Embedding of Unimodular Row Vectors

Tao Wu 1, Jinwang Liu 2,* and Jiancheng Guan 2

1 School of Computer Science and Engineering, Hunan University of Science and Technology, Xiangtan 411201, China; wt@mail.hnust.edu.cn
2 School of Mathematics and Computational Science, Hunan University of Science and Technology, Xiangtan 411201, China; 1070113@hnust.edu.cn
* Correspondence: jwliu64@aliyun.com; Tel.: +86-18784585284

Abstract: In this paper, we mainly study the embedding problem of unimodular row vectors, focusing on avoiding the identification of polynomial zeros. We investigate the existence of the minimal syzygy module of the ZLP polynomial matrix and demonstrate that the minimal syzygy module has structural properties that are similar to the fundamental solution system of homogeneous linear equations found in linear algebra. Finally, we provide several embedding methods for unimodular vectors in certain cases.

Keywords: ZLP polynomial matrix; unimodular vector; minimal syzygy module; fundamental solution system; embedding of ZLP polynomial matrix

MSC: 15A23; 15A83

1. Introduction

In the fields of mathematical theory and engineering calculations, algorithmic algebra and symbolic computation hold significant scientific importance and application value. Some mathematical and engineering problems, such as algebraic topology, circuit analysis, multi-dimensional control, signal processing, multi-dimensional systems, algebraic geometry, and computer algebra can be transformed into problems in terms of multivariate polynomial matrices. Algorithmic algebra and symbolic computing play important roles in solving these problems [1–12]. Topics related to polynomial matrix decomposition, equivalence, and embedding problems have consistently been hot topics in this field.

In 1955, French mathematician J.P. Serre proposed a conjecture: Finitely generated projective modules are free in a polynomial ring over a field. After more than 20 years of research and exploration [13–22], in 1976, Quillen [23] and Suslin [24] independently and almost simultaneously solved Serre’s conjecture. The solution to this conjecture is equivalent to proving that any ZLP polynomial matrix can be embedded into a polynomial invertible matrix. The solution of Serre’s conjecture has profoundly impacted algebraic geometry, computational algebra, symbolic computing, and topology. But this only proves the correctness of the conjecture in theory. In mathematical theory and engineering calculations, there is a need to obtain the specific polynomial invertible matrix after embedding. In 1992, Logar [25] presented an algorithm for the Quillen–Suslin theorem, focusing on the embedding problem of the ZLP polynomial matrix A(x). The core of the algorithm first requires identifying the polynomial invertible matrix U(x), so that A(x) · U(x) = (E 0). Then it is demonstrated that the final columns of the multivariate polynomial matrix U(x) constitute a group of free bases for the syzygy module of the polynomial matrix A(x); that is, the minimal syzygy module. Finally, a method to compute the free bases of the syzygy module is provided, but the algorithm needs to calculate the zeros of the polynomial. In 2007, Fabianska [26] presented the QS algorithm for the commutative ring. The algorithm first finds the normalized component in the last variable of the polynomial ring. Subsequently, it
finds the qualified matrix on the local ring and finally promotes the local solution to obtain the matrix \( U(x) \). But the algorithm needs to calculate the maximum ideal containing a specified ideal, which equates to calculating the zeros of the polynomial. As is well known, Abel and Galois proved that there is no formulaic solution for a univariate quintic equation, and the challenge of determining zeros of higher-order univariate polynomials has not been fully solved.

The Quillen–Suslin theorem has established an important foundation for the study of multivariate polynomial matrices, strongly promoting advancements in related theories and applications in related fields. Although the Quillen–Suslin theorem theoretically proves that any ZLP polynomial matrix can be embedded into a polynomial invertible matrix, how to find the above polynomial invertible matrix has always been a difficult problem for mathematical workers. According to research, the embedding problem of the ZLP matrix and the challenge of determining zeros of higher-order univariate polynomials has not been fully solved.

2. Preliminaries and Results

In the following, let \( R = K[x_1, x_2, \cdots, x_n] \) denote the set of polynomial rings in \( n \) variables \( x_1, x_2, \cdots, x_n \) with coefficients in the field \( K \); \( R^{t \times m} \) denotes the set of \( t \times m \) matrices with entries in \( R \), \( E_r \) denotes the \( r \times r \) identity matrix, and \( 0_{r \times t} \) denotes the \( r \times t \) zero matrix.

