On Edge-Primitive Graphs of Order as a Product of Two Distinct Primes

Renbing Xiao 1, Xiaojiao Zhang 1 and Hua Zhang 2,*

1 School of Mathematics and Information Science, Nanchang Normal University, Nanchang 330032, China; xiaorenbing@ncnu.edu.cn (R.X.); xiaojiaozhang@ncnu.edu.cn (X.Z.)
2 School of Mathematics, Yunnan Normal University, Kunming 650091, China
* Correspondence: zhhdahu@gmail.com

Abstract: A graph is edge-primitive if its automorphism group acts primitively on the edge set of the graph. Edge-primitive graphs form an important subclass of symmetric graphs. In this paper, edge-primitive graphs of order as a product of two distinct primes are completely determined. This depends on non-abelian simple groups with a subgroup of index pq being classified, where p > q are odd primes.

Keywords: edge-primitive graphs; primitive permutation group; symmetric graphs

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1. Introduction

Throughout this paper, all graphs considered are assumed to be finite, connected and undirected. Let \( G = (V_G, E_G) \) be a graph with vertex set \( V_G \) and edge set \( E_G \). The size \(|V_G|\) is called the order of the graph \( G \). Define an arc as a pair of ordered adjacent vertices, let \( A_G \) be the set of the arcs of \( G \). Each edge \( \{a, \beta\} \) corresponds to two arcs \( (a, \beta) \) and \( (\beta, a) \).

Let \( G \) be a graph. For an integer \( s \geq 1 \), an s-arc in \( G \) is an \( (s + 1) \)-tuple \((v_0, v_1, \ldots, v_s)\) of vertices such that \( \{v_i, v_{i+1}\} \in E_G \), and \( v_i \neq v_{i+1} \) for \( 0 \leq i \leq s - 1 \). A permutation of \( V_G \) that preserves the adjacency of \( G \) is called an automorphism of \( G \), and all automorphisms of \( G \) form a group which is called the full automorphism group of \( G \), denoted by \( Aut(G) \). Let \( G \) be a subgroup of \( Aut(G) \), and denoted by \( G \leq Aut(G) \). Let \( G \leq Aut(G) \) act on \( V_G \) and \( E_G \). We say that \( G \) as the subgroup of \( G \) is a vertex-stabilizer if \( G \) is fixing the vertex \( a \). (Similarly, let \( e = (a, \beta) \in E_G \). We may define the edge-stabilizer and arc-stabilizer of \( G \), denoted by \( G_e \) and \( G_{\alpha\beta} \), respectively). Moreover, the group \( Aut(G) \) has a natural action on \( E_G \). Then, the graph \( G \) is said to be G-edge-transitive if \( E_G \neq 0 \) and for each pair of edges there exists some \( g \in G \leq Aut(G) \) mapping one of these two edges to the other one. So, the graph \( G \) is called G-vertex-transitive if \( G \leq Aut(G) \) is transitive on \( V_G \) or \( A_G \), respectively. A graph \( G \) that is a G-arc-transitive graph for some \( G \leq Aut(G) \) is also known as a symmetric graph.

A graph \( G \) is G-edge-primitive if \( G \leq Aut(G) \) acts primitively on \( E_G \), that is, if \( G \) preserves no nontrivial partition of the edge set. A G-edge-transitive graph \( G \) is G-edge-primitive if some edge-stabilizer, the subgroup of its automorphism group which fixes a given edge, is a maximal subgroup of the automorphism [1]. Additionally, \( G \) is called edge-primitive if it is \( Aut(G) \)-edge-primitive. In this paper, the original motivation was the problem of classifying all edge-primitive graphs of order as a product of two distinct primes. The study of edge-primitive graphs was initiated by R. M. Weiss. In 1973, Weiss [2] confirmed all edge-primitive graphs of valency 3. These graphs are the Heawood graph of order 14, the complete bipartite graph \( K_{3,3} \), the Levi graph and the Biggs–Smith cubic distance-transitive graph of order 102. Giudici and Li [3] systematically studied the O’Nan–Scott primitive types of the automorphism groups of edge-primitive graphs, and the G-edge-primitive...
graphs for $G$, an almost simple group with socle $\text{PSL}_2(q)$, are classified. We use $\text{soc}(G)$ to denote the socle of a group $G$, that is, the subgroup of $G$ generated by all minimal normal subgroups of $G$. In case $G$ is finite, the socle is the product of all minimal normal subgroups of $G$. A two dimensional projective group is denoted by $\text{PSL}_2(q)$. Li and Zhang [3] analyzed edge-primitive $s$-arc-transitive graphs for $s \geq 4$. Guo et al. classified edge-primitive tetravalent and pentavalent graphs in [5] and [6]. Pan et al. discussed edge-primitive graphs of prime valency in [7], and edge-primitive Cayley graphs on abelian groups and dihedral groups in [8]. Lu [9] proved that a finite 2-arc-transitive edge-primitive graph has an almost simple automorphism group if it is neither a cycle nor a complete bipartite graph. Recently, Giudici and King [10] classified edge-primitive 3-arc-transitive graphs.

The work of studying edge-primitive graphs of specific orders is also attractive. Pan et al. studied all edge-primitive graphs of prime power order in [11], and edge-primitive graphs of order twice as a prime power in [12]. The main work of this paper is to classify all edge-primitive graphs of order as a product of two distinct primes.

