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Keller–Osserman Phenomena for Kardar–Parisi–Zhang-Type Inequalities

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Abstract: For coercive quasilinear partial differential inequalities containing nonlinearities of the Kardar–Parisi–Zhang type, we find conditions guaranteeing the absence of global positive solutions. These conditions extend both the classical result of Keller and Osserman and its recent Kon'kov–Shishkov generalization. Additionally, they complement the results for the noncoercive case, which had been previously established by the same author.

Keywords: coercive quasilinear inequalities; KPZ nonlinearities; blow-up

MSC: 35R45; 35J62

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1. Introduction

According to the Pokhozhaev paradigm, blow-up phenomena are equivalent to the absence of global solutions. This approach is based on the following reasoning. It is quite frequent for real models of mathematical physics that an equation (or inequality) is resolvable "in small" (i.e., in a neighborhood of the ground state). In this case, if we can prove that no global solutions exist, then there exists a point where the solution is destroyed, which means the blow-up phenomenon. For a thorough explanation of this approach, readers are addressed to the famous monograph [1] providing the foundation of the global nonexistence theory and containing a lot of blow-up results for various semilinear and quasilinear equations, inequalities, and boundary-value problems.

In this paper, the said phenomena are investigated for coercive inequalities with nonlinearities of the Kardar–Parisi–Zhang-type (KPZ-type nonlinearities), i.e., coercive inequalities containing the second power of the first derivative of the desired function (note that this kind of nonlinearity is not covered by the authors of [1]). The motivation to study KPZ-type nonlinearities is well known; for instance, a comprehensive list of recent publications illustrating their applications not covered by other kinds of nonlinearities (for example, interface dynamics and directed polymer models) is provided in [2], which is a review of various results about equations and inequalities with KPZ-type nonlinearities. Their theoretical value is caused by the following circumstance: the second power is the greatest one such that a priori L_∞ estimates of first-order derivatives of the solution via the L_∞-norm of the solution itself hold (see, for example, [3–5]).

For noncoercive KPZ-type nonlinearities, i.e., for the case where the highest-order linear part is dominated by the (low-order) nonlinear one, the above phenomenon is investigated earlier (see [2] and references therein). The coercive case was an open problem up to now, though the result for semilinear coercive inequalities (studied in regards to the problem of the equilibrium of a charged gas in a container) has been known longer than six decades: in [6,7], a sufficient condition of the absence of global solutions is proved for inequalities of the kind

$$\Delta v \geq \mu(v), \tag{1}$$

where Δ denotes the Laplacian: $\Delta = \dfrac{\partial^2}{\partial x_1^2} + \cdots + \dfrac{\partial^2}{\partial x_n^2}$.

2. Regular Case

In $\mathbb{R}^n$, consider the inequality

$$\Delta u + \sum_{j=1}^{n} g_j(x,u)\left(\frac{\partial u}{\partial x_j}\right)^2 \geq \omega(u), \tag{2}$$

assuming that there exists a function g continuous and locally summable over the positive semiaxis and such that $g_j(x,s) \leq g(s)$ in $\mathbb{R}^n \times (0,\infty), j = 1,2,\dots,n$.

Introduce the function

$$f(s) := \int_0^s e^{\int_0^x g(\tau)d\tau} dx. \tag{3}$$

Then, $f'(s) = e^{\int_0^s g(\tau)d\tau} > 0$, i.e., f is monotone. Hence, the function f^{-1} is well defined on the range of the function f and is monotone as well. Denote f^{-1} by ψ.

The main result of this section is preceded by the following classical theorem (see [6,7]).

Theorem 1 (Keller–Osserman)**.** *If*

$$\int_1^\infty \frac{d\tau}{\sqrt{\int_1^\tau \mu(s)ds}} < \infty, \tag{4}$$

then inequality (1) *has no positive global solutions.*

The following assertion is valid.

