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On the Geometry of Kobayashi–Nomizu Type and Yano Type Connections on the Tangent Bundle with Sasaki Metric

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Abstract: In this paper, we address the study of the Kobayashi–Nomizu type and the Yano type connections on the tangent bundle TM equipped with the Sasaki metric. Then, we determine the curvature tensors of these connections. Moreover, we find conditions under which these connections are torsion-free, Codazzi, and statistical structures, respectively, with respect to the Sasaki metric. Finally, we introduce the mutual curvature tensor on a manifold. We investigate some of its properties; furthermore, we study mutual curvature tensors on a manifold equipped with the Kobayashi–Nomizu type and the Yano type connections.

Keywords: Codazzi manifold; Kobayashi–Nomizu type connection; mutual curvature; Yano type connection

MSC: 53C15; 53C25; 53D05



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1. Introduction

Let M be a differentiable manifold, with $p \in M$ and $T_p M$ the tangent space to M at p . The geometry of the tangent bundle $TM = \cup_{p \in M} T_p M$ has a significant number of applications in geometric mechanics, Lagrangian geometry, and mathematical physics. As an example, it can be regarded as a state space. In fact, the first n -tuples of the coordinate system of TM are given by the position of the object and the second n -tuples of its coordinate system are given by the velocity of the object.

The most remarkable geometric object in the Riemannian geometry of TM is the Levi-Civita connection. In the present paper, we consider other two important connections on TM . One of them is the Kobayashi–Nomizu type connection and another is the Yano type connection; then, we study their geometric properties. We also emphasize that these connections are fundamentally defined on a $(J^2 = \pm 1)$ -metric manifold [1].

In this paper, we assume TM is equipped with the Sasaki metric g^S ; then, we construct a $(J^2 = \pm 1)$ -(metric) structure on TM and according to this structure we apply the Kobayashi–Nomizu type and Yano type connections on TM and study their geometric consequences.

The geometry of tangent bundles endowed with certain Riemannian lift metrics has been studied extensively (see [2–6]). The purpose of this paper is to investigate the Kobayashi–Nomizu type and the Yano type connections on TM and their geometric properties [1]. Then, as mentioned in [7], we study the mutual curvatures of TM and their relations with the curvature of M and mutual curvatures of M . We also consider the pair of connections $(\tilde{\nabla}^{kn}, \tilde{\nabla}^C)$ and $(\tilde{\nabla}^{kn}, \tilde{\nabla}^y)$ on TM and study the mutual curvatures of these pairs,

where $\tilde{\nabla}^{kn}$, $\tilde{\nabla}^y$ and $\tilde{\nabla}^C$ are the Kobayashi–Nomizu type, the Yano type, and the complete lift connections on TM , respectively.

Information geometry is a branch of mathematics which relates to the differential geometry and statistics [8]. Moreover, it is connected to applied sciences and pure sciences [8,9]. In this field, we use the methods of differential geometry in probability theory. This new approach of information geometry was initiated by C. R. Rao. He has shown that a statistical model is a Riemannian manifold, endowed with the Fisher information matrix. This means that, on any space of probability distributions, one can define a Riemannian metric. In fact, information geometry is the study of natural geometric structures on families of probability distributions. An important tool in this respect are the statistical connections and statistical manifolds. They can also be applied in computer science, physics, machine learning, neural networks, image processing, statistical mechanics, etc. More precisely, a statistical manifold is a manifold whose points are probability distributions (see [8–11], for more details).

A statistical structure on a (pseudo-)Riemannian manifold (M, g) is a pairing (g, ∇) , where ∇ is a torsion-free affine connection such that ∇g is totally symmetric. A (pseudo-)Riemannian manifold (M, g) endowed with the Levi-Civita connection ∇ with respect to g is the simplest example of a statistical manifold. Then, the statistical manifolds can be regarded as generalizations of (pseudo-)Riemannian manifolds.

This paper is organized as follows: first, we introduce the concept of $(J^2 = \pm 1)$ -(metric) manifolds and then, from a Sasaki type connection, we construct the Kobayashi–Nomizu type connection $\tilde{\nabla}^{kn}$ and the Yano type connection $\tilde{\nabla}^y$ on TM . Then, we study the couples $(g^S, \tilde{\nabla}^{kn})$ and $(g^S, \tilde{\nabla}^y)$ and their geometric consequences on TM . We also find conditions under which these connections are torsion-free, Codazzi, or statistical structures, respectively, on (TM, g^S) . In the second part, we compute all the curvature components of the Kobayashi–Nomizu type and the Yano type connections. Furthermore, we also prove that (TM, g^S) equipped with these connections is a flat space if and only if (M, g) is a flat space with respect to the Levi-Civita connection. Finally, we introduce the mutual curvature of a Riemannian manifold (M, g) and we obtain some interesting properties of it. Then, we study the mutual curvatures of TM for the pairs of connections $(\tilde{\nabla}^{kn}, \tilde{\nabla}^C)$ and $(\tilde{\nabla}^{kn}, \tilde{\nabla}^y)$, where ∇ , ∇^1 and ∇^2 are torsion-free connections on M . In our future work, we will explore how the main results in this paper can be applied in conjunction with soliton theory, submanifold theory, and other related fields (as discussed in [12–23]) to yield more new results.

2. Preliminaries

We recall some basic facts which we use in this paper.

Let (M, g) be an n -dimensional Riemannian manifold and ∇ an affine connection on M . A Codazzi couple on M is a pairing (g, ∇) such that the cubic tensor field $C = \nabla g$ is totally symmetric, i.e., the Codazzi equations hold:

$$(\nabla_X g)(Y, Z) = (\nabla_Y g)(Z, X) = (\nabla_Z g)(X, Y), \quad \forall X, Y, Z \in \chi(M).$$

The triple (M, g, ∇) is called a Codazzi manifold and ∇ is called a Codazzi connection. Furthermore, if ∇ is torsion-free, then we say that (M, g, ∇) is a statistical manifold, (g, ∇) is a statistical couple, and ∇ is a statistical connection. If we denote by $K_{\nabla, \hat{\nabla}}$ the difference tensor of ∇ and the Levi-Civita connection $\hat{\nabla}$ ($K_{\nabla, \hat{\nabla}}(X, Y) = \nabla_X Y - \hat{\nabla}_X Y$), then it follows that $K_{\nabla, \hat{\nabla}}$ is symmetric (i.e., $K_{\nabla, \hat{\nabla}}(X, Y) = K_{\nabla, \hat{\nabla}}(Y, X)$) if and only if ∇ is torsion-free. Furthermore, if ∇ is a statistical connection on (M, g) , then we have $C(X, Y, Z) = -2g(K_{\nabla, \hat{\nabla}}(X, Y), Z)$.

For any torsion-free affine connection ∇ on a Riemannian manifold (M, g) , the dual (conjugate) connection ∇^* is defined by

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X^* Z), \quad \forall X, Y, Z \in \chi(M). \quad (1)$$

Then, we say that (g, ∇, ∇^*) is a dualistic structure on M . It is easy to see that ∇ is torsion-free if and only if ∇^* is torsion-free. Moreover, if ∇ is a statistical connection on (M, g) , then so is ∇^* . In this case, if we consider the cubic tensor of ∇^* by C^* , then it is easy to see that $C^* = -C$, and consequently $K_{\nabla^*, \hat{\nabla}} = -K_{\nabla, \hat{\nabla}}$. We have also $K_{\nabla, \nabla^*} = 2K_{\nabla, \hat{\nabla}}$.

If (g, ∇, ∇^*) is a dualistic structure on M , we can define a family of α -connections [21] $\{\nabla^{(\alpha)}\}_{\alpha \in \mathbb{R}}$ as follows:

$$\nabla^{(\alpha)} = \frac{1+\alpha}{2}\nabla + \frac{1-\alpha}{2}\nabla^*, \quad \forall \alpha \in \mathbb{R}. \quad (2)$$

It is known that, for any $\alpha \in \mathbb{R}$, $\nabla^{(\alpha)}$ is a statistical connection on (M, g) . Obviously, $\nabla^{(1)} = \nabla$, $\nabla^{(-1)} = \nabla^*$ and $\nabla^{(0)} = \hat{\nabla}$. Let R, R^* and $\hat{R}$ be the curvature tensors of ∇, ∇^* , and $\hat{\nabla}$, respectively. Then, we have

$$R(X, Y)Z = \hat{R}(X, Y)Z + (\hat{\nabla}_X K)(Y, Z) - (\hat{\nabla}_Y K)(X, Z) + [K_{\nabla, \hat{\nabla}}, K_{\nabla, \hat{\nabla}}](X, Y)Z,$$

$$R(X, Y)Z + R^*(X, Y)Z = 2\hat{R}(X, Y)Z + 2[K_{\nabla, \hat{\nabla}}, K_{\nabla, \hat{\nabla}}](X, Y)Z,$$

where

$$[K_{\nabla, \hat{\nabla}}, K_{\nabla, \hat{\nabla}}](X, Y)Z = K_{\nabla, \hat{\nabla}}(X, K_{\nabla, \hat{\nabla}}(Y, Z)) - K_{\nabla, \hat{\nabla}}(Y, K_{\nabla, \hat{\nabla}}(X, Z)),$$

for all $X, Y, Z \in \chi(M)$. It is known that

$$g(\hat{R}(X, Y)Z, W) = -g(\hat{R}(X, Y)W, Z), \quad g(\hat{R}(X, Y)Z, W) = g(\hat{R}(Z, W)X, Y),$$

for all $X, Y, Z, W \in \chi(M)$.