To avoid confusion, and for simplicity, \( A(x) \) will be denoted by \( A \).

Firstly, we provide several basic definitions.

**Definition 1.** Let \( A \in R^{s \times t} (s < t) \); \( A \) is said to be a ZLP (zero left prime) matrix if the \( s \times s \) minors of \( A \) generate the unit ideal \( R \).

**Definition 2.** Let \( A \in R^{s \times t} (s < t) \) be a ZLP matrix,

1. If there exists a ZRP matrix \( Q_1 \in R^{r \times (t-s)} \), such that \( A \cdot Q_1 = 0_{s \times (t-s)} \), then \( Q_1 \) is termed the right zero matrix of \( A \).
2. If there exists a matrix \( Q_2 \in R^{r \times s} \), such that \( A \cdot Q_2 = E_s \), then \( Q_2 \) is termed the right inverse matrix of \( A \).
3. If there exists a matrix \( Q \in R^{r \times t} \), such that \( A \cdot Q = (E_s, 0_{s \times (t-s)}) \), then \( Q \) is termed the right unitization matrix of \( A \).

We know that the right inverse matrix is easy to calculate [17], but the right zero matrix involves calculating free bases, and it is difficult.

**Definition 3.** In the ring \( R^{n \times 1} \), a solution vector that satisfies \( A \cdot y = 0 \) is termed a syzygy of the equations \( A \cdot y = 0 \), and the set of all solution vectors is termed the syzygy module of matrix \( A \), denoted as \( \text{Syz}(A) \). The set with the least number of generating elements in the syzygy module is called the minimal syzygy module.

**Definition 4.** The embedding of a ZLP matrix \( A \in R^{s \times t} (s < t) \) refers to completing the ZLP matrix \( A \) to a square invertible matrix \( B \in R^{l \times l} \), i.e., the rows of \( A \) in \( B \) be ordered in the same manner as in \( A \); we say that the embedding of \( A \) can be implemented.

**Definition 5.** Let \( A \in R^{s \times t} (s < t) \) be a ZLP matrix; the solution set vector \( a_1, a_2, \ldots, a_r \) of the homogeneous linear equation system \( A \cdot y = 0 \) is termed the fundamental solution system of \( A \cdot y = 0 \) if the following two conditions are met:

1. \( a_1, a_2, \ldots, a_r \) is linearly independent on ring \( R \), i.e., if there exists \( u_1, u_2, \ldots, u_r \in R \), such that \( u_1a_1 + u_2a_2 + \ldots + u_ra_r = 0 \), then \( u_1 = u_2 = \ldots = u_r = 0 \).
For any solution vector $\beta$ with $A \cdot y = 0$, there exists $u_1, u_2, \ldots, u_r \in R$, such that

$$\beta = u_1 \alpha_1 + u_2 \alpha_2 + \ldots + u_r \alpha_r.$$  

If the above equation holds, we can say that $\beta$ can be represented as an $R$-combination of $\alpha_1, \alpha_2, \ldots, \alpha_r$. 

From the definitions, it is easy to see that the solution vector of the homogeneous linear equation system $A \cdot y = 0$ is syzygy of $A$, and the fundamental solution system of $A \cdot y = 0$ corresponds to the minimal syzygy module of $A$.

**Lemma 1.** Let $A \in R^{s \times t}(s < t)$ be a ZLP matrix, then the right zero matrix of $A$ must exist.

**Proof.** Since $A$ is a ZLP matrix, by the Quillen–Suslin theorem, there exists $B \in A^{t \times s}$, such that

$$A \cdot B = (E_s \ 0_{s \times (t-s)}),$$

and block $B$ into $B = (P \ Q)$, where $P \in A^{t \times s}$, $Q \in A^{t \times (t-s)}$, then we have $A \cdot Q = 0_{s \times (t-s)}$, and $Q$ is a ZRP matrix, i.e., $Q$ is a right zero matrix of $A$. □

**Proposition 1.** Let $F \in R^{s \times 1}$ be a ZLP matrix. The embedding of $F$ is equivalent to finding an invertible matrix $U \in R^{t \times 1}$, such that $F \cdot U = (E_t \ 0)$.