In this paper, the notations used are standard [1]. For a positive integer $n$, we usually use $K_n$ and $K_{n,n}$ to denote the complete graph of order $n$ and the complete bipartite graph of order $2n$, respectively. $\mathbb{Z}_n$ is defined as the cyclic group of order $n$, and $D_{2n}$ as the dihedral group of order $2n$. As in Atlas [13], sometimes we simply use $n$ to denote a cyclic group of order $n$. For the two groups $K$ and $H$, we use $K \times H$ and $K : H$ to denote the direct product of $K$ and $H$ and the semidirect product of $K$ by $H$, respectively. The general linear group $\text{GL}_n(q)$ consists of all the $n \times n$ matrices with entries in $F_q$ that have a non-zero determinant. The special linear group $\text{SL}_n(q)$ is the subgroup of all matrices of determinant 1. The projective linear group $\text{PGL}_n(q)$ and projective special linear group $\text{PSL}_n(q)$ are the groups obtained from $\text{GL}_n(q)$ and $\text{SL}_n(q)$ on factoring by the scalar matrices contained in those groups. We use $\text{PSU}_n(q)$, $\text{PSp}_n(q)$, and $\text{PΩ}_n(q)$ to denote the projective symplectic group, the projective unitary group, the projective orthogonal group, respectively. See [3] for details.

The classification of graph theory is closely related to the classification of group theory. The application of group theory in graph theory is mainly achieved through the role of groups on graphs. The symmetry of a graph is mainly described by the role of the automorphism group of the graph on each subgraph of the graph, such as the transitivity and primitivity of the automorphism group on the vertex set and edge set of the graph. The edge-primitive graph discussed in this paper is one of them. Specifically, the construction, characterization, and classification of various edge-primitive graphs with additional conditions have become some of the main issues discussed in algebraic graph theory. This paper completes the classification of specific orders in the edge-primitive graph, that is, edge-primitive graphs of order as a product of two distinct primes are completely determined.

The main result of this paper is shown as follows. Some of the graphs that appear in Table 1 will be explained in Section 2.

**Theorem 1.** Let $\Gamma$ be a $G$-edge-primitive graph of order $pq$, where $G \leq \text{Aut} \Gamma$, and $p > q$ are odd primes. Then, one of the following holds:

1. $\Gamma$ is a star.
2. $\Gamma$, $G$ are listed in Table 1, where for $\alpha \in V\Gamma$ and $e \in E\Gamma$, $G_\alpha$ and $G_e$ is the stabilizer of $\alpha$ and $e$, respectively.

**Table 1.** Edge-primitive graphs of order as a product of two distinct primes.

| $G$          | $G_e$          | $G_\alpha$ | $|V\Gamma|$ | Remark                  |
|--------------|----------------|-------------|-------------|-------------------------|
| $M_{22}$     | $M_{10}$       | $2^4 : A_6$ | 77          | $G(77,6)$               |
| $A_{pq}$     | $S_{pq-2}$     | $A_{pq-1}$  | $pq$        | $K_{pq}, pq \geq 15$   |
| $\text{PSL}_2(19)$ | $D_{20}$       | $A_5$       | 57          | $G(57,6)$               |
| $\text{PSL}_2(25)$ | $D_{24}$       | $S_5$       | 65          | $G(65,10)$              |
The layout of this paper is as follows. We collect some basic properties of edge-primitive graphs and some examples for edge-primitive graphs in Section 2. The most important theorem was proved in the last section.

2. Preliminary and Examples

The simplest examples of edge-primitive graphs are the stars $K_{1,n}$, the cycles with prime numbers of vertices, and the complete graphs $K_n$. Following [3], we call an edge-primitive graph trivial if it is a star or a cycle. In this paper, non-trivial edge-primitive graphs are our main research object. We first collect some preliminary results of edge-primitive graphs for later use.

**Lemma 1** ([3], Lemma 3.4). Let $\Gamma$ be a non-trivial $G$-edge-primitive graph for some $G \leq \text{Aut}(\Gamma)$. Then, $\Gamma$ is $G$-arc-transitive.

Arc transitive graphs can be represented using the group theory method of constructing a coset graph. Let $G$ be a finite group, and let $H \leq G$. We say that the set $|G : H|$ is the right coset of $H$ in $G$ if $|G : H| = \{Hx \mid x \in G\}$. For an element $g \in G$ with $g^2 \in H$, $Hx, Hy \in |G : H|$, we say that $\text{Cos}(G, H)$ is a coset graph of $G$ with respect to $H$ and $g$ if $Hx$ and $Hy$ are adjacent if and only if $yx^{-1} \in HgH$. The graph $\Gamma$ is connected if and only if $(H, g) = G$. Let $\alpha$ be the vertex $H$ of the coset graph. Moreover, the valency of $\Gamma$ is $|H : H \cap Hg|$, and the stabilizer of the edge $\{H, Hg\}$ is $\langle H \cap Hg, g \rangle$. See [3] for details.

Let $e = \{\alpha, \beta\} \in E\Gamma$, denote by $G_e, G_{\alpha}, G_{\beta}$ as the vertex-stabilizer, edge-stabilizer, and arc-stabilizer of $G$, respectively.

**Lemma 2** ([12], Lemma 2.5). Let $\Gamma$ be a graph, $1 \neq N \triangleleft G \leq \text{Aut}\Gamma$, and $e = \{\alpha, \beta\} \in E\Gamma$. Then, the following statements hold.

1. If $\Gamma$ is $G$-edge-primitive, then $G_e \cong N_e(G/N)$. In particular, $|G_e| = |N_e||G : N|$.
2. If $\Gamma$ is $G$-arc-transitive, then $G_e \cong G_{\alpha}\mathbb{Z}_2$.
3. If $\Gamma$ is $G$-edge-transitive but not $G$-arc-transitive, then $G_e \cong G_{\alpha\beta}$.

The valency of a regular graph $\Gamma$ is denoted by $\text{val}(\Gamma)$.

