Article

Bounds for Toeplitz Determinants and Related Inequalities for a New Subclass of Analytic Functions

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Abstract: In this article, we use the $q$-derivative operator and the principle of subordination to define a new subclass of analytic functions related to the $q$-Ruscheweyh operator. Sufficient conditions, sharp bounds for the initial coefficients, a Fekete–Szegö functional and a Toeplitz determinant are investigated for this new class of functions. Additionally, we also present several established consequences derived from our primary findings.

Keywords: analytic functions; $q$-derivative operator; $q$-Ruscheweyh operator; subordination; Hankel determinant; Toeplitz determinant

MSC: 30C45; 30C50

1. Introduction

Let $A$ represent the set of all complex valued mappings $u$ of the form

$$u(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

which are analytic in the open unit disc $\Omega = \{z \in \mathbb{C} : |z| < 1\}$. A function $u$ is classified as univalent if it never repeats the same value. The subclass of $A$ containing univalent functions is denoted as $S$.

We represent by $P$ the set comprising all analytic functions $h$ of the form

$$h(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

such that $\text{Re}(h(z)) > 0$ in $\Omega$. A function $u$ belonging to the class $A$ is referred to as a starlike function (represented as $u \in S^*$) and a convex function (represented as $u \in C$) if it meets the following inequality conditions:

$$\text{Re}\left(\frac{z u'(z)}{u(z)}\right) > 0, \ (z \in \Omega)$$

and

$$\text{Re}\left(1 + \frac{z u''(z)}{u'(z)}\right) > 0, \ (z \in \Omega).$$
The sets $S^*(\delta)$ and $C(\delta)$ of starlike and convex functions of order $\delta$ (where $0 \leq \delta < 1$), respectively, are defined as follows:

$$S^*(\delta) = \left\{ u \in A : \Re \left( \frac{z u'(z)}{u(z)} \right) > \delta, \ (z \in \mathbb{D}) \right\}$$

and

$$C(\delta) = \left\{ u \in A : \Re \left( 1 + \frac{z u'(z)}{u'(z)} \right) > \delta, \ (z \in \mathbb{D}) \right\}.$$

For $\gamma = 0$, it can be seen that $S^*(0) = S^*$ and $C(0) = C$.

The families $t - UC\mathcal{V}$ and $t - US^*$ of $t$-uniformly convex functions and $t$-uniformly starlike functions are defined by Kanas and Winiowska [1,2] as follows:

$$t - UC\mathcal{V} = \left\{ u \in A : t \left( \frac{|zu'(z)|}{u(z)} - 1 \right) < \Re \left( \frac{|u'(z)|}{u'(z)} \right), \ z \in \mathbb{D}, \ t \geq 0 \right\}$$

and

$$t - US^* = \left\{ u \in A : t \left( \frac{|zu'(z)|}{u(z)} - 1 \right) < \Re \left( \frac{|u'(z)|}{u'(z)} \right), \ z \in \mathbb{D}, \ t \geq 0 \right\}.$$

These two classes are generalizations of the classes of convex univalent functions and uniformly starlike functions, defined by Goodman [3]. In a similar way, Wang et al. [4] studied the subfamilies $S^*(\gamma, \delta)$ and $C(\gamma, \delta)$ of analytic functions defined by the following inequalities, respectively:

$$\left| \frac{zu'(z)}{u(z)} - 1 \right| < \delta \left| \frac{zu'(z)}{u(z)} + 1 \right|, \ z \in \mathbb{D}$$

and

$$\left| \frac{(zu'(z))'}{u'(z)} - 1 \right| < \delta \left| \frac{(zu'(z))'}{u'(z)} + 1 \right|, \ z \in \mathbb{D},$$

where $0 < \gamma \leq 1$, $0 < \delta \leq 1$.