Note that the above equations do not hold for R and R^* . But we have the following equations for them:

$$g(R(X, Y)Z, W) = -g(R^*(X, Y)W, Z),$$

and

$$g(R(X, Y)Z, W) + g(R(Y, X)W, Z) = g(R(Z, W)X, Y) + g(R(W, Z)Y, X).$$

Let (x^i, y^i) be an induced coordinate system of TM and $\{\frac{\partial}{\partial x^i} |_{(x,y)}, \frac{\partial}{\partial y^i} |_{(x,y)}\}$ be the natural basis of $T_{(x,y)}TM$. Then, various lifts of a vector field $X = X^i \partial_i$ on M (complete lift, horizontal lift, and vertical lift, respectively) are defined as follows:

$$X^C = X^i \frac{\partial}{\partial x^i} + y^a (\partial_a X^i) \frac{\partial}{\partial y^i}, \quad X^H = X^i \frac{\partial}{\partial x^i} - y^a \Gamma_{ai}^k X^i \frac{\partial}{\partial y^k}, \quad X^V = X^i \frac{\partial}{\partial y^i},$$

where $\partial_a X^i = \frac{\partial X^i}{\partial x^a}$ and Γ_{ki}^j denote the Christoffel symbols with respect to g .

It is known that $T_{(x,y)}TM$ can be decomposed as $H_{(x,y)}TM \oplus V_{(x,y)}TM$, where $H_{(x,y)}TM$ is spanned by $\{\frac{\delta}{\delta x^i} |_{(x,y)} := (\frac{\partial}{\partial x^i})^h = \frac{\partial}{\partial x^i} |_{(x,y)} - y^k \Gamma_{ki}^j(x) \frac{\partial}{\partial y^j} |_{(x,y)}\}$ and $V_{(x,y)}$ is spanned by $\{\frac{\partial}{\partial y^i} |_{(x,y)} := (\frac{\partial}{\partial x^i})^v\}$, where Γ_{ki}^j are the coefficients of an affine connection ∇

on M . According to [22], the Lie brackets of horizontal and vertical lifts of vector fields are given by

$$[X^H, Y^H] = [X, Y]^H - (R(X, Y)y)^V, \quad [X^H, Y^V] = (\nabla_X Y)^V - T(X, Y)^V, \quad [X^V, Y^V] = 0, \quad (3)$$

where T and R are the torsion and curvature tensors of ∇ , respectively.

On a Riemannian manifold (M, g) , one can define the Sasaki lift metric g^S on TM as follows (see [23]):

$$g_{(x,y)}^S(X^H, Y^H) = g_x(X, Y), \quad g_{(x,y)}^S(X^V, Y^H) = 0, \quad g_{(x,y)}^S(X^V, Y^V) = g_x(X, Y). \quad (4)$$

The components of the Levi-Civita connection of the Sasaki metric g^S are given by

$$(\nabla_{X^H}^S Y^H)_{(x,y)} = (\hat{\nabla}_X Y)_{(x,y)}^H - \frac{1}{2}(\hat{R}_x(X, Y)y)^V, \quad (\nabla_{X^V}^S Y^H)_{(x,y)} = \frac{1}{2}(\hat{R}_x(y, X)Y)^H, \quad (5)$$

$$(\nabla_{X^H}^S Y^V)_{(x,y)} = (\hat{\nabla}_X Y)_{(x,y)}^V + \frac{1}{2}(\hat{R}_x(y, Y)X)^H, \quad (\nabla_{X^V}^S Y^V)_{(x,y)} = 0, \quad (6)$$

for all vector fields X, Y on M and $(x, y) \in TM$, where $\hat{\nabla}$ is the Levi-Civita connection of (M, g) and $\hat{R}$ is its curvature tensor. ∇^S gives us the idea of introducing a new affine connection on TM . To construct this affine connection, we first assume that ∇ is a torsion-free affine connection on (M, g) . Therefore, we consider the following affine connection on TM (which is called a Sasaki type connection):

$$(\tilde{\nabla}_{X^H} Y^H)_{(x,y)} = (\nabla_X Y)_{(x,y)}^H - \frac{1}{2}(R_x(X, Y)y)^V, \quad (\tilde{\nabla}_{X^V} Y^H)_{(x,y)} = \frac{1}{2}(R_x(y, X)Y)^H, \quad (7)$$

$$(\tilde{\nabla}_{X^H} Y^V)_{(x,y)} = (\nabla_X Y)_{(x,y)}^V + \frac{1}{2}(R_x(y, Y)X)^H, \quad (\tilde{\nabla}_{X^V} Y^V)_{(x,y)} = 0, \quad (8)$$

for all vector fields X, Y on M and $(x, y) \in TM$, where R is the curvature tensor of ∇ . Now, we will obtain the dual connection of $\tilde{\nabla}$. Before this, we need the following remark:

Remark 1. Let ∇ be a torsion-free connection on (M, g) . We set

$$g(\bar{R}(X, Y)Z, W) = g(R(Z, W)X, Y), \quad g(\bar{R}^*(X, Y)Z, W) = g(R(Z, W)X, Y), \quad (9)$$

for any $X, Y, Z, W \in \chi(M)$, where R^* is the curvature tensor of the dual connection ∇ . Then, $\bar{R}$ and $\bar{R}^*$ are $(1, 3)$ -tensor fields on M and satisfy

$$g(\bar{R}(X, Y)Z, W) = -g(\bar{R}(X, Y)W, Z) = -g(\bar{R}^*(Y, X)Z, W). \quad (10)$$

Lemma 1. The components of the dual connection $\tilde{\nabla}^*$ of $\tilde{\nabla}$ are given by

$$(\tilde{\nabla}_{X^H}^* Y^H)_{(x,y)} = (\nabla_X^* Y)_{(x,y)}^H + \frac{1}{2}(\bar{R}^*_x(Y, X)y)^V, \quad (11)$$

$$(\tilde{\nabla}_{X^V}^* Y^H)_{(x,y)} = \frac{1}{2}(R_x^*(y, X)Y)^H,$$

$$(\tilde{\nabla}_{X^H}^* Y^V)_{(x,y)} = (\nabla_X^* Y)_{(x,y)}^V - \frac{1}{2}(\bar{R}_x^*(Y, y)X)^H, \quad (\tilde{\nabla}_{X^V}^* Y^V)_{(x,y)} = 0. \quad (12)$$

Proof. We prove only one of them. Using (1), (7), (8) and (10) we obtain

$$\begin{aligned} g^S(\tilde{\nabla}_{X^H}^* Y^H, Z^H) &= X^H g(Y^H, Z^H) - g^S(\tilde{\nabla}_{X^H} Z^H, Y^H) \\ &= X^H g(Y^H, Z^H) - g^S((\nabla_X Z)^H, Y^H) \\ &= Xg(Y, Z) - g(\nabla_X Z, Y) = g(\nabla_X^* Y, Z) = g^S((\nabla_X^* Y)^H, Z^H), \end{aligned} \quad (13)$$

and

$$\begin{aligned} g^S(\tilde{\nabla}_{X^H}^* Y^H, Z^V) &= -g^S(\tilde{\nabla}_{X^H} Z^V, Y^H) = -\frac{1}{2}g^S((R(y, Z)X)^H, Y^H) \\ &= -\frac{1}{2}g(R(y, Z)X, Y) = -\frac{1}{2}g(\bar{R}(X, Y)y, Z) \\ &= \frac{1}{2}g(\bar{R}^*(Y, X)y, Z) = \frac{1}{2}g^S((\bar{R}^*(Y, X)y)^V, Z^V). \end{aligned} \quad (14)$$

The two above equations imply the first Equation (11). $\square$

Remark 2. Due to the fact that ∇ is torsion-free, it is easy to verify that $\tilde{\nabla}$ (also $\tilde{\nabla}^*$) is torsion-free. It is obvious that $\tilde{\nabla}$ and $\tilde{\nabla}^*$ cannot be compatible with g^S (unless they reduce to the Levi-Civita connection ∇^S). But the most important question is whether these connections can be statistical connections or not.

Proposition 1. Let ∇ be a torsion-free connection on (M, g) . Then, the Sasaki type connection $\tilde{\nabla}$ is a statistical connection on (TM, g^S) if and only if it reduces to the Levi-Civita connection ∇^S .

Proof. Using (7) and (8), we obtain $(\tilde{\nabla}_{X^V} g^S)(Y^H, Z^V) = 0$ and $(\tilde{\nabla}_{Y^H} g^S)(X^V, Z^V) = (\nabla_Y g)(X, Z)$. Then, $\tilde{\nabla}$ is a statistical connection on (TM, g^S) if and only if $(\nabla_Y g)(X, Z) = 0$, for all $X, Y, Z \in \chi(M)$. Therefore, ∇ reduces to the Levi-Civita connection $\hat{\nabla}$ of (M, g) . Consequently, $\tilde{\nabla}$ is the Levi-Civita connection ∇^S of (TM, g^S) . $\square$

3. Kobayashi–Nomizu Type and Yano Type Connections on TM

In this section, we recall the concept of $(J^2 = \pm 1)$ -metric manifold. Then, we study the geometry of (TM, g^S) with the Kobayashi–Nomizu type connection $\tilde{\nabla}^{kn}$ and Yano type connection $\tilde{\nabla}^y$.

Definition 1 ([1]). A smooth manifold M having a tensor field J of type $(1, 1)$ with $J^2 = \alpha Id$, where Id denotes the identity tensor field and $\alpha = \pm 1$, is said to be a $(J^2 = \pm 1)$ manifold or a manifold endowed with an α -structure. Furthermore, if g is a Riemannian metric on M such that $g(JX, JY) = \epsilon g(X, Y)$, for all $X, Y \in \chi(M)$, where $\epsilon = \pm 1$, then we say that M has an (α, ϵ) -structure or M is a $(J^2 = \pm 1)$ -metric manifold.