**Lemma 3.** Let $\Gamma$ be a nontrivial $G$-edge-primitive graph and $e = \{\alpha, \beta\} \in E\Gamma$. Then $|G_\alpha| > |G_e|$.

**Proof.** It can be easily concluded that $\Gamma$ is non-trivial, $\text{val}(\Gamma) = |G_\alpha : G_{\alpha\beta}| \geq 3$, and $\Gamma$ is $G$-arc-transitive, so $|G_e| = 2 \cdot |G_{\alpha\beta}|$. \(\square\)

We define a transitive permutation group $G \leq \text{Sym}(\Omega)$ as quasiprimitive if every minimal normal subgroup of $G$ is transitive on $\Omega$. Moreover, we say that the group $G$ is biquasiprimitive if each of its minimal normal subgroups has at most two orbits, and there is a minimal normal subgroup with exactly two orbits $\Omega$ on it.

Let $\Gamma$ be a $G$-edge-primitive graph with $G \leq \text{Aut}(\Gamma)$, and let $N$ be a nontrivial normal subgroup of $G$. If $N$ is transitive on edges, then $\Gamma$ is either transitive on vertices or $\Gamma$ is bipartite and $N$ has two orbits on the vertex set. This simple observation leads to the following assertion.

**Lemma 4.** Let $\Gamma$ be a non-trivial $G$-edge-primitive graph with $G \leq \text{Aut}(\Gamma)$. Then, $G$ is either quasiprimitive or biquasiprimitive on $V\Gamma$.

Therefore, we need some relevant information for (quasi)primitive permutation groups. Let $G$ be a quasiprimitive group. Utilizing the structure and the action of $\text{soc}(G)$, the quasiprimitive permutation group is divided into eight types by O’Nan–Scott–Praeger theorem, namely HA, HS, HC, AS, SD, CD, PA, and TW. See Praeger [14] for details. As an application of the O’Nan–Scott–Praeger theorem, it is straightforward to obtain the following results.
Lemma 5. Let $G$ be a quasiprimitive permutation group of degree $pq$, where $p > q$ are odd primes. Then, $G$ is an almost simple group.

By Theorem 2.1 of [3], Giudici and Li have classified the groups which act edge-primitively on a complete graph.

Lemma 6 ([8], Lemma 2.4). Let $G$ be an almost simple group with $\text{soc}(G) = \text{PSL}_n(k)$ and $n \geq 3$. Then, the action of $G$ on a complete graph is not edge-primitive.

To construct edge-primitive graphs, the most important results are as follows:

Proposition 1 ([3], Proposition 2.5). Let $G$ be a finite group with a maximal subgroup $E$. Then, there exists a $G$-edge-primitive, arc-transitive graph $\Gamma$ with an edge stabiliser $E$ if and only if $E$ has a subgroup $A$ and $|E : A| = 2$. In addition, $G$ has a core-free subgroup $H$ such that $A < H \neq E$. In this case, $\Gamma = \text{Cos}(G, H, HgH)$ for some $g \in E \setminus A$.

Example 1. Let $\Gamma = K_{pq}$, where $p > q$ are odd primes. Then, $\text{Aut}\Gamma = S_{pq}$ contains a subgroup $G = A_{pq}$. This subgroup $G$ has a maximal subgroup $E = S_{pq-2}$, and $E$ has a subgroup $A = A_{pq-2}$ of index two. The group $G$ also contains a maximal subgroup isomorphic to $H = A_{pq-1}$, and $H$ contains the subgroup $A$. So, the graph $\Gamma$ is $G$-vertex-primitive, and by Proposition 1, the graph $\Gamma$ is also $G$-edge-primitive.

Example 2. Let $T \cong M_{22}$. According to Atlas [13], $M_{22}$ has two maximal subgroups $H \cong 2^4 : A_6$ and $E \cong M_{10}$ such that $H \cap E \cong A_6$. Define a coset graph

$$G(77, 16) := \text{Cos}(T, H, HgH),$$

with $g \in E \setminus H$ an involution.

By Proposition 1, $G(77, 16)$ is an $T$-edge-primitive graph, with valency $|H : H \cap E| = 16$, and $|G(77, 16)| = |T : H| = 77$. Based on the calculation of the Magma [15], it can be concluded that any $T$-arc-transitive graph with vertex stabilizer $2^4 : A_6$ and valency 77 is isomorphic to $G(77, 16)$ and has the automorphism group $M_{22} \Z_2$. So, $G(77, 16)$ is $G$-edge-primitive with $M_{22} \leq G \leq M_{22} \Z_2$.

Example 3. Let $T \cong \text{PSL}_2(19)$. Then, following Proposition 8.4 of [3], $T$ has a subgroup $H \cong A_5$ and a maximal subgroup $E \cong D_{20}$ of one conjugate class such that $H \cap E \cong D_{10}$. Define a coset graph

$$G(57, 6) := \text{Cos}(T, H, HgH),$$

with $g \in E \setminus H$ an involution.

By Proposition 1, this graph is $T$-edge-primitive, with valency $|H : H \cap E| = 6$ and $|G(57, 6)| = |T : H| = 57$. Furthermore, after calculation in Magma [15], it can be concluded that any $T$-arc-transitive graph with vertex stabilizer $A_5$ and valency 57 is isomorphic to $G(57, 6)$, and has automorphism group $\text{PSL}_2(19).Z_2$. So, $G(57, 6)$ is $G$-edge-primitive with $\text{PSL}_2(19) \leq G \leq \text{PGL}_2(19)$.

Example 4. Let $T \cong \text{PSL}_2(25)$. Then, from Proposition 8.4 of [3], $T$ has a subgroup $H \cong S_5$ and of a maximal subgroup $E \cong D_{24}$ of one conjugate class such that $H \cap E \cong D_{12}$. Define a coset graph

$$G(65, 10) := \text{Cos}(T, H, HgH),$$

with $g \in E \setminus H$ an involution.