Let $u, v \in A$, define their convolution by

$$(u * v)(z) = z + \sum_{n=2}^{\infty} b_n a_n z^n,$$

where, $u$ is given by (1) and

$$v(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ (z \in \mathbb{D}).$$

The investigation of integral and differential operators has been a valuable area of research since the inception of the theory of analytic functions. The introduction of the first integral operator can be attributed to Alexander [5] in 1915. These operators have been examined from various perspectives, including the incorporation of quantum calculus. The study of $q$-calculus has recently gained attention due to its wide applications in applied sciences. Jackson [6,7] was the pioneer in defining $q$-analogues of derivatives using $q$-calculus. Using $q$-derivatives, Ismail et al. [8] defined and investigated $q$-starlike functions, prompting the further exploration of $q$-calculus within the domain of geometric functions theory (GFT). Subsequently, several $q$-extensions of integral and differential operators have been defined. Kanas and Răducanu [9] defined the $q$-Ruscheweyh differential operator, and Noor et al. in [10] studied the $q$-Bernardi integral operator. Furthermore, as a $q$-analogue of the operator given in [11], Govindaraj and Sivasubramanian [12] defined the $q$-Salagean operator. The studies by authors [13–15] also highlight significant contributions to the $q$-generalizations of certain subclasses of analytic functions. Recently, Srivastava [16]
published a comprehensive review article that serves as a valuable resource for researchers and scholars involved in GFT and \( q \)-calculus.

Now, let us revisit some definitions and details regarding \( q \)-calculus, which will enhance our understanding of this new article.

**2. Preliminaries**

**Definition 1 ([17]).** The \( q \)-number \( [s]_q \) for \( 0 < q < 1 \) is defined as

\[
[s]_q = \frac{1 - q^s}{1 - q}, \quad (s \in \mathbb{C}).
\]

In particular, for \( s = n \in \mathbb{N} = \{1, 2, 3, \ldots\} \),

\[
[n]_q = \sum_{k=0}^{n-1} q^k.
\]

The \( n \)th \( q \)-factorial \( [n]_q! \) is defined as

\[
[n]_q! = \prod_{k=1}^{n} [k]_q, \quad (n \in \mathbb{N}).
\]

For \( n = 0 \), we have \([0]_q! = 1\).

**Definition 2 ([17]).** For any complex number \( s \), the \( q \)-generalized Pochhammer symbol \( [s]_q, n \) is defined as

\[
[s]_q, n = [s]_q [s+1]_q [s+2]_q \cdots [s+n-1]_q, \quad (n \in \mathbb{N}).
\]

For \( n = 0 \), \([s]_q, 0 = 1\).

**Definition 3 ([6]).** For \( 0 < q < 1 \), the \( q \)-difference operator \( \mathcal{D}_q : \mathcal{A} \to \mathcal{A} \) for \( u \in \mathcal{A} \) is defined as follows:

\[
(\mathcal{D}_q u)(z) = \begin{cases} 
\frac{u(z) - u(qz)}{z - qz}, & \text{if } z \neq 0, \\
u'(0), & \text{if } z = 0.
\end{cases}
\]

For \( u \) of the form (1), we have

\[
(\mathcal{D}_q u)(z) = 1 + \sum_{n=2}^{\infty} [n]_q b_n z^{n-1}.
\]

It can be noted that

\[
\lim_{q \to 1^-} [n]_q = n
\]

and

\[
\lim_{q \to 1^-} (\mathcal{D}_q u)(z) = u'(z).
\]

**Definition 4 ([9]).** For \( u \in \mathcal{A} \), the \( q \)-Ruscheweyh differential operator \( L_q^\lambda : \mathcal{A} \to \mathcal{A} \) is defined as

\[
L_q^\lambda (u(z)) = \phi(q, \lambda + 1; z) * u(z)
\]

\[
= z + \sum_{n=2}^{\infty} \psi_n(u^\lambda) b_n z^n,
\]
The operator $L^\lambda$ where $0 < \lambda < 1$,

$$\phi(q, \lambda + 1; z) = z + \sum_{n=2}^{\infty} \psi_{n, \lambda, q} z^n$$

and

$$\psi_{n, \lambda, q} = \frac{\Gamma_q(\lambda + n)}{(n - 1)! \Gamma_q[\lambda + 1]} = \frac{[\lambda + 1]_q [\lambda + 2]_q \cdots [\lambda + n - 1]_q}{[n - 1]_q!}.$$  

Remark 1. For specific values of $\lambda$, $\lambda = \phi_{n, \lambda, q}$ holds:

$$(\lambda)_q = \frac{\Gamma_q(\lambda + n)}{(n - 1)! \Gamma_q[\lambda + 1]} = \frac{[\lambda + 1]_q [\lambda + 2]_q \cdots [\lambda + n - 1]_q}{[n - 1]_q!}.$$

