Asymptotic Stability and Dependency of a Class of Hybrid Functional Integral Equations

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Abstract: Here, we discuss the solvability of a class of hybrid functional integral equations by applying Darbo’s fixed point theorem and the technique of the measure of noncompactness (MNC). This study has been located in space $BC(R_+)$. Furthermore, we prove the asymptotic stability of the solution of our problem on $R_+$. We introduce the idea of asymptotic dependency of the solutions on some parameters for that class. Moreover, general discussion, examples, and remarks are demonstrated.

Keywords: hybrid functional equation; measure of noncompactness; asymptotic stability and dependency

MSC: 34L30; 34K06

1. Introduction

The study of delay functional integral equations has received much consideration over the last few decades. Further studies and results for such kinds of problems may be found in [1–3] and the references therein.

The technique of MNC [4] in the Banach space $BC(R_+)$ had been effectively utilized by J. Banas (see [5,6]) for demonstrating that asymptotic stable solutions for various functional equations have been established (see [7,8]).

Numerous practical real-world applications of quadratic integral equations are established. For various studies on the solvability of different classes of nonlinear equations, see [1,9–13].

Quadratic integral equations continuously emerge in numerous problems, such as the theory of radiative transfer, the kinetic theory of gases, the theory of neutron transport, the queuing theory, and the traffic theory.

Although the existence results in each of these monographs are included [1,11–13], their primary goal was to show a unique method or strategy as well as results pertaining to different existence for particular quadratic integral equations.

The significance of the investigations of hybrid functional integral and quadratic functional integral problems locates within the reality that this type involves different dynamic systems in particular cases. This class of hybrid differential equations involves the perturbations of original differential equations in several ways. A sharp classification of distinctive sorts of perturbations of differential equations shows up in Dhage [14], which can be treated with hybrid fixed point theory.

The authors in [13,15] discussed the two equations

$$\frac{d}{dt}\left(\frac{\theta(t) - h(t, \theta(\phi_1(t)))}{g(t, \theta(\phi_2(t)))}\right) = f\left(t, D^{\alpha}\left(\frac{\theta(t) - h(t, \theta(\phi_1(t)))}{g(t, \theta(\phi_2(t)))}\right)\right) \text{ a.e.}$$
\[
\frac{\vartheta(t) - h(t, \vartheta(t))}{g(t, \vartheta(t))} = f_1 \left( t, \frac{\vartheta(t) - h(t, \vartheta(t))}{g(t, \vartheta(t))}, \int_{0}^{\phi(t)} f_2 \left( s, \frac{\vartheta(s) - h(s, \vartheta(s))}{g(s, \vartheta(s))} \right) ds \right),
\]

in a bounded interval. Also, they studied the solvability of these problems using the technique of MNC in a finite interval and also discussed the continuous dependency.

Existence and stability results for Atangana–Baleanu fractional problems are established in [16], BVP for a nonlinear Hadamard fractional differential equation is discussed in [17,18], and, for fractional systems, see [19–21].

Here, consider the class of hybrid functional integral equation

\[
\begin{align*}
\frac{\vartheta(t) - h(t, \vartheta(t))}{g(t, \vartheta(t))} &= f(t, g_1(t, \vartheta(t)), \lambda \int_{0}^{\beta_2(t)} f_1 \left( \xi, \frac{\vartheta(\xi) - h(\xi, \vartheta(\xi))}{g(\xi, \vartheta(\xi))} \right) d\xi, \\
g_2 \left( t, \frac{\vartheta(\beta_3(t)) - h(t, \vartheta(\beta_3(t)))}{g(t, \vartheta(\beta_3(t)))} \right) &= f_2 \left( t, \xi, \frac{\vartheta(\xi) - h(\xi, \vartheta(\xi))}{g(\xi, \vartheta(\xi))} \right) d\xi, \quad t \geq 0.
\end{align*}
\] (1)

Our aim here is to establish the solvability and discuss some asymptotic stability facts of the solution \( \vartheta \in \text{BC}(R_+) \) of (1). The main tool in our study is applying Darbo’s fixed point [4] and MNC technique.

Furthermore, the asymptotic dependency of \( \vartheta \in \text{BC}(R_+) \) on the parameter \( \lambda \geq 0 \) and on the functions \( g_1, g_2, \beta_2, \) and \( \beta_4 \) has been studied. Some special cases and examples have been discussed.

Let

\[
\frac{\vartheta(t) - h(t, \vartheta(t))}{g(t, \vartheta(t))} = v(t),
\]

and, easily, we obtain

\[
\vartheta(t) = h(t, \vartheta(t)) + v(t)g(t, \vartheta(t)), \quad t \geq 0
\] (2)

as a solution of (1), which implies that \( v \) is a solution of

\[
v(t) = f(t, g_1(t, v(\beta_1(t))), \lambda \int_{0}^{\beta_2(t)} f_1(t, \xi, v(\xi)) d\xi, \quad g_2(t, v(\beta_3(t))), \quad t \geq 0.
\] (3)

We arrange our article just like that: we conclude the solvability of (3) in \( \text{BC}(R_+) \), and then the asymptotic stability of the solution \( v \in \text{BC}(R_+) \) of (3) is discussed in Section 2. The main theorems for the existence of the solutions \( \vartheta \in \text{BC}(R_+) \) and the asymptotic stability and dependency of the solution \( \vartheta \in \text{BC}(R_+) \) on the parameter \( \lambda \geq 0 \), and on the functions \( g_1, g_2, \beta_2 \) and \( \beta_4 \) have been established. Finally, some general remarks and comments will be provided.

The class \( \text{BC}(R_+) \) of all bounded and continuous functions in \( R_+ \), with an internal composition law is noted by

\[
(\cdot) : X \times X \rightarrow X, \quad (x, y) \rightarrow xy,
\]

which is associative and bilinear.

A normed algebra is an algebra endowed with a norm satisfying the following property:

For all \( x, y \in X \), we have

\[
\|xy\| \leq \|x\|\|y\|.
\]

A complete normed algebra is called a Banach algebra.

Now, let \( \theta \in Y \subseteq \text{BC}(R_+) \) and \( \varepsilon \geq 0 \) be provided, defined as \( \omega_\varepsilon^{\text{BC}}(\theta, \varepsilon), \varepsilon \geq 0 \), the modulus of continuity of the function \( \theta \) on \( [0, T] \).
\[
\omega^\tau(\theta, \epsilon) = \sup\{||\theta(t) - \theta(\zeta)|| : t, \zeta \in [0, \tau], |t - \zeta| \leq \epsilon\}
\]
and
\[
\omega^\tau(Y, \epsilon) = \sup\{\omega^\tau(\theta, \epsilon) : \theta \in Y\}.
\]

Also,
\[
\omega_0^\tau(Y) = \lim_{\epsilon \to 0} \omega^\tau(Y, \epsilon), \quad \omega_0(Y) = \lim_{\tau \to \infty} \omega_0^\tau(Y)
\]
and
\[
diam Y(t) = \sup\{|\theta(t) - \nu(t)|, \theta, \nu \in Y\}.
\]

The MNC on BC(R+) has the form [22, 23]
\[
\mu(Y) = \omega_0(Y) + \lim_{t \to \infty} \sup diam Y(t).
\]

**Theorem 1** ([4]). Let C be a nonempty, bounded, closed, and convex subset of a Banach space and let Y : C \rightarrow C be a continuous mapping. Assume that there exists a constant K \in [0, 1) such that \mu(Y) \leq K \mu(\Lambda) for any nonempty subset \Lambda of C, where \mu is an MNC defined in \epsilon. Then, Y has at least one fixed point in C.

### 2. Existence of Solutions

To achieve our goals, assume that

(i) \beta_i : R_+ \rightarrow R_+, \beta_i(t) \leq t, i = 1, 2, 3, 4 are continuous.