By Proposition 1, this graph is $T$-edge-primitive, with valency $|H : H \cap E| = 10$ and $|G(65, 10)| = |T : H| = 65$. Furthermore, a computation by Magma [15] shows that any $T$-arc-transitive graph with vertex stabilizer $A_5$ and valency 65 is isomorphic to $G(65, 10)$, and has automorphism group $\text{PSL}_2(25).Z_2$. So, $G(65, 10)$ is $G$-edge-primitive with $\text{PSL}_2(25) \leq G \leq \text{PGL}_2(25)$.
3. Proof of Theorem 1

Let $\Gamma$ be a non-trivial $G$-edge-primitive graph for some $G \leq \text{Aut}(\Gamma)$. Further assume that the order of this graph is $pq$, where $p > q$ are odd primes. By Lemmas 1 and 4, $\Gamma$ is $G$-arc-transitive and $G$ is either quasiprimitive or biquasiprimitive on $V\Gamma$.

Therefore, we need to consider two types of cases when $G$ is quasiprimitive on $V\Gamma$. By Lemma 5, $G$ can only be an almost simple group. Thus, $\text{soc}(G) = T$ is non-abelian simple and transitive on $V\Gamma$, so $|T : T_a| = pq$. Non-abelian simple groups with a subgroup of index $pq$ have been classified in [16] (Theorem 1.1) (see also [17] [THEOREM]). The result can be read off as follows.

**Lemma 7.** Let $T$ be a non-abelian simple group with a subgroup $H$ of index $pq$, where $p > q$ are odd primes. Then, the tuple $(T, H)$ is listed in Table 2, where $P_1$ is the stabilizer of the classical group acting naturally on the 1-subspaces.

### Table 2. Non-abelian simple groups with a subgroups of index $pq$

| Row | $T$ | $H$ | $|T : H|$ | Conditions |
|-----|-----|-----|----------|------------|
| 1   | $A_5$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | 3 · 5 | $\mathbb{Z}_2 \times \mathbb{Z}_2 \leq A_4$ |
| 2   | $A_7$ | $\text{PSL}_2(7)$ | 3 · 5 | $\mathbb{Z}_2 \times \mathbb{Z}_2 < A_4$ |
| 3   | $A_8$ | $2^4 : \text{PSL}_3(2)$ | 5 · 7 | $\mathbb{Z}_2 \times \mathbb{Z}_2 < A_4$ |
| 4   | $M_{11}$ | $M_{23} : 2$ | 5 · 7 | $\mathbb{Z}_2 \times \mathbb{Z}_2 < A_4$ |
| 5   | $M_{22}$ | $2^4 : A_6$ | 7 · 11 | $\mathbb{Z}_2 \times \mathbb{Z}_2 < A_4$ |
| 6   | $M_{23}$ | $\text{PSL}_3(4) : 2$ | 11 · 23 | $\mathbb{Z}_2 \times \mathbb{Z}_2 < A_4$ |
| 7   | $\text{PSL}_2(11)$ | $A_4$ | 5 · 11 | $\mathbb{Z}_2 \times \mathbb{Z}_2 < A_4$ |
| 8   | $\text{PSL}_2(19)$ | $A_5$ | 3 · 19 | $\mathbb{Z}_2 \times \mathbb{Z}_2 < A_4$ |
| 9   | $\text{PSL}_2(23)$ | $S_4$ | 11 · 23 | $\mathbb{Z}_2 \times \mathbb{Z}_2 < A_4$ |
| 10  | $\text{PSL}_2(25)$ | $S_5$ | 5 · 13 | $\mathbb{Z}_2 \times \mathbb{Z}_2 < A_4$ |
| 11  | $\text{PSL}_2(29)$ | $A_5$ | 7 · 29 | $\mathbb{Z}_2 \times \mathbb{Z}_2 < A_4$ |
| 12  | $\text{PSL}_2(59)$ | $A_5$ | 29 · 59 | $\mathbb{Z}_2 \times \mathbb{Z}_2 < A_4$ |
| 13  | $\text{PSL}_2(61)$ | $A_5$ | 31 · 61 | $\mathbb{Z}_2 \times \mathbb{Z}_2 < A_4$ |
| 14  | $\text{PSL}_2(2)$ | $2^6 : (\mathbb{S}_3 \times \mathbb{S}_l(2))$ | 5 · 31 | $\mathbb{Z}_2 \times \mathbb{Z}_2 < A_4$ |
| 15  | $\text{PO}_4^+(2)$ | $2^6 : \text{PSU}_4(2)$ | 7 · 17 | $\mathbb{Z}_2 \times \mathbb{Z}_2 < A_4$ |
| 16  | $\text{PO}_5^-(2)$ | $2^8 : \text{PSL}_6(2)$ | 17 · 31 | $\mathbb{Z}_2 \times \mathbb{Z}_2 < A_4$ |
| 17  | $A_{pq}$ | $A_{pq-1}$ | $pq$ | $p q \geq 15$ |
| 18  | $A_p$ | $S_{p-2}$ | $p^{q-1} : p$ | $p \geq 11$, $p^{q-1}$ prime |
| 19  | $A_{p+1}$ | $S_{p-1}$ | $p^{q-1} : p$ | $p \geq 5$, $p^{q-1}$ prime |
| 20  | $\text{PSL}_2(p)$ | $D_{p+1}$ | $p^{q-1} : p$ | $p^{q-1}$ odd prime |
| 21  | $\text{PSL}_2(q)$ | $P_1$ | $p^{q-1} : p$ | $n \geq 3$, $p^{q-1} = pq$ |
| 22  | $\text{PSU}_4(2)$ | $P_1$ | $(2^q + 1)(2^{q+1} + 1)$ | $q = 2^q + 1$, $p = 2^{q+1} + 1$ |
| 23  | $\text{PSU}_3(2^q)$ | $P_1$ | $(2^q + 1)(2^{q+1} - 2^q + 1)$ | $q = 2^q + 1$, $p = 2^{q+1} - 2^q + 1$ |
| 24  | $\text{PO}_5^+(2^{q+1})$ | $P_1$ | $(2^q + 1)(2^{q+1} - 1)$ | $q = 2^q + 1$, $p = 2^{q+1} - 1$ |
| 25  | $\text{PO}_5^-(2^{q+1})$ | $P_1$ | $(2^q - 1)(2^{q+1} + 1)$ | $q = 2^q - 1$, $p = 2^{q+1} + 1$ |