**Proof.** Since $F \cdot U = (E_t \ 0)$, let $T = (0 \ E_{t-s}) \cdot U^{-1}$, then

$$\begin{pmatrix} F \\ T \end{pmatrix} \cdot U = \begin{pmatrix} E_t & 0 \\ 0 & E_{t-s} \end{pmatrix}.$$  

Thus $\begin{pmatrix} F \\ T \end{pmatrix} = U^{-1}$. Thus, $F$ can be embedded into the invertible matrix $U^{-1}$. □

Since the right inverse matrix of $F$ can be easily found, the embedding problem of $F$ can be transformed into finding the right zero matrix of $F$.

**Theorem 1.** Let $A \in R^{s \times t}(s < t)$ be a ZLP matrix, then the column vector of the right zero matrix $Q$ of $A$ constitutes the fundamental solution system of $Ax = 0$.

**Proof.** Assume $\alpha_1, \alpha_2, \ldots, \alpha_{t-s}$ is the column vector of $Q$. Thus, $\alpha_1, \alpha_2, \ldots, \alpha_{t-s}$ are all solution vectors of $A \cdot y = 0$, if there exists $u_1, u_2, \ldots, u_{t-s} \in R$, such that $u_1 \alpha_1 + u_2 \alpha_2 + \ldots + u_{t-s} \alpha_{t-s} = 0$, and let $u = (u_1, u_2, \ldots, u_{t-s})^T$, then $Q \cdot u = 0$. Because $Q$ is a ZRP matrix, there exists $P \in R^{(t-s) \times s}$, such that $P \cdot Q = E_{t-s}$, then $0 = P \cdot Q \cdot u = E_{t-s} \cdot u = u$, so $u = 0$, i.e., $u_1 = u_2 = \ldots = u_{t-s} = 0, \alpha_1, \alpha_2, \ldots, \alpha_{t-s}$ is linearly independent.

Next, we prove that any solution vector $\beta$ with $A \cdot y = 0$ can be represented as an $R$-combination of $\alpha_1, \alpha_2, \ldots, \alpha_{t-s}$. Assuming $\beta$ is a solution of the original equation system $A \cdot y = 0$, then $\beta \in R^{t \times 1}$, consider $A \cdot y = 0$ as a linear equation system of the fraction field $K(x_1, x_2, \ldots, x_n)$, because the rank of $A$ is $s$, then the number of vectors contained in the fundamental solution system of $A \cdot y = 0$ is $t-s$, and $\alpha_1, \alpha_2, \ldots, \alpha_{t-s}$ is a fundamental solution system of the equation system, then $\beta = u_1 \alpha_1 + u_2 \alpha_2 + \ldots + u_{t-s} \alpha_{t-s}$, where $u_i \in K(x_1, x_2, \ldots, x_n), i = 1, 2, \ldots, t-s$. We hereby certify that $u_i \in R$, thereby proving that $\beta$ can be represented as an $R$-combination of $\alpha_1, \alpha_2, \ldots, \alpha_{t-s}$. Because $Q = (\alpha_1, \alpha_2, \ldots, \alpha_{t-s}) \in R^{t \times (t-s)}$ is a ZRP matrix, there exists $P \in R^{(t-s) \times t}$, such that $P \cdot Q = E_{t-s}$. Let $u = (u_1, u_2, \ldots, u_{t-s})^T$, then $\beta = u_1 \alpha_1 + u_2 \alpha_2 + \ldots + u_{t-s} \alpha_{t-s} = (\alpha_1, \alpha_2, \ldots, \alpha_{t-s}) \cdot u = Q \cdot u$, $P \cdot \beta = P \cdot Q \cdot u = E_{t-s} \cdot u$, because $P \in R^{(t-s) \times 1}$, $\beta \in R^{t \times 1}$, then $u = P \cdot \beta \in R^{(t-s) \times 1}$, so $\beta$ can be expressed as an $R$-combination of $(\alpha_1, \alpha_2, \ldots, \alpha_{t-s})$, and the column vector $(\alpha_1, \alpha_2, \ldots, \alpha_{t-s})$ of the right zero matrix of $A$ constitutes the fundamental solution system of the original equation system $A \cdot y = 0$. □
Lemma 2. Let \( A \in R^{s \times t} \) be a ZLP matrix, \( Q_1 \in R^{t \times (t-s)} \) is the right zero matrix of \( A \), then there exists \( P \in R^{s \times s} \), such that \((P \cdot Q_1)\) is an invertible matrix, and \( A \cdot (P \cdot Q_1) = (E_s \cdot 0_{s \times (t-s)})\).

