Adaptive Global Synchronization for a Class of Quaternion-Valued Cohen-Grossberg Neural Networks with Known or Unknown Parameters

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Abstract: In this paper, the adaptive synchronization problem of quaternion-valued Cohen–Grossberg neural networks (QVCGNNs), with and without known parameters, is investigated. On the basis of constructing an appropriate Lyapunov function, and utilizing parameter identification theory and decomposition methods, two effective adaptive feedback schemes are proposed, to guarantee the realization of global synchronization of CGQVNNs. The control gain of the above schemes can be obtained using the Matlab LMI toolbox. The theoretical results presented in this work enrich the literature exploring the adaptive synchronization problem of quaternion-valued neural networks (QVNNs). Finally, the reliability of the theoretical schemes derived in this work is shown in two interesting numerical examples.

Keywords: Cohen–Grossberg neural networks; quaternion; adaptive control; synchronization; linear matrix inequality (LMI)

MSC: 34D06

1. Introduction

Over recent decades, Cohen–Grossberg neural networks (CGNNs), which were introduced and explored by Cohen and Grossberg [1], have aroused widespread interest for their potential applications in image processing, control problems, and optimizations, and a variety of outcomes utilizing CGNNs have been reported [2–8]. However, these results have mainly been concentrated in the real (complex) number field, and there are few reports on the exploration of CGNNs with quaternion parameters in the literature [9,10].

A quaternion, initially introduced and explored by Hamilton [11], consists of a real unit and three imaginary units. If we regard a quaternion as an extension of plural space into real space with higher dimensions, that is, treat a quaternion as four-dimensional real space, then problems that cannot be solved in three-dimensional space can be solved by mapping to four-dimensional space, such as gimbal lock, image compression, and so on. It is worth mentioning that, differently from the real number and plural, the multiplication exchange law no longer supports quaternions, which makes it more difficult to explore the dynamics of QVNNs. Therefore, studying the dynamic characteristics of QVNNs is a challenging topic. Recently, many results such as the μ-stability [12,13], stability [14–18], anti-synchronization [19,20], and synchronization [21–26] of QVNNs have been reported. However, to the best of our knowledge, there are few reports on the dynamic characteristics of QVCNNs in the literature. Therefore, it is necessary to further explore QVCNNs.

Synchronization involves coupled systems reaching identical dynamical behaviors at the same time. Since the synchronization of two chaotic systems was first achieved in 1990 by Pecora and Carrol [27] using the drive–response method, chaos synchronization began to
enter the field of vision of researchers. To solve the synchronization problem, scholars have
designed many control methods and obtained various outcomes \[19–26,28–32\]. Based on the
Lyapunov stability theory and introducing a new definition for global quasi-synchronization,
refs. \[21,29\] explored the quasi-synchronization problem of chaotic systems with parameter
mismatch. Based on parameter identification and utilizing adaptive control and Lyapunov
functionals, Ref. \[24\] successfully explored exponential synchronization of CGNNs. For
memristive neural networks (MNNs), Ref. \[22,26\] designed different controllers to explore
fixed-time synchronization in a quaternion field. Through dividing the model into equivalent
real-valued systems, utilizing fixed-time theory, and constructing novel nonlinear feedback
controllers, fixed-time synchronization of a neural network with delays was investigated
in \[23\]. Based on the adaptive control theory and by designing a linear feedback controller
with an update law, Ref. \[32\] achieved the synchronization of two coupled neural networks in
the case of known or unknown parameters. However, reports about the synchronization of
CGQVNNs are rare, let alone adaptive synchronization.

Motivated by the aforementioned factor, the goal of this work was to investigate the
adaptive synchronization control of CGQVNNs, with and without known parameters. The
main contributions of this work are outlined as follows:

(1) Reports focusing on the adaptive synchronization of QVCGNNs are rare. This is
the first exploration of the adaptive synchronization of QVCGNNs with known or
unknown parameters;

(2) Different easy to implement feedback control schemes are designed, to guarantee
adaptive synchronization in the case of known and unknown parameters, utilizing
parameter identification theory, decomposition methods, and Lyapunov theory;

(3) QVNNs are an expansion of RVNNs and CVNNs; hence, the theoretical results
presented in this paper can also be applied to RVNNs and CVNNs.

The remaining contents of this work are summarized as follows: In Section 2, we
introduce the network model and preliminaries. The schemes of adaptive control are given in
Section 3. In Section 4, simulation examples are presented, to demonstrate the correctness of
the results.

Notations: \(Q\) denotes a quaternion-valued field. \(Q^n\) and \(Q^{n\times m}\) denote the \(n\)-dimensional
quaternion-valued vector space and the quaternion-valued matrix space with \(n \times m\) dimen-
sions, respectively. \(\mathbb{R}^n\) and \(\mathbb{R}^{n\times m}\) denote the \(n\)-dimensional real-valued vector space
the real-valued matrix space with \(n \times m\) dimensions, respectively. \(C < 0 (C > 0)\) means that \(C\)
is a negative (positive) definite matrix. The smallest eigenvalue of matrix \(A\) is denoted by \(\lambda_{\min}(A)\).

2. Model Description and Preliminaries

A quaternion \(q \in Q\), consisting of a real part and three imaginary parts, can be
described as
\[ q = q^K + q^I i + q^J j + q^K k, \]
where \(q^K, q^I, q^J, q^K \in \mathbb{R}\). \(i, j, k\) are imaginary units. \(i, j, k\) follow the Hamilton rules:
\[ i^2 = j^2 = k^2 = ijk = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j. \]

For any \(q_i = q_{i1} + q_{i2} i + q_{i3} j + q_{i4} k \in \mathbb{Q}\), \(\overline{q}_i\) means conjugate with \(\overline{q}_i = q_{i1} - q_{i2} i - q_{i3} j - q_{i4} k\), \(|q| = \sqrt{q_{i1}^2 + q_{i2}^2 i + q_{i3}^2 j + q_{i4}^2 k}\) means modulus. \(\|q\| = \sqrt{\sum_{i=1}^{n} |q_i|^2}\) means the
norm, where \(q = (q_1, q_2, ..., q_n) \in \mathbb{Q}^n\).
Thus, model (1) is a special case of the classical CGNNs. Assume that for Assumption 1.