The operator $L^\lambda_q$ satisfies the following identity:

$$z \mathcal{D}_q(L^\lambda_q u(z)) = \left(1 + \frac{[\lambda]_q}{q^\lambda}\right)L^\lambda_q + 1 u(z) - \frac{[\lambda]_q}{q^\lambda} L^\lambda_q u(z).$$

In particular, for $0 < n \leq \infty$, $\{0, 1, 2, 3, \ldots\}$,

$$L^0_q u(z) = u(z),$$

$$L^1_q u(z) = z \mathcal{D}_q u(z),$$

$$L^2_q u(z) = \frac{z}{[n]_q!} \mathcal{D}_q^n \left(z^{n-1} u(z)\right).$$

Note that

$$\mathcal{D}_q(L^\lambda_q u(z)) = 1 + \sum_{n=2}^{\infty} [n]_q \psi_{n, \lambda, q} b_n z^{n-1},$$

and

$$\lim_{q \to 1^-} L^\lambda_q u(z) = L^\lambda(u(z)),$$

which shows that when $q \to 1^-$, the operator defined in Definition 4 reduces to the operator defined by Ruscheweyh [18].

**Definition 5.** A function $u \in A$ is said to belong to the class $S(\lambda, q)$ if the following inequality holds:

$$\left| \frac{z \mathcal{D}_q(L^\lambda_q u(z))}{L^\lambda_q u(z)} - 1 \right| < q \left| \frac{z \mathcal{D}_q(L^\lambda_q u(z))}{L^\lambda_q u(z)} + 1 \right|, \quad z \in \mathbb{D},$$

where $0 < q \leq 1$. Equivalently,

$$\frac{z \mathcal{D}_q(L^\lambda_q u(z))}{L^\lambda_q u(z)} \leq \frac{1 + qz}{1 - q^2z}.$$  

(6)

**Remark 1.** For specific values of $\lambda$ and $q \to 1$, we have the following special cases:

- If $\lambda = 0$ and $q \to 1$, then $S(\lambda, q) = S^*$, which is the familiar subclass of starlike functions.
- If $\lambda = 1$ and $\lambda \to 1$, then $S(\lambda, q) = C$, the family of normalized univalent convex functions.

Noonan and Thomas [19] introduced the following $m^{th}$ Hankel determinant of $u$ given by (1), where $m \geq 1$, $n \geq 1$ and $b_1 = 1$.

$$H^m(n) = \begin{vmatrix} b_n & b_{n+1} & \cdots & b_{n+m-1} \\ b_{n+1} & b_{n+2} & \cdots & b_{n+m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n+m-1} & b_{n+m} & \cdots & b_{n+2m-2} \end{vmatrix}.$$  

(7)
In particular, for \( m = 2 \) and \( n = 1 \), the Hankel determinant \( H_m(n) \) reduces to the famous Fekete–Szegö functional:

\[
H_2(1) = \begin{vmatrix}
1 & b_2 \\
b_2 & b_3
\end{vmatrix} = |b_3 - b_2^2|.
\]

This functional is further generalized as

\[
|b_3 - \mu b_2^2|
\]

where \( \mu \) is any complex or real number.

The significance of the Hankel determinant is evident in the field of singularity theory [20], and it proves beneficial in the examination of power series with integer coefficients (see [21–23]). Several researchers have established upper bounds for \( H_m(n) \) across various values of \( m \) and \( n \), for numerous subclasses of analytic functions (see, for example, [24–28]).

