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Stability of Vertical Rotations of an Axisymmetric Ellipsoid on a Vibrating Plane

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Abstract: In this paper, we address the problem of an ellipsoid with axisymmetric mass distribution rolling on a horizontal absolutely rough plane under the assumption that the supporting plane performs periodic vertical oscillations. In the general case, the problem reduces to a system with one and a half degrees of freedom. In this paper, instead of considering exact equations, we use a vibrational potential that describes approximately the dynamics of a rigid body on a vibrating plane. Since the vibrational potential is invariant under rotation about the vertical, the resulting problem with the additional potential is integrable. For this problem, we analyze the influence of vibrations on the linear stability of vertical rotations of the ellipsoid.

Keywords: axisymmetric ellipsoid; vibrating plane; nonholonomic constraint; permanent rotations; vertical rotations; stability; vibrational potential; integrability

MSC: 37N05; 70E18; 70E50; 70K20

1. Introduction

In this paper, we consider the dynamics of an ellipsoid of revolution with axisymmetric mass distribution rolling on an absolutely rough plane. We assume that the supporting plane performs periodic motions. The analysis of rigid body dynamics on a moving plane is motivated by problems of modeling the motion of a rigid body under various external influences, for example, inside a moving vehicle. The modeling of this motion usually involves oscillations or rotations of the supporting surface, which are considered as such external influences. In this paper, we focus on vertical vibrations of the supporting plane.

The problem of the motion of a rigid body on a moving plane goes back to the classical problem of the Kapitsa pendulum [1–4], a mathematical pendulum with a vibrating suspension point. This is one of the simplest problems demonstrating the dynamical stability: under sufficiently fast oscillations of the suspension point the pendulum stabilizes in the upper position, which is unstable in the absence of vibrations. Later, the problems of the motion of rigid bodies rolling on vibrating surfaces were addressed. Without claiming to cite all relevant literature, we mention [5–12]. In particular, [5] examines the motion of a homogeneous sphere on a vibrating surface of complex form. In [6], the dynamics of a rattleback on a vibrating plane is investigated. The authors of [7,8,10] analyze the dynamics of spherical bodies with a displaced center of mass on a vertically vibrating plane. The paper [9] examines the dynamics of a spherical robot with an axisymmetric pendulum on a vibrating plane and, in particular, analyzes the influence of vibrations on vertical rotations of the pendulum. The dynamics of the Chaplygin sphere on a horizontally oscillating plane and the control of its motion using the gyrostatic momentum is studied in [11,12].

One of the efficient approaches to investigating the dynamics of bodies on a vibrating surface is the application of the methods of averaging theory. This approach, which consists of averaging equations over a period, considerably facilitates analysis of the dynamics of...
bodies on a vibrating plane. Under some restrictions on the oscillation parameters the systems obtained using this approach provide a fairly accurate description of the dynamics of a rigid body on a plane. Using this method, P. L. Kapitsa [2] showed that the oscillations of the suspension point of the pendulum cause a torque whose magnitude is determined only by the mass of the pendulum and by the square of the oscillation velocity of the suspension point. This torque was called by the author the *vibrational torque*. The procedure of averaging the equations of motion and deriving the vibrational torque is described in detail for a rigid body with a vibrating suspension point in [13]. The recent paper [14] presents a bifurcation analysis of the dynamics of the Lagrange top with an additional vibrational potential that approximately describes the vertical oscillations of the suspension point. The averaging of the equations of motion for a sphere with a displaced center of mass which rolls without slipping on a vibrating plane was performed in [7,10]. In this paper, we perform a similar procedure of averaging for an ellipsoid of revolution rolling on a vibrating plane.

One of the problems treated in this paper is the investigation of the influence of the vibrations of the plane on the stability of partial solutions (vertical rotations) of an ellipsoid. In the case of a fixed supporting plane, the stability of partial solutions of an ellipsoid was widely explored in the literature for bodies rolling with and without slipping. Much effort was directed at analyzing the stability of permanent rotations of geometrically and dynamically symmetric bodies [15–18]. The ellipsoid of revolution is a special case of such bodies. In [19] the stability of permanent rotations of a homogeneous triaxial ellipsoid on a smooth plane was investigated. Permanent rotations of an ellipsoid with axisymmetric mass distribution on a smooth plane were analyzed in detail in [20]. The paper [21] also analyzes permanent rotations of an ellipsoid of revolution on a smooth plane, but from the viewpoint of the problem of elevation of Jellett’s egg (see also [22]). This paper explores the influence of the form of an (oblate or prolate) ellipsoid and the displacement of the center of mass on the possibility of lifting it to the vertical position by means of dissipative forces. In [22], only the prolate ellipsoid is considered.

The problem of an ellipsoid moving on an absolutely rough plane, as well as the search for and analysis of permanent rotations, has also received a great deal of attention in the literature. In [23,24], the stability of permanent (including vertical) rotations of an arbitrary convex body of revolution was investigated and conditions for the stability of such rotations were derived. In the work of A. P. Markeev [25,26] on the dynamics of a triaxial ellipsoid, an assumption was made about similarity of the form of the ellipsoid to the form of the sphere. Interesting results of the search for and analysis of permanent rotations were obtained in the work of A. V. Karapetyan [27]. He showed that there exist mass-geometric parameters of the ellipsoid such that it performs permanent rotations about an arbitrary axis passing through the center of mass. A detailed stability analysis of such rotations was made in [28]. The stability of vertical rotations of an axisymmetric ellipsoid about the principal axis which is not an axis of revolution was investigated in [29], where the instability of such solutions was shown. In [30], a parametric analysis of the stability of permanent rotations of a homogeneous triaxial ellipsoid was carried out: stability regions on the parameter plane (the ratios of two semiaxes to the third) were constructed depending on the velocity of rotation. In [31], bifurcation diagrams in the space of first integrals were constructed for a balanced ellipsoid of revolution in the case of a smooth and a rough plane.