(ii) f : R_+ \times R \times R \times R \rightarrow R is continuous in t \in R_+, \forall \xi, v, w \in R and satisfies Lipschitz condition,
\[
|f(t, \xi, v, w) - f(t, \xi_1, v_1, w_1)| \leq b(|\xi - \xi_1| + |v - v_1| + |w - w_1|),
\]
\[
\forall (t, \xi, v, w), (t, \xi_1, v_1, w_1) \in R_+ \times R \times R, b > 0.
\]

(iii) g_i : R_+ \times R \rightarrow R, i = 1, 2 are continuous in t \in R_+, \forall \xi \in R and satisfy Lipschitz condition,
\[
|g_i(t, \xi) - g_i(t, \xi_1)| \leq b_i|\xi - \xi_1|,
\]
\[
\forall (t, \xi), (t, \xi_1) \in R_+ \times R, b_i > 0.
\]

(iv) f_i : R_+ \times R_+ \times R \rightarrow R, i = 1, 2 are Carathéodory functions, which are measurable in t, \zeta \in R_+ \times R_+, \forall v \in R and continuous in v \in R, \forall t, \zeta \in R_+ \times R_+, and there exist measurable and bounded functions k_i, c_i : R_+ \times R_+ \rightarrow R, where
\[
|f_i(t, \xi, v)| \leq k_i(t, \xi) + c_i(t, \xi)|v|, \forall (t, \xi) \in R_+ \times R_+
\]
and
\[
\lim_{t \to \infty} \int_0^{\beta_i(t)} k_i(t, \zeta)d\zeta = 0, \lim_{t \to \infty} \int_0^{\beta_i(t)} c_i(t, \zeta)d\zeta = 0, \quad j = 2, 4,
\]
\[
\sup_{t \in R_+} \int_0^{\beta_i(t)} k_i(t, \zeta)d\zeta = k_i, \sup_{t \in R_+} \int_0^{\beta_i(t)} c_i(t, \zeta)d\zeta = c_i, \quad j = 2, 4.
\]

(v) For a positive constant t satisfying the equation
\[ bb_2 c_2 r^2 + (bb_1 + b\lambda c_1 + bc_2 m_2^* + bb_2 k_2 - 1)r + m^* + bm_1^* + b\lambda k_1 + bk_2 m_2^* = 0. \]

From Equation (4), we receive
\[
|f(t, \xi, u, w)| - |f(t, 0, 0, 0)| \leq |f(t, \xi, u, w) - f(t, 0, 0, 0)| \leq b(|\xi| + |u| + |w|),
\]
\[
|f(t, \xi, u, w)| \leq |f(t, 0, 0, 0)| + b(|\xi| + |u| + |w|)
\]
\[
|f(t, \xi, u, w)| \leq m(t) + b(|\xi| + |u| + |w|),
\]
where
\[
m(t) = |f(t, 0, 0, 0)|, \ m \in BC(R_+) \ \text{and} \ m^* = \sup_{t \in R_+} |m(t)| < \infty.
\]

In the same manner, from Equation (5), we receive
\[
|g_i(t, \xi)| \leq |m_i(t)| + b_i|\xi|,
\]
where
\[
m_i(t) = |g_i(t, 0)|, \ m_i \in BC(R_+) \ \text{and} \ m_i^* = \sup_{t \in R_+} |m_i(t)| < \infty.
\]

**Theorem 2.** Suppose that (i) – (iv) hold. Then, we have a solution \( v \in BC(R_+) \) for (3).

**Proof.** Let
\[
Q_r = \{ v \in BC(R_+) : ||v|| \leq r \},
\]
\[
r = m^* + bm_1^* + bb_1 r + b\lambda k_1 + b\lambda c_1 r + bm_2^* k_2 + bm_2^* c_2 r + bb_2 k_2 r + bb_2 c_2 r^2.
\]

Associate the operator
\[
F v(t) = f\left(t, g_1(t, v(\beta_1(t))), \lambda \int_0^{\beta_1(t)} f_1(t, \xi, v(\xi))d\xi, g_2(t, v(\beta_3(t))) \cdot \int_0^{\beta_3(t)} f_2(t, \xi, v(\xi))d\xi \right), \ \ t \geq 0.
\]

For \( v \in Q_r \), then
\[
|Fv(t)| = |f\left(t, g_1(t, v(\beta_1(t))), \lambda \int_0^{\beta_1(t)} f_1(t, \xi, v(\xi))d\xi, g_2(t, v(\beta_3(t))) \cdot \int_0^{\beta_3(t)} f_2(t, \xi, v(\xi))d\xi \right)|
\]
\[
\leq |m(t)| + b\left(|m_1(t)| + b_1|v(\beta_1(t))|\right) + b\left( \int_0^{\beta_1(t)} (k_1(t, \xi) + c_1(t, \xi)|v(\xi)|)d\xi \right)
\]
\[
+ b\left(|m_2(t)| + b_2|v(\beta_3(t))|\right) \cdot \int_0^{\beta_3(t)} (k_2(t, \xi) + c_2(t, \xi)|v(\xi)|)d\xi,
\]
then
\[
\|Fv\| \leq m^* + bm_1^* + bb_1 ||v|| + b\lambda \left( \int_0^{\beta_1(t)} k_1(t, \xi)d\xi + ||v|| \int_0^{\beta_1(t)} c_1(t, \xi)d\xi \right)
\]
\[
+ b\left( m_2^* + b_2||v|| \right) \cdot \left( \int_0^{\beta_1(t)} k_2(t, \xi)d\xi + ||v|| \int_0^{\beta_1(t)} c_2(t, \xi)d\xi \right)
\]
\[
\leq m^* + bm_1^* + bb_1 r + b\lambda (k_1 + c_1 r) + (bm_2^* + bb_2 r) \cdot (k_2 + c_2 r)
\]
\[
\leq m^* + bm_1^* + bb_1 r + b\lambda k_1 + b\lambda c_1 r + bm_2^* k_2 + bm_2^* c_2 r + bb_2 k_2 r + bb_2 c_2 r^2.
\]

Thus, the mapping \( F \) draws the set \( Q_r \) into \( Q_r \).
Then, take $\delta > 0$ and take $v_1, v_2 \in Q_r$, such that $\|v_2 - v_1\| \leq \delta$, and then

$$|Fv_2(t) - Fv_1(t)| =$$

$$f(t, g_1(t, v_2(\beta_1(t)))), \lambda \int_0^{\beta_2(t)} f_1(t, \zeta, v_2(\zeta)) d\zeta, g_2(t, v_2(\beta_3(t))) \cdot \int_0^{\beta_4(t)} f_2(t, \zeta, v_2(\zeta)) d\zeta$$

$$- f(t, g_1(t, v_1(\beta_1(t)))), \lambda \int_0^{\beta_2(t)} f_1(t, \zeta, v_1(\zeta)) d\zeta, g_2(t, v_1(\beta_3(t))) \cdot \int_0^{\beta_4(t)} f_2(t, \zeta, v_1(\zeta)) d\zeta$$

$$\leq \lambda \int_0^{\beta_2(t)} \|f_1(t, \zeta, v_2(\zeta)) - f_1(t, \zeta, v_1(\zeta))\| d\zeta$$

$$+ \lambda \int_0^{\beta_4(t)} \|f_2(t, \zeta, v_2(\zeta)) - f_2(t, \zeta, v_1(\zeta))\| d\zeta$$

$$\cdot (\beta_2(t) - \beta_4(t)) \cdot (\beta_3(t) - \beta_3(t))$$

Therefore,

$$\|Fv_2 - Fv_1\| \leq$$

$$bb_1 \|v_2 - v_1\| + 2b \lambda \left( \int_0^{\beta_2(t)} k_1(t, \zeta) d\zeta + r \int_0^{\beta_2(t)} c_1(t, \zeta) d\zeta \right)$$