Theorem 2. *If*

$$\int_1^\infty \frac{d\tau}{\sqrt{\int_1^\tau \frac{\omega[\psi(s)]}{\psi'(s)}ds}} < \infty,$$

then inequality (2) *has no positive global solutions.*

Proof. Suppose, to the contrary, that the assumptions of the theorem are satisfied, but there exists a positive function $u(x)$ satisfying inequality (2) in $\mathbb{R}^n$. Then, the function $u(x)$ satisfies the inequality

$$\Delta u + g(u)|\nabla u|^2 \geq \omega(u) \tag{5}$$

in $\mathbb{R}^n$ as well.

Introduce $v(x) := f[u(x)]$, where the function f is defined by relation (3). Then,

$$\frac{\partial v}{\partial x_j} = f'(u)\frac{\partial u}{\partial x_j} \text{ and } \frac{\partial^2 v}{\partial x_j^2} = f''(u)\left(\frac{\partial u}{\partial x_j}\right)^2 + f'(u)\frac{\partial^2 u}{\partial x_j^2}, j = \overline{1,n}.$$

Further, $f''(s) = g(s)e^{\int_0^s g(\tau)d\tau}$, i.e., $g(s) = \frac{f''(s)}{f'(s)}$ and, therefore,

$$\Delta v = f'(u)\left[\Delta u + g(u)|\nabla u|^2\right], \text{ i.e., } \psi'(v)\Delta v = \left[\Delta u + g(u)|\nabla u|^2\right].$$

Now, taking into account that $v(x)$ is positive provided that $u(x)$ is positive, we conclude that $v(x)$ is a positive solution of inequality (1) with $\mu(s) = \dfrac{\omega[\psi(s)]}{\psi'(s)}$. However, due to Theorem 1, inequality (1) has no global positive solutions under Condition (4).

The obtained contradiction completes the proof. □

3. Singular Case

The case where $g(s) = \dfrac{\text{const}}{s}$ is not covered by the previous section because the local integrability condition is violated. In that case, another ansatz is used. More exactly, the following assertion is valid.

Theorem 3. *If there exists α from $(-1,\infty)$ such that $g_j(x,s) \le \dfrac{\alpha}{s}$ in $\mathbb{R}^n \times (0,\infty)$, $j = 1,2,\dots,n$, and*

$$\int_1^\infty \frac{d\tau}{\sqrt{\int_1^{\tau^{\frac{1}{\alpha+1}}} s^{2\alpha}\omega(s)\,ds}} < \infty, \tag{6}$$

then inequality (2) has no positive global solutions.

Proof. Suppose, to the contrary, that the assumptions of the theorem are satisfied, but there exists a positive function $u(x)$ satisfying inequality (2) in $\mathbb{R}^n$. Then, the function $u(x)$ satisfies inequality

$$\Delta u + \frac{\alpha}{u}|\nabla u|^2 \ge \omega(u) \tag{7}$$

in $\mathbb{R}^n$ as well.

Denoting $u^{\alpha+1}(x)$ by $v(x)$, we see that

$$\frac{\partial v}{\partial x_j} = (\alpha+1)u^\alpha \frac{\partial u}{\partial x_j},\ \frac{\partial^2 v}{\partial x_j^2} = \alpha(\alpha+1)u^{\alpha-1}\left(\frac{\partial u}{\partial x_j}\right)^2 + (\alpha+1)u^\alpha \frac{\partial^2 u}{\partial x_j^2}\ j = \overline{1,n},\ \Delta v = (\alpha+1)u^\alpha\left[\Delta u + \frac{\alpha}{u}|\nabla u|^2\right],$$

and, therefore,

$$\frac{\Delta v}{(\alpha+1)u^\alpha} \ge \omega(u).$$

Now, taking into account that $u(x) = v^{\frac{1}{\alpha+1}}(x)$, and $u(x)$ is positive everywhere, we conclude that $v(x)$ is a positive solution of the inequality

$$\Delta v \ge (\alpha+1)v^{\frac{\alpha}{\alpha+1}}\omega\left(v^{\frac{1}{\alpha+1}}\right). \tag{8}$$

Due to [6,7], the last inequality has no global positive solutions provided that

$$\int_1^\infty \frac{d\tau}{\sqrt{\int_1^\tau \rho^{\frac{\alpha}{\alpha+1}}\omega\left(\rho^{\frac{1}{\alpha+1}}\right)d\rho}} < \infty. \tag{9}$$

Now, consider inequality (6) and use the substitution $s = \rho^{\frac{1}{\alpha+1}}$ in its internal integral. We see that Condition (6) implies the validity of inequality (9). Hence, inequality (8) has no global positive solutions.