**Lemma 8.** Assume that $G$ is an almost simple quasiprimitive group on $V\Gamma$. Then, $\Gamma$ and $G$ are listed in Table 1 in Theorem 1.

**Proof.** Based on the assumption, $G \cong T \cdot o$, where $o \leq \text{Out}(T)$ and $|T : T_a| = pq$. Hence, the tuple $(T, T_a)$ (as $(T, H)$ there) is listed in Table 2. We analyze each candidate in the following.

**Row 1.** In this case, $T \cong A_5$, $T_a = \mathbb{Z}_2 \times \mathbb{Z}_2$, $|V\Gamma| = 15$ and $\text{Out}(T) = \mathbb{Z}_2$. Hence, $G \cong A_5$ or $\text{Aut}(A_5) \cong S_5$, and $|G_a| = 4$ or 8, respectively. By Lemma 3, $|G_a| < 4$ or 8. However, by the Atlas [13], $A_5$ or $S_5$ have no such maximal subgroup $G_a$. So, $\Gamma$ is not edge-primitive in this case.
Row 2. In this case, $T \cong A_7$, if $T_a \cong \text{PSL}(2,7)$, $|V_T| = 15$ and $G \cong T.o$, where $o \leq \text{Out}(A_7) \cong Z_2$. Then, by [1] (p. 308, TABLE B.2), $T$ is of rank 2 on $V_T$, so one non-trivial suborbit is of lengths 14. Hence, $\Gamma$ is $T$-arc-transitive of valency 14. Assume $\text{val}(\Gamma) = 14$, then $|T_a\beta| = \frac{|T_a|}{|\Gamma|} = 12$ and $|T| = 24$, so $|G_e| = 24|o|$. However, by the Atlas [13], there is no maximum subgroup of order 24|o| in group $A_7.o$, which is a contradiction. If $T \cong (A_4 \times 3) : 2$, $|V_T| = 35$, then by Lemma 3, $|G_e| < 72|o|$. However, by the Atlas [13], $G \cong T.o$ has no such maximal subgroup $G_e$, which is a contradiction.

Row 3. Assume that $T \cong A_8$, if $T_a \cong 2^3 : \text{PSL}_3(2)$, $|V_T| = 15$ and $G \cong T.o$, where $o \leq \text{Out}(A_8) \cong Z_2$. By [1] (p. 308, TABLE B.2), $T$ is of rank 2 on $V_T$, so one non-trivial suborbit is of lengths 14. Then, $\Gamma$ is $T$-arc-transitive of valency 14. Assume $\text{val}(\Gamma) = 14$, then $|T_a\beta| = \frac{|T_a|}{|\Gamma|} = 96$ and $|T| = 192$, so $|G_e| = 192|o|$. However, by the Atlas [13], there is no maximum subgroup of order 192|o| in group $A_8.o$, which is a contradiction. If $T \cong 2^4 : (S_3 \times S_3)$, $|V_T| = 35$, and by Lemma 3, $|G_e| < 576|o|$. By the Atlas [13], $G_e \cong (A_5 \times 3) : 2$ or $S_5 \times S_3$, then $|G_a\beta| = \frac{|G_e|}{|o|} = 360$ or 720. However, a computation by Magma [15] shows that $2^3 : (S_3 \times S_3)$ has no subgroup with order 360 or 720, a contradiction.

Row 4. Assume that $T \cong M_{11}$, $|V_T| = 55$, and $\text{Out}(M_{11}) = 1$. By Lemma 3, $|T_e| < |T_a| = 144$. By the Atlas [13], $T_e \cong S_9$ or $M_8 : S_3$, then $|T_a\beta| = \frac{|T_a|}{|\Gamma|} = 120$ or 48. However, according to the calculation in Magma [15], it can be concluded that $T_a \cong M_9 \colon 2$ has no subgroup with order 120 or 48, which is a contradiction.

Row 5. In this case, $T$ is primitive on $V_T$, $|V_T| = 77$, and $G \cong T.o$, where $o \leq \text{Out}(M_{23}) \cong Z_2$. By [1] (p. 321, TABLE B.2), $T$ is of rank 3 on $V_T$, and it is easy to compute out that the lengths of its two non-trivial suborbits are 16 and 60. So, $\Gamma$ is $T$-arc-transitive of valency 16 and 60. If $\text{val}(\Gamma) = 16$, by Example 2, $\Gamma = G(77,16)$. If $\text{val}(\Gamma) = 60$, then $|T_a\beta| = \frac{|T_a|}{|\Gamma|} = 96$, and by Lemma 6, $|G_e| = |T_e.o| = |T_a\beta.Z_2.o| = 192|o|$, hence, $|G_e| \leq 384$. However, by the Atlas [13], all maximal subgroups of $T$ and $\text{Aut}(T)$ are of order at least 660, which is a contradiction.