$$+ \|v_2 - v_1\| \cdot \left( \int_0^{\beta_4(t)} k_2(t, \zeta) d\zeta + r \int_0^{\beta_4(t)} c_2(t, \zeta) d\zeta \right)$$

$$(i*) \text{ Choose } \tau > 0, \text{ satisfying } t \geq \tau, \text{ and }$$

$$\|Fv_2 - Fv_1\| \leq$$

$$bb_1 \|v_2 - v_1\| + 2b \lambda \left( \int_0^{\beta_2(t)} k_1(t, \zeta) d\zeta + r \int_0^{\beta_2(t)} c_1(t, \zeta) d\zeta \right)$$

$$+ \|v_2 - v_1\| \cdot \left( \int_0^{\beta_4(t)} k_2(t, \zeta) d\zeta + r \int_0^{\beta_4(t)} c_2(t, \zeta) d\zeta \right)$$

$$+ b(m^2 + b^2) \left( \int_0^{\beta_4(t)} k_2(t, \zeta) d\zeta + r \int_0^{\beta_4(t)} c_2(t, \zeta) d\zeta \right)$$

$$\leq bb_1 \delta + 2b \lambda (\epsilon_1 + \epsilon_2) + bb_2 \delta (\epsilon_3 + \epsilon_4) + 2b(m^2 + b^2)(\epsilon_3 + \epsilon_4 r) = e.$$
Hence, we obtain
\[
diamFY(t) \leq bb_1 \ diamY(t) + 2b\lambda (k_1 + c_1r) + bb_2(k_2 + c_2r) \ diamY(t) + 2b(m_2^2 + b_2r)(k_2 + c_2r)
\]
\[
\lim_{n \to 0} \sup diamFY(t) \leq \left( bb_1 + bb_2(k_2 + c_2r) \right) \lim_{n \to 0} \sup diamY(t).
\]

Let $\Sigma > 0$ and $\delta > 0$ be taken and select a mapping $v \in Y$ and $t \in [0, \Sigma]$, where $|t_2 - t_1| < \delta$, $t_1 \leq t_2$. Take $\theta_f(\delta)$, $\theta_g(\delta)$, $\theta_j(\delta)$ as defined in [9,13]. Therefore,
\[
|Fv(t_2) - Fv(t_1)| =
\]
\[
\left| f(t_2, g_1(t_2, v(\beta_1(t_2))), \lambda \int_0^{\beta_2(t_2)} f_1(t_2, \xi, v(\xi))d\xi, g_2(t_2, v(\beta_3(t_2))) \cdot \int_0^{\beta_4(t_2)} f_2(t_2, \xi, v(\xi))d\xi \right|
\]
\[
- f(t_1, g_1(t_1, v(\beta_1(t_1))), \lambda \int_0^{\beta_2(t_1)} f_1(t_1, \xi, v(\xi))d\xi, g_2(t_1, v(\beta_3(t_1))) \cdot \int_0^{\beta_4(t_1)} f_2(t_1, \xi, v(\xi))d\xi \right|
\]
\[
\leq \left| f(t_2, g_1(t_2, v(\beta_1(t_2))), \lambda \int_0^{\beta_2(t_2)} f_1(t_2, \xi, v(\xi))d\xi, g_2(t_2, v(\beta_3(t_2))) \cdot \int_0^{\beta_4(t_2)} f_2(t_2, \xi, v(\xi))d\xi \right|
\]
\[
- f(t_1, g_1(t_1, v(\beta_1(t_1))), \lambda \int_0^{\beta_2(t_1)} f_1(t_1, \xi, v(\xi))d\xi, g_2(t_1, v(\beta_3(t_1))) \cdot \int_0^{\beta_4(t_1)} f_2(t_1, \xi, v(\xi))d\xi \right|
\]
\[
+ f(t_1, g_1(t_2, v(\beta_1(t_2))), \lambda \int_0^{\beta_2(t_2)} f_1(t_2, \xi, v(\xi))d\xi, g_2(t_2, v(\beta_3(t_2))) \cdot \int_0^{\beta_4(t_2)} f_2(t_2, \xi, v(\xi))d\xi \right|
\]
\[
- f(t_1, g_1(t_1, v(\beta_1(t_1))), \lambda \int_0^{\beta_2(t_1)} f_1(t_1, \xi, v(\xi))d\xi, g_2(t_1, v(\beta_3(t_1))) \cdot \int_0^{\beta_4(t_1)} f_2(t_1, \xi, v(\xi))d\xi \right|
\]
\[
+ f(t_1, g_1(t_1, v(\beta_1(t_1))), \lambda \int_0^{\beta_2(t_1)} f_1(t_1, \xi, v(\xi))d\xi, g_2(t_1, v(\beta_3(t_1))) \cdot \int_0^{\beta_4(t_1)} f_2(t_1, \xi, v(\xi))d\xi \right|
\]
\[
- f(t_1, g_1(t_1, v(\beta_1(t_1))), \lambda \int_0^{\beta_2(t_1)} f_1(t_1, \xi, v(\xi))d\xi, g_2(t_1, v(\beta_3(t_1))) \cdot \int_0^{\beta_4(t_1)} f_2(t_1, \xi, v(\xi))d\xi \right|
\]
\[
+ f(t_1, g_1(t_1, v(\beta_1(t_1))), \lambda \int_0^{\beta_2(t_1)} f_1(t_1, \xi, v(\xi))d\xi, g_2(t_1, v(\beta_3(t_1))) \cdot \int_0^{\beta_4(t_1)} f_2(t_1, \xi, v(\xi))d\xi \right|
\]
\[
- f(t_1, g_1(t_1, v(\beta_1(t_1))), \lambda \int_0^{\beta_2(t_1)} f_1(t_1, \xi, v(\xi))d\xi, g_2(t_1, v(\beta_3(t_1))) \cdot \int_0^{\beta_4(t_1)} f_2(t_1, \xi, v(\xi))d\xi \right|
\]
\[
+ f(t_1, g_1(t_1, v(\beta_1(t_1))), \lambda \int_0^{\beta_2(t_1)} f_1(t_1, \xi, v(\xi))d\xi, g_2(t_1, v(\beta_3(t_1))) \cdot \int_0^{\beta_4(t_1)} f_2(t_1, \xi, v(\xi))d\xi \right|
\]
\[
- f(t_1, g_1(t_1, v(\beta_1(t_1))), \lambda \int_0^{\beta_2(t_1)} f_1(t_1, \xi, v(\xi))d\xi, g_2(t_1, v(\beta_3(t_1))) \cdot \int_0^{\beta_4(t_1)} f_2(t_1, \xi, v(\xi))d\xi \right|
\]
\[
+ f(t_1, g_1(t_1, v(\beta_1(t_1))), \lambda \int_0^{\beta_2(t_1)} f_1(t_1, \xi, v(\xi))d\xi, g_2(t_1, v(\beta_3(t_1))) \cdot \int_0^{\beta_4(t_1)} f_2(t_1, \xi, v(\xi))d\xi \right|
\]
\[
- f(t_1, g_1(t_1, v(\beta_1(t_1))), \lambda \int_0^{\beta_2(t_1)} f_1(t_1, \xi, v(\xi))d\xi, g_2(t_1, v(\beta_3(t_1))) \cdot \int_0^{\beta_4(t_1)} f_2(t_1, \xi, v(\xi))d\xi \right|
\]
\[
- f(t_1, g_1(t_1, v(\beta_1(t_1)))), \lambda \int_0^{\beta_1(t_1)} f_1(t_1, \xi, v(\xi)) d\xi, g_2(t_1, v(\beta_3(t_1))) \cdot \int_0^{\beta_3(t_1)} f_2(t_2, \xi, v(\xi)) d\xi \\
+ f(t_1, g_1(t_1, v(\beta_1(t_1)))), \lambda \int_0^{\beta_1(t_1)} f_1(t_1, \xi, v(\xi)) d\xi, g_2(t_1, v(\beta_3(t_1))) \cdot \int_0^{\beta_3(t_1)} f_2(t_2, \xi, v(\xi)) d\xi \\
- f(t_1, g_1(t_1, v(\beta_1(t_1)))), \lambda \int_0^{\beta_1(t_1)} f_1(t_1, \xi, v(\xi)) d\xi, g_2(t_1, v(\beta_3(t_1))) \cdot \int_0^{\beta_3(t_1)} f_2(t_2, \xi, v(\xi)) d\xi
\]