In this paper, we study the dynamics of an ellipsoid of revolution on a vertically vibrating absolutely rough plane. The paper is structured as follows. In Section 2, we derive equations of motion for an ellipsoid with a displaced center of mass on a vibrating plane and, by averaging them, obtain the vibrational torque. In Section 3, we present invariants of the system considered and perform a reduction in the equations of motion to a system with one degree of freedom. In Section 4, we investigate the stability of vertical rotations of the ellipsoid and analyze the influence of vibrations of the supporting plane on their stability.
2. Equations of Motion

2.1. Model Assumptions

Consider the motion of a dynamically symmetric ellipsoid of revolution with a displaced center of mass on a horizontal plane performing periodic oscillations. We will examine this problem under the following assumptions:

1. The ellipsoid moves with one point in contact with the supporting plane without losing contact with it;
2. The velocity of the point of contact $P$ of the ellipsoid with the supporting plane is zero;
3. In the general case, the center of mass of the ellipsoid is offset from its geometric center and lies on its symmetry axis;
4. The supporting plane performs vertical periodic oscillations according to the law $\xi(t)$.

2.2. Configuration Space and Kinematic Relations

Consider the motion of a dynamically symmetric ellipsoid of revolution of mass $m$ with semiaxes $b_1$ and $b_3$ on an absolutely rough horizontal plane (Figure 1). Assume that the center of mass of the ellipsoid is offset from its geometric center along the symmetry axis by distance $a$.

![Diagram of an ellipsoid on a vibrating plane and image of the ellipsoid on a plane in a meridian section.](image)

**Figure 1.** (a) Diagram of an ellipsoid on a vibrating plane and (b) image of the ellipsoid on a plane in a meridian section.

To describe the motion of the ellipsoid, we introduce two coordinate systems:

- A fixed (inertial) coordinate system $Oxyz$ with unit vectors $\alpha, \beta, \gamma$, where $\gamma$ is the vertical unit vector;
- A moving coordinate system $C_{x_1x_2x_3}$ with unit vectors $e_1, e_2, e_3$ attached to the ellipsoid, with origin at its center of mass $C$ and with the axis $Cx_3$ directed along the symmetry axis.

Referred to the moving frame $C_{x_1x_2x_3}$, the vector of displacement of the center of mass can be written as $a = ae_3 = (0, 0, a)$.

Let us specify the position of the ellipsoid by the coordinates of its center of mass $r_e = (x_e, y_e, z_e)$ in the fixed coordinate system, and its orientation in space, by the rotation matrix $Q$ whose columns are the unit vectors $\alpha, \beta, \gamma$ referred to the moving coordinate system:

$$Q = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} \in SO(3).$$

Hence, the configuration space of the problem of the free motion of the ellipsoid is the product $\mathcal{N} = \{(r_e, Q)\} = \mathbb{R}^3 \times SO(3)$. 

By definition, the vectors \( \alpha, \beta, \) and \( \gamma, \) which form an orthonormal basis, satisfy the relations

\[
\begin{align*}
(\alpha, \alpha) &= 1, & (\beta, \beta) &= 1, & (\gamma, \gamma) &= 1, \\
(\alpha, \beta) &= 0, & (\beta, \gamma) &= 0, & (\gamma, \alpha) &= 0.
\end{align*}
\]

We will describe the dynamics of the system by the vectors of the angular velocity \( \omega = (\omega_1, \omega_2, \omega_3) \) and the velocity of motion of the center of mass \( v = (v_1, v_2, v_3) \) and refer these vectors to the moving coordinate system \( Cx_1x_2x_3 \) (here and in what follows, unless otherwise specified, all vectors will be referred to the moving coordinate system \( Cx_1x_2x_3 \)). The vectors \( \omega \) and \( v \) are quasi-velocities and are related to the derivatives of the configuration variables by the following kinematic relations:

\[
\begin{align*}
\dot{\alpha} &= \alpha \times \omega, \\
\dot{\beta} &= \beta \times \omega, \\
\dot{\gamma} &= \gamma \times \omega, \\
\dot{r}_c &= Q^\top v.
\end{align*}
\]

2.3. Constraint Equations and Dynamical Equations

Let us write the constraint equations corresponding to the assumptions described in Section 2.1. The condition that the ellipsoid must move without loss of contact with the plane involves imposing the following holonomic constraint on the system:

\[
z_c + (r, \gamma) + \xi(t) = 0, \tag{2}
\]

where \( r \) is the radius vector of the point of contact (see Figure 1), which can be expressed in terms of the vector \( \gamma \) using the relation

\[
r = -\frac{B\gamma}{\sqrt{(\gamma, B\gamma)}} - ae_3, \tag{3}
\]

where \( B = \text{diag}(b_1^2, b_1^2, b_3^2) \).

**Remark 1.** In the case of an arbitrary axisymmetric surface, the dependence \( r(\gamma) \) can be represented as

\[
r(\gamma) = (R_1(\gamma_3)\gamma_1, R_1(\gamma_3)\gamma_2, R_2(\gamma_3)),
\]

where \( R_1(\gamma_3) \) and \( R_2(\gamma_3) \) are arbitrary functions related to each other by

\[
\frac{dR_2}{d\gamma_3} = R_1 - \frac{1 - \gamma_3^2}{\gamma_3} \frac{dR_1}{d\gamma_3}.
\]

In the case of the ellipsoid considered here, explicit expressions for \( R_1(\gamma_3), R_2(\gamma_3) \) have the form

\[
R_1(\gamma_3) = -\frac{b_1^2}{\sqrt{b_1^2(1 - \gamma_3^2) + b_3^2\gamma_3^2}}, \quad R_2(\gamma_3) = -\frac{b_3^2\gamma_3^2}{\sqrt{b_1^2(1 - \gamma_3^2) + b_3^2\gamma_3^2}} - a.
\]

The condition that there be no slipping at the point of contact is described by the nonholonomic constraint

\[
f = v + \omega \times r + \xi(t)\gamma = 0. \tag{4}
\]

**Remark 2.** In fact, two components of the vector Equation (4) are autonomous nonholonomic constraints (projections of \( f \) onto the horizontal plane) and the third component is a nonautonomous holonomic constraint. Integrating the latter, we obtain (2).
In the chosen quasi-velocities, the Lagrangian function of the system, with the holonomic constraint (2) taken into account, has the form

\[ \mathcal{L} = \frac{1}{2} (\omega, J\omega) + \frac{1}{2} m(v, v) + mg(r, \gamma) + mg\xi(t), \]

where \( I = \text{diag}(i_1, i_1, i_3) \) is the principal central tensor of inertia of the ellipsoid and \( g \) is the free-fall acceleration.

