the information dimension is based on Shannon’s information entropy (see [19]). The entropy and fractal dimension are also measures of spatial complexity (see [20–25]).
Stanley and Meakin [26] showed that there are analogies between thermodynamics and multifractals. They also proved in the same paper that Legendre transform is an analog of entropy. Ryabko [27] proved that Hausdorff dimension and Kolmogorov complexity are equivalent to Shannon entropy.

For more information about the relationship between entropy and fractal dimension, readers can consult [28].

Among other applications in many areas, the orthogonal polynomials also constitute an important tool in information theory. The interest began with the modern density functional theory (see [29–31]), which says that the physical and chemical properties of fermionic systems (atoms, molecules, nuclei, solids) can be described using the single-particle probability density. Also, it is known that the wave function of many important systems (for example, $D$–dimensional harmonic oscillator and hydrogen atom) can be expressed using the families of orthogonal polynomials.

There are several discrete measures which are associated with a sequence of orthogonal polynomials. Many times, the analysis of such measures needs the use of entropy (see [32,33]).

The concept of entropy has many generalizations: Tsallis entropy, Rényi entropy, Varma entropy, Kaniadakis entropy, fractional entropy, fractal entropy, natural time entropy, etc. The motivation to study these entropies are their applications, such as earthquakes (see [34–39]), stock exchanges (see [40,41]), plasma (see [42–44]), Markov chains (see [45–47]), astrophysics (see [48,49]), model selection (see [50,51]), combinatorics (see [52,53]), finance (see [54–57]), and Lie symmetries (see [58,59]). Other theoretical results, similar to those proved in this paper can be found in [60–64].

In the literature, there are some papers that study few entropies and divergences (Shannon, Kullback–Leibler, Tsallis, and Rényi) of orthogonal polynomials (see [65–71]).

Dehesa et al. [65] and Yáñez et al. [66] showed that the computation of some entropies can be reduced, many times, to integrals involving orthogonal polynomials.

Aptekarev et al. [67] obtain asymptotic expansions of discrete Shannon entropy corresponding to Chebyshev orthonormal polynomials of the first and second kind. In the proofs can be found many connections with relevant objects from number theory.

Buyarov et al. [68] study the asymptotic behavior of Shannon entropy of Jacobi and Laguerre polynomials with respect to a Jacobi and Laguerre weight function, respectively.

Buyarov et al. [69] develop an algorithm for an effective and accurate numerical computation of Shannon entropy of polynomials that are orthogonal on a segment of the real axis. They pay attention to Gegenbauer polynomials, both for their own interest (as a very representative class of polynomials) and also because of their many applications. The results of some numerical experiments are discussed, illustrating both the accuracy and efficiency of the algorithm proposed here and comparing it with other computing strategies used so far.

Martínez-Finkelshtein et al. [70] are concerned with the asymptotic behavior of Kullback–Leibler divergence (and of Shannon entropy) of generalized Jacobi polynomials. For the particular case of Chebyshev polynomials of the first and second kind, they compare the obtained results with those from [67].

Sfetcu [71] studies the asymptotic behavior of Tsallis and Rényi divergences (and of Tsallis and Rényi entropies, respectively) of generalized Jacobi polynomials.

Dehesa [72] deals with the spread of the hypergeometric orthogonal polynomials along their orthogonality interval. Fisher information, Shannon entropy, Rényi entropy, and their corresponding spreading lengths are analytically expressed in terms of the degree and the parameter(s) of the orthogonality weight function, being closely related to the gradient functional (Fisher) and $L_q$–norms (Shannon and Rényi) of the polynomials. Moreover, there are degree asymptotics for these entropy-like functionals of the three canonical families of the hypergeometric orthogonal polynomials, namely Hermite, Laguerre and Jacobi polynomials. Finally, a number of open related issues, whose solutions are both physico-mathematically and computationally relevant, are identified.
Sobrino and Dehesa [73] solve the various asymptotics of the unweighted and weighted $L_q$ norms of the hypergeometric orthogonal polynomials, taking into account their close connection to the entropy and complexity-like quantities because of their relevance in the information theory of special functions and quantum systems and technologies to facilitate their numerical and symbolic computation.