\[
\leq \theta_f(\delta) + b\theta_{g_1}(\delta) + bb_1 [v(\beta_1(t_2)) - v(\beta_1(t_1))] + b\lambda \int_0^{\beta_1(t_1)} |f_1(t_2, \xi, v(\xi))| d\xi \\
+ b\lambda \int_0^{\beta_1(t_1)} |f_1(t_2, \xi, v(\xi)) - f_1(1, \xi, v(\xi))| d\xi \\
+ b\theta_{g_2}(\delta) \cdot \int_0^{\beta_1(t_1)} |f_2(t_2, \xi, v(\xi))| d\xi + bb_2 [v(\beta_3(t_2)) - v(\beta_3(t_1))] \cdot \int_0^{\beta_1(t_1)} |f_2(t_2, \xi, v(\xi))| d\xi \\
+ b(m_2^2 + b\alpha) \int_0^{\beta_1(t_1)} |f_2(t_2, \xi, v(\xi))| d\xi + b(m_2^2 + b\alpha) \int_0^{\beta_1(t_1)} |f_2(t_2, \xi, v(\xi)) - f_2(t_1, \xi, v(\xi))| d\xi
\]

Now, let \( t_1, t_2 \in [0, \Xi], |t_2 - t_1| < \delta \). Then, we deduce that

\[
\omega^\Xi(FY, \epsilon) \leq \theta_f(\delta) + b\theta_{g_1}(\delta) + bb_1 \omega^\Xi(Y, \omega^\Xi(\beta_1, \epsilon)) + b\lambda \int_0^{\beta_1(t_1)} |f_1(t_2, \xi, v(\xi))| d\xi \\
+ b\lambda \theta_{f_1}(\delta) \beta_2(\Xi) + b\theta_{g_2}(\delta) (k_2 + c_2r) + bb_2 (k_2 + c_2r) \omega^\Xi(Y, \omega^\Xi(\beta_3, \epsilon)) \\
+ b(m_2^2 + b\alpha) \int_0^{\beta_1(t_1)} |f_2(t_2, \xi, v(\xi))| d\xi + b(m_2^2 + b\alpha) \theta_{f_2}(\delta) \beta_4(\Xi).
\]

From our assumptions, \( f, g_i, i = 1, 2 \) are uniform continuity, and \( \omega^\Xi(\beta_1, \epsilon) \to 0 \), \( \omega^\Xi(\beta_3, \epsilon) \to 0 \) as \( \epsilon \to 0 \), we have achieved that \( \theta_f(\delta) \to 0, \theta_{g_1}(\delta) \to 0, \theta_{f_1}(\delta) \to 0, i = 1, 2 \) as \( \delta \to 0 \) independent of \( \nu \in Q_r \). Consequently, we obtain

\[
\omega^\Xi_0(FY) \leq \left( bb_1 + bb_2 (k_2 + c_2r) \right) \omega^\Xi_0(Y)
\]

and as \( \Xi \to \infty \)

\[
\omega_0(FY) \leq \left( bb_1 + bb_2 (k_2 + c_2r) \right) \omega_0(Y).
\] (8)

Now, from (7) and (8),

\[
\mu(FY) = \left( bb_1 + bb_2 (k_2 + c_2r) \right) \mu(Y).
\]
Since \((bb_1 + bb_2(k_2 + c_2r)) < 1\), \(F\) is a contraction regarding MNC \((\mu)\), which implies that \(\nu \in Q_r\) is a solution of (3). \(\square\)

Now, we discuss the asymptotic stability of the solution of (3).

**Theorem 3.** The solution \(\nu \in BC(R_+)\) of Equation (3) is asymptotically stable; that is, for any \(e^* > 0\), there exist \(T(e^*) > 0\) and \(r > 0\), satisfying for any two solutions \(\nu, \bar{\nu} \in Q_r\), and then \(|\nu(t) - \bar{\nu}(t)| \leq e^*\) for \(t \geq T(e^*)\).

**Proof.** For two solutions \(\nu, \bar{\nu} \in Q_r\) of Equation (3), therefore,

\[
|\nu(t) - \bar{\nu}(t)| = |f(t, g_1(t, \nu(\beta_1(t)))), \lambda \int_0^{\beta(t)} f_1(t, t, \nu(\xi))d\xi, g_2(t, \nu(\beta_3(t))) \cdot \int_0^{\beta(t)} f_2(t, t, \nu(\xi))d\xi - f(t, g_1(t, \nu(\beta_1(t)))), \lambda \int_0^{\beta(t)} f_1(t, t, \bar{\nu}(\xi))d\xi, g_2(t, \bar{\nu}(\beta_3(t))) \cdot \int_0^{\beta(t)} f_2(t, t, \bar{\nu}(\xi))d\xi| \leq bb_1|\nu(\beta_1(t)) - \bar{\nu}(\beta_1(t))| + 2b\lambda \left( \int_0^{\beta(t)} k_1(t, t, )d\xi + \int_0^{\beta(t)} c_1(t, t, )d\xi \right) + bb_2|\nu(\beta_3(t)) - \bar{\nu}(\beta_3(t))| \cdot (k_2 + c_2r) + 2b(m_2 + b_2r) \cdot \left( \int_0^{\beta(t)} k_2(t, t, )d\xi + \int_0^{\beta(t)} c_2(t, t, )d\xi \right),
\]

and then

\[
||\nu - \bar{\nu}|| \leq bb_1||\nu - \bar{\nu}|| + 2b\lambda (e_1 + re_2) + bb_2||\nu - \bar{\nu}|| \cdot (k_2 + c_2r) + 2b(m_2 + b_2r) \cdot (e_3 + re_4) \leq \frac{2b\lambda (e_1 + re_2) + 2b(m_2 + b_2r) \cdot (e_3 + re_4)}{1 - (bb_1 + bb_2(k_2 + c_2r))} \leq e^*,
\]

that is,

\[
|\nu(t) - \bar{\nu}(t)| \leq ||\nu - \bar{\nu}|| \leq e^*.
\]

Therefore \(\nu \in BC(R_+)\) is asymptotically stable. \(\square\)

### 3. Main Theorems

Here, we examine the existence of \(\vartheta \in BC(R_+)\) as a solution for the functional Equation (2). Take into account the following assumptions:

\(\text{(vi)}\) \(h : R_+ \times R \rightarrow R\) and \(g : R_+ \times R \rightarrow R \setminus \{0\}\) satisfy the following:

they are continuous in \(t \in R_+, \forall \vartheta \in R\) satisfy Lipschitz condition,

\[
|h(t, \vartheta) - h(t, \vartheta_1)| \leq l_1|\vartheta - \vartheta_1| \quad \text{and} \quad |g(t, \vartheta) - g(t, \vartheta_1)| \leq l_2|\vartheta - \vartheta_1|,
\]

\(\forall (t, \vartheta), (t, \vartheta_1) \in R_+ \times R, l_1, l_2 > 0.\)