Definition 2 ([1]). Let (M, J) be a $(J^2 = \pm 1)$ manifold and ∇ an affine connection on M . The affine connection ∇^0 given by

$$\nabla_X^0 Y = \nabla_X Y + \frac{(-\alpha)}{2}(\nabla_X J)JY, \quad \forall X, Y \in \chi(M), \quad (15)$$

is called the ∇^0 type connection on (M, J) . Also, the Kobayashi–Nomizu type connection ∇^{kn} of (M, J) is defined by

$$\nabla_X^{kn} Y = \nabla_X^0 Y + \frac{(-\alpha)}{4}\{(\nabla_Y J)JX - (\nabla_{JY} J)X\}, \quad \forall X, Y \in \chi(M). \quad (16)$$

Furthermore, the Yano type connection ∇^y of (M, J) is given by

$$\nabla_X^y Y = \nabla_X Y + \frac{(-\alpha)}{2}(\nabla_Y J)JX + \frac{(-\alpha)}{4}\{(\nabla_X J)JY - (\nabla_{JX} J)Y\}, \quad \forall X, Y \in \chi(M), \quad (17)$$

Remark 3. If (M, J, g) is a $(J^2 = \pm 1)$ -metric manifold and $\hat{\nabla}$ is the Levi-Civita connection of g , then $\hat{\nabla}^0$, $\hat{\nabla}^{kn}$, and $\hat{\nabla}^y$ are called first canonical, Kobayashi–Nomizu, and Yano connections, respectively.

Now, we consider the almost complex structure $\tilde{J}$ on TM defined by

$$\tilde{J}(X^H) = -X^V, \quad \tilde{J}(X^V) = X^H, \quad \forall X \in \chi(M). \quad (18)$$

It is easy to see that $(TM, \tilde{J}, g^S)$ is an almost Hermitian manifold. In fact, (TM, g^S) has a $(-1, 1)$ -structure, i.e., $\alpha = -1$ and $\epsilon = 1$. We assume in the following that TM is equipped with the almost complex structure $\tilde{J}$ defined above.

Using (5), (6), (15) and (18), the components of $\tilde{\nabla}^0$ type connection on TM are as follows:

$$\tilde{\nabla}_{X^H}^0 Y^H = (\nabla_X Y)^H - \frac{1}{4}\{R(X, Y)y + R(y, Y)X\}^V, \quad (19)$$

$$\tilde{\nabla}_{X^H}^0 Y^V = (\nabla_X Y)^V + \frac{1}{4}\{R(y, Y)X + R(X, Y)y\}^H, \quad (20)$$

$$\tilde{\nabla}_{X^V}^0 Y^H = \frac{1}{4}(R(y, X)Y)^H, \quad \tilde{\nabla}_{X^V}^0 Y^V = \frac{1}{4}(R(y, X)Y)^V. \quad (21)$$

From (16) we have

$$\tilde{\nabla}_{\tilde{X}}^{kn} \tilde{Y} = \tilde{\nabla}_{\tilde{X}}^0 \tilde{Y} + \frac{1}{4}\mathcal{A}(\tilde{X}, \tilde{Y}), \quad \forall \tilde{X}, \tilde{Y} \in \chi(TM), \quad (22)$$

where $\mathcal{A}(\tilde{X}, \tilde{Y}) = (\tilde{\nabla}_{\tilde{Y}} \tilde{J})\tilde{J}(\tilde{X}) - (\tilde{\nabla}_{\tilde{J}(\tilde{Y})} \tilde{J})\tilde{X}$. Using (5), (6) and (15), it is easy to verify that $\mathcal{A}(X^H, Y^H) = \mathcal{A}(X^V, Y^V) = \mathcal{A}(X^V, Y^H) = \mathcal{A}(X^H, Y^V) = 0$. Therefore, we obtain $\tilde{\nabla}^{kn} = \tilde{\nabla}^0$. Therefore, we state the following interesting result:

Lemma 2. Let M be a manifold with a torsion-free connection ∇ . Then, the Kobayashi–Nomizu type connection of $(TM, \tilde{\nabla})$ coincides with $\tilde{\nabla}^0$, i.e.,

$$\tilde{\nabla}_{X^H}^{kn} Y^H = (\nabla_X Y)^H + \frac{1}{4}\{R(Y, X)y + R(Y, y)X\}^V, \quad (23)$$

$$\tilde{\nabla}_{X^H}^{kn} Y^V = (\nabla_X Y)^V + \frac{1}{4}\{R(y, Y)X + R(X, Y)y\}^H, \quad (24)$$

$$\tilde{\nabla}_{X^V}^{kn} Y^H = \frac{1}{4}(R(y, X)Y)^H, \quad \tilde{\nabla}_{X^V}^{kn} Y^V = \frac{1}{4}(R(y, X)Y)^V. \quad (25)$$

Proposition 2. Let M be a manifold with a torsion-free connection ∇ and let TM be its tangent bundle equipped with the Kobayashi–Nomizu type connection. Then, the following assertions hold:

- (1) The Kobayashi–Nomizu type connection $\tilde{\nabla}^{kn}$ is torsion-free if and only if M is a flat space. In this case, $\tilde{\nabla}^{kn}$ reduces to $\tilde{\nabla}$.
- (2) The Kobayashi–Nomizu type connection $\tilde{\nabla}^{kn}$ is a Codazzi connection if and only if ∇ reduces to $\hat{\nabla}$. In this case, $\tilde{\nabla}^{kn}$ is a metric connection for (TM, g^S) .

Proof. According to (23)–(25) and (3), the components of the torsion tensor of the Kobayashi–Nomizu type connection $\tilde{\nabla}^{kn}$ are

$$\tilde{T}^{kn}(X^H, Y^H) = \tilde{T}^{kn}(X^V, Y^H) = -\frac{1}{4}(R(Y, X)y)^V, \quad (26)$$

$$\tilde{T}^{kn}(X^H, Y^V) = \frac{1}{4}(R(X, Y)y)^V, \quad \tilde{T}^{kn}(X^V, Y^V) = \frac{1}{4}(R(Y, X)y)^V. \quad (27)$$

Then, (26) and (27) imply item (1).

Now, we consider the Riemannian manifold (M, g) equipped with a torsion-free connection ∇ . Direct computations and (23)–(25) give

$$(\tilde{\nabla}_{X^V}^{kn} g^S)(Y^H, Z^V) = 0, \quad (\tilde{\nabla}_{Y^H}^{kn} g^S)(X^V, Z^V) = (\nabla_Y g)(X, Z).$$

Then, $(\tilde{\nabla}_{X^V}^{kn} g^S)(Y^H, Z^V) = (\tilde{\nabla}_{Y^H}^{kn} g^S)(X^V, Z^V)$ if and only if $(\nabla_Y g)(X, Z) = 0$, for all $X, Y, Z \in \chi(M)$ (i.e., ∇ reduces to $\hat{\nabla}$). In this case, we obtain

$$(\tilde{\nabla}_{X^H}^{kn} g^S)(Y^H, Z^H) = (\tilde{\nabla}_{X^V}^{kn} g^S)(Y^V, Z^V) = (\tilde{\nabla}_{X^H}^{kn} g^S)(Y^H, Z^V) = (\tilde{\nabla}_{X^V}^{kn} g^S)(Y^H, Z^H) = 0.$$

Thus, $\tilde{\nabla}^{kn}$ is a metric connection with respect to g^S . $\square$

Corollary 1. *The Kobayashi–Nomizu type connection is a statistical connection on (TM, g^S) if and only if M is a flat manifold and $\tilde{\nabla}^{kn}$ reduces to ∇^S .*

Remark 4. *The item (2) of Proposition 2 states that the Kobayashi–Nomizu type connection is Codazzi if and only if it is the Kobayashi–Nomizu connection.*

Next, we study the Yano type connection on the tangent bundle of a manifold. According to (17) and using (5) and (6), we deduce that the components of the Yano type connection $\tilde{\nabla}^y$ of $(TM, \tilde{J})$ are given by

$$\tilde{\nabla}_{X^H}^y Y^H = (\nabla_X Y)^H + \frac{1}{4}\{2R(Y, X)y + R(Y, y)X\}^V, \quad (28)$$

$$\tilde{\nabla}_{X^H}^y Y^V = (\nabla_X Y)^V + \frac{1}{4}\{R(y, X)Y + R(X, Y)y\}^H, \quad (29)$$

$$\tilde{\nabla}_{X^V}^y Y^H = \frac{1}{4}\{R(y, X)Y - R(X, Y)y\}^H, \quad (30)$$

$$\tilde{\nabla}_{X^V}^y Y^V = \frac{1}{4}\{R(y, X)Y + R(X, Y)y\}^V. \quad (31)$$

Proposition 3. *Let (M, g) be a Riemannian manifold with a torsion-free connection ∇ and let TM be its tangent bundle equipped with the Sasaki metric and the Yano type connection. Then, the following assertions hold:*

- (1) *The Yano type connection is torsion-free, if and only if M is a flat space. In this case, $\tilde{\nabla}^y$ reduces to $\hat{\nabla}$.*
- (2) *$(g^S, \tilde{\nabla}^y)$ is a Codazzi couple for TM if and only if ∇ reduces to $\hat{\nabla}$ and*

$$\hat{R}(X, Y, y, Z) + \hat{R}(X, Z, y, Y) = \hat{R}(Y, X, y, Z) + \hat{R}(Y, Z, y, X), \quad \forall X, Y, Z \in \chi(M). \quad (32)$$

Proof. Applying (28)–(31) and (3), it is easy to see that the components of the torsion tensor of the Yano type connection $\tilde{\nabla}^y$ are

$$\tilde{T}^y(X^H, Y^H) = \frac{1}{4}(R(Y, X)y)^V, \quad \tilde{T}^y(X^V, Y^V) = \frac{1}{4}(R(X, Y)y)^V, \quad (33)$$

$$\tilde{T}^y(X^H, Y^V) = \tilde{T}^y(X^V, Y^H) = \frac{1}{4}(R(Y, X)y)^H. \quad (34)$$

Then, (33) and (34) imply (1).