Therefore, how to choose different methods for exploring the dynamic behaviors of QVNNs. According to the Hamilton rules, we can see that the commutative law does not hold for quaternion multiplication. For this reason, natures that hold for the real and complex fields may not be applicable to quaternions, which greatly increases the difficulty of calculation and analysis, and the previous methods dealing with RVNNS and CVNNs cannot be applied directly to QVNNs. Therefore, how to choose different methods for exploring the dynamic behaviors of QVNNs is a challenging problem.

Consider the following simplified QVCGNNs with time delays:

\[
\dot{s}(t) = -C[k(s(t)) - Af(s(t)) - Bg(s(t - \nu)) + \bar{3}(t)],
\]

where \( s(\cdot) = (s_1(\cdot), s_2(\cdot), \ldots, s_n(\cdot)) \) \( \in \mathbb{Q}^n \) represents the state vector. \( C = \text{diag}(c_1, c_2, \ldots, c_n) \) \( \in \mathbb{Q}^{n \times n} \) stands for the amplification gain. \( k(s(\cdot)) = (k_1(s_1(\cdot)), \ldots, k_n(s_n(\cdot))) \) \( : \mathbb{Q}^n \rightarrow \mathbb{Q}^n \) denotes the vector-valued behaved function. \( A, B \in \mathbb{Q}^{n \times n} \) denote the weight matrix and the delayed weight matrix of quaternion, respectively. \( f(s(\cdot)) = (f_1(s_1(\cdot)), \ldots, f_n(s_n(\cdot))) \) \( : \mathbb{Q}^n \rightarrow \mathbb{Q}^n \) represent activation function without and with time delays, respectively. \( \nu \) is the time delay with \( \nu_i \geq 0 \). \( \bar{3}(t) = (\bar{3}_1(t), \bar{3}_2(t), \ldots, \bar{3}_n(t)) \) \( \in \mathbb{Q}^n \) stands for the external input vector.

Remark 2. Compared with typical CGNNs, the activation function in model (1) is constant. Thus, model (1) is a special case of the classical CGNNs.

For research purposes, the following assumptions are given:

**Assumption 1.** Assume that for \( v \in \mathbb{Q}, s \in \mathbb{Q} \), there exist real diagonal matrices \( M^{ij} \in \mathbb{R}^{n \times n} \) (\( i, j = R, I, J, K \)) with \( M^{ij} > 0 \), such that

\[
\begin{align*}
|f^R(s) - f^R(\bar{s})| &\leq M^{RR}|s^R - \bar{s}^R| + M^{RI}|s^I - \bar{s}^I| + M^{RJ}|s^J - \bar{s}^J| + M^{RK}|s^K - \bar{s}^K|, \\
|f^I(s) - f^I(\bar{s})| &\leq M^{IR}|s^R - \bar{s}^R| + M^{II}|s^I - \bar{s}^I| + M^{IJ}|s^J - \bar{s}^J| + M^{IK}|s^K - \bar{s}^K|, \\
|f^J(s) - f^J(\bar{s})| &\leq M^{JR}|s^R - \bar{s}^R| + M^{JI}|s^I - \bar{s}^I| + M^{JJ}|s^J - \bar{s}^J| + M^{JK}|s^K - \bar{s}^K|, \\
|f^K(s) - f^K(\bar{s})| &\leq M^{KR}|s^R - \bar{s}^R| + M^{KI}|s^I - \bar{s}^I| + M^{KJ}|s^J - \bar{s}^J| + M^{KK}|s^K - \bar{s}^K|, 
\end{align*}
\]

Similarly, for \( g(\cdot) \), there exist real diagonal matrices \( N^{ij} \in \mathbb{R}^{n \times n} \) (\( i, j = R, I, J, K \)) with \( N^{ij} > 0 \) such that

\[
\begin{align*}
|g^R(s) - g^R(\bar{s})| &\leq N^{RR}|s^R - \bar{s}^R| + N^{RI}|s^I - \bar{s}^I| + N^{RJ}|s^J - \bar{s}^J| + N^{RK}|s^K - \bar{s}^K|, \\
|g^I(s) - g^I(\bar{s})| &\leq N^{IR}|s^R - \bar{s}^R| + N^{II}|s^I - \bar{s}^I| + N^{IJ}|s^J - \bar{s}^J| + N^{IK}|s^K - \bar{s}^K|, \\
|g^J(s) - g^J(\bar{s})| &\leq N^{JR}|s^R - \bar{s}^R| + N^{JI}|s^I - \bar{s}^I| + N^{JJ}|s^J - \bar{s}^J| + N^{JK}|s^K - \bar{s}^K|, \\
|g^K(s) - g^K(\bar{s})| &\leq N^{KR}|s^R - \bar{s}^R| + N^{KI}|s^I - \bar{s}^I| + N^{KJ}|s^J - \bar{s}^J| + N^{KK}|s^K - \bar{s}^K|.
\end{align*}
\]

**Assumption 2.** Assume that for \( v \in \mathbb{Q} \), \( k(s) \) can be expressed as

\[
k(s) = k^R(s) + k^I(s)i + k^J(s)j + k^K(s)k.
\]
Lemma 1 ([33]). Assume that $\Omega_1, \Omega_2, \text{and} \Omega_3$ are real matrices with appropriate dimensions and $\Omega_3$ satisfies $\Omega_3 > 0$, then for $\forall x, y \in \mathbb{R}^n$, one has

$$2x^T \Omega_1^T \Omega_2 y \leq x^T \Omega_1^T \Omega_3 \Omega_1 x + y^T \Omega_2^T \Omega_3^{-1} \Omega_2 y.$$ 

Then, the response system is designed as follows:

$$\dot{s}(t) = -C[k(\tilde{s}(t)) - A\hat{f}(\tilde{s}(t)) - B\tilde{g}(\tilde{s}(t - \nu)) + \mathcal{J}(t)] + U(t),$$

where $\tilde{s}(t) = (\tilde{s}_1(t), \tilde{s}_2(t), \ldots, \tilde{s}_n(t)) \in \mathbb{Q}^n$ is the response state vector, $U(t) = (U_1(t), U_2(t), \ldots, U_n(t)) \in \mathbb{Q}^n$ is the external control input.