\(\text{(vii)}\) \(l_1 + Ml_2 < 1.\)

Equation (9) implies

\[
|h(t, \vartheta)| - |h(t, 0)| \leq |h(t, \vartheta) - h(t, 0)| \leq l_1|\vartheta|,
\]

\[
|h(t, \vartheta)| \leq |h(t, 0)| + l_1|\vartheta|
\]

and

\[
|h(t, \vartheta)| \leq |a_1(t)| + l_1|\vartheta|,
\]
where
\[ a_1(t) = |h(t, 0)|, \quad a_1 \in BC(R_+) \quad \text{and} \quad a^*_1 = \sup_{t \in R_+} |a_1(t)| < \infty. \]

Analogously, we can obtain
\[ |g(t, \vartheta)| \leq |a_2(t)| + l_2|\vartheta|, \]
where
\[ a_2(t) = |g(t, 0)|, \quad a_2 \in BC(R_+) \quad \text{and} \quad a^*_2 = \sup_{t \in R_+} |a_2(t)| < \infty. \]

**Theorem 4.** Suppose that Theorem 2 is verified. Let (vi) and (vii) occur and \( \nu \in BC(R_+) \) is a solution of (3), and then (2) has a solution \( \vartheta \in BC(R_+) \).

**Proof.** Assume that \( \nu \in BC(R_+) \), the solution of (3), exists and the set \( B_\rho \) in form
\[ B_\rho = \{ \vartheta \in BC(R_+) : \|\vartheta\| \leq \rho \}, \quad \rho = \frac{a^*_1 + Ma^*_2}{1 - (l_1 + Ml_2)}. \]

Define \( A \) by
\[ A\vartheta(t) = h(t, \vartheta(t)) + \nu(t)g(t, \vartheta(t)), \quad t \geq 0. \]

Let \( \vartheta \in BC(R_+) \) and \( M = \sup_{t \in R_+} |\nu(t)| = r \), and then
\[
|A\vartheta(t)| = |h(t, \vartheta(t)) + \nu(t)g(t, \vartheta(t))| \\
\leq |a_1(t)| + l_1|\vartheta(t)| + |\nu(t)||a_2(t)| + l_2|\vartheta(t)|
\]
and
\[
\|A\vartheta\| \leq a^*_1 + l_1\|\vartheta\| + M(a^*_2 + l_2\|\vartheta\|) \\
\leq a^*_1 + l_1\rho + M(a^*_2 + l_2\rho) = \rho.
\]

Then, the operator \( A \) maps \( B_\rho \) into itself.

Now, let \( \rho > 0 \) be given and take \( \vartheta_1, \vartheta_2 \in B_\rho \), such that \( \|\vartheta_2 - \vartheta_1\| \leq \delta \), and then, for every \( \nu \in BC(R_+) \), we have
\[
|A\vartheta_2(t) - A\vartheta_1(t)| = |h(t, \vartheta_2(t)) + \nu(t)g(t, \vartheta_2(t)) - h(t, \vartheta_1(t)) - \nu(t)g(t, \vartheta_1(t))| \\
\leq |h(t, \vartheta_2(t)) - h(t, \vartheta_1(t))| + |\nu(t)||g(t, \vartheta_2(t)) - g(t, \vartheta_1(t))|. \tag{10}
\]

(i*) Choose \( \Upsilon > 0 \), where \( t \geq \Upsilon \), and then
\[
|A\vartheta_2(t) - A\vartheta_1(t)| \leq l_1\|\vartheta_2 - \vartheta_1\| + Ml_2\|\vartheta_2 - \vartheta_1\| \\
\leq (l_1 + Ml_2)\|\vartheta_2 - \vartheta_1\| \\
\leq (l_1 + Ml_2)\delta = \epsilon.
\]

(ii*) Also, for \( \Upsilon > 0 \), \( t \in [0, \Upsilon] \), from (10), we receive
\[
|A\vartheta_2(t) - A\vartheta_1(t)| \leq l_1\|\vartheta_2 - \vartheta_1\| + Ml_2\|\vartheta_2 - \vartheta_1\| \\
\leq (l_1 + Ml_2)\delta = \epsilon.
\]

Then, the mapping \( A \) is continuous.
Now, take nonempty $X \subset B_p$. Then, for any $\vartheta_1, \vartheta_2 \in X$ and $\varpi > 0, t \geq \varpi$, (10) yields

$$
|A\vartheta_2(t) - A\vartheta_1(t)| \leq l_1|\vartheta_2(t) - \vartheta_1(t)| + Ml_2|\vartheta_2(t) - \vartheta_1(t)|
$$

$$
\leq l_1\sup\{|\vartheta_2(t) - \vartheta_1(t)|, \vartheta_1, \vartheta_2 \in X\} + Ml_2\sup\{|\vartheta_2(t) - \vartheta_1(t)|, \vartheta_1, \vartheta_2 \in X\}
$$

and then

$$
diam AX(t) \leq (l_1 + Ml_2) \text{diam} X(t)
$$

$$
\lim_{n \to \infty} \sup diam AX(t) \leq (l_1 + Ml_2) \lim_{n \to \infty} \sup diam X(t). \tag{11}
$$

 Arbitrarily take $\vartheta \in X, t \in [0, \varpi)$, where $|t_2 - t_1| < \delta, t_1 \leq t_2$, and define $\theta_0(\delta), \theta_\infty(\delta)$ as in $[9,13]$. Then, we have

$$
|A\vartheta(t_2) - A\vartheta(t_1)| = \left| h(t_2, \vartheta(t_2)) + v(t_2)g(t_2, \vartheta(t_2)) - h(t_1, \vartheta(t_1)) - v(t_1)g(t_1, \vartheta(t_1)) \right|
$$

$$
\leq \theta_0(\delta) + l_1|\vartheta(t_2) - \vartheta(t_1)| + |v(t_2) - v(t_1)|(|a_2^2 + l_2\rho) + M\theta_\infty(\delta)
$$

$$
+ Ml_2|\vartheta(t_2) - \vartheta(t_1)|,
$$

and then

$$
|A\vartheta(t_2) - A\vartheta(t_1)| \leq \theta_0(\delta) + l_1\sup\{|\vartheta(t_2) - \vartheta(t_1)|\} + (a_2^2 + l_2\rho)\epsilon
$$

$$
+ M\theta_\infty(\delta) + Ml_2\sup\{|\vartheta(t_2) - \vartheta(t_1)|\}.
$$

Now, we deduce that

$$
\omega^\varpi(AX, \epsilon) \leq (l_1 + Ml_2)\omega^\varpi(X, \epsilon) + \theta_0(\delta) + M\theta_\infty(\delta) + (a_2^2 + l_2\rho)\epsilon
$$

$$
\omega_0^\varpi(AX) \leq (l_1 + Ml_2)\omega_0^\varpi(X)
$$

and as $\varpi \to \infty$

$$
\omega_0(AX) \leq (l_1 + Ml_2)\omega_0(X). \tag{12}
$$

Now, from (11) and (12), we obtain

$$
\mu(AX) = (l_1 + Ml_2)\mu(X).
$$

Since $l_1 + Ml_2 < 1$, $A$ is a contraction operator regarding MNC; thus, Equation (2) has a solution $\vartheta \in B_p$. \( \Box \)

Consequently, we deduce the next result.