By using (28)–(31), after direct computations, we obtain

$$(\tilde{\nabla}_{X^V}^y g^S)(Y^H, Z^V) = 0, \quad (\tilde{\nabla}_{Y^H}^y g^S)(X^V, Z^V) = (\nabla_Y g)(X, Z) = 0, \quad \forall X, Y, Z \in \chi(M).$$

Therefore, $(\tilde{\nabla}_{X^V}^y g^S)(Y^H, Z^V) = (\tilde{\nabla}_{Y^H}^y g^S)(X^V, Z^V)$ if and only if ∇ reduces to the Levi-Civita connection $\hat{\nabla}$ of (M, g) . According to this fact, we obtain

$$(\tilde{\nabla}_{X^H}^y g^S)(Y^V, Z^V) = (\tilde{\nabla}_{Y^V}^y g^S)(X^H, Z^V) = (\tilde{\nabla}_{X^H}^y g^S)(Y^H, Z^H) = 0. \quad (35)$$

On the other hand, we obtain

$$(\tilde{\nabla}_{X^V}^y g^S)(Y^V, Z^V) = -(\tilde{\nabla}_{X^H}^y g^S)(Y^V, Z^H) = -\frac{1}{4}\{\hat{R}(X, Y, y, Z) + \hat{R}(X, Z, y, Y)\},$$

and

$$(\tilde{\nabla}_{Y^V}^y g^S)(X^H, Z^H) = \frac{1}{4}\{\hat{R}(Y, X, y, Z) + \hat{R}(Y, Z, y, X)\}.$$

Then,

$$(\tilde{\nabla}_{X^V}^y g^S)(Y^V, Z^V) = (\tilde{\nabla}_{Y^V}^y g^S)(X^V, Z^V), \quad (\tilde{\nabla}_{X^H}^y g^S)(Y^V, Z^H) = (\tilde{\nabla}_{Y^V}^y g^S)(X^H, Z^H),$$

if and only if

$$\hat{R}(X, Y, y, Z) + \hat{R}(X, Z, y, Y) = \hat{R}(Y, X, y, Z) + \hat{R}(Y, Z, y, X), \quad \forall X, Y, Z \in \chi(M).$$

Therefore, we have item (2). $\square$

Corollary 2. Let (M, g) be a Riemannian manifold with a torsion-free connection ∇ . Then, $(TM, g^S, \tilde{\nabla}^y)$ is a statistical manifold if and only if M is flat and $\tilde{\nabla}^y$ reduces to ∇^S .

Remark 5. From Propositions 2 and 3, we conclude that the Kobayashi–Nomizu type connection cannot be a pure Codazzi connection (non-metric connection), but this is possible for the Yano type connection.

4. Curvatures of Kobayashi–Nomizu Type and Yano Type Connections on TM

In this section, we compute all the components of the curvature tensors of the Kobayashi–Nomizu type and Yano type connections of the $(\tilde{J}^2 = -1)$ manifold $(TM, \tilde{J})$ and we study some of their properties.

Lemma 3. Let M be a smooth manifold with a torsion-free connection ∇ and TM be its tangent bundle equipped with an almost complex structure $\tilde{J}$. The components of the curvature tensor $\tilde{R}^{kn}$ of the Kobayashi–Nomizu type connection are

$$\begin{aligned} \tilde{R}^{kn}(X^H, Y^H)Z^H &= \left\{ R(X, Y)Z + \frac{1}{16}\{R(y, R(Z, Y)y)X + R(X, R(Z, Y)y)y \right. \\ &\quad \left. + R(y, R(Z, y)Y)X + R(X, R(Z, y)Y)y\}_{X \leftrightarrow Y} + \frac{1}{4}(R(y, R(X, Y)y)Z) \right\}^H \\ &\quad + \frac{1}{4}\left\{ (\nabla_X R)(Z, Y)y + R(Z, Y)\nabla_X y \right. \\ &\quad \left. + (\nabla_X R)(Z, y)Y + R(Z, \nabla_X y)Y \right\}_{X \leftrightarrow Y}^V, \end{aligned} \quad (36)$$

$$\begin{aligned}\tilde{R}^{kn}(X^H, Y^H)Z^V &= \left\{ R(X, Y)Z + \frac{1}{16} \{ R(y, R(Z, Y)y)X + R(X, R(Z, Y)y)y \right. \\ &\quad + R(y, R(Z, y)Y)X + R(X, R(Z, y)Y)y \}_{X \leftrightarrow Y} + \frac{1}{4} (R(y, R(X, Y)y)Z) \Big\}^V \\ &\quad + \frac{1}{4} \left\{ (\nabla_X R)(Z, Y)y + R(Z, Y)\nabla_X y \right. \\ &\quad \left. + (\nabla_X R)(Z, y)Y + R(Z, \nabla_X y)Y \right\}_{X \leftrightarrow Y'}^H,\end{aligned}\quad (37)$$

$$\tilde{R}^{kn}(X^V, Y^V)Z^H = \frac{1}{16} \left\{ R(y, X)R(y, Y)Z \right\}_{X \leftrightarrow Y'}^H \quad (38)$$

$$\tilde{R}^{kn}(X^V, Y^V)Z^V = \frac{1}{16} \left\{ R(y, X)R(y, Y)Z \right\}_{X \leftrightarrow Y'}^V \quad (39)$$

$$\begin{aligned}\tilde{R}^{kn}(X^H, Y^V)Z^H &= \frac{1}{4} \{ (\nabla_X R)(y, Y)Z + R(\nabla_X y, Y)Z \}^H + \frac{1}{16} \{ R(R(y, Y)Z, X)y \\ &\quad + R(R(y, Y)Z, y)X - R(y, Y)R(Z, X)y - R(y, Y)R(Z, y)X \}^V,\end{aligned}\quad (40)$$

$$\begin{aligned}\tilde{R}^{kn}(X^H, Y^V)Z^V &= -\frac{1}{16} \{ R(y, Y)R(y, Z)X + R(y, Y)R(X, Z)y - R(y, R(y, Y)Z)X \\ &\quad - R(X, R(y, Y)Z)y \}^H + \frac{1}{4} \{ (\nabla_X R)(y, Y)Z + R(\nabla_X y, Y)Z \}^V,\end{aligned}\quad (41)$$

where $\{\mathcal{B}(X, Y)\}_{X \leftrightarrow Y} = \mathcal{B}(X, Y) - \mathcal{B}(Y, X)$.

Proof. Using the formula of the curvature tensor and (23)–(25), we obtain the components of $\tilde{R}^{kn}$. $\square$

It is known that a Riemannian manifold (M, g) with constant (sectional) curvature is called a space form. If κ denotes the constant value of the sectional curvature, then the curvature tensor field has the expression

$$\hat{R}(X, Y)Z = \kappa(g(Y, Z)X - g(X, Z)Y), \quad \forall X, Y, Z \in \chi(M),$$

where $\hat{R}$ is the curvature tensor field of the Levi-Civita connection $\hat{\nabla}$ of g .

We can generalize this definition for Codazzi manifolds as follows:

Definition 3. We say that a Codazzi manifold (M, g, ∇) has constant curvature if there is a constant κ such that the curvature tensor field R of ∇ is given by

$$R(X, Y)Z = \kappa(g(Y, Z)X - g(X, Z)Y), \quad \forall X, Y, Z \in \chi(M)$$

According to Proposition 2, if we consider the Levi-Civita connection $\hat{\nabla}$ on (M, g) , then $(TM, g^S, \tilde{\nabla}^{kn})$ is a Codazzi manifold. The following theorem shows that $(TM, g^S, \tilde{\nabla}^{kn})$ cannot have non-zero constant curvature.

Theorem 1. Let (M, g) be a Riemannian manifold with the Levi-Civita connection $\hat{\nabla}$. If $(TM, g^S, \tilde{\nabla}^{kn})$ has constant curvature, then TM is flat.

Proof. Assume that $(TM, g^S, \tilde{\nabla}^{kn})$ has constant curvature. Then, there is a constant κ such that

$$\tilde{R}^{kn}(\tilde{X}, \tilde{X})\tilde{Z} = \kappa(g^S(\tilde{Y}, \tilde{Z})\tilde{X} - g^S(\tilde{X}, \tilde{Z})\tilde{Y}), \quad \forall \tilde{X}, \tilde{Y}, \tilde{Z} \in \chi(TM).$$

Using (39), we obtain

$$\begin{aligned} \kappa(g_{(x,y)}^S(Y^V, Z^V)X^V - g_{(x,y)}^S(X^V, Z^V)Y^V) \\ = \tilde{R}_{(x,y)}^{kn}(X^V, Y^V)Z^V = \frac{1}{16} \left\{ \hat{R}(y, X)\hat{R}(y, Y)Z \right\}_{X \leftrightarrow Y}^V. \end{aligned}$$

If we take $y = 0$ in the above equation, we obtain

$$\kappa(g_x(Y, Z)X_{(x,0)}^V - g_x(X, Z)Y_{(x,0)}^V) = 0,$$

which implies $\kappa = 0$, and so $\tilde{R}^{kn} = 0$. $\square$

Now, we investigate the flatness of $(TM, g^S, \tilde{\nabla}^{kn})$.