The obtained contradiction completes the proof. □

4. Critical Case

If $\alpha = -1$ in inequality (7), then the substitution from the previous section cannot be used. However, the following (weaker) result is still valid for this critical case.

Theorem 4. *Let* $g_j(x,s) \le -\frac{1}{s}$ *in* $\mathbb{R}^n \times (0,\infty), j = 1,2,\ldots,n$, *and there exists a positive constant* β *such that*

$$\int_1^{\infty} \frac{d\tau}{\sqrt{\int\limits_{\beta e}^{\beta e^{\tau}} \frac{\omega(s)}{s^2} ds}} < \infty. \tag{10}$$

Then, inequality (2) *has no global solutions exceeding* β *everywhere.*

Proof. Suppose, to the contrary, that the assumptions of the theorem are satisfied, but there exists a function $u(x)$ satisfying inequality (2) in $\mathbb{R}^n$ such that $u(x) > \beta$ in $\mathbb{R}^n$. Then, the function $u(x)$ satisfies the inequality

$$\Delta u - \frac{|\nabla u|^2}{u} \ge \omega(u) \tag{11}$$

in $\mathbb{R}^n$ as well.

Denoting $v(x) = \ln \frac{u(x)}{\beta}$ by $v(x)$, we see that $v(x)$ is positive everywhere,

$$\frac{\partial v}{\partial x_j} = \frac{1}{u(x)}\frac{\partial u}{\partial x_j} = \frac{\beta}{u(x)}\frac{1}{\beta}\frac{\partial u}{\partial x_j}, \quad \text{and} \quad \frac{\partial^2 v}{\partial x_j^2} = \frac{\frac{\partial^2 u}{\partial x_j^2}u - \left(\frac{\partial u}{\partial x_j}\right)^2}{u^2(x)} = \frac{1}{u(x)}\left[\frac{\partial^2 u}{\partial x_j^2} - \frac{1}{u(x)}\left(\frac{\partial u}{\partial x_j}\right)^2\right], j = \overline{1,n}.$$

Therefore,

$$\Delta v = \frac{1}{u}\left(\Delta u - \frac{|\nabla u|^2}{u}\right),$$

i.e., the left-hand side of inequality (11) is equal to $u\Delta v$.

Now, taking into account that $e^v = \frac{u}{\beta}$, we conclude that $u(x) = \beta e^v$. Thus, $\beta e^v \Delta v \ge \omega\left(\beta e^v\right)$, i.e., $v(x)$ is a positive solution of the inequality

$$\Delta v \ge \frac{e^{-v}}{\beta}\omega\left(\beta e^v\right). \tag{12}$$

Due to [6,7], this inequality has no global positive solutions provided that

$$\int_1^{\infty} \frac{d\tau}{\sqrt{\int\limits_1^{\tau} \frac{\omega\left(\beta e^{\rho}\right)}{e^{\rho}} d\rho}} < \infty. \tag{13}$$

Now, using the substitution $s = \beta e^{\rho}$, we conclude that Condition (13) is equivalent to Condition (10). Hence, inequality (12) has no global positive solutions.