Proof. Since \( A \) is a ZLP matrix, according to the Quillen–Suslin theorem, there is an invertible matrix \((P \cdot Q)\), such that \( A \cdot (P \cdot Q) = (E_s \cdot 0_{s \times (t-s)})\), where \( P \in R^{s \times s} \), \( Q \in A^{t \times (t-s)} \).

Since \((P \cdot Q)\) is invertible, then \( Q \) is a ZRP matrix, and \( A \cdot Q = 0 \); therefore, \( Q \) is the right zero matrix of \( A \). And if the column vectors of \( Q \), \( Q_1 \) are the fundamental solution system of \( A \cdot y = 0 \), then there exists \( U, V \in R^{(t-s) \times (t-s)} \), such that \( Q = Q_1 \cdot U \), \( Q_1 = Q \cdot V \), then \( Q_1 = Q_1 \cdot U \cdot V \). Because \( Q_1 \) is a ZRP matrix, there is a matrix \( F \in R^{(t-s) \times t} \), such that \( F \cdot Q_1 = E_{t-s} \), then \( E_{t-s} = F \cdot Q_1 = F \cdot Q_1 \cdot U \cdot V = U \cdot V \), so \( U \cdot V \) is an invertible matrix, then

\[
(P \cdot Q_1) = (P \cdot Q) = (P \cdot Q) \cdot \text{diag}(E_s \cdot V)
\]

Since \((P \cdot Q), \text{diag}(E_s \cdot V)\) are both invertible matrices, then \((P \cdot Q_1)\) is also an invertible matrix, \( A \cdot (P \cdot Q_1) = A \cdot (P \cdot Q) = (A \cdot P \cdot A) \cdot Q = (E_s \cdot 0) \), and the conclusion is true. □

Theorem 2. Let \( A \in R^{s \times t} \) be a ZLP matrix, if \( Q_1 \in R^{t \times (t-s)} \) is a matrix composed of a column vector based on the fundamental solution system of \( A \cdot y = 0 \), then \( Q_1 \) is the right zero matrix of \( A \).

Proof. According to the Quillen–Suslin theorem, there is an invertible matrix \((P \cdot Q)\), such that \( A \cdot (P \cdot Q) = (E_s \cdot 0_{s \times (t-s)})\). It can be seen that \( Q \) is the right zero matrix of \( A \). According to theorem 1, \( Q \) is the fundamental solution system of \( A \cdot y = 0 \). Then there exists \( U, V \in R^{(t-s) \times (t-s)} \), such that \( Q = Q_1 \cdot U \), \( Q_1 = Q \cdot V \), as shown by the proof of Lemma 2, \( A \cdot (P \cdot Q_1) = (E_s \cdot 0) \). Therefore, \( Q_1 \) is the right zero matrix of \( A \). □

We know that the embedding challenge for a ZLP matrix can be transformed into the embedding problems of unimodular rows. By addressing the embedding of a single unimodular row, we can recursively solve the embedding of a ZLP matrix.

Now, thinking about a matrix

\[
E_n - u \cdot v = \begin{bmatrix}
1 - u_1 v_1 & -u_1 v_2 & \ldots & -u_1 v_n \\
-u_2 v_1 & 1 - u_2 v_2 & \ldots & -u_2 v_n \\
\vdots & \vdots & \ddots & \vdots \\
-u_n v_1 & -u_n v_2 & \ldots & 1 - u_n v_n
\end{bmatrix},
\]

where \( u = (u_1, u_2, \ldots, u_n)^T \in R^{n \times 1} \), \( v = (v_1, v_2, \ldots, v_n) \in R^{1 \times n} \), \( v \cdot u = 1 \), and \( u, v \) is a unimodular.

Below, we present an important theorem.