Theorem 2. Let M be a smooth manifold with a torsion-free connection ∇ and let TM be its tangent bundle. Then, TM equipped with the Kobayashi–Nomizu type connection $\tilde{\nabla}^{kn}$ is a flat space if and only if M is a flat space.

Proof. According to the above description, it follows that if M is a flat space, then all of the components of the curvature tensor of the Kobayashi–Nomizu type connection in TM are zero. Moreover, the Equation (36) holds for every $(x, y) \in TM$, in particular for $y = 0$. Thus, we derive that all terms of (36) are zero except $R(X, Y)Z$. Then, if (TM, g^S) is a flat space equipped with $\tilde{\nabla}^{kn}$, then from (36) we derive that M is a flat space. $\square$

Lemma 4. Let M be a smooth manifold with a torsion-free connection ∇ and TM its tangent bundle equipped with an almost complex structure $\tilde{J}$. The components of the curvature tensor $\tilde{R}^y$ of the Yano type connection are as follows:

$$\begin{aligned} \tilde{R}^y(X^H, Y^H)Z^H = \frac{1}{4} \left\{ 4R(X, Y)Z + \{R(R(Y, Z)y, y) + R(R(y, Z)Y, y)X\}_{X \leftrightarrow Y} \right. \\ \left. + R(y, R(X, Y)y)Z - R(R(X, Y)y, Z)y \right\}^H \\ + \frac{1}{4} \left\{ 2\{(\nabla_X R)(Z, Y)y + R(Z, Y)\nabla_X y\} \right. \\ \left. + (\nabla_X R)(Z, y)Y + R(Z, \nabla_X y)Y \right\}_{X \leftrightarrow Y}^V, \end{aligned} \quad (42)$$

$$\begin{aligned} \tilde{R}^y(X^H, Y^H)Z^V = \left\{ R(X, Y)Z + \frac{1}{16} \{2R(R(y, Y)Z, X)y + R(R(y, Y)Z, y)X \right. \\ \left. + 2R(R(Y, Z)y, X)y + R(R(Y, Z)y, y)X\}_{X \leftrightarrow Y} + \frac{1}{4} \{R(y, Z)R(X, Y)y\} \right\}^V \\ + \frac{1}{4} \left\{ (\nabla_X R)(y, Z)Y + R(\nabla_X y, Z)Y \right\}_{X \leftrightarrow Y}^H, \end{aligned} \quad (43)$$

$$\begin{aligned} \tilde{R}^y(X^V, Y^V)Z^H = \frac{1}{16} \left\{ R(y, X)R(y, Y)Z - R(X, R(y, Y)Z)y - R(y, X)R(Y, Z)y \right. \\ \left. + R(X, R(Y, Z)y)y \right\}_{X \leftrightarrow Y}^H, \end{aligned} \quad (44)$$

$$\tilde{R}^y(X^V, Y^V)Z^V = \frac{1}{16} \left\{ R(R(Z, y)Y, y)X \right\}_{X \leftrightarrow Y}^V, \quad (45)$$

$$\begin{aligned}\tilde{R}^y(X^H, Y^V)Z^H &= \frac{1}{4} \left\{ (\nabla_X R)(y, Y)Z + R(\nabla_X y, Y)Z - (\nabla_X R)(Y, Z)y - R(Y, Z)\nabla_X y \right\}^H \\ &\quad + \frac{1}{16} \left\{ 2R(R(y, Y)Z, X)y + R(R(y, Y)Z, y)X - 2R(R(Y, Z)y, X)y \right. \\ &\quad \left. - R(R(Y, Z)y, y)X + 2R(R(Z, X)y, y)Y + R(R(Z, y)X, y)Y \right\}^V,\end{aligned}\quad (46)$$

$$\begin{aligned}\tilde{R}^y(X^H, Y^V)Z^V &= \frac{1}{4} \left\{ (\nabla_X R)(y, Z)Y + R(\nabla_X y, Z)Y \right\}^V - \frac{1}{16} \left\{ \{R(Y, y)R(Z, y)X\}_{X \leftrightarrow Y} \right. \\ &\quad \left. + R(Y, R(Z, y)X)y + R(X, R(Z, y)Y)y \right\}^H.\end{aligned}\quad (47)$$

According to Proposition 3, if we consider the Levi-Civita connection $\hat{\nabla}$ on (M, g) and assume that the curvature tensor of $\hat{\nabla}$ satisfies (32), then $(TM, g^S, \hat{\nabla}^y)$ is a Codazzi manifold. The following theorem shows that $(TM, g^S, \hat{\nabla}^y)$ cannot have non-zero constant curvature.

Theorem 3. *Let (M, g) be a Riemannian manifold with the Levi-Civita connection $\hat{\nabla}$. If $(TM, g^S, \hat{\nabla}^y)$ has constant curvature, then TM is flat.*

Proof. The proof is similar to the proof of Theorem 1. $\square$

In the similar way of Theorem 2, we can deduce the following:

Theorem 4. *Let M be a smooth manifold with a torsion-free connection ∇ and let TM be its tangent bundle. Then, (TM, g^S) equipped with the Yano type connection $\tilde{\nabla}^y$ is flat if and only if M is flat.*

5. Mutual Curvature Tensor

Mutual (or relative) curvature tensor is a concept related to two affine connections on a manifold M . If these two connections are equal with the Levi-Civita connection of a Riemannian manifold (M, g) , we must have the Riemannian curvature tensor. Mutual (or relative) curvature was previously introduced by O. Calin and C. Udriște in [24] and S. Puechmorel in [7] in two different ways. Then, D. Iosifidis showed in [25] that none of these are tensor. He also presented a new and correct definition of mutual curvature tensor. Recently, D. Iosifidis and K. Pallikaris in [26] formulated a bi-connection theory of gravity whose gravitational action consists of a mutual curvature scalar. This shows the important application of mutual curvature in mathematical physics.

Definition 4 ([25]). *Let M be a manifold endowed with a pair of connections (∇^1, ∇^2) . Then, their mutual curvature tensor R_{∇^1, ∇^2} is a tensor of type $(1, 3)$ defined by*

$$R_{\nabla^1, \nabla^2}(X, Y)Z = \frac{1}{2} \{ \nabla_X^1 \nabla_Y^2 Z - \nabla_Y^1 \nabla_X^2 Z - \nabla_{[X, Y]}^1 Z + \nabla_X^2 \nabla_Y^1 Z - \nabla_Y^2 \nabla_X^1 Z - \nabla_{[X, Y]}^2 Z \}, \quad (48)$$

for all $X, Y, Z \in \chi(M)$.

It is easy to verify that the mutual curvature has the following properties [27]:

$$R_{\nabla^1, \nabla^2}(X, Y)Z = R_{\nabla^2, \nabla^1}(X, Y)Z, \quad R_{\nabla^1, \nabla^2}(X, Y)Z = -R_{\nabla^1, \nabla^2}(Y, X)Z,$$

$$R_{\nabla^1, \nabla^2}(X, Y)Z + R_{\nabla^1, \nabla^2}(Y, Z)X + R_{\nabla^1, \nabla^2}(Z, X)Y = 0,$$

for all $X, Y, X \in \chi(M)$.

Lemma 5. Let M be a manifold and (∇^1, ∇^2) a pair of torsion-free connections. Then, we have

$$R_{\nabla^1, \nabla^2} = \frac{1}{2}(R_{\nabla^1} + R_{\nabla^2} - [K_{\nabla^1, \nabla^2}, K_{\nabla^1, \nabla^2}]), \quad (49)$$

where $[K_{\nabla^1, \nabla^2}, K_{\nabla^1, \nabla^2}]$ is the $(1, 3)$ -tensor given by

$$[K_{\nabla^1, \nabla^2}, K_{\nabla^1, \nabla^2}](X, Y)Z = K_{\nabla^1, \nabla^2}(X, K_{\nabla^1, \nabla^2}(Y, Z)) - K_{\nabla^1, \nabla^2}(Y, K_{\nabla^1, \nabla^2}(X, Z)), \quad (50)$$

for all $X, Y, Z \in \chi(M)$.

Proof. Considering $K_{\nabla^1, \nabla^2} = \nabla^1 - \nabla^2$ in (48), the proof is complete. $\square$

It is easy to verify that $[K_{\nabla^1, \nabla^2}, K_{\nabla^1, \nabla^2}]$ satisfies the following properties:

$$[K_{\nabla^1, \nabla^2}, K_{\nabla^1, \nabla^2}](X, Y)Z = -[K_{\nabla^1, \nabla^2}, K_{\nabla^1, \nabla^2}](Y, X)Z,$$

$$[K_{\nabla^1, \nabla^2}, K_{\nabla^1, \nabla^2}](X, Y)Z + [K_{\nabla^1, \nabla^2}, K_{\nabla^1, \nabla^2}](Y, Z)X + [K_{\nabla^1, \nabla^2}, K_{\nabla^1, \nabla^2}](Z, X)Y = 0,$$

for all $X, Y, Z \in \chi(M)$. Moreover, if (M, g) is a Riemannian manifold and K_{∇^1, ∇^2} is symmetric relative to g , i.e., $g(X, K_{\nabla^1, \nabla^2}(Y, Z)) = g(Y, K_{\nabla^1, \nabla^2}(X, Z))$, then we have

$$g([K_{\nabla^1, \nabla^2}, K_{\nabla^1, \nabla^2}](X, Y)Z, W) = -g([K_{\nabla^1, \nabla^2}, K_{\nabla^1, \nabla^2}](X, Y)W, Z), \quad \forall X, Y, Z, W \in \chi(M).$$

The above properties state that $[K_{\nabla^1, \nabla^2}, K_{\nabla^1, \nabla^2}]$ is a curvature-like tensor.