Sometimes, we cannot examine all the information about complex systems as the physical or observable probabilities become incomplete and do not sum to one. Wang introduced in [74] the fractal entropy, i.e., a generalized entropy, by using a so-called incomplete normalization with an empirical parameter $q$, which is intended to "absorb" the effect of complex correlations or interaction and can be related to the energy of the studied system, namely $S = \sum_{i=1}^{n} p_i^q \log \left( \frac{1}{p_i} \right)$. In general, $q < 1$ and $q > 1$ imply, respectively, the repulsive and attractive effect of the complex correlations. This extensive incomplete statistic is shown to be able to reproduce very well the quantum distributions of correlated heavy electrons in the weak coupling regime.

Motivated by the applications mentioned above, we deal, in this paper, with fractal divergences (they can be also called relative fractal entropies) between the equiprobability distribution and some probability distribution, which is defined with the help of a particular sequence of orthonormal polynomials, namely generalized Jacobi polynomials. More exactly, for any $a \in (1, \infty)$ and a sequence of probability distributions $(\psi_n(x))_n$, where $x \in (-1, 1)$, there are defined fractal Kullback–Leibler divergence $(D_a(\psi_n(x)))_n$, fractal Tsallis divergence $(D_a^T(\psi_n(x)))_n$, and fractal Rényi divergence $(D_a^R(\psi_n(x)))_n$, which prove that the first two sequences are convergent to 0 and the last has infinite limit being divergent.

2. Preliminaries

We denote by $\mathbb{N}$, the set of positive integer numbers; by $\mathbb{Q}$, the set of rational numbers; by $\mathbb{R}$, the set of real numbers; by $\text{GCD}(x, y)$, the greatest common divisor of two positive integer numbers $x$ and $y$; and by log, the natural logarithm function.

Let $n \in \mathbb{N}$, $p = (p_1, \ldots, p_n)$ such that $\sum_{i=1}^{n} p_i = 1$ (i.e. $p$ is a probability distribution) and $a \in (1, \infty)$ (if $a = 1$, readers can consult [70,71]).

Fractal Kullback–Leibler divergence between the probability distributions $p = (p_1, \ldots, p_n)$ and $p^* = \left( \frac{1}{n}, \ldots, \frac{1}{n} \right)$ is given via the following:

$$D_a(p) = \sum_{i=1}^{n} p_i^a \log (np_i).$$

Let $a \in \mathbb{R} \setminus \{1\}$. Tsallis logarithm is given by the following:

$$\log^T(x) = \begin{cases} x^{a-1} - 1, & \text{if } x > 0 \\ \frac{a}{a-1} x, & \text{if } x = 0. \end{cases}$$

Fractal Tsallis divergence between the probability distributions $p = (p_1, \ldots, p_n)$ and $p^* = \left( \frac{1}{n}, \ldots, \frac{1}{n} \right)$ is defined via the following:

$$D_a^T(p) = \sum_{i=1}^{n} p_i^a \log^T(n p_i).$$

Fractal Rényi divergence between the probability distributions $p = (p_1, \ldots, p_n)$ and $p^* = \left( \frac{1}{n}, \ldots, \frac{1}{n} \right)$ is given via the following:
\[ D^R_n(p) = \frac{1}{1 - \alpha} \log \left( \sum_{i=1}^{n} p_i^n (np_i)^{\alpha - 1} \right) = \frac{1}{1 - \alpha} \log \left( \sum_{i=1}^{n} p_i^{n+\alpha-1} n^{\alpha-1} \right). \]

From now on, it is assumed that \( \alpha \geq 0 \).

Let \( u, v \in (-1, \infty), h : (-1, 1) \to (0, \infty) \) an analytic function and
\[ w(x) \overset{\text{def}}{=} (1 - x)^a (1 + x)^b h(x) \text{ for any } x \in (-1, 1). \]

Let \( x = \cos \theta \in (-1, 1) \), with \( \theta \in (0, \pi) \).