For a function \( u \) represented by Equation (1), the symmetric Toeplitz determinant \( T_m(n) \) is defined as

\[
T_i(j) = \begin{vmatrix}
\vdots & \vdots & \cdots & \vdots \\
\vdots & \vdots & \cdots & \vdots \\
\vdots & \vdots & \cdots & \vdots \\
1 & b_2 & b_3 \\
b_2 & b_3 & b_4 \\
b_3 & b_4 & b_5 \\
b_4 & b_5 & b_6 \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\end{vmatrix}, \quad (8)
\]

where \( m \geq 1, n \geq 1 \) and \( b_1 = 1 \). In particular,

\[
T_2(1) = \begin{vmatrix}
1 & b_2 \\
b_2 & 1
\end{vmatrix}, \quad T_2(2) = \begin{vmatrix}
b_2 & b_3 \\
b_3 & b_4
\end{vmatrix}, \quad T_2(3) = \begin{vmatrix}
b_3 & b_4 \\
b_4 & b_5
\end{vmatrix},
\]

\[
T_3(1) = \begin{vmatrix}
1 & b_2 & b_3 \\
b_2 & 1 & b_2 \\
b_3 & b_2 & 1
\end{vmatrix}, \quad T_3(2) = \begin{vmatrix}
b_2 & b_3 & b_4 \\
b_3 & b_4 & b_5 \\
b_4 & b_5 & b_6
\end{vmatrix}.
\]

In recent times, a number of researchers have focused on exploring the bounds of the Toeplitz determinant \( T_m(n) \) for various families of analytic functions (see, for example, [29–32]). In the investigation of Toeplitz determinants, the research conducted in [33,34] incorporates elements of quantum calculus, while [35] explores a set of analytic functions introduced through the utilization of the Borel distribution.

Our main results rely on the utilization of the following Lemmas for their proof.

**Lemma 1 ([36]).** If \( h(z) = 1 + p_1 z + p_2 z^2 + \cdots \) is an analytic function with a positive real part in \( \mathbb{D} \) and \( \mu \) is a complex number, then

\[
|p_2 - \mu p_1|^2 \leq 2 \max\{1, |2\mu - 1|\}.
\]

The inequality is sharp for the functions given by

\[
h_1(z) = \frac{1 + z}{1 - z}, \quad h_2(z) = \frac{1 + z^2}{1 - z^2}, \quad z \in \mathbb{D}.
\]

**Lemma 2 ([37]).** If \( h(z) = 1 + p_1 z + p_2 z^2 + \cdots \) is an analytic function with a positive real part in \( \mathbb{D} \), then

\[
|p_n| \leq 2.
\]

The inequality is sharp for the function given by

\[
h(z) = \frac{1 + z}{1 - z}, \quad z \in \mathbb{D}.
\]
Lemma 3 ([38]). If \( h(z) = 1 + p_1z + p_2z^2 + \cdots \) is an analytic function with a positive real part in \( \Omega \), and \( \mu \) is a complex number, then
\[
|p_n - \mu p_{n-k}| \leq 2 \max(1, |2\mu - 1|),
\]
for all \( n, k \in \mathbb{N} \) and \( k < n \). The inequality is sharp for the function given by
\[
h(z) = \frac{1+z}{1-z}, \quad z \in \Omega.
\]

3. Main Results

The following result gives the sufficient conditions for the functions belonging to the class \( S(\lambda, q) \).

Theorem 1. Let \( u \in A \) be given by Equation (1), satisfying
\[
\sum_{n=2}^{\infty} \psi_{n,\lambda,q} \left\{ (1 + q^2) |n|_q - (1 - q) \right\} |b_n| \leq q(1 + q).
\]
Then, \( u \in S(\lambda, q) \).

Proof. Let
\[
F(z) = \frac{zD_q(L_q^u(z))}{L_q^u(z)}.
\]
Then,
\[
\left| \frac{F(z) - 1}{q + q^2 F(z)} \right| = \frac{\left| \frac{zD_q(L_q^u(z))}{L_q^u(z)} - 1 \right|}{q + q^2 \frac{zD_q(L_q^u(z))}{L_q^u(z)}}
= \frac{zD_q(L_q^u(z)) - L_q^u(z)}{qL_q^u(z) + q^2zD_q(L_q^u(z))}
= \frac{(z + \sum_{n=2}^{\infty} |n|_q \psi_{n,\lambda,q} b_n z^n) - (z + \sum_{n=2}^{\infty} \psi_{n,\lambda,q} b_n z^n)}{q(z + \sum_{n=2}^{\infty} \psi_{n,\lambda,q} b_n z^n) + q^2(z + \sum_{n=2}^{\infty} \psi_{n,\lambda,q} |n|_q b_n z^n)}
\leq \frac{\sum_{n=2}^{\infty} |n|_q |b_n| b_n z^n - 1}{\sum_{n=2}^{\infty} (q + q^2 |n|_q) \psi_{n,\lambda,q} |b_n| z^n - 1}
\leq \frac{q + q^2 - \sum_{n=2}^{\infty} (q + q^2 |n|_q) \psi_{n,\lambda,q} |b_n| z^n - 1}{(q + q^2) - \sum_{n=2}^{\infty} (q + q^2 |n|_q) \psi_{n,\lambda,q} |b_n| z^n - 1}
\leq \frac{1}{1 - q^2},
\]
which shows
\[
\frac{F(z) - 1}{q + q^2 F(z)} = w(z)
\]
is a Schwarz function and we can write
\[
F(z) = \frac{1 + qw(z)}{1 - q^2w(z)}
\implies F(z) \prec \frac{1 + qz}{1 - q^2z},
\]
Hence, the result follows. \( \Box \)
Theorem 2. Let $u \in S(\lambda, q)$ be given by Equation (1), then