Define $X(t) = \tilde{s}(t) - s(t)$ as the synchronization error signal between the drive and response system, then

$$\dot{X}(t) = -C[\tilde{k}(X(t)) - A\tilde{f}(X(t)) - B\tilde{g}(X(t - \nu))] + U(t),$$

where $\tilde{k}(X(t)) = k(\tilde{s}(t)) - k(s(t)), \tilde{f}(X(t)) = f(\tilde{s}(t)) - g(s(t)), \tilde{g}(X(t - \nu)) = g(\tilde{s}(t - \nu)) - g(s(t - \nu)).$

For reasons of simplicity, we show $X = X(t), X_f = X(t - \nu), \tilde{k}(X(t)) = \tilde{k}(X), U = U(t).$ Let $X(t) = X^R(t) + X^I(t) + iX^I(t) + X^K(t)k$. Then, the error system can be divided into the following equivalent system:

$$
\begin{align*}
\dot{X}^R &= K^R(X) + A^R \tilde{f}^R(X) + A^I \tilde{f}^I(X) + A^K \tilde{f}^K(X) + B^R \tilde{g}^R(X) + B^I \tilde{g}^I(X) + B^K \tilde{g}^K(X) + U^R, \\
\dot{X}^I &= K^I(X) - A^I \tilde{f}^R(X) + A^R \tilde{f}^I(X) + A^K \tilde{f}^K(X) - B^I \tilde{g}^R(X) + B^I \tilde{g}^I(X) + B^K \tilde{g}^K(X) + U^I, \\
\dot{X}^K &= K^K(X) - A^K \tilde{f}^R(X) + A^I \tilde{f}^I(X) + A^K \tilde{f}^K(X) - B^K \tilde{g}^R(X) + B^K \tilde{g}^I(X) + B^K \tilde{g}^K(X) + U^K,
\end{align*}
$$

where

$$
\begin{align*}
K^R(X) &= -C^R \tilde{k}^R(X) + C^I \tilde{k}^I(X) + C^K \tilde{k}^K(X) + C^{K^R} \tilde{k}^R(X), \\
K^I(X) &= -C^R \tilde{k}^R(X) - C^I \tilde{k}^I(X) - C^K \tilde{k}^K(X) + C^{K^I} \tilde{k}^I(X), \\
K^K(X) &= -C^R \tilde{k}^R(X) - C^I \tilde{k}^I(X) + C^K \tilde{k}^K(X) - C^{K^K} \tilde{k}^K(X), \\
B^R &= C^B R - C^I B^I - C^K B^K, \quad B^I = -C^B R - C^I B^I - C^K B^K, \quad B^K = -C^B R - C^I B^I - C^K B^K, \\
U^R &= \xi^R(t)X^R(t) - K^R(X(t)), \\
U^I &= \xi^I(t)X^I(t) - K^I(X(t)), \\
U^K &= \xi^K(t)X^K(t) - K^K(X(t))
\end{align*}
$$

To achieve global synchronization, we design a control scheme, as follows:
where $X'(t) = (X_1'(t), X_2'(t), \ldots, X_n'(t)) \in \mathbb{R}^n$, $\zeta_i'(t) = \text{diag}(\zeta_1'(t), \zeta_2'(t), \ldots, \zeta_n'(t)) \in \mathbb{R}^{n \times n}$, $(i = R, I, J, K)$ are the coupling strength matrices that are updated on the basis of the following adaptive law:

$$
\begin{align*}
\begin{cases}
\dot{\zeta}_R(t) &= -\omega_R (X_R(t))^2 \\
\dot{\zeta}_I(t) &= -\omega_I (X_I(t))^2 \\
\dot{\zeta}_J(t) &= -\omega_J (X_J(t))^2 \\
\dot{\zeta}_K(t) &= -\omega_K (X_K(t))^2
\end{cases}
\end{align*}
$$

(7)

where $\omega_R, \omega_I, \omega_J, \omega_K$ are positive constants to be determined later, for $p = 1, 2, \ldots, n$. In addition, let $\omega^j = \text{diag}(\omega_{1}^j, \omega_{2}^j, \ldots, \omega_{n}^j)$ ($i = R, I, J, K$).

3. Main Results

In this section, several criteria are proposed for achieving global synchronization of systems (1) and (4) with known and unknown parameters through designing effective controllers.