Theorem 3. The column vectors of the matrix \( E_n - u \cdot v \) is the generator set of the syzygy of unimodular row vector \( v \), where \( v \cdot u = 1 \).

Proof. Thus, each column of \( E_n - u \cdot v \) belongs to \( \text{Syz}(v) \). Thinking about the matrix \((E_n - u \cdot v \cdot u)\), where \( u \) is a ZRP matrix if we perform a column transformation on it, we have that \((E_n - u \cdot v \cdot u)\) is equivalent to \((E_n \cdot u)\); thus, \((E_n \cdot u)\) is a ZLP matrix. Therefore, \((E_n - u \cdot v \cdot u)\) is a ZLP matrix. According to the proposition of Lin [27], there exist unimodular matrices \( M, N \in R^{n \times n} \), such that

\[
M \cdot (E_n - u \cdot v) \cdot N = (E_n \cdot 0).
\]

Let \( M^{-1} = (m_1, m_2, \ldots, m_n) \), we have \((E_n - u \cdot v) \cdot N = (m_1, m_2, \ldots, m_{n-1}, 0)\). Setting \( Q \in R^{n \times (n-1)} = (m_1, m_2, \ldots, m_{n-1}) \); then, \( Q \) is a ZRP matrix. By calculation, we have \( v \cdot (E_n - u \cdot v) = (0, 0, \ldots, 0) \) and \( v \cdot (E_n - u \cdot v) \cdot N = (0, 0, \ldots, 0) \). Therefore, each column of \( Q \) is a solution vector of \( v \cdot y = 0 \) and \( Q \) is the right zero matrix of \( v \). By theorem 1, \( Q \) is the fundamental solution system of \( v \cdot y = 0 \). Since \( Q \) can be linearly represented by the
column vectors of \( v \), we have that any \( r \in \text{Syz}(v) \) can be linearly represented by the column vectors of \( E_n - u \cdot v \) and the column vectors of the matrix \( E_n - u \cdot v \) form the generating set for the syzygy of the unimodular row \( v \).

**Lemma 3.** If \( v \cdot u = v \cdot \pi = 1 \), then \( E_n - u \cdot v \) is equivalent to \( E_n - \pi \cdot v \), where \( u, \pi \in \mathbb{R}^{n \times 1} \) are unimodular columns, and \( v \in \mathbb{R}^{1 \times n} \) is a unimodular row.

**Proof.** Since \((E_n - u \cdot v)(E_n - (\pi - u) \cdot v) = E_n - u \cdot v - (\pi - u) \cdot v + u \cdot v(\pi - u) \cdot v = E_n - (u + \pi - u) \cdot v \), we have that the matrix \( E_n - (u + \pi - u) \cdot v \) is an invertible matrix, \((E_n - (\pi - u) \cdot v)(E_n + (\pi - u) \cdot v) = E_n - (\pi - u) \cdot v(\pi - u) \cdot v = E_n\); therefore, we have that the matrix \( E_n - (\pi - u) \cdot v \) is an invertible matrix, and \( E_n - u \cdot v \) is equivalent to \( E_n - \pi \cdot v \). \( \square \)

**Theorem 4.** Let \( v = (v_1, v_2, \ldots, v_n) \) be a unimodular row, if there exists a certain \( v_i \) \( (i = 1, 2, \ldots, n) \), such that

\[
v_i \mod (v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n) \equiv c, c \in K, c \neq 0,
\]

then the embedding of \( v = (v_1, v_2, \ldots, v_n) \) can be implemented.

**Proof.** Without loss of generality, let \( i = 1 \), and then, we have

\[
v_1 + u_2v_2 + \ldots + u_nv_n = c,
\]

where \( u_2, \ldots, u_n \in R \). By Formula (1), completing unimodular row \( v \) to a square invertible matrix

\[
V_1 = \begin{pmatrix}
v_1 & v_2 & \cdots & v_n \\
-u_2 & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
-u_n & 0 & \cdots & 1
\end{pmatrix}
\]

where \( \det V_1 = c \), and then, we have

\[
V = \begin{pmatrix}
v_1 & v_2 & \cdots & v_n \\
-u_2 & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
-(1/c)u_n & 0 & \cdots & 1/c
\end{pmatrix}
\]

where \( \det V = 1 \). \( \square \)

**Theorem 5.** Let \( v = (v_1, v_2, \ldots, v_n) \) be a unimodular row, if there exists a certain \( v_i \) \( (i = 1, 2, \ldots, n) \), such that

\[
v_i \mod (v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n) \equiv 0,
\]

then the embedding of \( v = (v_1, v_2, \ldots, v_n) \) can be implemented.