We write the equations of motion of the system in the absence of external forces in the form of Lagrange equations in quasi-velocities with the constraints [32,33]

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial v} \right) + \omega \times \frac{\partial L}{\partial v} = \left( \frac{\partial f}{\partial v} \right)^\top \lambda, \\
\frac{d}{dt} \left( \frac{\partial L}{\partial \omega} \right) + \omega \times \frac{\partial L}{\partial \omega} + v \times \frac{\partial L}{\partial v} + \gamma \times \frac{\partial L}{\partial \gamma} = \left( \frac{\partial f}{\partial \omega} \right)^\top \lambda,
\]

where \( \lambda \) are undefined multipliers which are found from the joint solution of Equation (6) and the time derivative of the constraint (4).

After eliminating the velocity \( v \) using the constraint Equation (4), the equations of motion (6) reduce to one vector equation

\[ J\dot{\omega} + \omega \times J\omega + mr \times (\omega \times r) + m(g - \ddot{\xi}(t))\gamma \times r = 0, \]

where \( J\omega = I\omega + mr \times (\omega \times r) \). Here and in what follows, by the vector \( r \) we mean its explicit expression (3) written in terms of the vector \( \gamma \). From the equation thus obtained it can be seen that the rolling motion on a vertically vibrating plane is similar to that on a fixed plane with time-dependent free-fall acceleration.

Equations (7) and (1), combined with the constraint (4), give a complete description of the dynamics of an axisymmetric ellipsoid on a vibrating horizontal plane.

We note that the equations for \( \omega \) and \( \gamma \) decouple from the complete system and form a closed system of two vector differential equations

\[
\begin{aligned}
\begin{cases}
J\dot{\omega} + \omega \times J\omega + mr \times (\omega \times r) + m(g - \ddot{\xi}(t))\gamma \times r = 0, \\
\dot{\gamma} = \gamma \times \omega.
\end{cases}
\end{aligned}
\]

In this case, the velocity of the center of mass is uniquely determined from Equation (4), the coordinate \( z_c \) is uniquely defined from the constraint Equation (2), and the remaining variables are defined from the quadratures (1).

2.4. Averaged Equations of Motion

To average the equations of motion and to obtain a vibrational potential, we apply the method employed in [10], which is based on the Boltzmann–Hamel Equations [34]. As quasi-velocities we need to use the angular velocities \( \omega \) and the constraints (4). After eliminating the total time derivative, the Lagrangian function takes the form

\[ \mathcal{L} = \frac{1}{2} (\omega, J\omega) + m\dot{\xi}(t)(\omega, r \times \gamma) - m(f, \omega \times r + \dot{\xi}(t)\gamma) + \frac{1}{2} mf^2 + mg(r, \gamma). \]

Next, we introduce the generalized momenta

\[
\begin{aligned}
\mathfrak{M} = \frac{\partial \mathcal{L}}{\partial \dot{\omega}} = J\omega - mr \times (f - \ddot{\xi}(t)\gamma), \\
\mathfrak{N} = \frac{\partial \mathcal{L}}{\partial \dot{f}} = m(f - \omega \times r - \dot{\xi}(t)\gamma).
\end{aligned}
\]
The equations of motion in the variables (9) can be written [34] in the quasi-Hamiltonian form

\[
\dot{M} = \left( M + (\gamma \times e_3) \frac{\partial r}{\partial \gamma_3} - \left( \frac{e_3 \times (r \times e_3)}{1 - \gamma_3^2} \right) (e_3 \times (\gamma \times M)) \times e_3 \right) \times \frac{\partial H}{\partial M} + N \times \frac{\partial H}{\partial N} + \gamma \times \frac{\partial H}{\partial \gamma}, \tag{10}
\]

\[
\dot{N} = \gamma \times (M - \tilde{m} \xi(t) r \times \gamma), \tag{11}
\]

where \(\lambda\) are new undefined multipliers.

**Remark 3.** To derive Equations (10) and (11), we have used the equation for changing the position of the center of mass of the rigid body relative to the moving coordinate system attached to the oscillating plane. In this case, the quasi-velocity \(f\) is the velocity of the point of contact, also relative to the moving plane.

By choosing the constraints \(f\) as quasi-velocities, Equation (11) decouples and is used only to find undefined multipliers. This makes it necessary to express \(\dot{N}\) and \(\dot{M}\) from Equation (10) and the derivative of the constraint Equation (4), which in the chosen variables takes the form

\[
\frac{\partial H}{\partial N} = 0, \tag{12}
\]

and to substitute them into Equation (11).

After applying the constraint (12), Equation (10), together with one of the kinematic relations (1), take the form

\[
\begin{aligned}
\dot{M} &= (M - m \xi(t) r \times \gamma) \times J^{-1}(M - \tilde{m} \xi(t) r \times \gamma) + \\
&\quad + m \gamma \times ((M - m \xi(t) r \times \gamma) \times r) - mg \gamma \times r + m \xi(t) (r \times \gamma) \times r, \\
\dot{\gamma} &= \gamma \times (M - \tilde{m} \xi(t) r \times \gamma)
\end{aligned} \tag{13}
\]

and completely describe the dynamics of the reduced system.