**Theorem 1.** Assume that Assumption 1 holds if a positive matrix $S$ and four positive diagonal matrices $\omega^R$, $\omega^I$, $\omega^J$, $\omega^K$ exist, such that

$$
\Lambda = \left(\begin{array}{cccccccc}
\Lambda_{11} & O & O & O & \Lambda_{15} & \Lambda_{16} & \Lambda_{17} & \Lambda_{18} \\
* & \Lambda_{22} & O & O & \Lambda_{25} & \Lambda_{26} & \Lambda_{27} & \Lambda_{28} \\
* & * & \Lambda_{33} & O & \Lambda_{35} & \Lambda_{36} & \Lambda_{37} & \Lambda_{38} \\
* & * & * & \Lambda_{44} & \Lambda_{45} & \Lambda_{46} & \Lambda_{47} & \Lambda_{48} \\
* & * & * & * & -S & O & O & O \\
* & * & * & * & * & -S & O & O \\
* & * & * & * & * & * & -S & O \\
* & * & * & * & * & * & * & -S
\end{array}\right) < 0,
$$

(8)

where

$$
\begin{align*}
\Lambda_{11} &= \frac{1}{2} \sum_{i=R,I}^{K} \sum_{j=R,I}^{K} \mathcal{A}^i(A^j)^T + 2(\sum_{i=R,I}^{K} \sum_{j=R,I}^{K} \mathcal{M}^{ij}) + S - \omega^R, \\
\Lambda_{22} &= \frac{1}{2} \sum_{i=R,I}^{K} \sum_{j=R,I}^{K} \mathcal{A}^i(A^j)^T + 2(\sum_{i=R,I}^{K} \sum_{j=R,I}^{K} \mathcal{M}^{ij}) + S - \omega^I, \\
\Lambda_{33} &= \frac{1}{2} \sum_{i=R,I}^{K} \sum_{j=R,I}^{K} \mathcal{A}^i(A^j)^T + 2(\sum_{i=R,I}^{K} \sum_{j=R,I}^{K} \mathcal{M}^{ij}) + S - \omega^J, \\
\Lambda_{44} &= \frac{1}{2} \sum_{i=R,I}^{K} \sum_{j=R,I}^{K} \mathcal{A}^i(A^j)^T + 2(\sum_{i=R,I}^{K} \sum_{j=R,I}^{K} \mathcal{M}^{ij}) + S - \omega^K, \\
\Lambda_{15} &= \frac{1}{2} |B^R|N_{RR} + \frac{1}{2} |B^I|N_{IR} + \frac{1}{2} |B^J|N_{JR} + \frac{1}{2} |B^K|N_{KR}, \\
\Lambda_{16} &= \frac{1}{2} |B^R|N_{RI} + \frac{1}{2} |B^I|N_{II} + \frac{1}{2} |B^J|N_{JI} + \frac{1}{2} |B^K|N_{KI}, \\
\Lambda_{17} &= \frac{1}{2} |B^R|N_{RJ} + \frac{1}{2} |B^I|N_{JI} + \frac{1}{2} |B^J|N_{JJ} + \frac{1}{2} |B^K|N_{JJ}, \\
\Lambda_{18} &= \frac{1}{2} |B^R|N_{RK} + \frac{1}{2} |B^I|N_{IK} + \frac{1}{2} |B^J|N_{JK} + \frac{1}{2} |B^K|N_{KK}, \\
\Lambda_{25} &= \frac{1}{2} |B^R|N_{IR} + \frac{1}{2} |B^I|N_{RR} + \frac{1}{2} |B^J|N_{KR} + \frac{1}{2} |B^K|N_{IR}, \\
\Lambda_{26} &= \frac{1}{2} |B^R|N_{II} + \frac{1}{2} |B^I|N_{RI} + \frac{1}{2} |B^J|N_{JI} + \frac{1}{2} |B^K|N_{II},
\end{align*}
$$

Then, under the control function of the controller (6) and corresponding adaptive rule (7), systems (1) and (4) will be globally synchronized.

**Proof.** First, we define the Lyapunov function through the following:

\[
V(t) = \frac{1}{2}((X(R)(t))^T X(R)(t) + (X(I)(t))^T X(I)(t) + (X(I)(t))^T X(I)(t)) \\
+ \int_{t-v}^{t} ((X(R)(v))^T S X(R)(v) + (X(I)(v))^T S X(I)(v) + (X(I)(v))^T S X(I)(v)) \\
+ (X(K)(v))^T S X(K)(v))dv + \frac{1}{2} \sum_{p=1}^{n} \frac{1}{\omega_p^R} (\xi^R_p(t) + \omega_p^R)^2 + \frac{1}{2} \sum_{p=1}^{n} \frac{1}{\omega_p^I} (\xi^I_p(t) + \omega_p^I)^2 \\
+ \frac{1}{\omega_p^K} (\xi^K_p(t) + \omega_p^K)^2}
\]

Combining with Lemma 2, the derivative of \(V(t)\) with respect to \(t\) through the solutions of system (5) is as follows:

\[
\dot{V}(t) \leq ((X(R)(t))^T (\Theta + S) X(R)(t) + (X(I)(t))^T (\Theta + S) X(I)(t) + (X(I)(t))^T (\Theta + S) X(I)(t)) \\
+ (X(K)(t))^T (\Theta + S) X(K)(t) + 2[\tilde{f}^R(X(t))]^T f^R(X(t)) + [\tilde{f}^I(X(t))]^T f^I(X(t)) \\
+ [\tilde{f}^J(X(t))]^T f^J(X(t)) + [\tilde{f}^K(X(t))]^T f^K(X(t))) - \sum_{p=1}^{n} \omega_p^R (X^R_p(t))^2 - \sum_{p=1}^{n} \omega_p^I (X^I_p(t))^2 \\
- \sum_{p=1}^{n} \omega_p^K (X^K_p(t))^2 - (X(R)(t))^T S X(R)(t-v) \\
- (X(I)(t-v))^T S X(I)(t-v) - (X(I)(t-v))^T S X(I)(t-v) - (X(K)(t-v))^T S X(K)(t-v) \\
+ (X(R)(t))^T B^R S^R(X(t-v)) + (X(R)(t))^T B^I S^R(X(t-v)) + (X(I)(t))^T B^R S^R(X(t-v)) \\
+ (X(I)(t))^T B^I S^R(X(t-v)) + (X(I)(t))^T B^I S^I(X(t-v)) + (X(I)(t))^T B^I S^I(X(t-v)) \\
- (X(K)(t))^T B^R S^K(X(t-v)) + (X(K)(t))^T B^I S^I(X(t-v)) - (X(K)(t))^T B^I S^I(X(t-v)) \\
+ (X(K)(t))^T B^R S^K(X(t-v)) + (X(K)(t))^T B^R S^K(X(t-v)) + (X(K)(t))^T B^R S^K(X(t-v))}
\]
where \( \Theta = \frac{1}{2}A^R (A^R)^T + \frac{1}{2}A^I (A^I)^T + \frac{1}{2}A^J (A^J)^T + \frac{1}{2}A^K (A^K)^T. \)