Consider a sequence of polynomials as follows:
\[ q_n(x) = k_n x^n + \ldots + k_1 x + k_0, \]
\( n \in \mathbb{N} \cup \{0\}, \ k_i \in \mathbb{R} \) for any \( i \in \{0, 1, \ldots, n\}, k_n > 0 \), such that
\[ \int_{-1}^{1} q_m(x) q_n(x) w(x) dx = \delta_{mn} \]
for any \( m, n \in \mathbb{N} \cup \{0\} \), where \( \delta_{mn} \) is Kronecker symbol given via the following:
\[ \delta_{mn} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n. \end{cases} \]

We use the same terminology as in [70] and call such kind of measures, as well the corresponding orthogonal polynomials, "generalized Jacobi".

The reciprocal of the \( n \)-th Christoffel function is given via the following:
\[ \lambda_n(x) \overset{\text{def}}{=} \frac{1}{\sum_{i=0}^{n-1} (q_i(x))^2}. \]

According to [75], the following relationship is valid uniformly on compact subsets of \((-1, 1)\):
\[ \lim_{n \to \infty} n \lambda_n(x) = \pi w(x) \sqrt{1 - x^2} \]

Let \( \psi_{n,j}(x) = \lambda_n(x) q_{j-1}^2(x) \) for any \( j = 1, 2, \ldots, n \), and the discrete probability distribution is expressed as follows:
\[ \psi_n(x) = (\psi_{n,1}(x), \ldots, \psi_{n,n}(x)). \]

This paper is dedicated to the study of the asymptotic behavior of fractal Kullback–Leibler divergence \( D_n(\psi_n(x)) \), fractal Tsallis divergence \( D^T_n(\psi_n(x)) \), and fractal Rényi divergence \( D^R_n(\psi_n(x)) \), as \( n \to \infty \).

For any \( i \in \{1, \ldots, n\} \), the following asymptotic formula is expressed as follows (see [76]):
\[ q_i(x) = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{w(x)(1 - x^2)}} \left( \cos \left( \left( i + \frac{1}{2} \right) \theta + \varphi(x) - \frac{\pi}{4} \right) + O \left( \frac{1}{t} \right) \right) \]

The above equation is valid uniformly on compact subsets of \((-1, 1)\), where the phase function \( \varphi(x) \) is given via the following:
\[ \varphi(x) \overset{\text{def}}{=} \frac{1}{2} ((u + v) \theta - u \pi) + \frac{\sqrt{1 - x^2}}{2\pi} \int_{-1}^{1} \frac{\log(h(t))}{(t - x) \sqrt{1 - t^2}} dt, \]

The integral is understood in the sense of its principal value, i.e.,
\[ \int_{-1}^{1} \frac{\log(h(t))}{(t - x) \sqrt{1 - t^2}} dt = \lim_{\epsilon \to 0} \left( \int_{-1}^{x-\epsilon} \frac{\log(h(t))}{(t - x) \sqrt{1 - t^2}} dt + \int_{x+\epsilon}^{1} \frac{\log(h(t))}{(t - x) \sqrt{1 - t^2}} dt \right). \]
Let us denote

$$y_i(x) = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{w(x)}(1-x^2)^{\frac{1}{2}}} \cos\left(\left(i + \frac{1}{2}\right)\theta + \varphi(x) - \frac{\pi}{4}\right)$$

for any \(i \in \{0, 1, \ldots, n - 1\}\).

The following lemma is very useful in this paper.

**Lemma 1** (see [70]). Let \(H : \mathbb{N} \to \mathbb{R}\) be a bounded and periodic function with period \(k\). Then, the following will yield:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} H(i) = \frac{1}{k} \sum_{i=0}^{k-1} H(i).$$