Row 6. In this case, $T \cong M_{23}$, if $T_a \cong \text{PSL}_3(4) : 2$, $|V_T| = 253$, and $\text{Out}(M_{23}) = 1$, $G \cong M_{23}$. By Lemma 3, $|G_e| < |G_a| = 40320$. By the Atlas [13], we can see that there are five possibilities for $G_e$, $2^4 : A_7$, $A_8$, $M_{11}$, $2^4 : (3 \times A_5) : 2$, $23 : 11$. Note that only the case $2^4 : (3 \times A_5) : 2$ contains an index two subgroup $2^4 : (3 \times A_5)$, hence $|G_a\beta| = 2880$. However, from Magma [15], $G_a \cong \text{PSL}_3(4) : 2$ has no subgroup with order 2880, which is a contradiction. Similarly, for the case $T_a \cong 2^4 : A_7$, we also get a contradiction.

Row 7–13. By [3] (Theorem 1.3), since $|V_T| = pq$, and $p > q$ are odd primes. Now, a direct computation can determine all the possibilities of $\Gamma$. If $T \cong \text{PSL}_2(19)$, by Example 3, $\Gamma = G(57,6)$. If $T \cong \text{PSL}_2(25)$, by Example 4, $\Gamma = G(65,5)$.

Row 14. In this case, $T \cong \text{PSL}_3(2)$, $|V_T| = 155$, and $G \cong T.o$, where $o \leq \text{Out}(\text{PSL}_3(2)) \cong Z_2$. Using the calculations in Magma [15], it can be concluded that $T$ is of rank 3 on $V_T$, with two non-trivial suborbits of lengths 42 or 112. Hence, val $(\Gamma) = 42$ or 112, and $\Gamma$ is $T$-arc-transitive. If $\text{val}(\Gamma) = 42$, then $|T_a\beta| = \frac{|T_a|}{|\Gamma|} = 1536$, by Lemma 2, $|G_e| = |T_e.o| = |T_a\beta.Z_2.o| = 3076|o|$. However, by the Atlas [13], there is no maximum subgroup of order 3076|o| in group $\text{PSL}_3(2).o$, which is a contradiction. If $\text{val}(\Gamma) = 112$, then $|T_a\beta| = \frac{|T_a|}{\text{val}(\Gamma)} = 576$, and $|G_e| = |T_e.o| = |T_a\beta.Z_2.o| = 1152|o|$. By the Atlas [13], there is no maximum subgroup of order 1152|o| in group $\text{PSL}_3(2).o$, which is a contradiction.

Row 15. If $T \cong P\Omega_8^-(2)$, $|V_T| = 119$, and $G \cong T.o$, where $o \leq \text{Out}(P\Omega_8^-(2)) \cong Z_2$. According to the calculation in Magma [15], it can be concluded that $T$ is of rank 3 on $V_T$, with two non-trivial suborbits of lengths 54 or 64. Hence, val $(\Gamma) = 54$ or 64, and $\Gamma$ is $T$-arc-
transitive. If \( \text{val}(\Gamma) = 54 \), then \(|T_{a\beta}| = \frac{T_{a\beta}}{\text{val}(\Gamma)} = 30720\), and \(|G_e| = |T_e.o| = |T_{a\beta}.\mathbb{Z}_2.o| = 61440|o|\). However, by Atlas [13], there is no maximum subgroup of order 61440|o| in group \( P\Omega_4^+(2).o \), which is a contradiction. If \( \text{val}(\Gamma) = 64 \), then \(|T_{a\beta}| = \frac{T_{a\beta}}{\text{val}(\Gamma)} = 25920\), and \(|G_e| = |T_e.o| = |T_{a\beta}.\mathbb{Z}_2.o| = 51840|o|\). However, by the Atlas [13], there is no maximum subgroup of order 51840|o| in group \( P\Omega_4^+(2).o \), which is a contradiction.

Row 16. In this case, \( T \cong P\Omega_4^+(2) \), \(|VT| = 527\), and \( G \cong T.o \), where \( o \leq \text{Out}(P\Omega_4^+(2)) \cong \mathbb{Z}_2 \). From Magma [15], a simple computation can determine that the rank of \( T \) on \( VT \) is 3, with two non-trivial suborbits of lengths 256 or 270. Hence, \( \text{val}(\Gamma) = 256 \) or 270, and \( \Gamma \) is \( T \)-arc-transitive. If \( \text{val}(\Gamma) = 256 \), then \(|T_{a\beta}| = \frac{T_{a\beta}}{\text{val}(\Gamma)} = 174182400\), and \(|G_e| = |T_e.o| = |T_{a\beta}.\mathbb{Z}_2.o| = 348364800|o|\). However, by Atlas [13], \( G \cong T.o \) has no maximal subgroup with order 348364800|o|, which is a contradiction. Similarly, if \( \text{val}(\Gamma) = 270 \), then \(|T_{a\beta}| = \frac{T_{a\beta}}{\text{val}(\Gamma)} = 165150720\), \(|G_e| = 330301440|o|\). By the Atlas [13], there is no maximum subgroup of order 330301440|o| in group \( P\Omega_4^+(2).o \), which is a contradiction.

Row 17. In this case, \( T \cong A_{pq} \) is 2-transitive on the set of right cosets of \( A_{pq-1} \), where \( p q \geq 15 \), so \( \Gamma = K_{pq} \) and \( T_a \cong A_{pq-1} \). Hence, \( T_{a\beta} \cong A_{pq-2} \), and \( T_e \cong S_{pq-2} \) is maximal in \( T \), so \( \Gamma \) is \( G \)-edge-primitive with \( G \cong A_{pq} \) and \( S_{pq} \).