**Corollary 1.** For each solution $\nu \in BC(R_+)$ of (3), consequently, the uniqueness of solution of (2) holds.

**Proof.** Take $\nu \in BC(R_+)$ as a solution of (3) and $\vartheta_1, \vartheta_2$ are two solutions of (2), and then

$$
|\vartheta_2(t) - \vartheta_1(t)| \leq \left| h(t, \vartheta_2(t)) + v(t)g(t, \vartheta_2(t)) - h(t, \vartheta_1(t)) - v(t)g(t, \vartheta_1(t)) \right|
$$

$$
\leq \left| h(t, \vartheta_2(t)) - h(t, \vartheta_1(t)) \right| + |v(t)||g(t, \vartheta_2(t)) - g(t, \vartheta_1(t))|,
$$

and then

$$
\|\vartheta_2 - \vartheta_1\| \leq l_1\|\vartheta_2 - \vartheta_1\| + Ml_2\|\vartheta_2 - \vartheta_1\|
$$
and

\[ \|\vartheta_2 - \vartheta_1\|(1 - (l_1 + Ml_2)) \leq 0. \]

Then, \( \vartheta_2 = \vartheta_1 \) and the solution of (2) are unique. \( \square \)

**Theorem 5.** The solution \( \vartheta \in BC(R_+) \) of (2) is asymptotically stable; that is, for any \( \varepsilon > 0 \), there exist \( T(\varepsilon) > 0 \) and \( \rho > 0 \). Moreover, for \( \vartheta, \bar{\vartheta} \in B_\rho \), there are any two solutions, and then \( |\vartheta(t) - \bar{\vartheta}(t)| \leq \varepsilon \) for \( t \geq T(\varepsilon) \).

**Proof.** Take \( \vartheta, \bar{\vartheta} \in B_\rho \), any two solutions of (2), and thus

\[
|\vartheta(t) - \bar{\vartheta}(t)| = \left| h(t, \vartheta(t)) + v(t)g(t, \vartheta(t)) - h(t, \bar{\vartheta}(t)) - v(t)g(t, \bar{\vartheta}(t)) \right|
\]

\[
\leq |h(t, \vartheta(t)) - h(t, \bar{\vartheta}(t))| + |v(t) - v(t)||g(t, \vartheta(t)) - g(t, \bar{\vartheta}(t))|
\]

\[
\leq l_1 |\vartheta(t) - \bar{\vartheta}(t)| + |v(t) - \bar{v}(t)||(a^*_2 + l_2\rho) + Ml_2|\vartheta(t) - \bar{\vartheta}(t)|. 
\]

Theorem 3 implies that

\[
|v(t) - \bar{v}(t)| \leq \varepsilon^*, \ t \geq T(\varepsilon^*),
\]

and then

\[
\|\vartheta - \bar{\vartheta}\| \leq l_1 \|\vartheta - \bar{\vartheta}\| + (a^*_2 + l_2\rho)\varepsilon^* + Ml_2 \|\vartheta - \bar{\vartheta}\|.
\]

Hence,

\[
\|\vartheta - \bar{\vartheta}\| \leq \frac{(a^*_2 + l_2\rho)\varepsilon^*}{1 - (l_1 + Ml_2)} = \varepsilon;
\]

that is,

\[
|\vartheta(t) - \bar{\vartheta}(t)| \leq \|\vartheta - \bar{\vartheta}\| \leq \varepsilon.
\]

Consequently, \( \vartheta \in BC(R_+) \) is asymptotically stable of Equation (2). \( \square \)

**Asymptotic Dependency**

Now, replace the assumption \( (iv)^* \) by \( (iv)^* \) as follows:

\( (iv)^* f_i : R_+ \times R_+ \times R \to R, \ i = 1, 2 \) are Carathéodory functions and satisfy Lipschitz condition,

\[
|f_i(t, \xi, \nu) - f_i(t, \xi, \nu_1)| \leq \zeta_i(t)|\nu - \nu_1|, \ i = 1, 2,
\]

\[
\forall (t, \xi, \nu), (t, \xi, \nu_1) \in R_+ \times R \times R.
\]

where

\[
\lim_{t \to \infty} \int_0^t \zeta_i(t)dt = 0, \ \sup_{t \in R_+} \int_0^t \zeta_i(t)dt = \zeta_i^*, \ i = 1, 2, \ j = 2, 4.
\]

**Theorem 6.** Suppose that Theorem 4 is verified and then the asymptotic dependency of the solution of (2) on the function \( g_1 \) occurred; that is,

\[
\forall \varepsilon > 0, \ \exists \ \delta(\varepsilon), \ \text{where} \ \left| g_1(t, \nu(\beta_1(t))) - g_1(t, \nu(\beta_1(t))) \right| < \delta, \ t > T(\varepsilon)
\]

and then \( \|\vartheta - \bar{\vartheta}\| < \varepsilon, \)
where $\theta^*$ is a solution of

$$\theta^*(t) = h(t, \theta^*(t)) + v^*(t)g(t, \theta^*(t)), \quad t \geq 0$$

and $v^*$ is the solution of

$$v^*(t) = f\left(t, g_1^*(t, v^*(\beta_1(t))), \lambda \int_0^{\beta_1(t)} f_1(t, \xi, v^*(\xi))d\xi, g_2(t, v^*(\beta_3(t)))\right) + \int_0^{\beta_2(t)} f_2(t, \xi, v^*(\xi))d\xi, \quad t \geq 0.$$

**Proof.** Assume that $\theta^*$ satisfies (2) blending with the function $g_1^*$, and then

$$|\theta(t) - \theta^*(t)| = \left|h(t, \theta(t)) + v(t)g(t, \theta(t)) - h(t, \theta^*(t)) - v^*(t)g(t, \theta^*(t))\right|$$

$$\leq |h(t, \theta(t)) - h(t, \theta^*(t))| + |v(t) - v^*(t)||g(t, \theta(t))| + |v^*(t)||g(t, \theta^*(t)) - g(t, \theta^*(t))|$$

$$\leq l_1|\theta(t) - \theta^*(t)| + |v(t) - v^*(t)||g(t, \theta(t))| + Ml_2|\theta(t) - \theta^*(t)|,$$

and then

$$\|\theta - \theta^*\| \leq l_1\|\theta - \theta^*\| + \|v - v^*\|(\|a_2\| + l_2\rho) + Ml_2\|\theta - \theta^*\|$$

$$\leq \|v - v^*\|(a_2^* + l_2\rho) \frac{1}{1 - (l_1 + Ml_2)}.$$

But,

$$|v(t) - v^*(t)| =$$

$$\left|f\left(t, g_1(t, v(\beta_1(t))), \lambda \int_0^{\beta_1(t)} f_1(t, \xi, v(\xi))d\xi, g_2(t, v^*(\beta_3(t)))\right) + \int_0^{\beta_1(t)} f_2(t, \xi, v(\xi))d\xi\right|$$

$$- f\left(t, g_1^*(t, v^*(\beta_1(t))), \lambda \int_0^{\beta_1(t)} f_1(t, \xi, v^*(\xi))d\xi, g_2(t, v^*(\beta_3(t)))\right) - \int_0^{\beta_1(t)} f_2(t, \xi, v^*(\xi))d\xi\right|$$

$$\leq \left|g_1(t, v(\beta_1(t))) - g_1^*(t, v(\beta_1(t)))\right| + bb_1\left|v(\beta_1(t)) - v^*(\beta_1(t))\right|$$

$$+ \lambda \int_0^{\beta_2(t)} |f_1(t, \xi, v(\xi)) - f_1(t, \xi, v^*(\xi))|d\xi + bb_2\left|v^*(\beta_3(t)) - v^*(\beta_3(t))\right| \cdot \int_0^{\beta_4(t)} |f_2(t, \xi, v^*(\xi))|d\xi$$

$$+ \left|g_2(t, v^*(\beta_3(t)))\right| \cdot \int_0^{\beta_4(t)} |f_2(t, \xi, v^*(\xi)) - f_2(t, \xi, v^*(\xi))|d\xi,$$

and then

$$\|v - v^*\| \leq b\delta + bb_1\|v - v^*\| + bb_1\|v - v^*\| + bb_2\|v - v^*\|(k_2 + c_2r) + b(m_2^* + b_2r)(\xi^*_2\|v - v^*\|)$$

$$\leq \frac{b\delta + bb_1\|v - v^*\| + bb_1\|v - v^*\| + bb_2\|v - v^*\|(k_2 + c_2r) + b(m_2^* + b_2r)(\xi^*_2\|v - v^*\|)}{1 - (bb_1 + bb_2\lambda + bb_2(k_2 + c_2r) + b(m_2^* + b_2r))} = \varepsilon_5.$$