Remark 6. In [28], Opozda considered a statistical connection ∇ on a Riemannian manifold (M, g) and introduced the concept of sectional K -curvature by using the curvature-like tensor $[K_{\nabla, \hat{\nabla}}, K_{\nabla, \hat{\nabla}}]$. According to Opozda's point of view, a statistical manifold (M, g, ∇) has constant (sectional) $K_{\nabla, \hat{\nabla}}$ -curvature (since she was dealing with a statistical connection, she used the symbol K instead of $K_{\nabla, \hat{\nabla}}$) if there is a constant number λ such that

$$[K_{\nabla, \hat{\nabla}}, K_{\nabla, \hat{\nabla}}](X, Y)Z = \lambda(g(Y, Z)X - g(X, Z)Y), \quad \forall X, Y, Z \in \chi(M). \quad (51)$$

Starting from this idea, we can introduce the concept of constant (sectional) K_{∇^1, ∇^2} -curvature, considering two statistical connections ∇^1 and ∇^2 on (M, g) .

Definition 5. Let ∇^1 and ∇^2 be statistical connections on (M, g) . We say that M has constant K_{∇^1, ∇^2} -curvature if there is a constant number λ such that

$$[K_{\nabla^1, \nabla^2}, K_{\nabla^1, \nabla^2}](X, Y)Z = \lambda(g(Y, Z)X - g(X, Z)Y), \quad \forall X, Y, Z \in \chi(M). \quad (52)$$

The study of the sectional K_{∇^1, ∇^2} -curvature of statistical manifolds can be an interesting and important topic for researchers, and many articles can be written on this concept in the future. We provide an example.

Example 1. Let (M, g) be a Riemannian manifold and η a 1-form on M . Considering the affine connection ∇ given by

$$\nabla_X Y = \hat{\nabla}_X Y + g(X, Y)\eta^\sharp + \eta(X)Y + \eta(Y)X, \quad \forall X, Y \in \chi(M),$$

it is easy to see that ∇ is a statistical connection on (M, g) (see [29], for more details) with

$$K_{\nabla, \hat{\nabla}}(X, Y) = g(X, Y)\eta^\sharp + \eta(X)Y + \eta(Y)X.$$

Using (50), we obtain

$$[K_{\nabla, \hat{\nabla}}, K_{\nabla, \hat{\nabla}}](X, Y)Z = g(Y, Z)\eta(X)\eta^\sharp - g(X, Z)\eta(Y)\eta^\sharp + g(Y, Z)\eta(\sharp)X - g(X, Z)\eta(\sharp)Y \\ + \eta(Z)\eta(Y)X - \eta(Z)\eta(X)Y.$$

The above equation can be written as

$$[K_{\nabla, \hat{\nabla}}, K_{\nabla, \hat{\nabla}}](X, Y)Z = \eta(\eta^\sharp)(g(Y, Z)X - g(X, Z)Y) + \eta(Z)(g(Y, \eta^\sharp)X - g(X, \eta^\sharp)Y) \\ + g(g(Y, Z)X - g(X, Z)Y, \eta^\sharp)\eta^\sharp.$$

Then, (M, g, ∇) has constant curvature λ if and only if

$$(\eta(\eta^\sharp) - \lambda)(g(Y, Z)X - g(X, Z)Y) + \eta(Z)(g(Y, \eta^\sharp)X - g(X, \eta^\sharp)Y) \\ + g(g(Y, Z)X - g(X, Z)Y, \eta^\sharp)\eta^\sharp = 0.$$

Putting $Z = \eta^\sharp$ in the above equation, we deduce that if (M, g, ∇) has constant curvature λ , then $\eta(\eta^\sharp) = \frac{\lambda}{2}$ (i.e., η has a constant norm).

Let (M, J, g) be a $(J^2 = \pm 1)$ -metric manifold equipped with a dualistic structure (∇, ∇^*) and $\{\nabla^{(\alpha)}\}_{\alpha \in \mathbb{R}}$ be a family of (α) -connections on M . Using (48) and (2), we obtain

$$R_{\nabla^{kn}, \nabla^{(\alpha)}} = (1 + \alpha)R_{\nabla^{kn}, \nabla} + (1 - \alpha)R_{\nabla^{kn}, \nabla^*}, \quad (53)$$

and

$$R_{\nabla^y, \nabla^{(\alpha)}} = (1 + \alpha)R_{\nabla^y, \nabla} + (1 - \alpha)R_{\nabla^y, \nabla^*}. \quad (54)$$

Evidently, if the mutual curvatures $R_{\nabla^{kn}, \nabla}$ and $R_{\nabla^{kn}, \nabla^*}$ are zero, then $R_{\nabla^{kn}, \nabla^{(\alpha)}}$ vanishes. Similar conclusion holds for $R_{\nabla^y, \nabla^{(\alpha)}}$.

In the following we present more results on $R_{\nabla^{kn}, \nabla^{(\alpha)}}$ and $R_{\nabla^y, \nabla^{(\alpha)}}$.

Proposition 4. Let (M, J, g) be a $(J^2 = \pm 1)$ -metric manifold equipped with a dualistic structure (∇, ∇^*) and $\{\nabla^{(\alpha)}\}_{\alpha \in \mathbb{R}}$ a family of (α) -connections on M . Then, the following assertions hold:

- (1) If the mutual curvatures $R_{\nabla^{kn}, \nabla}$ and $R_{\nabla^{kn}, \nabla^*}$ (respectively, $R_{\nabla^y, \nabla}$ and R_{∇^y, ∇^*}) are parallel with respect to the ∇ and ∇^* , then the mutual curvature $R_{\nabla^{kn}, \nabla^{(\alpha)}}$ (respectively, $R_{\nabla^y, \nabla^{(\alpha)}}$) is parallel with respect to the (β) -connection $\nabla^{(\beta)}$.
- (2) The mutual curvature $R_{\nabla^{(\alpha)}, \nabla^{(\beta)}}$ is parallel with respect to the Kobayashi–Nomizu type (respectively, the Yano type) connection whenever, R_{∇} , R_{∇^*} and the mutual curvature R_{∇, ∇^*} are parallel with respect to the Kobayashi–Nomizu type (respectively, the Yano type) connection.
- (3) The (α) -curvature $R^{(\alpha)}$ of an (α) -connection $\nabla^{(\alpha)}$ is parallel with respect to the Kobayashi–Nomizu type (respectively, the Yano type) connection whenever, R_{∇} , R_{∇^*} and the mutual curvature R_{∇, ∇^*} are parallel with respect to the Kobayashi–Nomizu type (respectively, the Yano type) connection.
- (4) If the mutual curvatures $R_{\nabla^{kn}, \nabla}$ and $R_{\nabla^{kn}, \nabla^*}$ (respectively, $R_{\nabla^y, \nabla}$ and R_{∇^y, ∇^*}) are parallel with respect to the Yano type (respectively, Kobayashi–Nomizu type) connection, then the mutual curvature $R_{\nabla^{kn}, \nabla^{(\alpha)}}$ (respectively, $R_{\nabla^y, \nabla^{(\alpha)}}$) is parallel with respect to the Yano type (respectively, Kobayashi–Nomizu type) connection.

Proof. For the mutual curvature $R_{\nabla^{kn}, \nabla^{(\alpha)}}$ and the (β) -connection $\nabla^{(\beta)}$ by direct computations we obtain the following:

$$\begin{aligned} (\nabla_X^{(\beta)} R_{\nabla^{kn}, \nabla^{(\alpha)}})(Y, Z)W &= \frac{(1+\alpha)(1+\beta)}{2} (\nabla_X R_{\nabla^{kn}, \nabla})(Y, Z)W \\ &+ \frac{(1-\alpha)(1+\beta)}{2} (\nabla_X R_{\nabla^{kn}, \nabla^*})(Y, Z)W \\ &+ \frac{(1+\alpha)(1-\beta)}{2} (\nabla_X^* R_{\nabla^{kn}, \nabla})(Y, Z)W \\ &+ \frac{(1-\alpha)(1-\beta)}{2} (\nabla_X^* R_{\nabla^{kn}, \nabla^*})(Y, Z)W, \end{aligned} \quad (55)$$

for every $X, Y, Z, W \in \chi(M)$. Replacing ∇^{kn} by ∇^y in (55), we derive that (55) holds too. Thus, (55) proves item (1). As mentioned in Lemma 2, in [27], we also have

$$\begin{aligned} (\nabla_X^{kn} R_{\nabla^{(\alpha)}, \nabla^{(\beta)}})(Y, Z)W &= \frac{(1+\alpha)(1+\beta)}{4} (\nabla_X^{kn} R_{\nabla})(Y, Z)W \\ &+ \frac{(1-\alpha\beta)}{2} (\nabla_X^{kn} R_{\nabla, \nabla^*})(Y, Z)W \\ &+ \frac{(1-\alpha)(1-\beta)}{4} (\nabla_X^{kn} R_{\nabla^*})(Y, Z)W. \end{aligned} \quad (56)$$

From (56), we derive that if R_{∇} , R_{∇^*} , and R_{∇, ∇^*} are parallel with respect to the Kobayashi–Nomizu type connection ∇^{kn} , then the mutual curvature $R_{\nabla^{(\alpha)}, \nabla^{(\beta)}}$ is parallel with respect to ∇^{kn} . Moreover, if we put the Yano type connection ∇^y instead of ∇^{kn} in (56), it implies that (56) is true too. Therefore, this proves item (2). Putting $\alpha = \beta$ in (56), the item (3) is proven. (53) and (54) give the following relations:

$$(\nabla_X^y R_{\nabla^{kn}, \nabla^{(\alpha)}})(Y, Z)W = (1+\alpha)(\nabla_X^y R_{\nabla^{kn}, \nabla})(Y, Z)W + (1-\alpha)(\nabla_X^y R_{\nabla^{kn}, \nabla^*})(Y, Z)W, \quad (57)$$

$$(\nabla_X^{kn} R_{\nabla^y, \nabla^{(\alpha)}})(Y, Z)W = (1+\alpha)(\nabla_X^{kn} R_{\nabla^y, \nabla})(Y, Z)W + (1-\alpha)(\nabla_X^{kn} R_{\nabla^y, \nabla^*})(Y, Z)W. \quad (58)$$

From (57), (58) it follows (4). $\square$

Mutual Curvatures on the Tangent Bundle

We study the complete, horizontal, and vertical lifts of the mutual curvature R_{∇_1, ∇_2} on the tangent bundle. Recall the definitions of the complete and the horizontal lifts of an affine connection.