**Proof.** Without loss of generality, let \( i = 1 \), we have

\[
v_1 = a_2v_2 + \ldots + a_nv_n, \quad u_1v_1 + u_2v_2 + u_3v_3 + \ldots + u_nv_n = 1,
\]

where \( a_2, \ldots, a_n, u_1, u_2, \ldots, u_n \in R \). By substituting \( v_1 \), we obtain

\[
(u_2 + u_1a_2) \cdot v_2 + (u_3 + u_1a_3) \cdot v_3 + \ldots + (u_n + u_1a_n) \cdot v_n = 1.
\]
Completing unimodular row \( v' = (0, v_2, v_3, \ldots, v_n) \) to a square invertible matrix

\[
N = \begin{bmatrix}
0 & v_2 & \cdots & v_n \\
-(u_2 + u_1 a_2) & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
-(u_n + u_1 a_n) & 0 & \cdots & 1 \\
\end{bmatrix},
\]

where \( \det N = 1 \). By Formula (2), perform a column transformation on matrix \( N \) to obtain matrix \( M \), we have

\[
M = \begin{bmatrix}
v_1 & v_2 & \cdots & v_n \\
-(u_2 + u_1 a_2 - a_2) & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
-(u_n + u_1 a_n - a_n) & 0 & \cdots & 1 \\
\end{bmatrix},
\]

where \( \det M = 1 \). \( \square \)

**Lemma 4.** Let \( v = (v_1, v_2, \ldots, v_n) \in R \), \( u = (u_1, u_2, \ldots, u_n)^T \in R \), \( v \) be a unimodular row, \( u \) is a unimodular column and \( v \cdot u = 1 \). If the embedding of \( v \) can be implemented, then, the embedding of \( u \) can be implemented.

**Proof.** Since the embedding of \( v \) can be implemented, by the Quillen–Suslin theorem, we assume that there is an invertible matrix \( Q_{n \times n} \in R \), such that

\[
Q = \begin{bmatrix}
v_1 & v_2 & \cdots & v_n \\
2a_1 & 2a_2 & \cdots & 2a_n \\
\cdots & \cdots & \cdots & \cdots \\
a_1 & a_2 & \cdots & a_n \\
\end{bmatrix},
\]

\[
Q \cdot u = \begin{bmatrix}
v_1 & v_2 & \cdots & v_n \\
u_21 & a_2 & \cdots & a_2n \\
\cdots & \cdots & \cdots & \cdots \\
u_n1 & a_n & \cdots & a_nn \\
\end{bmatrix},
\]

\[
\begin{bmatrix}
u_1 \\
u_2 \\
\vdots \\
u_n \\
\end{bmatrix} = \begin{bmatrix}
1 \\
x_1 \\
\vdots \\
x_n \\
\end{bmatrix}.
\]

We perform a linear row transformation on matrix \( Q \cdot u \); we have \( P \cdot Q \cdot u = [1, 0, \ldots, 0]^T \) where \( P \) is the elementary matrix product. Since \( P \) and \( Q \) are invertible matrices, by Proposition 1, we have solved the embedding of \( u \). \( \square \)

**Theorem 6.** If there are \( n - 1 \) components in \( v = (v_1, v_2, \ldots, v_n) \), we generate the principal ideal, then the embedding of \( v \) can be implemented.