Then,

$$\|\theta - \theta^*\| \leq \frac{\varepsilon_5(a_2^* + l_2\rho)}{1 - (l_1 + Ml_2)} = \varepsilon.$$

By the same manner, we can prove the asymptotic dependency on the function $\gamma_2$. \[\square\]
Example 1. An example of $g_1$ can be

$$g_1(t, u) = \gamma t e^{-\tau} + b_2 u,$$

and then

$$g_1^*(t, u) = \gamma^* t e^{-\tau} + b_2 u;$$

the function $g_1$ satisfies Lipschitz condition

$$|g_1(t, u) - g_1(t, v)| \leq b_2 |u - v|$$

and

$$|g_1(t, u) - g_1^*(t, u)| \leq |\gamma - \gamma^*| t e^{-\tau} \leq |\gamma - \gamma^*| \leq \delta.$$

Theorem 7. Presume that Theorem 4 is established; therefore, the asymptotic dependency of the solution of (2) on $\beta_2$ yields

$$\forall \epsilon > 0, \quad \exists \delta(\epsilon),$$

where $|\beta_2 - \beta_2^*| < \delta, \ t > T(\epsilon)$ and then $\|\theta - \theta^*\| < \epsilon$

where $\theta^*$ satisfies

$$\theta^*(t) = h(t, \theta^*(t)) + v^*(t) g(t, \theta^*(t)), \ t \geq 0$$

and $v^*$ is the solution of

$$v^*(t) = f \left( t, g_1(t, v^*(\beta_1(t))), \lambda \int_0^{\beta_1(t)} f_1(t, \xi, v^*(\xi)) d\xi,  \\
g_2(t, v^*(\beta_3(t))) \cdot \int_0^{\beta_3(t)} f_2(t, \xi, v^*(\xi)) d\xi \right), \ t \geq 0.$$

Proof. Given $\delta > 0$, where $|\beta_2 - \beta_2^*| < \delta$, take $\theta^*$ as satisfying (2) regarding the function $\beta_2^*$, so

$$|\theta(t) - \theta^*(t)| = \left| h(t, \theta(t)) + v(t) g(t, \theta(t)) - h(t, \theta^*(t)) - v^*(t) g(t, \theta^*(t)) \right| \leq |h(t, \theta(t)) - h(t, \theta^*(t))| + |v(t) - v^*(t)| |g(t, \theta(t))| + |v^*(t)| |g(t, \theta^*(t)) - g(t, \theta^*(t))| \leq |l_1| \theta(t) - \theta^*(t)| + |v(t) - v^*(t)| |g(t, \theta(t))| + Ml_2 |\theta(t) - \theta^*(t)|,$$

and then

$$\|\theta - \theta^*\| \leq l_1 \|\theta - \theta^*\| + \|v - v^*\| (a_2^* + l_2\rho) + Ml_2 \|\theta - \theta^*\| \leq \frac{\|v - v^*\| (a_2^* + l_2\rho)}{1 - (l_1 + Ml_2)}.$$

But,

$$|v(t) - v^*(t)| = \left| f \left( t, g_1(t, v^*(\beta_1(t))), \lambda \int_0^{\beta_1(t)} f_1(t, \xi, v^*(\xi)) d\xi, g_2(t, v^*(\beta_3(t))) \cdot \int_0^{\beta_3(t)} f_2(t, \xi, v^*(\xi)) d\xi \right) \right|$$
- \( f(t, g_1(t, \nu^*(\beta_1(t))), \lambda \int_0^\beta_2(t) f_1(t, \zeta, \nu^*(\zeta)) d\zeta, g_2(t, \nu^*(\beta_3(t))) \cdot \int_0^\beta_4(t) f_2(t, \zeta, \nu^*(\zeta)) d\zeta) \)

\[ \leq bb_1 |v(\beta_1(t)) - \nu^*(\beta_1(t))| + b \lambda \int_0^\beta_2(t) |f_1(t, \zeta, \nu^*(\zeta))| d\zeta + bb_2 |v(\beta_3(t)) - \nu^*(\beta_3(t))| \cdot \int_0^\beta_4(t) |f_2(t, \zeta, \nu^*(\zeta))| d\zeta + b(m_2^2 + b_2r) \cdot \int_0^\beta_4(t) |f_2(t, \zeta, \nu^*(\zeta)) - f_2(t, \zeta, \nu^*(\zeta))| d\zeta, \]

and then

\[ \|v - \nu^*\| \leq bb_1 \|v - \nu^*\| + b \lambda (k_1 + c_1r) |\beta_2 - \beta_2^*| + b \lambda \xi_1^2 \|v - \nu^*\| + bb_2 \|v - \nu^*\| (k_2 + c_2r) \leq bb_1 \|v - \nu^*\| + b \lambda (k_1 + c_1r) |\nu^* - \nu^*| + bb_2 \|v - \nu^*\| (k_2 + c_2r) \]

Then,

\[ \|\theta - \theta^*\| \leq \frac{e_6 (a_2^* + b_2 \rho)}{1 - (l_1 + Ml_2)} = \epsilon. \]

Analogously, the asymptotic dependency on the function \( \beta_4 \) can be proved. \( \square \)

**Example 2.** An example of \( \beta_2 \) in the form

\[ \beta_2(t) = t - \gamma t e^{-\gamma} \]
and then

\[ \beta_2^*(t) = t - \gamma^* t e^{-\gamma^*} \]

It is clear that our assumption of \( \beta_2 \) satisfies Lipschitz condition and

\[ |\beta_2(t) - \beta_2^*(t)| \leq |\gamma - \gamma^*| t e^{-\gamma} \leq |\gamma - \gamma^*| \leq \delta. \]

**Theorem 8.** Let Theorem 4 be verified, and then the asymptotic dependency of the solution of (2) on the parameter \( \lambda \) occurred; that is,

\[ \forall \epsilon > 0, \quad \exists \delta(\epsilon), \text{where} \quad |\lambda - \lambda^*| < \delta \Rightarrow \|\theta - \theta^*\| < \epsilon, \quad t > T(\epsilon) \]

and where \( \theta^* \) is the solution of

\[ \theta^*(t) = h(t, \theta^*(t)) + \nu^*(t) \xi(t, \theta^*(t)), \quad t \geq 0 \]

and \( \nu^* \) is the solution of

\[ \nu^*(t) = f \left( t, g_1(t, \nu^*(\beta_1(t))), \lambda^* \int_0^\beta_2(t) f_1(t, \zeta, \nu^*(\zeta)) d\zeta, g_2(t, \nu^*(\beta_3(t))), \cdot \int_0^\beta_4(t) f_2(t, \zeta, \nu^*(\zeta)) d\zeta \right), \quad t \geq 0. \]
**Proof.** Let $\delta > 0$ be given such that $|\lambda - \lambda^*| \leq \delta$ and take $\vartheta^*$ as a solution of (2) regarding parameter $\lambda^*$; thus,
\[
\|\vartheta - \vartheta^*\| \leq \frac{\|\nu - \nu^*\| (a_2^* + lb)}{1 - (l_1 + Ml_2)} - \epsilon,
\]
and, as in Theorem 6, we obtain
\[
\|\nu - \nu^*\| \leq \frac{b\delta(k_1 + c_1r)}{1 - (bb_1 + b\lambda^* \xi_1^* + bb_2(k_2 + c_2r) + b\xi_2^* (m_2^* + b_2r))} = \epsilon \gamma
\]
and
\[
\|\vartheta - \vartheta^*\| \leq \frac{\epsilon \gamma (a_1^* + lb)}{1 - (l_1 + Ml_2)} = \epsilon.
\]
\]

4. Comments and Remarks

The problem (1) contains several key problems that appear in classical analysis. Here, we provide a few special problems.