In what follows, we assume that the oscillations of the plane are performed with small amplitude \(\delta\) and high frequency \(\Omega \sim \frac{1}{\varepsilon}\). We introduce a “fast” time variable, \(d\tau = \Omega dt\), and average the resulting system (13) over it in accordance with the Bogolyubov theorem [35] and taking into account \(\langle \dot{\xi}(\tau) \rangle_\tau = 0\). Comparing the result obtained by averaging with the equations of a similar system without vibrations of the plane, we arrive at the conclusion that fast vibrations lead to the appearance of a vibrational potential of the form

\[
U_v = \frac{1}{2} m^2 \left( J^{-1}(r \times \gamma) \cdot r \times \gamma \right) \Xi^2, \tag{14}
\]

where \(\Xi^2\), the average of the squared velocity of the plane’s motion

\[
\Xi^2 = \langle \dot{\xi}(t)^2 \rangle_\tau,
\]

is a constant depending on the law of oscillations. In the special case of harmonic oscillations \(\xi(t) = \delta \cos \Omega t\)

\[
\langle \dot{\xi}(t)^2 \rangle = \frac{\delta^2 \Omega^2}{2}.
\]

As stated above, we will consider small oscillations of the plane with high frequency. In this case, in accordance with the Bogolyubov theorem, the solutions of the averaged system will be similar to the solution of an exact system on time intervals of order \(\frac{1}{\varepsilon}\).
We also note that in the case of axisymmetric mass distribution the vibrational potential depends only on the component $\gamma_3$. Thus, after averaging we have obtained the problem of an axisymmetric ellipsoid rolling on an absolutely rough plane in a potential possessing axial symmetry. The integrability of this problem was established already in [36]. However, in the general case, additional integrals are not expressed in terms of elementary functions.

Next, we derive equations of motion of the averaged system not by using the Boltzmann–Hamel equations, but by adding the resulting vibrational potential to the Lagrangian (5).

The Lagrangian function of the averaged system with the vibrational potential has the form

$$L = \frac{1}{2} (\omega, I_\omega) + \frac{1}{2} m(v, v) + mg(r, \gamma) - \frac{m^2 \Xi^2}{2} \left( \mathbf{I}^{-1}(r \times \gamma), r \times \gamma \right).$$

In this case, the system is subject to the (autonomous) constraint—the condition that there be no slipping at the point of contact

$$v + \omega \times r = 0. \tag{15}$$

Substituting the resulting Lagrangian function into Equation (6), eliminating the undefined multipliers using the constraint (15), we obtain a reduced system of equations

$$\begin{align*}
J \dot{\omega} + \omega \times J \omega + m r \times (\omega \times \dot{r}) + mg \gamma \times r - m^2 \Xi^2 \gamma \times \left( \mathbf{I}^{-1}(r \times \gamma) \times r \right) &= 0, \\
\dot{\gamma} &= \gamma \times \omega, \tag{16}
\end{align*}$$

where the expression

$$M_v = m^2 \Xi^2 \gamma \times \left( \mathbf{I}^{-1}(r \times \gamma) \times r \right)$$

is the vibrational torque.

Thus, Equations (16) and (1), combined with the constraint (15), give an approximate description of the dynamics of an axisymmetric ellipsoid on a vibrating absolutely rough horizontal plane.

For further analysis, we introduce the angular momentum vector

$$M = I_\omega = I_\omega + m r \times (\omega \times r) \tag{17}$$

and rewrite Equation (16) in terms of the angular momenta

$$\begin{align*}
\dot{M} &= M \times \omega + m \dot{r} \times (\omega \times r) + m g r \times \gamma + M_v, \\
\dot{\gamma} &= \gamma \times \omega, \tag{18}
\end{align*}$$

where the angular velocity $\omega$ is expressed in terms of the angular momentum as $\omega = \mathbf{I}^{-1} M$.

Equations (18) contain 8 parameters. We rescale the variables and time as

$$\frac{M}{mb_3^2} \sqrt{\frac{g}{b_3}} \rightarrow M, \quad \omega \sqrt{\frac{g}{b_3}} \rightarrow \omega, \quad \frac{r}{b_3} \rightarrow r, \quad \frac{t}{\sqrt{\frac{g}{b_3}}} \rightarrow t. \tag{19}$$

The resulting system of equations depends on five independent dimensionless parameters, which we denote as follows:

$$\alpha = \frac{a}{b_3} \in [0, 1], \quad \beta = \frac{b_1}{b_3} \in (0, \infty), \quad \nu = \frac{i_3}{i_1} \in (0, 2],$$

$$\eta = \frac{mb_3^2}{i_1} \in (0, \infty), \quad \Theta = \frac{1}{\sqrt{bg}} \Xi \in [0, \infty). \tag{20}$$

The values of the parameter $\beta \in (0, 1)$ correspond to an “prolate” ellipsoid, and $\beta \in (1, \infty)$, to an “oblate” one. The boundary values of the parameter $\beta$ correspond either
to a thin rod ($\beta = 0$) or to a thin disk ($\beta \to \infty$), but these cases correspond to the motion of a body with a sharp edge, so we do not consider them in this paper.

**Remark 4.** The scaling transformation (19) is equivalent to choosing units of measurement of dimensional quantities so that the following relations are satisfied:

$$m = 1, \quad g = 1, \quad b_3 = 1.$$  

In what follows, unless otherwise specified, we will perform all calculations for dimensionless quantities.

3. Integrability of the Averaged System (18)

3.1. Invariants

The equations of motion (18) admit the obvious geometric integral

$$F_0 = (\gamma, \gamma) = 1$$

and the energy integral

$$E = \frac{1}{2} (M, \omega) - (r, \gamma) + \frac{\Theta^2}{2} \left(J^{-1}(r \times \gamma), r \times \gamma\right)$$

and preserve the invariant measure $\mu_0 dM d\gamma$ with density [37]

$$\mu_0 = \frac{1}{\sqrt{\nu + \eta (r, \Omega^2)}}. \quad (21)$$

In addition, Equation (18) admit two integrals linear in angular momenta which are due to the axial symmetry of the body and to the invariance under rotation about the vertical. Moreover, these integrals do not collapse when vertical oscillations of the supporting plane are added in explicit form, i.e., Equation (8) also preserve two first integrals, linear in angular momenta $M$, and the invariant measure (21). For the case of an ellipsoid rolling on an absolutely rough plane, these integrals are not expressed in terms of elementary functions.