The obtained contradiction completes the proof. □

5. Examples

5.1. Inequalities with Constant Coefficients at Principal Nonlinear Terms

Since 1957, the classical result of Keller and Osserman was substantially strengthened. In particular, it is extended for the case of variable principal coefficients, i.e., for the case where the left-hand side of inequality (1) is changed for

$$\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} a_{i,j}(x,u), \tag{14}$$

where each principal coefficient satisfies the following restriction for the growth with respect to the second independent variable: $a_{i,j}(x,s) \leq \text{const}|s|$ (see [8]). Since only sufficient blow-up conditions are provided in [8], the question whether the above growth restriction is essential remained open up to now. However, using the above results for KPZ-type inequalities, one can show that the said coefficients are allowed to grow much faster. To do that, it suffices to consider inequality (5), assigning the coefficient $g(s)$ to be equal to a positive constant (denote it by α). That inequality can be represented in the form

$$\sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j} \left(\delta_i^j \frac{1}{\alpha} e^{\alpha u} \right) \geq e^{\alpha u} \omega(u).$$

Its left-hand side is represented in form (14), but the growth restriction is not satisfied: the coefficients are allowed to grow exponentially. However, Theorem 2 provides the following sufficient blow-up condition:

$$\int_1^\infty \frac{d\tau}{\sqrt{\int_1^\tau (\alpha s+1)^2 \omega\left[\frac{\ln(\alpha s+1)}{\alpha}\right] ds}} < \infty.$$

Indeed, in the considered case, we have the relations $f(s) = \int_0^s e^{\alpha x} dx = \frac{e^{\alpha s}-1}{\alpha}$, i.e., $\psi(s) = \frac{\ln(\alpha s+1)}{\alpha}$, $\psi'(s) = \frac{1}{\alpha s+1}$, and, therefore, the function $\mu(s)$ from Condition (4) takes the form

$$e^{\ln(\alpha s+1)} \omega\left[\frac{\ln(\alpha s+1)}{\alpha}\right](\alpha s+1) = (\alpha s+1)^2 \omega\left[\frac{\ln(\alpha s+1)}{\alpha}\right].$$

5.2. Case of Emden–Fowler Nonlinearities at Right-Hand Sides

Consider the following singular case (see Section 3 above), where the right-hand side of the equation is a power function (the so-called nonlinearity of the Emden–Fowler kind):

$$\Delta u + \frac{\alpha}{u}|\nabla u|^2 \geq u^p. \tag{15}$$

According to Theorem 3, this inequality has no global positive solutions provided that

$$\int_1^\infty \frac{d\tau}{\sqrt{\int_1^{\tau^{\frac{1}{\alpha+1}}} s^{2\alpha+p} ds}} < \infty.$$

The internal integral is equal to

$$\left.\frac{s^{2\alpha+p+1}}{2\alpha+p+1}\right|_1^{\tau^{\frac{1}{\alpha+1}}} = \frac{\tau^{\frac{2\alpha+p+1}{\alpha+1}}-1}{2\alpha+p+1} = \frac{\tau^{1+\frac{\alpha+p}{\alpha+1}}-1}{2\alpha+p+1} = \frac{\tau^{q+1}-1}{2\alpha+p+1},$$

where $q = \dfrac{\alpha+p}{\alpha+1}$.

Then, the left-hand side of the last inequality is equal to $\displaystyle\int_1^\infty \frac{d\tau}{\sqrt{\dfrac{\tau^{q+1}-1}{2\alpha+p+1}}}$. Its convergence is equivalent to the convergence of the integral $\displaystyle\int_1^\infty \frac{d\tau}{\sqrt{\tau^{q+1}-1}}$. Apply the substitution $z := \tau^{q+1}-1$. Then, $\tau^{q+1} = z+1$, $\tau = (z+1)^{\dfrac{1}{q+1}}$, and, therefore, $d\tau = \dfrac{(z+1)^{\frac{1}{q+1}-1}}{q+1}dz = \dfrac{dz}{(q+1)(z+1)^{\frac{q}{q+1}}}$. Thus, the last integral is equal to $\frac{1}{(q+1)}\displaystyle\int_0^\infty \frac{dz}{\sqrt{z}(z+1)^{\frac{q}{q+1}}}$. The singularity of the integrand function at the origin is integrable. Its singularity at infinity is integrable under the assumption that $\dfrac{1}{2}+\dfrac{q}{q+1} > 1$, i.e., $q > 1$, which is equivalent to the inequality $p > 1$.