- If $f(\tau, \xi, u, w) = f(\tau, \xi, u)$ taking $g_2 = 0$, then we have
  \[
  \frac{\vartheta(\tau) - h(\tau, \vartheta(\tau))}{g(\tau, \vartheta(\tau))} = f(\tau, g_1 \left( \vartheta(\beta_1(\tau)) - h(\tau, \vartheta(\beta_1(\tau))) \right)), \quad \tau \geq 0.
  \]

- If $f(\tau, \xi, u, w) = f(\tau, \xi)$ taking $\lambda$, $g_1(\tau, \vartheta) = 0$, then we have
  \[
  \frac{\vartheta(\tau) - h(\tau, \vartheta(\tau))}{g(\tau, \vartheta(\tau))} = f(\tau, g_2 \left( \vartheta(\beta_2(\tau)) - h(\tau, \vartheta(\beta_2(\tau))) \right)), \quad \tau \geq 0.
  \]

- For $\lambda$ and $g_2(\tau, \vartheta) = 0$, then we have
  \[
  \frac{\vartheta(\tau) - h(\tau, \vartheta(\tau))}{g(\tau, \vartheta(\tau))} = f(\tau, g_1 \left( \vartheta(\beta_1(\tau)) - h(\tau, \vartheta(\beta_1(\tau))) \right)), \quad \tau \geq 0.
  \]

- For $g_1(\tau, \vartheta) = g_2(\tau, \vartheta) = 0$, then we have
  \[
  \frac{\vartheta(\tau) - h(\tau, \vartheta(\tau))}{g(\tau, \vartheta(\tau))} = f(\tau, \lambda \int_{0}^{\beta_2(\tau)} f_1 \left( \tau, \vartheta (\xi) - h(\tau, \vartheta(\xi)) \right) d\xi), \quad \tau \geq 0.
  \]

- For $f(\tau, \xi, u, w) = 1$, therefore
  \[
  \vartheta(\tau) = h(\tau, \vartheta(\tau)) + g(\tau, \vartheta(\tau)), \quad \tau \geq 0
  \]

Moreover, let $h = 0$, and then we have
\[
\vartheta(\tau) = g(\tau, \vartheta(\tau)), \quad \tau \geq 0.
\]

**Example 3.** Taking into account the next problem
\[
\begin{align*}
\theta(t) - \left( e^{-\frac{s\sin t}{20}} + \frac{\sqrt{\theta(t)}}{2} \right) &= \cos\frac{\theta(t)}{e^{\frac{t}{3}}} + \frac{1}{3} \left( e^{-2\tau + \frac{2}{3}\frac{\theta(\gamma t)}{\theta(t)}} - \left( e^{-\frac{s\sin t}{20}} + \frac{\sqrt{\theta(t)}}{2} \right) \right), \\
+ \frac{1}{4} \int_0^{\alpha t} e^{-(\tau + s)} \left( 1 + \frac{1}{2} \left( \theta(s) - \left( e^{-\frac{s\sin t}{20}} + \frac{\sqrt{\theta(t)}}{2} \right) \right) \right) ds, \\
&+ \left( e^{-\frac{t}{5}} + \frac{1}{2} \left( \theta(\omega t) - \left( e^{-\frac{s\sin t}{20}} + \frac{\sqrt{\theta(t)}}{2} \right) \right) \right) \cdot \int_0^{\phi t} \sin e^{-(\tau + s)} \left( 1 + \frac{1}{2} \left( \theta(s) - \left( e^{-\frac{s\sin t}{20}} + \frac{\sqrt{\theta(t)}}{2} \right) \right) \right) ds, \quad \tau \geq 0.
\end{align*}
\]

Set
\[
\begin{align*}
h(t, \theta) &= \frac{e^{-\frac{t}{2}} + \theta(t)}{2}, \\
g(t, \theta) &= \frac{\cos\theta(t)}{e^{\frac{t}{3}}} + \frac{1}{3} (u(t), z(t), w(t)),
\end{align*}
\]

where
\[
\begin{align*}
u(t) &= e^{-2\tau + \frac{1}{2} \left( \theta(\gamma t) - \left( e^{-\frac{s\sin t}{20}} + \frac{\sqrt{\theta(t)}}{2} \right) \right)}, \quad \gamma(t) < 1, \\
z(t) &= \frac{1}{4} \int_0^{\alpha t} e^{-(\tau + s)} \left( 1 + \frac{1}{2} \left( \theta(s) - \left( e^{-\frac{s\sin t}{20}} + \frac{\sqrt{\theta(t)}}{2} \right) \right) \right) ds, \quad \alpha(t) < 1, \\
w(t) &= \left( e^{-\frac{t}{5}} + \frac{1}{2} \left( \theta(\omega t) - \left( e^{-\frac{s\sin t}{20}} + \frac{\sqrt{\theta(t)}}{2} \right) \right) \right) \\
&\cdot \int_0^{\phi t} \sin e^{-(\tau + s)} \left( 1 + \frac{1}{2} \left( \theta(s) - \left( e^{-\frac{s\sin t}{20}} + \frac{\sqrt{\theta(t)}}{2} \right) \right) \right) ds, \quad \omega(t), \psi(t) < 1.
\end{align*}
\]

Putting
\[
\begin{align*}
m^* &= \frac{1}{e^t}, \quad m_1^* = \frac{1}{e^t}, \quad m_2^* = \frac{1}{e^t}, \quad b = \frac{1}{e^t} \quad b_1 = b_2 = \frac{1}{e^t}, \quad k_1 = k_2 = \frac{1}{e^t}, \quad \lambda = \frac{1}{e^t}, \quad c_1 = c_2 = \frac{1}{e^t}, \\
a_1^* = \frac{1}{20}, \quad a_2^* = \frac{1}{3}, \quad l_1 = \frac{1}{2}, \quad l_2 = \frac{1}{2}, \quad \text{and} \quad M = \frac{1}{e^t}.
\end{align*}
\]

We can find that
\[
l_1 + Ml_2 = 0.6839397206 < 1 \quad \text{and} \quad bb_1 + bb_2(k_2 + c_2r) = 0.9739931049 < 1.
\]

5. Conclusions

The solvability of various problems in some spaces of bounded continuous functions defined on the half-axis has been discussed by many scholars by applying MNC [24,25], for example, [3,26,27], and for global asymptotic stability (see [24]).

In this investigation, the asymptotic stability and dependency of the solutions for an implicit delays hybrid quadratic functional integral equation have been established on \( R_+ \).

Firstly, we proved the existence of the solutions \( v \in BC(R_+) \) of (3), and then we studied the asymptotic stability of the solutions \( v \in BC(R_+) \) of (3). Next, we investigated the existence and the stability of \( \theta \in BC(R_+) \) on \( R_+ \), by applying MNC in \( BC(R_+) \), and then we studied some asymptotic dependency of \( \theta \in BC(R_+) \) on the parameter \( \lambda \geq 0 \), and on
the functions $g_1$, $g_2$, $\beta_2$ and $\beta_4$. Furthermore, we can discuss other asymptotic dependency results on the other parameters of (3).

Finally, we discussed the exceptional cases, and examples are provided to illustrate our results.

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