The equations of motion (18) also possess the symmetry field

$$\zeta = M_1 \frac{\partial}{\partial M_2} - M_2 \frac{\partial}{\partial M_1} + \gamma_1 \frac{\partial}{\partial \gamma_2} + \gamma_2 \frac{\partial}{\partial \gamma_1}, \quad (22)$$

which corresponds to invariance of the system under rotations about the axis of dynamical symmetry of the ellipsoid.

3.2. Reduction by the Symmetry Field

Following [37], we choose as new variables the integrals of the vector field (22) in the form

$$\begin{cases}
  k_1 = M_1 \gamma_1 + M_2 \gamma_2 + \frac{D}{\beta^2} M_3, \\
  k_2 = \frac{\mu}{B^2} \left(\eta \beta^2 D (M_1 \gamma_1 + M_2 \gamma_2) + (\eta D^2 + B^2) M_3\right), \\
  k_3 = \frac{\gamma_1 M_2 - \gamma_2 M_1}{\sqrt{1 + \eta R^2}}, \\
  R^2 = \frac{1}{B^2} (\beta^2 (1 - \gamma_3^2) + D^2), \quad \mu = \frac{1}{\sqrt{\nu + \eta B^2 (\beta^2 (1 - \gamma_3^2) + \nu D^2)}},
\end{cases} \quad (23)$$
where \( B = \sqrt{(1 - \gamma_3^2)\beta^2 + \gamma_3^2} \) is the height of the geometric center of the ellipsoid, \( D = \gamma_3 + aB \), and the symbol \( R^2 \) is introduced to abbreviate some of our formulae.

In the new variables \( \gamma_3, k_1, k_2 \) and \( k_3 \) the equations of motion describing the dynamics of the system take the form

\[
\begin{align*}
\dot{\gamma}_3 &= kk_3, \\
\dot{k}_1 &= -k \mu \frac{(\beta^2 - 1)(\alpha \gamma_3 + B)}{B^2} k_2 k_3, \\
\dot{k}_2 &= -k \eta \frac{\beta^4(B^2 - 1)}{B^4} k_1 k_3, \\
\dot{k}_3 &= -\frac{k}{1 - \gamma_3^2} \left( \frac{D}{B}(\alpha \gamma_3 + B) \left( \eta k_2^3 + \frac{\nu \beta^4}{B^2} k_2^3 \right) - \frac{1}{\mu \beta^2} (B^2 + 2B\alpha \gamma_3 + \gamma_3^2) k_1 k_2 \\
&\quad + \gamma_3 (k_1^2 + k_2^3) + \eta \mu D \left( 1 - \gamma_3^2 \right)(\gamma_3^2 \beta^2 - \nu D) k_1 k_2 \right) - k(1 - \gamma_3^2) \frac{\partial U(\gamma_3)}{\partial \gamma_3},
\end{align*}
\]  

where \( k = \sqrt{1 + \eta R^2} \), \( U(\gamma_3) = (B + \alpha \gamma_3) + U_0(\gamma_3) \) is the potential energy of the system.

To reconstruct the dynamics of the system (18), it is necessary to add to Equations (24)–(27) a quadrature for the angle of proper rotation \( \phi = \arctan(\gamma_1, \gamma_2) \). The function \( \arctan(\cdot, \cdot) \) with two arguments calculates the arctan value of the quotient of its arguments and takes into account their signs in determining the value of the function in the interval \([0, 2\pi]\).

\[
\phi = \frac{\gamma_2 \gamma_1 - \gamma_1 \gamma_2}{\gamma_1^2 + \gamma_2^2}.
\]

Here, the right-hand side can be expressed in terms of the variables \( k_1, k_2, k_3 \) and \( \gamma_3 \) using relations (1), (17) and (23). The explicit expression for \( \phi \) is rather cumbersome, and so we do not present it here.

The resulting system of Equations (24)–(27) admits the energy integral

\[
E = \frac{1}{2(1 - \gamma_3^2)} \left( k_1^2 - \nu B^2 k_2^2 + \eta \frac{k_2^2}{B^2} \left( Dk_1 - \frac{B^2}{\eta \mu \beta^2} k_2 \right)^2 + k_3^2 \right) + U(\gamma_3)
\]  

and possesses an invariant measure with density \( \mu_k = (1 - \gamma_3^2)^{-1} k^{-1} \).

### 3.3. Additional Integrals of Motion and Reduction to a System with One Degree of Freedom

In addition to the energy integral (28), Equations (24)–(27) admit two first integrals of motion. To find them, we divide Equations (25) and (26) by (24) and obtain a system of two linear nonautonomous first-order equations

\[
\begin{align*}
\frac{dk_1}{d\gamma_3} &= -\mu \nu \left( \frac{\beta^2 - 1)(\alpha \gamma_3 + B)}{B^2} \right) k_2, \\
\frac{dk_2}{d\gamma_3} &= -\mu \eta \frac{\beta^4(B^2 - 1)}{B^4} k_1.
\end{align*}
\]

The general solution of this system of equations can be represented as

\[
k = \Phi(\gamma_3) C,
\]

where \( k = (k_1, k_2) \), \( \Phi(\gamma_3) \) is the matrix of the fundamental solutions of the system (29) with initial conditions \( \Phi(\gamma_3 = 0) = \hat{E} \), and \( C = (C_1, C_2) \) are constants of integration that are the required first integrals. As a result, the first integrals of motion have the form

\[
C = \Phi^{-1}(\gamma_3) k
\]

and are functions linear in momenta, with coefficients that are nonalgebraic functions of \( \gamma_3 \).
As is well known [20], the problem of an ellipsoid of revolution with a displaced center of mass that moves on a smooth plane admits two additional integrals of motion: the area integral \((M, \gamma)\) and the Lagrangian integral \(M = I \omega_3\). Analyzing the expressions (23) for defining \(k_1, k_2\) and the chosen initial conditions of the matrix of the fundamental solutions \(\Phi(\gamma = 0) = E\) and comparing them with the integrals in the case of a smooth plane, one can conclude that the integral \(C_1\) is an analog of the area integral and \(C_2\) is an analog of the Lagrangian integral.