In the light of (2), for \( i = R, I, J, K, \) we can obtain:

\[
2[\tilde{f}(X(t))]^T \tilde{f}(X(t)) \leq 2(M^R + M^I + M^J + M^K) ([X^R(t)]^T X^R(t) + [X^I(t)]^T X^I(t) + [X^J(t)]^T X^J(t))
\]


In the light of (3), the following inequality holds:

\[
(X^R(t))^T B^R \tilde{g}^R(X(t-v)) + (X^I(t))^T B^J \tilde{g}^J(X(t-v)) + (X^J(t))^T B^K \tilde{g}^K(X(t-v)) \leq (X^R(t))^T \{B^R N^{RR} + B^I N^{RI} + B^J N^{RJ} + B^K N^{RK}\} X^R(t-v) + (X^I(t))^T \{B^R N^{RI} + B^I N^{II} + B^J N^{IJ} + B^K N^{IK}\} X^I(t-v) + (X^J(t))^T \{B^R N^{RJ} + B^I N^{II} + B^J N^{IJ} + B^K N^{IK}\} X^J(t-v) + (X^K(t))^T \{B^R N^{RK} + B^I N^{IK} + B^J N^{IK} + B^K N^{KK}\} X^K(t-v)
\]

Based on (10)–(12), we obtain:

\[
V(t) \leq (X^R(t))^T (\Theta + 2M + S) X^R(t) + (X^I(t))^T (\Theta + 2M + S) X^I(t) + (X^J(t))^T (\Theta + 2M + S) X^J(t)
\]

\[
+ (X^K(t))^T (\Theta + 2M + S) X^K(t) + (X^R(t))^T \left( \sum_{i=R}^{K} B^i N^{Ri} \right) X^R(t-v) + (X^I(t))^T \left( \sum_{i=R}^{K} B^i N^{IIi} \right) X^I(t-v)
\]

\[
+ (X^J(t))^T \left( \sum_{i=R}^{K} B^i N^{IKi} \right) X^J(t-v) + (X^K(t))^T \left( \sum_{i=R}^{K} B^i N^{KKi} \right) X^K(t-v)
\]
\[ + (X^l(t))^T \{ B^R N^{iR} + B^I N^{iR} + B^R N^{Kl} + B^K N^{iR} \} X^R(t - \nu) + (X^l(t))^T \{ B^R N^{iI} + B^I N^{iI} + B^K N^{iI} \} X^I(t - \nu) + (X^l(t))^T \{ B^R N^{iK} + B^I N^{iK} + B^K N^{iK} \} X^K(t - \nu) + (X^l(t))^T \{ B^R N^{lR} + B^K N^{lR} + B^R N^{lI} + B^I N^{lI} \} X^R(t - \nu) + (X^l(t))^T \{ B^R N^{lI} + B^K N^{lI} + B^R N^{lK} + B^I N^{lK} \} X^I(t - \nu) + (X^l(t))^T \{ B^R N^{lK} + B^K N^{lK} + B^R N^{lI} + B^I N^{lI} \} X^K(t - \nu) + (X^k(t))^T \{ B^R N^{iR} + B^I N^{iR} + B^K N^{iR} \} X^R(t - \nu) + (X^k(t))^T \{ B^R N^{iI} + B^I N^{iI} + B^K N^{iI} \} X^I(t - \nu) + (X^k(t))^T \{ B^R N^{iK} + B^I N^{iK} + B^K N^{iK} \} X^K(t - \nu) + (X^k(t))^T \{ B^R N^{lR} + B^I N^{lR} + B^K N^{lR} \} X^R(t - \nu) + (X^k(t))^T \{ B^R N^{lI} + B^I N^{lI} + B^K N^{lI} \} X^I(t - \nu) + (X^k(t))^T \{ B^R N^{lK} + B^I N^{lK} + B^K N^{lK} \} X^K(t - \nu) - (X^R(t - \nu))^T S X^R(t - \nu) - (X^I(t - \nu))^T S X^I(t - \nu) - (X^K(t - \nu))^T S X^K(t - \nu) - \sum_{p=1}^n \omega_p^R(X_p^R(t))^2 - \sum_{p=1}^n \omega_p^I(X_p^I(t))^2 - \sum_{p=1}^n \omega_p^K(X_p^K(t))^2 = (\xi(t))^T \Lambda \xi(t) \]

where

\[ \xi(t) = (X^R(t), X^I(t), X^K(t), X^R(t - \nu), X^I(t - \nu), X^K(t - \nu)), \]

\[ \Omega = \sum_{i=R,I} \sum_{j=R,I} M^{ij}. \]

According to (8), we know that \( \Lambda \) is a negative definite matrix; thus, there exists a constant \( \rho > 0 \) such that

\[ \dot{V}(t) \leq -\rho(\xi(t))^T \xi(t) < 0 \quad (13) \]

where \( \rho = \lambda_{\min}(-\Lambda) > 0 \).

Therefore, the proof is completed and the global synchronization problem of QVCGNNs with known parameters is finally solved.