Let ∇ be an affine connection on a Riemannian manifold M . The horizontal lift connection ∇^H and the complete lift connection ∇^C of ∇ are defined by [22]:

$$\begin{aligned} (\nabla^H_{X^H} Y^H) &= (\nabla_X Y)^H, \quad (\nabla^H_{X^H} Y^V) = (\nabla_X Y)^V, \quad (\nabla^H_{X^V} Y^H) = (\nabla_{X^V} Y^H) = 0, \\ (\nabla^C_{X^H} Y^H) &= (\nabla_X Y)^H + (R(Y, X)Y)^V, \quad (\nabla^C_{X^V} Y^H) = (\nabla_{X^V} Y^H) = 0, \\ (\nabla^C_{X^H} Y^V) &= (\nabla_X Y)^V, \quad \nabla^C_{X^C} Y^C = (\nabla_X Y)^C, \quad \nabla^C_{X^C} Y^V = \nabla^C_{X^V} Y^C = (\nabla_X Y)^V. \end{aligned} \quad (59)$$

Lemma 6. Let (M, g) be a Riemannian manifold and (∇_1, ∇_2) a pair of connections. Then, the following identities hold:

$$(R_{\nabla_1, \nabla_2}(X, Y)Z)^C = \tilde{R}_{\nabla_1^C, \nabla_2^C}(X^C, Y^C)Z^C,$$

$$(R_{\nabla_1, \nabla_2}(X, Y)Z)^H = \tilde{R}_{\nabla_1^H, \nabla_2^H}(X^H, Y^H)Z^H,$$

$$\begin{aligned}(R_{\nabla^1, \nabla^2}(X, Y)Z)^V &= \tilde{R}_{\nabla^1{}^C, \nabla^2{}^C}(X^H, Y^H)Z^V = \tilde{R}_{\nabla^1{}^H, \nabla^2{}^H}(X^H, Y^H)Z^V \\ &= \tilde{R}_{\nabla^1{}^H, \nabla^2{}^C}(X^H, Y^H)Z^V,\end{aligned}$$

for all $X, Y, Z \in \chi(M)$.

Proof. We prove only the first equation. Using $[X^C, Y^C] = [X, Y]^C$, (48) and (59), we obtain

$$\begin{aligned}(R_{\nabla^1, \nabla^2}(X, Y)Z)^C &= \left(\frac{1}{2}\left\{\nabla^1_X \nabla^2_Y Z - \nabla^1_Y \nabla^2_X Z - \nabla^1_{[X, Y]} Z + \nabla^2_X \nabla^1_Y Z - \nabla^2_Y \nabla^1_X Z \right.\right. \\ &\quad \left.\left. - \nabla^2_{[X, Y]} Z\right\}\right)^C = \left(\frac{1}{2}\left\{\nabla^1{}^C_X \nabla^2{}^C_Y Z^C - \nabla^1{}^C_Y \nabla^2{}^C_X Z^C - \nabla^1{}^C_{[X^C, Y^C]} Z^C \right.\right. \\ &\quad \left.\left. + \nabla^2{}^C_X \nabla^1{}^C_Y Z^C - \nabla^2{}^C_Y \nabla^1{}^C_X Z^C - \nabla^2{}^C_{[X^C, Y^C]} Z^C\right\}\right) \\ &= \tilde{R}_{\nabla^1{}^C, \nabla^2{}^C}(X^C, Y^C)Z^C.\end{aligned}$$

□

Lemma 7. Let M be a manifold with a torsion-free connection ∇ and TM its tangent bundle equipped with an almost complex structure $\tilde{J}$. If we denote by ∇^C the complete lift connection on TM , then the components of the mutual curvature tensor $\tilde{R}_{\tilde{\nabla}^{kn}, \nabla^C}$ are as follows:

$$\begin{aligned}\tilde{R}_{\tilde{\nabla}^{kn}, \nabla^C}(X^H, Y^H)Z^H &= \frac{1}{8}\left\{8R(X, Y)Z + \{R(y, R(y, Y)Z)X + R(X, R(y, Y)Z)y\}_{X \leftrightarrow Y} \right. \\ &\quad \left. + R(y, R(X, Y)y)Z\right\}^H + \frac{1}{8}\left\{(\nabla_X R)(Z, Y)y + (\nabla_X R)(Z, y)Y \right. \\ &\quad \left. + 4(\nabla_X R)(y, Y)Z + 5R(Z, Y)\nabla_X y - 3R(Z, \nabla_X y)Y\right\}_{X \leftrightarrow Y'}^V,\end{aligned}\quad (60)$$

$$\begin{aligned}\tilde{R}_{\tilde{\nabla}^{kn}, \nabla^C}(X^H, Y^H)Z^V &= \frac{1}{8}\left\{8R(X, Y)Z + \{R(y, X)R(y, Z)Y + R(y, X)R(Y, Z)y\}_{X \leftrightarrow Y} \right. \\ &\quad \left. + R(y, R(X, Y)y)Z\right\}^V + \frac{1}{8}\left\{(\nabla_X R)(y, Z)Y + R(\nabla_X y, Z)Y \right. \\ &\quad \left. + (\nabla_X R)(Y, Z)y + R(Y, Z)\nabla_X y\right\}_{X \leftrightarrow Y'}^H,\end{aligned}\quad (61)$$

$$\tilde{R}_{\tilde{\nabla}^{kn}, \nabla^C}(X^V, Y^V)Z^H = \tilde{R}_{\tilde{\nabla}^{kn}, \tilde{\nabla}^C}(X^V, Y^V)Z^V = 0, \quad (62)$$

$$\begin{aligned}\tilde{R}_{\tilde{\nabla}^{kn}, \nabla^C}(X^H, Y^V)Z^H &= \frac{1}{8}\left\{(\nabla_X R)(y, Y)Z + R(\nabla_X y, Y)Z\right\}^H \\ &\quad - \frac{1}{8}\left\{R(y, Y)R(y, X)Z\right\}_{Y \leftrightarrow X'}^V,\end{aligned}\quad (63)$$

$$\tilde{R}_{\tilde{\nabla}^{kn}, \nabla^C}(X^V, Y^H)Z^V = -\frac{1}{8}\left\{(\nabla_Y R)(y, X)Z + R(\nabla_Y y, X)Z\right\}^V. \quad (64)$$

In the similar way as Theorem 2, we can deduce the following:

Theorem 5. Let M be a manifold with a torsion-free connection ∇ and TM its tangent bundle equipped with an almost complex structure $\tilde{J}$. The mutual curvature $\tilde{R}_{\tilde{\nabla}^{kn}, \nabla^C}$ vanishes if and only if M is a flat manifold.

Next, we consider two torsion-free connections ∇^1 and ∇^2 on a manifold M , then the pair of connections $(\tilde{\nabla}^{1^{kn}}, \tilde{\nabla}^{2^y})$, where $\tilde{\nabla}^{1^{kn}}$ is the Kobayashi–Nomizu type connection and $\tilde{\nabla}^{2^y}$ is the Yano type connection on TM .

Lemma 8. Let M be a manifold equipped with torsion-free connections ∇^i , $i = 1, 2$, and TM its tangent bundle equipped with an almost complex structure $\tilde{J}$. Then, we have

$$\begin{aligned} \tilde{R}_{\tilde{\nabla}^1 kn, \tilde{\nabla}^2 y}(X^H, Y^H)Z^H &= \frac{1}{2} \left\{ 2R_{\nabla^1, \nabla^2}(X, Y)Z + \frac{1}{16} \{ 2R_{\nabla^1}(y, R_{\nabla^2}(Z, Y)y)X \right. \\ &\quad + 2R_{\nabla^1}(X, R_{\nabla^2}(Z, Y)y)y + R_{\nabla^1}(y, R_{\nabla^2}(Z, Y)Y)X \\ &\quad + R_{\nabla^1}(X, R_{\nabla^2}(Z, Y)Y)y + R_{\nabla^2}(y, X)R_{\nabla^1}(Z, Y)y \\ &\quad + R_{\nabla^2}(X, R_{\nabla^1}(Z, Y)y)y + R_{\nabla^2}(y, X)R_{\nabla^1}(Z, Y)Y \\ &\quad \left. + R_{\nabla^2}(X, R_{\nabla^1}(Z, Y)Y)y \}_{X \leftrightarrow Y} \right. \\ &\quad + \frac{1}{4} \{ R_{\nabla^2}(y, R_{\nabla^2}(X, Y)y)Z - R_{\nabla^2}(R_{\nabla^2}(X, Y)y, Z) \\ &\quad + R_{\nabla^1}(y, R_{\nabla^1}(X, Y)y)Z \}^H + \frac{1}{8} \{ 2(\nabla^1_X R_{\nabla^2})(Z, Y)y \\ &\quad + 2R_{\nabla^2}(Z, Y)\nabla^1_X y + (\nabla^1_X R_{\nabla^2})(Z, Y)Y + R_{\nabla^2}(Z, \nabla^1_X y)Y \\ &\quad + (\nabla^2_X R_{\nabla^1})(Z, Y)y + R_{\nabla^1}(Z, Y)\nabla^2_X y + (\nabla^2_X R_{\nabla^1})(Z, Y)Y \\ &\quad \left. + R_{\nabla^1}(Z, \nabla^2_X y)Y \right\}_{X \leftrightarrow Y}^V, \end{aligned} \quad (65)$$