**Remark 5.** Similar integrals in the case of a body of revolution moving on a smooth plane or in the case of a rubber body rolling on a plane are described by elementary functions. In the problem of a body of revolution rolling on an absolutely rough plane, the first integrals are expressed in elementary functions probably only in the case of a sphere with axisymmetric mass distribution [38]. In the case of a balanced axisymmetric disk, additional integrals are expressed in terms of hypergeometric functions [39,40], and in the case of a balanced ellipsoid of revolution, in terms of the Heun functions [31].

After fixing the level set of the first integrals \(C_1\) and \(C_2\), we obtain a system with one degree of freedom

\[
\begin{align*}
\dot{\gamma}_3 &= k k_3, \\
\dot{k}_3 &= -\frac{k}{1 - \gamma_3^2} \left( \frac{D}{B} (a \gamma_3 + B) \left( \eta k_1^2 + \frac{\nu \beta^2 k_3^2}{B^2} \right) - \frac{1}{\mu B^2} \left( B^2 + 2B \gamma_3 + \gamma_3^2 \right) k_1 k_2 \right. \\
& \quad + \left. \gamma_3 (k_1^2 + k_3^2) + \frac{\eta \mu D}{B^2} (1 - \gamma_3^2) (\gamma_3 \beta^2 - \nu D) k_1 k_2 \right) - k (1 - \gamma_3^2) \frac{\partial U(\gamma_3)}{\partial \gamma_3},
\end{align*}
\]

which preserves the energy integral

\[
\mathcal{E} = \frac{1}{2(1 - \gamma_3^2)} \left( k_1^2 - \frac{\nu B^2}{\eta \beta^4} k_3^2 + \frac{\eta}{B^2} \left( D k_1 - \frac{B^2}{\eta \mu \beta^2} k_2 \right)^2 + k_3^2 \right) + U(\gamma_3). \tag{32}
\]

In this case, in the expressions (31) and (32) the variables \(k_1\) and \(k_2\) are expressed in terms of the integrals \(C_1, C_2\) and the variable \(\gamma_3\) using (30).

**Remark 6.** After rescaling time as \(k dt = d\tau\) the system (31) becomes Hamiltonian.

Figure 2 shows an example of the phase portrait of the system (31). As can be seen from the figure, for the chosen parameters the phase portrait has three fixed points. Depending on the parameters, the number of fixed points can decrease or increase. Moreover, even in the case of an absolutely smooth plane and in the absence of any additional forces (except for the force of gravity), the bifurcation diagram of the system is a rather complicated surface in the 3-dimensional space of first integrals. The sections formed by the intersection of this surface with a plane corresponding to a fixed value of one of the integrals can be very diverse [20]. Thus, the bifurcation analysis of the system considered here is a challenge in its own right beyond the scope of this paper. In this paper, we will not present a detailed classification of diagrams for the system under consideration, but only analyze the stability of vertical rotations.
Figure 2. An example of the phase portrait of the system (31), constructed for the parameters \( \beta = 0.5, \eta = 1, \nu = 1.5, \alpha = 0.5, \Theta^2 = 10 \) and the values of the integrals \( C_1 = 0.5, C_2 = 0.4 \).

4. Vertical Rotations and Analysis of Their Stability

4.1. Vertical Rotations of an Ellipsoid

As is well known [40], for any body of revolution there exist two partial solutions which correspond to vertical rotations of the body about the symmetry axis. By vertical rotations we mean rotations under which the symmetry axis of the ellipsoid is vertical and fixed and the body itself rotates about this axis. By upper vertical rotations we mean rotations under which the center of mass lying on the symmetry axis is above the geometric center of the ellipsoid (this case corresponds to \( \gamma_3 = 1 \)), and by lower vertical rotations we mean rotations under which the center of mass is below the geometric center (\( \gamma_3 = -1 \)).

These solutions are two one-parameter families of fixed points of the system (18)

\[
\sigma^\pm : M = (0, 0, c), \quad \gamma = (0, 0, \pm 1),
\]

where \( c \in (-\infty, \infty) \) is the parameter of the families which corresponds to the projection of the angular momentum vector onto the vertical symmetry axis, the sign “+” corresponds to the upper vertical rotations, and the sign “−”, to the lower ones.

Next, we investigate the stability of the above-mentioned fixed points \( \sigma^\pm \) depending on the system parameters and the parameter \( c \).

We note that Equation (18) are invariant under the transformation

\[
\gamma_3 \rightarrow -\gamma_3, \quad \alpha \rightarrow -\alpha.
\]

Thus, analysis of the lower vertical rotations is equivalent to analysis of the upper vertical rotations with negative values of the parameter \( \alpha \). Therefore, in what follows we will investigate the stability only of the upper vertical rotations \( \sigma^+ \), assuming that the parameter \( \alpha \) changes in the interval \( \alpha \in [-1, 1] \).

4.2. Linear Stability of Vertical Rotations

We will perform the linear stability analysis of the vertical rotations in the variables \( M \) and \( \gamma \). Despite the larger number of equations, this is simpler to do. This is due to the fact that, in investigating the stability, in the variables \( k_1, k_2, k_3, \gamma_3 \) it is necessary to first perform a reduction of Equations (24)–(27) to the common level set of the integrals \( C_1 = C_2 \) on which the vertical rotations lie. And since the dependence \( C_1(\gamma_3) \) is given by nonalgebraic functions, this is more difficult to do. Similar difficulties arise in the analysis of the stability of solutions \( \sigma^\pm \) for the reduced system (31).

To analyze the stability of the fixed points (33), we represent the system of differential Equation (18) as

\[
\dot{q} = f(q),
\]

where \( q = (M_1, M_2, M_3, \gamma_1, \gamma_2, \gamma_3) \), \( f(q) \) is the vector whose components are functions of \( q \) and the system parameters. Next, we linearize the system (18) near the fixed point \( q_0 = (0, 0, c, 0, 0, 1) \), which corresponds to the solution \( \sigma^+ \). This yields the system
\[ \dot{q} = L(q_0)\dot{q}, \quad L(q_0) = \frac{\partial f(q)}{\partial q} \bigg|_{q=q_0}, \]

where \( \tilde{q} = q - q_0 \).