### 3. The Asymptotic Behavior of Fractal Kullback–Leibler Divergence

**Theorem 1.** The sequence \((D_n(\psi_n(x)))_n\) is convergent to 0 for any \(x = \cos \theta \in (-1, 1)\).

**Proof.** Let \(x = \cos \theta \in (-1, 1)\). Then

$$D_n(\psi_n(x)) = \sum_{i=1}^{n-1} \left(\lambda_n(x)q_i^2(x)\right)^a \log(n\lambda_n(x)q_i^2(x)) = 
\sum_{i=0}^{n-1} \left(\lambda_n(x)q_i^2(x)\right)^a \left(\log(n) + \log(\lambda_n(x)) + \log\left(q_i^2(x)\right)\right) = 
(\lambda_n(x))^a \log(n) \sum_{i=0}^{n-1} (q_i^2(x))^a + 
(\lambda_n(x))^a \log(\lambda_n(x)) \sum_{i=0}^{n-1} (q_i^2(x))^a + (\lambda_n(x))^a \sum_{i=0}^{n-1} (q_i^2(x))^a \log\left(q_i^2(x)\right) = 
(\lambda_n(x))^a \log(n\lambda_n(x)) \sum_{i=0}^{n-1} (q_i^2(x))^a + (\lambda_n(x))^a \sum_{i=0}^{n-1} (q_i^2(x))^a \log\left(q_i^2(x)\right).$$

In order to obtain the conclusion, it is sufficient to show that the sequences

$$\left(\frac{1}{n} \sum_{i=0}^{n-1} (q_i^2(x))^a\right)_n \text{ and } \left(\frac{1}{n} \sum_{i=0}^{n-1} (q_i^2(x))^a \log\left(q_i^2(x)\right)\right)_n$$

are convergent.

Let \(f_1 : \mathbb{R} \to \mathbb{R}, f_1(x) = x^a\).

Because \(f_1\) is uniformly continuous on compact subsets of \(\mathbb{R}\) and \(\varepsilon_i(x) = o(1)\), the following relationship is obtained

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f_1(y_i(x) + \varepsilon_i(x)) - \frac{1}{n} \sum_{i=0}^{n-1} f_1(y_i(x)) = 0.$$

Let us consider

$$g_n^1(x) = \frac{1}{n} \sum_{i=0}^{n-1} f_1\left(\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{w(x)}(1-x^2)^{\frac{1}{2}}} \cos\left((i + \frac{1}{2})\theta + \varphi(x) - \frac{\pi}{4}\right)\right).$$

It is sufficient to prove that the sequence \((g_n^1(\cos \theta))_n\) is convergent.

**Case 1.** Assume that \(x = \cos \theta \in (-1, 1)\), with \(\frac{\theta}{\pi} \in \mathbb{Q}\). Then \(\frac{\theta}{\pi} = \frac{s}{k}\), where \(s, k \in \mathbb{N}\) with the property that \(s < k\) and such that \(GCD(s, k) = 1\).

Hence, there exist \(p, q\) in \(\mathbb{N} \cup \{0\}\) with the property that \(0 \leq q \leq k - 1\) and such that \(n - 1 = pk + q\).

Applying Lemma 1 for the function
Theorem 2. The sequence $H(n) = f_1 \left( \sqrt{\frac{1}{\pi}} \frac{1}{\sqrt{w(\cos(\pi S/k))}} \cos \left( \frac{1}{2} \left( n + \frac{1}{2} \right) \pi S \frac{k}{\pi} + \phi \left( \cos \left( \frac{\pi S}{k} \right) \right) - \frac{\pi}{4} \right) \right)$

the following equality is obtained

$$
\lim_{n \to \infty} g_n^a \left( \cos \left( \frac{\pi S}{k} \right) \right) = \frac{1}{k} \sum_{i=0}^{k-1} f_1 \left( \sqrt{\frac{1}{\pi}} \frac{1}{\sqrt{w(\cos(\pi S/k))}} \cos \left( \frac{1}{2} \left( i + \frac{1}{2} \right) \pi S \frac{k}{\pi} + \phi \left( \cos \left( \frac{\pi S}{k} \right) \right) - \frac{\pi}{4} \right) \right) \overset{\text{def}}{=} S_{k,a}.
$$

Combining these facts, we conclude that

$$
\left( \frac{1}{n} \sum_{i=0}^{n-1} (q_1^2(x))^a \right)_n
$$

is a convergent sequence.