Row 18–19. Assume that \( T \cong A_p \), \(|VT| = \frac{p-1}{2} \cdot p \), and \( G \cong T.o \), where \( o \leq \text{Out}(A_p) \). According to the calculation in Magma [15], \( T \) can be concluded that \( T \) is of rank 3 on \( VT \), with two non-trivial suborbits of lengths 2\( (p - 2) \) or \( \frac{(p - 2)(p - 3)}{2} \), and \( \Gamma \) is \( T \)-arc-transitive. If \( \text{val}(\Gamma) = 2(p - 2) \), then \(|T_{a\beta}| = \frac{T_{a\beta}}{\text{val}(\Gamma)} = \frac{(p - 3)!}{2} \), and \(|G_e| = |T_e.o| = |T_{a\beta}.\mathbb{Z}_2.o| = (p - 3)!|o|\). By [18] (Theorem 1.1), there exists no subgroup \( G_e \), which is a maximal subgroup of \( G \cong T.o \), such that \( G \) has a subgroup \( G_{a\beta} \) of index two. Thus, there is no \( G \)-edge-primitive graph arising in this case. If \( \text{val}(\Gamma) = \frac{(p - 2)(p - 3)}{2} \), then \(|T_{a\beta}| = \frac{T_{a\beta}}{\text{val}(\Gamma)} = (p - 4)!|o|\). By [18] (Theorem 1.1), there is no \( G \)-edge-primitive graph. Assume that \( T \cong A_{p+1} \), similar to the discussion above, \(|G_e| = (p - 2)!|o| \) or \( (p - 3)!|o| \), by [18] (Theorem 1.1), there is no \( G \)-edge-primitive graph occurring in this case.

Row 20. Then, \( T \cong PSL_2(p) \), \( T_a \cong D_{p \pm 1} \). By [3] (Theorem 1.3), no graph \( \Gamma \) exists in this case.

Row 21. Then, \( T \cong PSL_n(k) \) is 2-transitive on \( VT = \{n \geq 3\} \), so \( \Gamma \) is a complete graph, contradicting Lemma 6.

Row 22. In this case, \( T \cong PSp_4(2^q) \), \(|VT| = (2^{2^q} + 1)(2^{2^{q+1}} + 1) \), and \( G \cong T.o \) with \( o \leq \text{Out}(PSp_4(2^q)) \). From Magma [15], it can be concluded that \( T \) is of rank 3 on \( VT \), with two non-trivial suborbits of lengths \( 2^{2^q} + 2^{2^{q+1}} \) or \( 2^{3^q} \). Hence, \( \text{val}(\Gamma) = 2^{2^q} + 2^{2^{q+1}} \), and \( \Gamma \) is \( T \)-arc-transitive. If \( \text{val}(\Gamma) = 2^{2^q} + 2^{2^{q+1}} \), then \(|T_{a\beta}| = \frac{|PSp_4(2^q)|}{|VT|\cdot\text{val}(\Gamma)} \), and \(|G_e| = |T_e.o| = |T_{a\beta}.\mathbb{Z}_2.o| = \frac{2|PSp_4(2^q)||o|}{(2^{2^q} + 1)(2^{2^{q+1}} + 1)} = 2^{2^q+1}(2^{2^q} - 1)^2|o|\). Therefore, by [19] (Tables 8.12–8.15), \( G \cong T.o \) has no maximal subgroup with order \( 2^{2^{q+1}}(2^{2^q} - 1)^2 \), which is a contradiction. If \( \text{val}(\Gamma) = 2^{3^q} \). Similarly, \(|G_e| = \frac{2|PSp_4(2^q)||o|}{(2^{2^q} + 1)(2^{2^{q+1}} + 1)(2^{3^q} + 1)} = 2^{2^q+1}(2^{2^q} - 1)(2^{3^q} - 1)|o| \). By [19] (Tables 8.12–8.15), \( T \) has a maximal subgroup \( GL_2(2^q)\mathbb{Z}_2 \) with order \( 2^{2^q+1}(2^{2^q} - 1)(2^{3^q} - 1) \). However, \( T \cong PSp_4(2^q) \), so \( 2^{2^q+1} \) is an even number, again a contradiction.
Row 23. In this case, \( T \cong \text{PSU}_3(2^2) \), \(|VT| = (2^2 + 1)(2^{2i+1} - 2^i + 1)\), and \( G \cong T.o \) with \( o \leq \text{Out}(\text{PSU}_3(2^2)) \). From Magma [15], a simple computation can determine that the rank of \( T \) on \( V \) is 2, with one non-trivial suborbit of lengths \( 2^3 \cdot 2^i \). Hence, \( \text{val}(\Gamma) = 2^3 \cdot 2^i \) and \( \Gamma \) is T-arc-transitive. If \( \text{val}(\Gamma) = 2^3 \cdot 2^i \), then \(|Ta_\beta| = |T_o|/|\text{val}(\Gamma)| = |\text{PSU}_3(2^2)|/|V|/|\text{val}(\Gamma)| \), and \( |Ge| = |T_o| = |Ta_\beta Z_{2^2}o| \), so \( |Ge| = 2|\text{PSU}_3(2^2)||o|/(2^i + 1)(2^{2i+1} - 2^i + 1)(2^3 \cdot 2^i) = 2(2^{2i+1} - 1)|o|. \) However, by [19] (Table 8.5–8.6), \( G \cong T.o \) has no maximal subgroup with order \( 2(2^{2i+1} - 1)|o| \), which is a contradiction.