Let us write the characteristic equation of the linearized system

\[ \det(L(q_0) - \lambda E) = 0, \]

where \( \lambda \) are the eigenvalues of the matrix \( L(q_0) \) and \( E \) is the 6 \( \times \) 6 identity matrix. This equation reduces to the form

\[ P_6(\lambda) = \lambda^2(\lambda^4 + a_2\lambda^2 + a_0) = 0, \quad (34) \]

where the coefficients \( a_0 \) and \( a_2 \) depend on the parameters of the problem and the family of fixed points. Two zero roots of this equation correspond to the parameter of the family, \( c_s \), and to the integral of motion, \( \gamma^2 = 1 \). Four nonzero eigenvalues are the roots of the biquadratic equation. In the general case, depending on the values of the coefficients \( a_2 \) and \( a_0 \), the fixed point can have one of four types of stability (see Figure 3). Stable rotations correspond only to one of these types, the stability of center-center type.

![Figure 3. Types of fixed points (33) depending on the values of the coefficients \( a_0 \) and \( a_2 \) of the characteristic Equation (34).](image)

Thus, for the linear stability of vertical rotations it is necessary that the following conditions on the coefficients of the characteristic polynomial be satisfied:

\[ a_0 > 0, \quad a_2 > 0, \quad D = a_2^2 - 4a_0 > 0. \quad (35) \]

Coefficients of the characteristic equation

In the case at hand, for the ellipsoid of revolution we have

\[ a_2 = \frac{A_1\eta^2}{v^2(1 + \eta \rho^2)^2} c^2 - \frac{2\eta R}{1 + \eta \rho^2} + \frac{2\eta^2 R}{(1 + \eta \rho^2)^2} \Theta^2, \]

\[ a_0 = \left( \frac{\eta (\eta v^2 R^2 \Theta^2 - \eta (1 + \eta \rho)(\eta \rho R + 1 - v)c^2 - Rv^2(1 + \eta \rho^2))}{v^2(1 + \eta \rho^2)^2} \right)^2, \quad (36) \]

\[ D = \frac{A_2\eta^3}{v^4(1 + \eta \rho^2)^4} \left( 4v^2 \eta R^2 \Theta^2 + \eta (v + \eta \beta^2 \rho)^2 c^2 - 4Rv^2(1 + \eta \rho^2) \right), \]

where

\[ A_1 = \eta \rho \left( \eta \rho (R^2 + \rho^2) + 2\beta^2 + 2(2 - v)R \right) + (v - 1)^2 + 1 > 0, \]

\[ A_2 = \eta \rho (\rho + R) + 2 - v > 0, \]
\[ R = \rho - \beta^2, \quad \text{and} \quad \rho = 1 + \alpha \in [0, 2] \] is the distance from the point of contact to the center of mass of the ellipsoid. The values \( \rho \in (1, 2] \) correspond to the upper rotations, and the values \( \rho \in [0, 1) \) to the lower ones.

As can be seen from the expressions presented above, \( a_0 > 0 \) for any admissible values of the parameters (20). Thus, for the stability of vertical rotations it is necessary to satisfy only the two remaining conditions \( a_2 > 0 \) and \( D > 0 \).

Below we check whether these conditions are satisfied for the upper and lower rotations.

### 4.2.1. Stability of Vertical Rotations in the Absence of a Vibrational Potential

Recall [23, 40] that vertical rotations of a body with axisymmetric mass distribution are stable in the case where the projection of the angular momentum onto the symmetry axis satisfies the relation

\[
 c^2 > c_0^2 = \frac{4 \nu^2 (1 + \eta \rho^2) (\rho - \beta^2)}{\eta (v + \eta \beta^2 \rho)^2}.
\] (37)

In this case, the parameter \( \beta^2 \) corresponds physically to the radius of the sphere approximating the surface of the body at the point of contact.

Inequality (37) defines in the parameter space \((\rho, \beta^2, c^2)\) the surface \( c^2 = c_0^2 \), which separates the stable vertical rotations from the unstable ones. Stable rotations correspond to a fixed point of center-center type, and unstable ones, to a fixed point of focus-focus type. The parameters \( \eta \) and \( \nu \) in the expression (37) do not qualitatively influence the form of the surface \( c^2 = c_0^2 \), and so we do not consider the dependence on these parameters. The surface \( c^2 = c_0^2 \) is shown in Figure 4 for the values \( \eta = 0.5 \) and \( \nu = 0.5 \).

![Figure 4. The surface \( c^2 = c_0^2 \), which separates the regions of stable and unstable vertical rotations of the ellipsoid in the case of absence of a vibrational potential. The surface is plotted for \( \eta = 0.5 \) and \( \nu = 0.5 \).](image)

We note that, when \( \rho < \beta^2 \), inequality (37) is satisfied for any values of \( c \). Thus, if the center of mass of the body lies below the center of the sphere approximating the surface of the body at the point of contact, the vertical rotations are stable for any rotational velocity.

In the case \( \rho > \beta^2 \) the vertical rotations are stable only for sufficiently large angular velocities (momenta) \( c^2 > c_0^2 \) (37).

Below, we consider the influence of a vibrational potential on the stability of the partial solutions.
4.2.2. Stability of Vertical Rotations in the Presence of a Vibrational Potential

As in the previous case, for the stability of vertical rotations in the presence of a vibrational potential it is necessary to satisfy three conditions on the coefficients of the characteristic polynomial and its discriminants. As shown above, \( a_0 > 0 \) regardless of the parameter values. Analysis of the expressions for the coefficients (36) has shown that, when the condition \( D > 0 \) is satisfied, the coefficient \( a_2 \) will also be positive.

Thus, for the stability of vertical rotations of an axisymmetric ellipsoid on a vibrating plane it is necessary to satisfy only one condition, \( D > 0 \), from which it follows that

\[
c^2 > c_2^* = c_0^* - \frac{4v^2(\rho - \beta^2)^2}{(v + \eta^2\nu)^2}\Omega^2.
\]  

(38)

This inequality also defines in the parameter space the surface \( c^2 = c_2^* \), which separates the regions of stable and unstable rotations.