**Proof.** Without loss of generality, we suppose that \( v_2, v_3, \ldots, v_n \) generates the principal ideal \( \langle d \rangle \), and we have \( q_2 v_2 + q_3 v_3 + \ldots + q_n v_n = d \) \((q_2, \ldots, q_n, d \in R)\). Since \( v_2, \ldots, v_n \in \langle d \rangle \), we have \( v_2 = m_2 d, v_3 = m_3 d, \ldots, v_n = m_n d \) \((m_2, \ldots, m_n \in R)\). Substitution to obtain

\[
q_2 m_2 d + q_3 m_3 d + \ldots + q_n m_n d = d \quad \text{and} \quad q_2 m_2 + q_3 m_3 + \ldots + q_n m_n = 1.
\]

Since \( u = (u_1, u_2, \ldots, u_n)^T \), and \( vu = 1 \), we set

\[
\overline{u} = (0, q_2, \ldots, q_n)^T, \quad v' = (v_1, m_2, \ldots, m_n), \quad u' = (u_1, du_2, \ldots, du_n)^T.
\]

Thus, \( \overline{v} \) and \( u' \) are unimodular columns and \( v' \) is a unimodular row.

Think about the embedding of \( \overline{v} \); by Theorem 5, we have

\[
\begin{bmatrix}
0 & m_2 & \cdots & m_n \\
-q_2 & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
-q_n & 0 & \cdots & 1 \\
\end{bmatrix}
\]

as an invertible matrix. Since \( v' \overline{v} = 1 \), by Lemma 4, we can obtain the invertible matrix, which includes \( v' \). Subsequently, since \( v' u' = 1 \), using Lemma 4 once more, we can solve the embedding of \( u' \).
Assuming that the embedding of \( u' = (u_1, du_2, \ldots, du_n)^T \) is

\[
M = \begin{bmatrix}
    u_1 & a_{12} & \cdots & a_{1n} \\
    du_2 & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    du_n & a_{n2} & \cdots & a_{nn}
\end{bmatrix},
\]

where \( a_{ij} \in R, i, j = 1, 2, \ldots, n \) and \( \det M = 1 \). Then by the properties of determinants, we have

\[
N = \begin{bmatrix}
    u_1 & da_{12} & \cdots & da_{1n} \\
    u_2 & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    u_n & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\]

\( \det N = \det M = 1 \), and \( N \) is an invertible matrix; therefore, we solved the embedding of \( u \). And because \( vu = 1 \), using Lemma 4, we can obtain the invertible matrix, which includes the \( v \).

3. Algorithm and Example

Based on a well-known result on the connection of stable self-modules with orbits of unimodular rows, we have the following Algorithm 1.

**Algorithm 1** Smith form \( F \).

**Require:**
- The unimodular row \( v = (v_1, v_2, \ldots, v_n) \), and \( (v_2, v_3, \ldots, v_n) = (d) \in R \).

**Ensure:**
- The invertible matrix \( N \), which includes \( v \).

1. Using Symbol calculation software Singular compute \( u = (u_1, u_2, \ldots, u_n)^T \), such that \( vu = 1 \), and \( q_2v_2 + q_3v_3 + \ldots + q_nv_n = d (q_2, \ldots, q_n, d \in R) \).
2. Obtain \( v_2 = m_2d, v_3 = m_3d, \ldots, v_n = m_n d (m_2, \ldots, m_n \in R), q_2m_2d + q_3m_3d + \ldots + q_nm_n d = d \) and \( q_2m_2 + q_3m_3 + \ldots + q_nm_n = 1 \).
3. Set \( \pi = (0, q_2, \ldots, q_n)^T, \nu' = (v_1, m_2, \ldots, m_n), u' = (u_1, du_2, \ldots, du_n)^T \). Thus, \( \pi \) and \( u' \) are unimodular columns and \( v' \) is a unimodular row.
4. By Theorem 5, compute the embedding of \( \pi \).
5. By Lemma 4, compute the embedding of \( v' \).
6. By Lemma 4, compute the embedding of \( u' \).
7. By properties of determinants, compute the embedding of \( u \).
8. By Lemma 4, compute the embedding of \( v \) and obtain the invertible matrix \( N \), which includes \( v \).
9. **Return** \( N \).

**Example 1.** Let \( v = (1 + xy \ x \ x^2y) \); thus, \( v \) is a unimodular row, and then, we calculate the invertible matrix after embedding vector \( v \).