$$|b_2| \leq \frac{1 + q}{\psi_{2,\lambda,q}},$$  \hspace{1cm} (10) $$|b_3| \leq \frac{1 + q + q^2}{\psi_{3,\lambda,q}},$$  \hspace{1cm} (11) $$|b_4| \leq \frac{(1 + q)(1 + q^2)}{\psi_{4,\lambda,q}}.$$  \hspace{1cm} (12)

All the inequalities are sharp for the function $g(z)$, given by the equation

$$z D_q(L_q^4 f(z)) = \frac{1 + qz}{1 - q^2 z}.$$  \hspace{1cm} (13)

Proof. If $u \in S(\lambda, q)$, then

$$\frac{z D_q(L_q^4 u(z))}{L_q^4 u(z)} = \frac{1 + qw(z)}{1 - q^2 w(z)}, \quad z \in \mathbb{C}.$$  \hspace{1cm} (14)

Let

$$\frac{z D_q(L_q^4 u(z))}{L_q^4 u(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n.$$  \hspace{1cm} (15)

Then, using Equations (4) and (5), we have

$$z + \sum_{n=2}^{\infty} \left[ n \right]_q \psi_{n,\lambda,q} b_n z^n = z + \sum_{n=2}^{\infty} \psi_{n,\lambda,q} b_n z^n + \sum_{n=1}^{\infty} c_n z^{n+1} + \left( \sum_{n=1}^{\infty} c_n z^n \right) \left( \sum_{n=2}^{\infty} \psi_{n,\lambda,q} b_n z^n \right).$$

Using Cauchy product formula [20] and simplifying it, we obtain

$$\sum_{n=2}^{\infty} \left( \sum_{k=1}^{n} c_{n-k+1} \psi_{k-1,\lambda,q} b_{k-1} \right) z^n = \sum_{n=3}^{\infty} \left( \sum_{k=3}^{n} c_{n-k+1} \psi_{k-1,\lambda,q} b_{k-1} \right) z^n.$$  \hspace{1cm} (16)

Comparing coefficients of $z^2, z^3, z^4$ and simplifying, we have

$$c_1 = (2)_q - 1 \psi_{2,\lambda,q} b_2,$$  \hspace{1cm} (17) $$c_2 = (3)_q - 1 \psi_{3,\lambda,q} b_3 - (2)_q - 1 \psi_{2,\lambda,q} b_2^2,$$  \hspace{1cm} (18) $$c_3 = (4)_q - 1 \psi_{4,\lambda,q} b_4 - \psi_{3,\lambda,q} \psi_{2,\lambda,q} b_3 b_2 (3)_q + (2)_q - 2) + (2)_q - 1 \psi_{2,\lambda,q} b_2^3.$$  \hspace{1cm} (19)

Using Equations (17)–(19) in Equation (15), we obtain

$$\frac{z D_q(L_q^4 u(z))}{L_q^4 u(z)} = 1 + ([2]_q - 1) \psi_{2,\lambda,q} b_2 z + \left( ([3]_q - 1) \psi_{3,\lambda,q} b_3 - (2)_q - 1 \psi_{2,\lambda,q} b_2^2 \right) z^2 + \left( ([4]_q - 1) \psi_{4,\lambda,q} b_4 - \psi_{3,\lambda,q} \psi_{2,\lambda,q} b_3 b_2 (3)_q + (2)_q - 2) + (2)_q - 1 \psi_{2,\lambda,q} b_2^3 \right) z^3 + \cdots$$  \hspace{1cm} (20)