We see that inequality (15) has no global solutions provided that $p > 1$.

To compare this coercive example with the noncoercive case, consider the noncoercive inequality

$$\Delta u + \frac{\alpha}{u}|\nabla u|^2 + u^p \le 0, \tag{16}$$

for positive values of α (assuming that $n \ge 3$).

As in Section 3, assume that there exist its global positive solution $u(x)$ and introduce the function $v(x) := u^{\alpha+1}(x)$. Then $\Delta u + \dfrac{\alpha}{u}|\nabla u|^2 = \dfrac{\Delta v}{(\alpha+1)u^\alpha}$ (see Section 3). Now, taking into account that $u(x) = v(x)^{\frac{1}{\alpha+1}}$, we conclude that

$$\frac{\Delta v}{(\alpha+1)v^{\frac{\alpha}{\alpha+1}}} + v^{\frac{p}{\alpha+1}} \le 0, \text{ i.e., } -\Delta v \ge (\alpha+1)v^{\frac{\alpha+p}{\alpha+1}}.$$

Since $\alpha+1 > 0$, it follows that $v(x)$ is a global positive solution of the inequality

$$-\Delta v \ge v^{\frac{\alpha+p}{\alpha+1}}.$$

According to [9], the last inequality has no global positive solutions provided that $1 < \dfrac{\alpha+p}{\alpha+1} < \dfrac{n}{n-2}$. This condition is equivalent to the condition $1 < p < \dfrac{n+2\alpha}{n-2}$ (cf. the condition $p > 1$ obtained for the *coercive* case above).

6. Conclusions

In this paper, we investigate quasilinear partial differential inequalities of kind (2), where the coefficients $g_i(x,s)$ at the principal nonlinear terms are majorized either by locally summable (with respect to s) functions or by functions with singularities of the kind $\dfrac{\text{const}}{s}$. For both cases, we provide sufficient conditions of the absence of global positive solutions (or, which is the same, necessary conditions of their existence). The obtained results generalize both the classical Keller–Osserman result in [6,7] and its recent Kon'kov–Shishkov extension (see [8]).

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References

1. Mitidieri, È.; Pokhozhaev, S.I. A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities. *Proc. Steklov Inst. Math.* **2001**, *234*, 1–362.
2. Muravnik, A.B. Qualitative properties of solutions of equations and inequalities with KPZ-type nonlinearities. *Mathematics* **2023**, *11*, 990. [CrossRef]
3. Amann, H.; Crandall, M.G. On some existence theorems for semi-linear elliptic equations. *Ind. Univ. Math. J.* **1978**, *27*, 779–790. [CrossRef]
4. Kazdan, I.L.; Kramer, R.I. Invariant criteria for existence of solutions to second-order quasilinear elliptic equations. *Comm. Pure Appl. Math.* **1978**, *31*, 619–645. [CrossRef]
5. Pohožaev, S. Equations of the type $\Delta u = f(x, u, Du)$. *Mat. Sb.* **1980**, *113*, 324–338.
6. Keller, J.B. On solutions of $\Delta u = f(u)$. *Comm. Pure Appl. Math.* **1957**, *10*, 503–510. [CrossRef]
7. Osserman, R. On the inequality $\Delta u \geq f(u)$. *Pac. J. Math.* **1957**, *7*, 1641–1647. [CrossRef]
8. Kon'kov, A.A.; Shishkov, A.E. Generalization of the Keller–Osserman theorem for higher order differential inequalities. *Nonlinearity* **2019**, *32*, 3012–3022. [CrossRef]
9. Gidas, B.; Spruck, J. Global and local behavior of positive solutions of nonlinear elliptic equations. *Comm. Pure Appl. Math.* **1981**, *34*, 525–598. [CrossRef]