Row 24. In this case, \( T \cong \text{PO}^+_{2i+1}(2) \), \(|VT| = (2^2 + 1)(2^{2i+1} - 1)\), and \( G \cong T.o \) with \( o \leq \text{Out}(\text{PO}^+_{2i+1}(2)) \). According to the calculation in Magma [15], it can be concluded that \( T \) is of rank 3 on \( V \), with two non-trivial suborbits of lengths \( 2^{2i+1} \) and \( 2(2^2 - 1) \) \((2^{2i+1} - 1)\). Hence, \( \text{val}(\Gamma) = 2^{2i+1} \), and \( 2(2^2 - 1)(2^{2i+1} - 1) \), and \( \Gamma \) is T-arc-transitive. If \( \text{val}(\Gamma) = 2^{2i+1} \), then \(|Ta_\beta| = |T_o|/|\text{val}(\Gamma)| = |\text{PO}^+_{2i+1}(2)|/|V|/|\text{val}(\Gamma)| \), and \( |Ge| = |T_o| = |Ta_\beta Z_{2^2}o| \), so \( |Ge| = 2|\text{PO}^+_{2i+1}(2)||o|/(2^2 + 1)(2^{2i+1} - 2^i + 1)(2^{2i+1}) = 2^{2^2 \cdot 2^i + 1}(2^2 - 1)^{2^{2i+1}}(2^2 - 1)^{2^{2i+1} - 1}. \) However, by [19], \( G \cong T.o \) has no maximal subgroup with order \( 2^{2^2 \cdot 2^i + 1}(2^2 - 1)^{2^{2i+1}}(2^2 - 1)^{2^{2i+1} - 1} \), which is a contradiction.

Row 25. In this last case, \( T \cong \text{PO}_{2i+1}^-(2) \), \(|VT| = (2^{2i+1} - 1)(2^2 + 1)\), and \( G \cong T.o \) with \( o \leq \text{Out}(\text{PO}_{2i+1}^-(2)) \). After the calculation in Magma [15], it can be concluded that \( T \) is of rank 3 on \( V \), with two non-trivial suborbits of lengths \( 2(2^2 - 1) + 1)(2^{2i+1} - 2^i - 2) \) or \( 2^{2i+1} - 2^i - 2. \) Hence, \( \text{val}(\Gamma) = 2(2^2 - 1 + 1)(2^{2i+1} - 2^i - 1) \), and \( \Gamma \) is T-arc-transitive. If \( \text{val}(\Gamma) = 2(2^2 - 1 + 1)(2^{2i+1} - 2^i - 1) \), then \(|Ta_\beta| = |T_o|/|\text{val}(\Gamma)| = |\text{PO}_{2i+1}^+(2)|/|V|/|\text{val}(\Gamma)| \), and \( |Ge| = |T_o| = |Ta_\beta Z_{2^2}o| \), so \( |Ge| = 2|\text{PO}_{2i+1}^+(2)||o|/(2^{2^2 - 1 - 1})(2^{2^2 - 1 + 1})(2^{2^2 - 2^i - 1})(2^{2^2 - 2^i - 2}) = 2^{2^2 - 1 - 1}(2^{2^2 - 1 - 1})(2^{2^2 - 2^i - 1})(2^{2^2 - 2^i - 2}). \) However, by [19], \( G \cong T.o \) has no maximal subgroup with order \( 2^{2^2 - 1 - 1}(2^{2^2 - 1 - 1})(2^{2^2 - 2^i - 1})(2^{2^2 - 2^i - 2}) \), which is a contradiction. If \( \text{val}(\Gamma) = 2^{2^2 - 1 - 1} \). Similarly, \( |Ge| = 2|\text{PO}_{2i+1}^+(2)||o|/(2^{2^2 - 1 - 1})(2^{2^2 - 1 + 1})(2^{2^2 - 2^i - 1})(2^{2^2 - 2^i - 2}) = 2^{2^2 - 2^i + 1}(2^{2^2 - 1 + 1})^{2^2 - 1}(2^2 - 1)|o|. \) However, by [19], \( G \cong T.o \) has no maximal subgroup with order \( 2^{2^2 - 2^i + 1}(2^{2^2 - 1 + 1})^{2^2 - 1}(2^2 - 1)|o| \), which is a contradiction.

Now, we are ready to complete the proof of Theorem 1.

Proof of Theorem 1. Suppose that the \( \Gamma \) is a G-edge-primitive with order \( pq \), where \( G \leq \text{Aut}(\Gamma) \), and \( p > q \) are odd primes. Let \( \Gamma \) be a G-edge-primitive graph of order \( pq \), where \( G \leq \text{Aut}(\Gamma) \), and \( p > q \) are odd primes. Suppose \( \Gamma \) is not a star. By Lemma 4, \( G \) is quasiprimitive or biquasiprimitive on \( VT \).
Firstly, suppose that $G$ is quasiprimitive on $V\Gamma$, by Lemma 8, and $\Gamma, G, s$ are listed in Table 1 in Theorem 1.

Secondly, suppose that $G$ is biquasiprimitive on $V\Gamma$, then $G$ has biparts $\Delta_1$ and $\Delta_2$ with $V\Gamma = \Delta_1 \cup \Delta_2$, so $|\Delta_1| = |\Delta_2| = \frac{pq}{2}$. However, as $p > q$ are odd primes, this is impossible.

This completes the proof of Theorem 1. □

4. Conclusions

Currently, the construction, classification, and characterization of various edge-primitive graphs with additional conditions have become some of the main issues discussed in algebraic graph theory. In this paper, edge-primitive graphs of order as a product of two distinct primes are completely determined. This depends on non-abelian simple groups with a subgroup of index $pq$ being classified, where $p > q$ are odd primes. It is meaningful for future research to classify edge-primitive graphs of other specific orders or degrees.

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