**Case 2.** Assume that $x = \cos \theta \in (-1, 1)$, with $\frac{\theta}{\pi} \notin \mathbb{Q}$.

Using Kronecker’s Theorem (also known as Kronecker–Weyl’s Theorem) (see [77,78]), the following equality is obtained

$$
\lim_{n \to \infty} g_n^a (\cos \theta) = \int_0^1 f_1 \left( \sqrt{\frac{1}{\pi}} \frac{1}{\sqrt{w(\cos \theta)}} (1 - \cos^2 \theta)^{\frac{1}{k}} \cos(y \pi) \right) dy.
$$

Let $f_2 : \mathbb{R} \to \mathbb{R}$,

$$
f_2(x) = \begin{cases} x^{2a} \log(x^2) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}
$$

Using the same ideas like above with the function $f_2$ instead of $f_1$, we obtain that the sequence

$$
\left( \frac{1}{n} \sum_{i=0}^{n-1} (q_1^2(x))^a \log (q_1^2(x)) \right)_n
$$

is also convergent.

This completes the proof of Theorem 1. $\square$

4. The Asymptotic Behavior of Fractal Tsallis Divergence

**Theorem 2.** The sequence $(D_a^T(\psi_n(x)))_n$ is convergent to 0 for any $x = \cos \theta \in (-1, 1)$.

**Proof.** Let $x = \cos \theta \in (-1, 1)$. Then

$$
D_a^T(\psi_n(x)) = \sum_{i=1}^{n} \left( \lambda_n(x) q_{i-1}^2(x) \right)^a \log^T (n \lambda_n(x) q_{i-1}^2(x)) = 
\sum_{i=0}^{n-1} (\lambda_n(x))^a (q_1^2(x))^a \frac{(n \lambda_n(x) q_{i}^2(x))^{a-1} - 1}{a - 1} = 
\sum_{i=0}^{n-1} (\lambda_n(x))^a (q_1^2(x))^a \frac{(n \lambda_n(x) q_{i}^2(x))^{a-1} - 1 + 1}{a - 1} = 
\sum_{i=0}^{n-1} n^{a-1} (\lambda_n(x))^{a+i-1} q_{i}^2(x) \log^T (q_1^2(x)) \sum_{i=0}^{n-1} \frac{n^{a-1} (\lambda_n(x))^{a+i-1} q_{i}^2(x)}{a - 1} - 
$$
\[
\sum_{i=0}^{n-1} \frac{(\lambda_n(x))^a q_i^2(x)}{\alpha - 1} = (n\lambda_n(x))^{a+\alpha-1} \cdot \frac{1}{n} \sum_{i=0}^{n-1} q_i^2(x) \log^T(q_i^2(x)) + \frac{(n\lambda_n(x))^{a+\alpha-1}}{\alpha - 1} \cdot \frac{1}{n} \sum_{i=0}^{n-1} q_i^2(x) \cdot \frac{1}{n} \sum_{i=0}^{n-1} q_i^2(x).
\]

In order to obtain the conclusion, it is sufficient to show that the sequences
\[
\left( \frac{1}{n} \sum_{i=0}^{n-1} (q_i^2(x))^a \log^T(q_i^2(x)) \right)_{n \to \infty} \quad \text{and} \quad \left( \frac{1}{n} \sum_{i=0}^{n-1} (q_i^2(x))^a \right)_{n \to \infty}
\]
are convergent.

Let \( F_1 : \mathbb{R} \to \mathbb{R}, F_1(x) = x^{2a} \log^T(x^2) \).

Because \( F_1 \) is uniformly continuous on compact subsets of \( \mathbb{R} \) and \( \epsilon_i(x) = o(1) \), the following relationship is obtained:
\[
\lim_{n \to \infty} \left( \frac{1}{n} \sum_{i=0}^{n-1} F_1(y_i(x) + \epsilon_i(x)) - \frac{1}{n} \sum_{i=0}^{n-1} F_1(y_i(x)) \right) = 0.
\]

Let us consider
\[
G_n^1(x) = \frac{1}{n} \sum_{i=0}^{n-1} F_1 \left( \sqrt{\frac{2}{\pi}} \frac{1}{w(x)(1-x^2)^{\frac{a}{2}}} \cos \left( i + \frac{1}{2} \theta + \varphi(x) - \frac{\pi}{4} \right) \right).
\]

It is sufficient to prove that the sequence \( (G_n^1(x))_{n \to \infty} \) is convergent.