$$\begin{aligned} \tilde{R}_{\tilde{\nabla}^1 kn, \tilde{\nabla}^2 y}(X^H, Y^H)Z^V &= \frac{1}{2} \left\{ 2R_{\nabla^1, \nabla^2}(X, Y)Z \right. \\ &\quad + \frac{1}{16} \{ R_{\nabla^1}(R_{\nabla^2}(y, Y)Z, X)y + R_{\nabla^1}(R_{\nabla^2}(y, Y)Z, y)X \\ &\quad + R_{\nabla^1}(R_{\nabla^2}(Y, Z)y, X)y + R_{\nabla^1}(R_{\nabla^2}(Y, Z)y, y)X + 2R_{\nabla^2}(R_{\nabla^1}(y, Z)Y, X)y \\ &\quad + R_{\nabla^2}(R_{\nabla^1}(y, Z)Y, y)X + 2R_{\nabla^2}(R_{\nabla^1}(Y, Z)y, X)y + R_{\nabla^2}(R_{\nabla^1}(Y, Z)y, y)X \}_{X \leftrightarrow Y} \\ &\quad + \frac{1}{4} \{ R_{\nabla^1}(y, R_{\nabla^1}(X, Y)y)Z + R_{\nabla^2}(y, R_{\nabla^2}(X, Y)y)Z + R_{\nabla^2}(R_{\nabla^2}(X, Y)y, Z)y \}^V \\ &\quad + \frac{1}{8} \{ (\nabla^1_X R_{\nabla^2})(y, Y)Z + R_{\nabla^2}(\nabla^1_X y, Y)Z + (\nabla^1_X R_{\nabla^2})(Y, Z)y + R_{\nabla^2}(Y, Z)\nabla^1_X y \\ &\quad \left. + (\nabla^2_X R_{\nabla^1})(y, Z)Y + R_{\nabla^1}(\nabla^2_X y, Z)Y + (\nabla^2_X R_{\nabla^1})(Y, Z)y + R_{\nabla^1}(Y, Z)\nabla^2_X y \right\}_{X \leftrightarrow Y}^H, \end{aligned} \quad (66)$$

$$\begin{aligned} \tilde{R}_{\tilde{\nabla}^1 kn, \tilde{\nabla}^2 y}(X^V, Y^V)Z^H &= \frac{1}{32} \left\{ R_{\nabla^1}(y, X)R_{\nabla^2}(y, Y)Z - R_{\nabla^1}(y, X)R_{\nabla^2}(Y, Z)y \right. \\ &\quad \left. + R_{\nabla^2}(y, X)R_{\nabla^1}(y, Y)Z - R_{\nabla^2}(X, R_{\nabla^1}(y, Y)Z)y \right\}_{X \leftrightarrow Y}^H, \end{aligned} \quad (67)$$

$$\begin{aligned} \tilde{R}_{\tilde{\nabla}^1 kn, \tilde{\nabla}^2 y}(X^V, Y^V)Z^V &= \frac{1}{32} \left\{ R_{\nabla^1}(y, X)R_{\nabla^2}(y, Y)Z \right. \\ &\quad + R_{\nabla^1}(y, X)R_{\nabla^2}(Y, Z)y + R_{\nabla^2}(y, X)R_{\nabla^1}(y, Y)Z \\ &\quad \left. + R_{\nabla^2}(X, R_{\nabla^1}(y, Y)Z)y \right\}_{X \leftrightarrow Y}^V, \end{aligned} \quad (68)$$

$$\begin{aligned}
\tilde{R}_{\tilde{\nabla}^1, \tilde{\nabla}^2}^{kn}(X^H, Y^V)Z^H &= \frac{1}{2} \left\{ \frac{1}{4} \nabla^1_X R_{\nabla^2}(y, Y)Z - \nabla^1_X R_{\nabla^2}(Y, Z)y \right. \\
&\quad - R_{\nabla^1}(y, Y) \nabla^2_X Z + \nabla^2_X R_{\nabla^1}(y, Y)Z \\
&\quad - R_{\nabla^2}(y, Y) \nabla^1_X Z + R_{\nabla^2}(Y, \nabla^1_X Z)y - R_{\nabla^2}(y, \nabla^2_X Y)Z + R_{\nabla^2}(\nabla^2_X Y, Z)y \Big\}^H \\
&\quad + \frac{1}{32} \left\{ R_{\nabla^1}(R_{\nabla^2}(y, Y)Z, X)y + R_{\nabla^1}(R_{\nabla^2}(y, Y)Z, y)X - R_{\nabla^1}(R_{\nabla^2}(Y, Z)y, X)y \right. \\
&\quad - R_{\nabla^1}(R_{\nabla^2}(Y, Z)y, y)X + 2R_{\nabla^2}(R_{\nabla^1}(y, Y)Z, X)y + R_{\nabla^2}(R_{\nabla^1}(y, Y)Z, y)X \\
&\quad - R_{\nabla^2}(y, Y)R_{\nabla^1}(Z, X)y - R_{\nabla^2}(Y, R_{\nabla^1}(Z, X)y)y - R_{\nabla^2}(y, Y)R_{\nabla^1}(Z, y)X \\
&\quad \left. - R_{\nabla^2}(Y, R_{\nabla^1}(Z, y)X)y - R_{\nabla^1}(y, Y)2R_{\nabla^2}(Z, X)y - R_{\nabla^1}(y, Y)R_{\nabla^2}(Z, y)X \right\}^V,
\end{aligned} \tag{69}$$

$$\begin{aligned}
\tilde{R}_{\tilde{\nabla}^1, \tilde{\nabla}^2}^{kn}(X^H, Y^V)Z^V &= -\frac{1}{32} \left\{ R_{\nabla^1}(y, Y)R_{\nabla^2}(y, X)Z + R_{\nabla^1}(y, Y)R_{\nabla^2}(X, Z)y \right. \\
&\quad - R_{\nabla^1}(y, R_{\nabla^2}(y, Y)Z)X \\
&\quad - R_{\nabla^1}(X, R_{\nabla^2}(y, Y)Z)y - R_{\nabla^1}(y, R_{\nabla^2}(Y, Z)y)X - R_{\nabla^1}(X, R_{\nabla^2}(Y, Z)y)y \\
&\quad + R_{\nabla^2}(y, Y)R_{\nabla^1}(y, Z)X - R_{\nabla^2}(Y, R_{\nabla^1}(y, Z)X)y + R_{\nabla^2}(y, Y)R_{\nabla^1}(X, Z)y \\
&\quad - R_{\nabla^2}(Y, R_{\nabla^1}(X, Z)y)y + R_{\nabla^2}(y, X)R_{\nabla^1}(y, Y)Z + R_{\nabla^2}(X, R_{\nabla^1}(y, Y)Z)y \Big\}^H \\
&\quad + \frac{1}{8} \left\{ (\nabla^1_X R_{\nabla^2})(y, Y)Z + R_{\nabla^2}(\nabla^1_X y, Y)Z + (\nabla^1_X R_{\nabla^2})(Y, Z)y + R_{\nabla^2}(Y, Z)\nabla^1_X y \right. \\
&\quad \left. + (\nabla^2_X R_{\nabla^1})(y, Y)Z + R_{\nabla^1}(\nabla^2_X y, Y)Z \right\}^V.
\end{aligned} \tag{70}$$

Using the above lemma, we derive the following:

Theorem 6. Let M be a manifold equipped with torsion-free connections ∇^i , $i = 1, 2$, and TM its tangent bundle equipped with an almost complex structure $\tilde{J}$. Then, the following assertions hold:

- (1) If the mutual curvature R_{∇^1, ∇^2} vanishes and M is a flat space with respect to ∇^1 and ∇^2 , then the mutual curvature $\tilde{R}_{\tilde{\nabla}^1, \tilde{\nabla}^2}^{kn}$ vanishes.
- (2) If the mutual curvature $\tilde{R}_{\tilde{\nabla}^1, \tilde{\nabla}^2}^{kn}$ vanishes, then the mutual curvature R_{∇^1, ∇^2} vanishes.
- (3) If $\nabla^1 = \nabla^2 = \nabla$, then the mutual curvature $\tilde{R}_{\tilde{\nabla}^{kn}, \tilde{\nabla}^y}$ vanishes if and only if M is flat.

6. Conclusions

This research has studied two important families of linear connections on statistical manifolds. Due to the fact that these connections are made with the help of complex structures (which are the basis of Kähler manifolds), they can have many applications in mathematical physics.

Considering the Kobayashi–Nomizu type and the Yano type connections on a Riemannian manifold and its tangent bundle (with Sasaki metric), we investigated conditions that they are torsion-free, Codazzi, and statistical structures, respectively. We also studied the relations of the curvature of a Riemannian manifold endowed with a statistical manifold with the curvature of Kobayashi–Nomizu type and the Yano type connections.

The study of the mutual curvature of these connections on the tangent bundle of a Riemannian (or statistical) manifold is an important part of this paper. The importance of the mutual curvature in mathematical physics was shown by formulating a bi-Connection Theory of Gravity with the contribution of D. Iosifidis and K. Pallikaris.

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