Analysis of inequality (38) leads to the following conclusions on the stability of vertical rotations in the presence of vibrations of the supporting plane:

1. When \( \rho < \beta^2 \), the vertical rotations are stable for any values of the angular momentum \( c^2 \) and for any parameters of oscillations of \( \Theta^2 \);
2. When \( \rho > \beta^2 \), the vertical rotations become stable for smaller values of the angular momentum \( c^2 \) than those in the absence of vibrations;
3. There exists a critical value of \( \Theta^2 \) such that, when the inequality

\[
\Theta^2 > \Theta_*^2 = \frac{1 + \eta \rho^2}{\eta (\rho - \beta^2)}
\]

(39)

is satisfied, the vertical position of the ellipsoid becomes stable even without rotation \( (c^2 = 0) \).

**Remark 7.** Using dimensional quantities, inequality (39) can be represented as

\[
\delta^2 \Omega^2 > 2g \frac{L^2 + K^2}{L - \ell},
\]

(40)

where \( L = b_3 + a \) is the height of the center of mass, \( K = \sqrt{\frac{L}{m}} \) is the radius of inertia of the meridian section, and \( \ell = \frac{\nu}{2} \) is the radius of curvature of the ellipsoid’s vertex on which the rotation occurs. We note that condition (40) in the limiting case of a rod \( (b_1 \to 0) \) completely coincides with the condition for the stability of a physical pendulum obtained by P. L. Kapitsa in [1].

When \( \Theta \neq 0 \), the surface \( c^2 = c_2^* \) depends on the parameters \( \rho, \beta^2, \Theta^2, \eta \) and \( v \). The analysis of inequalities (37) and (38) implies that the regions of stability do not qualitatively change as the parameters \( \rho, \eta \) and \( v \) are varied. Therefore, for ease of visualization we will depict the regions of stability in the parameter space \( (\Theta^2, \beta^2, c^2) \) for fixed values of the three remaining parameters.

Figure 5 shows the surface \( c^2 = c_2^* \) for the fixed values \( \rho = 0.9, \eta = 0.5 \) and \( v = 0.5 \). If the parameter values lie below the surface, the vertical rotations are unstable, and if they lie above it, the vertical rotations are stable. It can be seen from the figure that, after the vibrations of the plane \( (\Theta^2 > 0) \) are added, the vertical rotations become stable for smaller values of the angular momentum \( c \).
Figure 5. The surface $c^2 = c_0^2$, which separates the regions of stable and unstable vertical rotations of the ellipsoid in the case of absence of a vibrational potential. The surface is plotted for $\eta = 0.5$, $\nu = 0.5$ and $\rho = 0.9$.

4.3. Analysis Of Reaction

In investigating the motion of rigid bodies on a plane, especially a moving one, special attention should be given to analysis of the normal reaction in order to avoid the loss of contact of the body with the plane during its motion. In the case at hand, the reaction of the plane, $N$, is expressed as follows:

$$N = mg\gamma + \lambda,$$

where $\lambda$ is the undetermined multiplier calculated in deriving the equations of motion. It is easy to verify that for the averaged system the normal reaction calculated for vertical rotations has the form

$$N_z = (N,\gamma)_{\sigma_h} = mg,$$

i.e., averaging smooths out the jumps of the normal reaction. Thus, within the framework of the averaged model, it is impossible to investigate the loss of contact of the body with the plane under vertical rotations which is due to vibrations of the plane.

For the exact system (8), in the case of vertical rotations, the normal reaction depends on time and has the form $N_z = m(g - \ddot{\xi}(t)^2)$, and its minimal value for a period is

$$N_{\min} = m(g - \delta \Omega^2).$$

For the body to move without loss of contact with the plane, the condition $\delta \Omega^2 < g$ must be satisfied. Since in the averaging procedure we assume that $\Omega \sim \frac{1}{\varepsilon}$, the motion without loss of contact with the plane is only possible for sufficiently small amplitudes $\delta < \frac{g}{\Omega^2} \sim \varepsilon^2$. In the averaged model, such amplitudes correspond to the small vibrational torque $\Xi \sim \varepsilon$.

A more complete analysis of the possibility of stabilization without loss of contact with the plane should be made using a complete (unaveraged) model. Such an analysis of the reaction for a sphere with an internal pendulum rolling on a rough vibrating plane is carried out in [9].

5. Conclusions

In this paper, we have treated the problem of an axisymmetric ellipsoid rolling on a vertically vibrating plane. It was shown that, for an arbitrary convex body of revolution with axisymmetric mass distribution, the equations describing the dynamics of the system will preserve two integrals which are linear in angular momenta and, in the general case, are nonalgebraic functions of $\gamma^3$. Following [10], we have obtained an averaged time-independent vibrational potential which is invariant under rotation about the vertical axis.
It is well known [37] that, in the presence of such a potential, the problem of a body of revolution rolling on a plane is integrable. Using the results of the above-mentioned paper, we have reduced the system under consideration to a system with one degree of freedom. As an example, we have constructed the phase portrait of the system.

To illustrate the influence of vibrations of the supporting plane on the ellipsoid’s dynamics, we have analyzed the stability of vertical rotations. The research results have shown that in the presence of a vibrational potential the vertical rotations of the ellipsoid become stable for smaller values of the angular velocity than those in the case of absence of vibrations. Moreover, there exists a critical value $\Theta_*$ of a magnitude of the vibrational potential such that the vertical position can become stable even in the absence of rotation. In the limiting case of a thin rod, the resulting value of $\Theta_*$ is the same as the value obtained by P. L. Kapitsa in [1].

To conclude, we point out the most interesting directions for the investigation of this problem. Further inquiry into the problem can involve the search for permanent rotations of the system and their bifurcation analysis. It would also be interesting to investigate the influence of a vibrational potential on the possibility of tumbling (similar to the tippe top inversion) of an ellipsoid with the center of mass displaced and friction forces added. A study of the tumbling of ellipsoidal bodies on a fixed plane was made in [21], where conditions on the system’s mass-geometric parameters for which a complete or partial inversion is possible were obtained.

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