It is remarkable that certain parameter values of the system cannot be accurately known in practical applications, and such uncertain factors will disrupt synchronization. Therefore, it is of great significance to explore the adaptive synchronization of two QVCGNNs in the case of unknown parameters. Therefore, the drive system defined by us remains the same as (1), and the response system with unknown parameters is as follows:

\[ \dot{s}(t) = -C[k(s(t)) - \tilde{A}f(\tilde{s}(t)) - \tilde{B}g(\tilde{s}(t - \nu)) + \mathcal{J}(t)] + \mathcal{U}(t) \quad (14) \]

where \( \tilde{A} = (\tilde{a}_{ij})_{n \times n} \), \( \tilde{B} = (\tilde{b}_{ij})_{n \times n} \) are uncertain or unknown matrices that we need to estimate.

Here, the synchronization error signal between the response system (14) and drive system (1) remains denoted by \( X(t) = \tilde{s}(t) - s(t) \). Then, the error dynamical system is as follows:
\[ X^R = K^R + A^R \tilde{f}(X) + A^I \tilde{f}(X) + A^K \tilde{f}(X) + B^R \tilde{g}(X) + B^I \tilde{g}(X) \\
A I^R - A^I \tilde{f}(X) + A^K \tilde{f}(X) + B^R \tilde{g}(X) + B^I \tilde{g}(X) \\
\eta \tilde{g}(X) + \eta \tilde{g}(X) + \tilde{g}(X) + U^R, \\
X^I = K^I - A^I \tilde{f}(X) + A^K \tilde{f}(X) - B^R \tilde{g}(X) + B^I \tilde{g}(X) \\
\eta \tilde{g}(X) + \eta \tilde{g}(X) - A^R \tilde{f}(X) + \eta \tilde{g}(X) + B^R \tilde{g}(X) \\
\eta \tilde{g}(X) + \eta \tilde{g}(X) + \tilde{g}(X) + U^I, \\
X^K = K^K - A^K \tilde{f}(X) + A^I \tilde{f}(X) + A^R \tilde{f}(X) - B^R \tilde{g}(X) + B^I \tilde{g}(X) \\
B^R \tilde{g}(X) - B^I \tilde{g}(X) + B^R \tilde{g}(X) + B^I \tilde{g}(X) \\
B^I \tilde{g}(X) + B^R \tilde{g}(X) + \tilde{g}(X) + U^K. \\
\]

where

\[ \begin{align*}
A^R &= C^R(\tilde{A} - A^R) - C^I(\tilde{A}^I - A^I) - C^K(\tilde{A}^K - A^K), \\
A^I &= C^R(\tilde{A}^I - A^I) - C^I(\tilde{A}^I - A^I) - C^K(\tilde{A}^K - A^K), \\
A^K &= C^R(\tilde{A} - A^R) - C^I(\tilde{A} - A^I) - C^K(\tilde{A}^K - A^K), \\
B^R &= C^R(\tilde{B}^R - B^R) - C^I(\tilde{B}^I - B^I) - C^K(\tilde{B}^K - B^K), \\
B^I &= C^R(\tilde{B}^I - B^I) - C^I(\tilde{B}^R - B^R) - C^K(\tilde{B}^K - B^K), \\
B^K &= C^R(\tilde{B}^K - B^K) - C^I(\tilde{B}^I - B^I) - C^K(\tilde{B}^R - B^R),
\end{align*} \]

and \( f(\tilde{s}) = f(\tilde{s}(t)), \ g(\tilde{s}_v) = g(\tilde{s}(t - v)). \)

\[ \square \]

**Theorem 2.** If Assumption 1 holds, the positive matrix \( S \) and four positive diagonal matrices \( \omega^R, \omega^I, \omega^K, \omega^K \) exist, such that inequality (8) holds and the adaptive rules of parameter matrices \( \hat{A} = (\hat{A}_{ij})_{n \times n}, \hat{B} = (\hat{B}_{ij})_{n \times n} \) are designed as

\[ \begin{align*}
\hat{A}^R_{pq} &= -\eta^R_{pq} c^R_{pq}(\tilde{s}) c^R_{pq}(\tilde{s}) - c^R_{pq}(\tilde{s}) c^R_{pq}(\tilde{s}) - c^R_{pq}(\tilde{s}) c^R_{pq}(\tilde{s}) + c^R_{pq}(\tilde{s}) c^R_{pq}(\tilde{s}) \\
\hat{A}^I_{pq} &= -\eta^R_{pq} c^I_{pq}(\tilde{s}) c^I_{pq}(\tilde{s}) + c^I_{pq}(\tilde{s}) c^I_{pq}(\tilde{s}) + c^I_{pq}(\tilde{s}) c^I_{pq}(\tilde{s}) - c^I_{pq}(\tilde{s}) c^I_{pq}(\tilde{s}) \\
\hat{A}^K_{pq} &= -\eta^R_{pq} c^K_{pq}(\tilde{s}) c^K_{pq}(\tilde{s}) - c^K_{pq}(\tilde{s}) c^K_{pq}(\tilde{s}) - c^K_{pq}(\tilde{s}) c^K_{pq}(\tilde{s}) + c^K_{pq}(\tilde{s}) c^K_{pq}(\tilde{s}) \\
\end{align*} \]
\[ \dot{b}^K_{pq} = -\sigma_p^R x^R_p[c^R_p s^R_q(\tilde{s}_v) + c^J_p l^J_q(\tilde{s}_v) - c^K_p s^K_q(\tilde{s}_v)] - \sigma_q^K x^K_q[c^K_q s^K_p(\tilde{s}_v) + c^R_q s^R_p(\tilde{s}_v)], \]

\[ \dot{b}^I_{pq} = -\sigma_p^I x^I_p[c^I_p s^I_q(\tilde{s}_v) + c^K_p s^K_q(\tilde{s}_v)] - \sigma_q^I x^I_q[c^I_q s^I_p(\tilde{s}_v) + c^K_q s^K_p(\tilde{s}_v)], \]

\[ \dot{b}^I_{pq} = -\sigma_p^I x^I_p[c^I_p s^I_q(\tilde{s}_v) + c^K_p s^K_q(\tilde{s}_v)] - \sigma_q^I x^I_q[c^I_q s^I_p(\tilde{s}_v) + c^K_q s^K_p(\tilde{s}_v)], \]

where parameters such as \( \xi \) defined in inequality (13).