**Case 1.** We assume that \( x = \cos \theta \in (-1, 1) \), with \( \frac{\theta}{\pi} \in \mathbb{Q} \). Then \( \frac{\theta}{\pi} = \frac{s}{k} \), where \( s, k \in \mathbb{N} \) with the property that \( s < k \) and such that \( \text{GCD}(s, k) = 1 \).

Hence, there exist \( p, q \) in \( \mathbb{N} \cup \{0\} \) with the property that \( 0 \leq q \leq k - 1 \) and such that \( n - 1 = pk + q \).

Applying Lemma 1 for the function
\[
H(n) = F_1 \left( \sqrt{\frac{2}{\pi}} \frac{1}{w(\cos(\pi s/k))} (1 - \cos^2(\pi s/k)) \right)^{\frac{1}{2}} \cos \left( n + \frac{1}{2} \frac{\pi s}{k} + \varphi(\cos(\pi s/k)) - \frac{\pi}{4} \right)
\]
the following equality is obtained:
\[
\lim_{n \to \infty} G_n^1 \left( \cos \left( \frac{\pi s}{k} \right) \right) = \frac{1}{k} \sum_{i=0}^{k-1} F_1 \left( \sqrt{\frac{2}{\pi}} \frac{1}{w(\cos(\pi s/k))} (1 - \cos^2(\pi s/k)) \right)^{\frac{1}{2}} \cos \left( i + \frac{1}{2} \frac{\pi s}{k} + \varphi(\cos(\pi s/k)) - \frac{\pi}{4} \right) \overset{def}{=} G_{k,a}^1.
\]

Combining these facts, we conclude that
\[
\left( \frac{1}{n} \sum_{i=0}^{n-1} (q_i^2(x))^a \log^T(q_i^2(x)) \right)_{n \to \infty}
\]
is a convergent sequence.

**Case 2.** Assume that \( x = \cos \theta \in (-1, 1) \), with \( \frac{\theta}{\pi} \notin \mathbb{Q} \).

Using Kronecker’s Theorem (also known as Kronecker–Weyl’s Theorem) (see [77,78]), the following equality is obtained:
\[
\lim_{n \to \infty} G_n^1(\cos \theta) = \int_0^1 F_1 \left( \sqrt{\frac{2}{\pi}} \frac{1}{w(\cos \theta)} (1 - \cos^2 \theta)^{\frac{1}{2}} \cos(y/\pi) \right) dy.
\]
Let $F_2 : \mathbb{R} \rightarrow \mathbb{R}$, $F_2(x) = x^{2a}$.

Using the same ideas like above with the function $F_2$ instead of $F_1$, we get that the sequence

$$\left( \frac{1}{n} \sum_{i=0}^{n-1} (q_i^2(x))^a \right)_n$$

is also convergent.

This completes the proof of Theorem 2. \(\square\)

5. The Asymptotic Behavior of Fractal Rényi Divergence

Theorem 3. The sequence $(D^R_a(\varphi_n(x)))_n$ is divergent for any $x = \cos \theta \in (-1, 1)$, more exactly

$$\lim_{n \to \infty} D^R_a(\varphi_n(x)) = \begin{cases} -\infty & \text{if } \alpha \in [0, 1) \\ +\infty & \text{if } \alpha \in (1, \infty). \end{cases}$$

Proof. Let $x = \cos \theta \in (-1, 1)$. Then

$$D^R_a(\varphi_n(x)) = \frac{1}{1 - \alpha} \log \left( \sum_{i=1}^{n} (\lambda_n(x)q_{i-1}^2(x))^{a} (n\lambda_n(x)q_{i-1}^2(x))^{a-1} \right) = \frac{1}{1 - \alpha} \log \left( \sum_{i=0}^{n-1} (\lambda_n(x)q_i^2(x))^{a+1} n^{-a} \right) = \frac{1}{1 - \alpha} \log \left( \sum_{i=0}^{n-1} (n\lambda_n(x))^{a+1} \cdot \frac{1}{n^a} (q_i^2(x))^{a+1} \right) + \frac{1}{1 - \alpha} \log \left( \frac{1}{n^{a-1}} \sum_{i=0}^{n-1} (q_i^2(x))^{a+1} \right).$$

In order to obtain the conclusions, it is sufficient to show that the sequence

$$\left( \frac{1}{n} \sum_{i=0}^{n-1} (q_i^2(x))^{a+1} \right)_n$$

is convergent.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^{2a+2\alpha-2}$.