By calculation, we have \( u = (1 - xy \ xy - 2 \ y)^T \) and \( vu = 1 \). Since \( (v_2, v_3) = (x) \), we have

\[
1 \cdot x + 0 \cdot x \cdot xy = x.
\]

By Theorem 6, we assume

\[
\pi = (0 \ 1 \ 0)^T, \nu' = (1 + xy + x \ 1 \ xy) u' = (1 - xy \ x(yx - 1) \ xy)^T.
\]

Thus, \( \pi \) and \( u' \) are unimodular columns and \( v' \) is a unimodular row.
Think about the embedding of $u$, by Theorem 5, we have

$$\begin{bmatrix}
0 & -1 & -x \\
1 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}$$

is an invertible matrix. Since $v'\pi = 1$, by Lemma 4, we can obtain the invertible matrix $A$, which includes $v'$, where

$$A = \begin{bmatrix}
1 + xy + x & 1 & xy \\
1 & 0 & 0 \\
0 & 0 & -1
\end{bmatrix}$$

And then, $v'u' = 1$, using Lemma 4 one more,

$$\begin{bmatrix}
1 + xy + x & 1 & xy \\
1 & 0 & 0 \\
0 & 0 & -1
\end{bmatrix} \cdot \begin{bmatrix}
1 - xy \\
x(xy - 1) \\
xy
\end{bmatrix} = \begin{bmatrix}
1 \\
1 - xy \\
x - xy
\end{bmatrix}.$$ 

We perform the row transformation on it, and we have

$$\begin{bmatrix}
1 & 0 & 0 \\
x - 1 & 1 & 0 \\
xy & 0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
1 + xy + x & 1 & xy \\
1 & 0 & 0 \\
0 & 0 & -1
\end{bmatrix} \cdot \begin{bmatrix}
1 - xy \\
x(xy - 1) \\
xy
\end{bmatrix} = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}.$$ 

Setting

$$P = \begin{bmatrix}
1 & 0 & 0 \\
x - 1 & 1 & 0 \\
xy & 0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
1 + xy + x & 1 & xy \\
1 & 0 & 0 \\
0 & 0 & -1
\end{bmatrix} = \begin{bmatrix}
1 + xy + x & 1 & xy \\
x^2y^2 + x^2y - xy & xy - 1 & x^2y^2 - xy \\
xy(1 + xy + x) & xy & x^2y^2 - 1
\end{bmatrix}.$$ 

By the Quillen–Suslin theorem, the inverse matrix of $P$ is the matrix embedded with vector $u'$, where

$$P^{-1} = \begin{bmatrix}
1 - xy \\
x(xy - 1) \\
xy
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
(1 + xy + x) & xy \\
0 & 1
\end{bmatrix}.$$ 

Using Lemma 4, we have that matrix $Q$ is the matrix embedded with vector $u$, where

$$Q = \begin{bmatrix}
1 - xy \\
x - (1 + xy + x) \\
y
\end{bmatrix}.$$ 

And because $vu = 1$, using Lemma 4, we can obtain the invertible matrix $M$ embedded with vector $v$, where

$$M = \begin{bmatrix}
1 + xy + x & x & x^2y \\
x^2y + xy - 1 & xy - 1 & x^2y^2 - xy \\
y(1 + xy + x) & xy & x^2y^2 - 1
\end{bmatrix},$$ 

and $\det M = 1$.

4. Conclusions

In this paper, we avoid the challenge of finding the zeros of multivariate polynomials by transforming the matrix embedding problem into studying matrix $E_n - uv$. For special unimodular vectors, we provide embedding methods, but there are also many problems that need to be solved. Investigating these problems holds significant theoretical value and scientific relevance in algebraic geometry, computer algebra, symbolic computing, and topology. It will also enrich the content, ideas, and methodologies of algorithmic algebra
and symbolic computation. Further research will also offer valuable applications in circuit analysis, signal processing, control, and multidimensional systems.

**Author Contributions:** Conceptualization, T.W.; methodology, J.L.; software, J.G. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research is supported by the National Natural Science Foundation of China (11971161, 12371507, 12271154, 12201204), the Hunan Provincial Natural Science Foundation of China (2022JJ30234, 2023JJ40275), and the Scientific Research Fund of Hunan Province Education Department (22A0334, 20C0790).

**Data Availability Statement:** Not applicable.

**Conflicts of Interest:** The authors declare no conflict of interest.

**References**

9. Quadrat, A. An introduction to constructive algebraic analysis and its applications. *OAI* 2010, 1, 281–471. [CrossRef]