Calculating the derivative of \( V(t) \) and combining it with the inequalities (11) and (12), as well as Lemma 1 and the adaptive rules of the parameter matrices (16), we obtain

\[ V(t) \leq -\rho(\tilde{\xi}(t))^T \tilde{\xi}(t) < 0, \]

where parameters such as \( \tilde{\xi}(t) \) and \( \rho \) are defined in exactly the same manner as those defined in inequality (13).

Therefore, global synchronization between system (1) and system (14) is realized. \( \Box \)
Remark 3. According to theorems 1 and 2, constructing an appropriate Lyapunov function, and utilizing parameter identification theory, two effective adaptive feedback schemes are proposed, to guarantee the realization of global synchronization for quaternion-valued Cohen–Grossberg neural networks, with and without unknown parameters.

Remark 4. It is worth mentioning that RVNNs and CVNNs are special cases of QVNNs, where $a^I, a^J, a^K = 0$ and $a^I, a^K = 0$, respectively. Therefore, the adaptive synchronization criteria proposed in this paper could be applied to the problem of synchronization of other RVNNs and CVNNs.

4. Numerical Examples

In order to demonstrate the effectiveness and reliability of the synchronization schemes proposed in this paper, two interesting examples are given in this section.

Example 1. Taking into account the following model as the drive system:

$$
\dot{x}(t) = -C[k(x(t)) - Af(x(t)) - Bg(x(t - \nu)) + 3(t)],
$$

the response system is

$$
\dot{\tilde{x}}(t) = -C[k(\tilde{x}(t)) - Af(\tilde{x}(t)) - Bg(\tilde{x}(t - \nu)) + 3(t)] + U(t)
$$

where

$$
C = \begin{pmatrix}
1 + 1i + 1j + 0.8k & 0 \\
0 & 0.8 + 1i + 1.5j + 0.9k
\end{pmatrix},
$$

$$
A = \begin{pmatrix}
-1.6 - 2.7i - 1.5j - 1.7k & 1.5 + 1.5i + 1.8j + 1.8k \\
1.5 + 1.5i + 1.2j & 0 - 1.3i + 1.6j - 1.6k
\end{pmatrix},
$$

$$
B = \begin{pmatrix}
1.3 + 1i + 1.2j + 0.9k & -1.2 - 1i - 1.2j - 1.8k \\
1.2 - 1.8i + 1j - 1.6k & -1.1 - 1i - 1.5j - 1.5k
\end{pmatrix},
$$

$$
k_1(x_1) = 1.5x_1^R - \sin(x_1^R) + (1.2x_1^I + \cos(x_1^I))i + (1.5x_1^J - 0.7 \sin(x_1^J))e + (1.6x_1^K - \sin(x_1^K))k;
$$

$$
k_2(x_2) = 1.6x_2^R - 1.2\cos(x_2^R) + (1.4x_2^I - 0.9\sin(x_2^I))i + (1.8x_2^J - 1.6\cos(x_2^J))e
$$

$$
+ (1.5x_2^K - 1.7\cos(x_2^K))k;
$$

$$
f_i(x_i) = \frac{1 - e^{-x_i^R}}{1 + e^{-x_i^R}} + \frac{1 - e^{-x_i^I}}{1 + e^{-x_i^I}}i + \frac{1 - e^{-x_i^J}}{1 + e^{-x_i^J}}j + \frac{1 - e^{-x_i^K}}{1 + e^{-x_i^K}}k (i = 1, 2);
$$

$$
g_i(x_i) = \frac{1}{1 + e^{-x_i^R}} + \frac{1}{1 + e^{-x_i^I}}i + \frac{1}{1 + e^{-x_i^J}}j + \frac{1}{1 + e^{-x_i^K}}k (i = 1, 2);
$$

$$
\gamma(t) = \tanh(-2t) - 3.5\cos(2t - 1) + \tanh(-4t - 1) - 1.2\cos(1.5t - 0.75);
$$

and $v_1 = 0.5, v_2 = 1$. $x_1(t) = 1 + 0.9i + 0.5j + 0.8k, \tilde{x}_1(t) = 8.65 + 4.5i - 3.7j + 2.1k, x_2(t) = 1.2 + 1.9i + 2.3j + 2.1k, \tilde{x}_2(t) = 3.2 - 2.5i + 4.2j + 5.9k$, for $t \in [-0.5, 0]$ and $x_3(t) = 1.2 + 1.9i + 2.3j + 2.1k, \tilde{x}_3(t) = 3.2 - 2.5i + 4.2j + 5.9k$, for $t \in [-1, 0]$ is given as initial values.

The error trajectories $X_1^R, X_1^I, X_1^J, X_1^K (i = 1, 2)$ for the drive and response system without control are depicted in Figure 1, from which we can easily observe that the drive–response system has not achieved synchronization.
Figure 1. The error trajectories $X^R_i, X^I_i, X^J_i, X^K_i (i = 1, 2)$ for the drive–response system without control.

In light of Theorem 1, and by utilizing the Matlab LMI toolbox, a set of solutions is obtained, as follows: $\omega^R_1 = 132.1068, \omega^R_2 = 132.1068, \omega^I_1 = 130.1655, \omega^I_2 = 130.1655, \omega^J_1 = 130.1682, \omega^J_2 = 130.1682, \omega^K_1 = 137.3688, \omega^K_2 = 137.3688$. Thus, the conclusion that the drive and response system will achieve global synchronization under the adaptive controller (6) can be drawn.