Because $f$ is uniformly continuous on compact subsets of $\mathbb{R}$ and $\varepsilon_i(x) = o(1)$, the following relationship is obtained:

$$\lim_{n \to \infty} \left( \frac{1}{n} \sum_{i=0}^{n-1} f(y_i(x) + \varepsilon_i(x)) - \frac{1}{n} \sum_{i=0}^{n-1} f(y_i(x)) \right) = 0.$$

Let us consider

$$g_n(x) = \frac{1}{n} \sum_{i=0}^{n-1} f \left( \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{\omega(x)}} (1 - x^2)^{1/2} \cos \left( \left( i + \frac{1}{2} \right) \theta + \varphi(x) - \frac{\pi}{4} \right) \right).$$

It is sufficient to prove that the sequence $(g_n(\cos \theta))_n$ is convergent.

Case 1. Assume that $x = \cos \theta \in (-1, 1)$, with $\theta \in \mathbb{Q}$. Then $\frac{\theta}{\pi} = \frac{s}{k}$, where $s, k \in \mathbb{N}$ with the property that $s < k$ and such that $\text{GCD}(s,k) = 1$.

Hence, there exist $p, q$ in $\mathbb{N} \cup \{0\}$ with the property that $0 \leq q \leq k - 1$ and such that $n - 1 = pk + q$.

Applying Lemma 1 for the function
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\[ H(n) = f \left( \frac{\sqrt{2}}{\pi} \frac{1}{\sqrt{w(\cos(\frac{\pi s}{k}))}} \cos \left( \left( n + \frac{1}{2} \right) \frac{\pi s}{k} + \varphi \left( \cos \left( \frac{\pi s}{k} \right) \right) - \frac{\pi}{4} \right) \right) \]

the following equality is obtained:

\[ \lim_{n \to \infty} g_n \left( \cos \left( \frac{\pi s}{k} \right) \right) = \frac{1}{k} \sum_{i=0}^{k-1} f \left( \frac{\sqrt{2}}{\pi} \frac{1}{\sqrt{w(\cos(\frac{\pi s}{k}))}} \cos \left( \left( i + \frac{1}{2} \right) \frac{\pi s}{k} + \varphi \left( \cos \left( \frac{\pi s}{k} \right) \right) - \frac{\pi}{4} \right) \right) \]

Combining these facts, we conclude that

\[ \left( \frac{1}{n} \sum_{i=0}^{n-1} (q_i^2(x))^{2} \right) \]

is a convergent sequence.

**Case 2.** Assume that \( x = \cos \theta \in (-1, 1) \), with \( \frac{\theta}{\pi} \notin \mathbb{Q} \).

Using Kronecker’s Theorem (also known as Kronecker–Weyl’s Theorem) (see [77,78]), the following equality is obtained

\[ \lim_{n \to \infty} g_n (\cos \theta) = \int_0^1 f \left( \frac{\sqrt{2}}{\pi} \frac{1}{\sqrt{w(\cos \theta)}} \left( 1 - \cos^2 \theta \right)^{\frac{1}{4}} \cos \left( y \frac{\pi}{4} \right) \right) dy. \]

This completes the proof of Theorem 3. \( \square \)

### 6. Conclusions

In this paper, the asymptotic behavior of three sequences of fractal divergences defined using a sequence of some generalized Jacobi polynomials (i.e., fractal Kullback–Leibler divergence, fractal Tsallis divergence, and fractal Rényi divergence) was studied. More precisely, we showed that the first two sequences are convergent to 0, and the last sequence is divergent (however, the limit of this divergent sequence exists).

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