Choose initial values of control gains $\zeta^R_1(0) = 0.5, \zeta^R_2(0) = 1.5, \zeta^I_1(0) = 0.5, \zeta^I_2(0) = 1.5, \zeta^J_1(0) = 0.3, \zeta^J_2(0) = 1.3, \zeta^K_1(0) = 0.3, \zeta^K_2(0) = 1.3$. Figure 2 shows the trajectories of the synchronization errors between $x^r_i$ and $\tilde{x}^r_i (r = R, I, J, K), (i = 1, 2)$. That is to say, system (1) and system (4) have evidently realized global synchronization.

Figure 2. The trajectories of synchronization errors between $x^r_i$ and $\tilde{x}^r_i (r = R, I, J, K), (i = 1, 2)$.
Example 2. Consider system (1) as the drive system, and give parameters as

\[
C = \begin{pmatrix}
0.6 - 0.2i & 0.8k \\
0 & 0.8 - 0.5i - 0.5j + 0.9k
\end{pmatrix},
\]

\[
A = \begin{pmatrix}
-0.2 - 0.5i - 0.5j - 0.5k & -0.5 + 0.3i + 0.8j + 0.6k \\
0.5 + 0.5i + 0.5j + 0.2k & -0.5 - 0.4i + 0.6j - 0.6k
\end{pmatrix},
\]

\[
B = \begin{pmatrix}
-0.4 + 0.4i - 0.6j + 0.4k & 0.2 - 0.5i - 0.5j - 0.6k \\
-0.8 - 0.6i + 0.6j - 0.2k & 0.6 - 0.4i + 0.5j - 0.8k
\end{pmatrix},
\]

\[
k_1(x_1) = 1.1x_1^R - 0.9\cos(x_1^R) + (1.8x_1^I - 2.7\sin(x_1^I))i + (1.4x_1^I - \cos(x_1^I))j + (1.5x_1^K + \sin(x_1^K))k;
\]

\[
k_2(x_2) = 1.2x_2^R + 0.9\sin(x_2^R) + (1.5x_2^I - 0.6\cos(x_2^I))i + (1.2x_2^I + 0.6\sin(x_2^I))j
\]

\[+ (1.6x_2^K + 0.7\sin(x_2^K))k;\]

\[
f_i(x_i) = \frac{1}{1 + e^{-x_{i1}}} + \frac{1}{1 + e^{-x_{i2}}} i + \frac{1}{1 + e^{-x_{i3}}} j + \frac{1}{1 + e^{-x_{i4}}} k + \tanh(-x_{i5})k \quad (i = 1, 2);\]

\[
g_1(x_1) = \frac{1}{1 + e^{-x_{11}}} + \frac{1}{1 + e^{-x_{12}}} i + \frac{1}{1 + e^{-x_{13}}} j + \frac{1}{1 + e^{-x_{14}}} k + \tanh(-x_{15})k;\]

\[
g_2(x_2) = \frac{1}{1 + e^{-x_{21}}} + \frac{1}{1 + e^{-x_{22}}} i + \frac{1}{1 + e^{-x_{23}}} j + \frac{1}{1 + e^{-x_{24}}} k;\]

\[
\hat{A} = \begin{pmatrix}
\hat{a}_{11} & -0.5 + 0.3i + 0.8j + 0.6k \\
0.5 + 0.5i + 0.5j + 0.2k & \hat{a}_{22}
\end{pmatrix},
\]

\[
\hat{B} = \begin{pmatrix}
\hat{b}_{11} & 0.2 - 0.5i - 0.5j - 0.6k \\
-0.8 - 0.6i + 0.6j - 0.2k & \hat{b}_{22}
\end{pmatrix}.
\]

Choose initial values as \( x_1(t) = 1.26 - 2.4i + 3.5j, x_2(t) = 2.8 + 2.9i - 2.4j - 2.5k, \) for \( t \in [-0.5, 0] \) and \( x_2(t) = -1.2 - 1.7i + 2.2j + 1k, \) for \( t \in [-0.8, 0] \). Then, the error trajectories \( X_i^R, X_i^I, X_i^K (i = 1, 2) \) for the drive and response system without control are depicted in Figure 3, which shows that the drive and response system cannot achieve synchronization.

In light of Theorem 2, and by utilizing the Matlab LMI toolbox, a set of solutions is obtained, as follows: \( \omega_1^R = 171.3329, \omega_2^R = 171.3329, \omega_1^I = 169.6663, \omega_2^I = 169.6663, \omega_1^K = 169.6663, \omega_2^K = 169.6663, \omega_1^I = 176.3329, \omega_2^I = 176.3329, \) which implies that system (1) and system (14) will be globally synchronized under the influence of the adaptive control scheme proposed in Theorem 2.

Choose initial values of control gains \( \zeta_i^R(0) = 4 \) (\( i = R, I, J, K \)), \((p = 1, 2)\). Figure 4 shows the trajectories of the synchronization errors between \( x'_r \) and \( \tilde{x}'_r \) \((r = R, I, J)\), \((K; i = 1, 2)\). Obviously, the drive system and response system have achieved global synchronization under the adaptive control scheme proposed in Theorem 2.
Figure 3. The trajectories of synchronization errors between $x_r^i$ and $\tilde{x}_r^i$ without control $(r = R, I, J, K), (i = 1, 2)$.

Figure 4. The trajectories of synchronization errors between $x_r^i$ and $\tilde{x}_r^i (r = R, I, J, K), (i = 1, 2)$.

5. Conclusions

In this paper, under the precondition of decomposing QVCGNNs into equivalent real-valued systems, the adaptive synchronization criteria of QVCGNNs with and without
known parameters were explored. Using Lyapunov theory and the LMI technique, easy to implement controllers with simple adaptive rules were designed in the case of known parameters. Moreover, aiming at QVCGNNs with unknown parameters, an adaptive feedback control scheme combining adaptive control theory and parameter identification was proposed. Differently from the existing results, this paper discussed the adaptive synchronization of QVCGNNs with known or unknown parameters for the first time. Finally, the correctness of the schemes was demonstrated through two examples and corresponding simulations.

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