Taking $w(z) = \omega_1 z + \omega_2 z^2 + \omega_3 z^3 + \cdots$, we have

$$\frac{1 + q w(z)}{1 - q^2 w(z)} = 1 + (q + q^2) \omega_1 z + (q + q^2)(\omega_2 + q^2 \omega_1^2) z^2 + (q + q^2)(\omega_3 + 2q^2 \omega_2 \omega_1 + q^4 \omega_1^3) z^3 + \cdots.$$  \hspace{1cm} (21)
We also know that for each Schwartz function \( w(z) = \omega_1 z + \omega_2 z^2 + \omega_3 z^3 + \cdots \), there is a function \( b(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \) with a positive real part such that
\[
\frac{1 + w(z)}{1 - w(z)} = b(z),
\]
which implies
\[
\begin{align*}
\omega_1 &= \frac{p_1}{2}, \\
\omega_2 &= \frac{1}{2} \left( p_2 - \frac{1}{2} p_1^2 \right), \\
\omega_3 &= \frac{1}{2} p_3 - \frac{1}{2} p_1 p_2 + \frac{1}{3} p_1^3.
\end{align*}
\]
Using values of \( \omega_1, \omega_2 \) and \( \omega_3 \) in Equation (21), we obtain
\[
\frac{1 + q w(z)}{1 - q^2 w(z)} = 1 + \frac{q + q^2}{2} p_1 z + \frac{q + q^2}{2} \left( p_2 - \frac{1}{2} p_1^2 \right) z^2 \\
+ \frac{q + q^2}{2} \left( p_3 - (1 - q^2) p_1 p_2 + \frac{(1 - q^2)^2}{4} p_1^3 \right) z^3 + \cdots. \tag{22}
\]
In view of (14), comparing the coefficients of \( z \) from (20) and (22), we have
\[
([2]_q - 1) \psi_{2,\lambda,q} b_2 = \frac{q + q^2}{2} - p_1,
\]
which implies
\[
b_2 = \frac{1 + q}{2\psi_{2,\lambda,q}} p_1. \tag{23}
\]
Taking the modulus and applying Lemma 2, we obtain the desired inequality (10).
Comparing the coefficients of \( z^3 \) from (20) and (22), we have
\[
([3]_q - 1) \psi_{3,\lambda,q} b_3 - ([2]_q - 1) \psi_{2,\lambda,q}^2 b_2^2 = \frac{q + q^2}{2} \left( p_2 - \frac{1 - q^2}{2} - p_1^2 \right).
\]
Using Equation (23) and simplifying it, we obtain
\[
b_3 = \frac{1}{2\psi_{3,\lambda,q}} \left( p_2 + \frac{q + q^2}{2} p_1^2 \right). \tag{24}
\]
Taking the modulus and applying Lemma 1, we obtain the desired inequality (11).
Comparing the coefficients of \( z^3 \) from Equations (20) and (22), we have
\[
([4]_q - 1) \psi_{4,\lambda,q} b_4 - \psi_{3,\lambda,q} \psi_{2,\lambda,q} b_3 b_2 ([3]_q + [2]_q - 2) + ([2]_q - 1) \psi_{2,\lambda,q}^3 b_2^3 \\
= \frac{q + q^2}{2} \left( p_3 - (1 - q^2) p_1 p_2 + \frac{(1 - q^2)^2}{4} p_1^3 \right).
\]
Using Equations (23) and (24) and simplifying the above equation, we obtain
\[
b_4 = \frac{1 + q}{2(1 + q + q^2)\psi_{4,\lambda,q}} \left( p_3 + \frac{q(1 + 2q)}{4} p_1 p_2 + \frac{q^2(1 + q)}{4} p_1^3 \right). \tag{25}
\]
The last equation can be rearranged as
\[
b_4 = \frac{1 + q}{2(1 + q + q^2)\psi_{4,\lambda,q}} \left\{ \left( p_3 + \frac{q(1 + 2q)}{4} p_1 p_2 \right) + \frac{q(1 + 2q)}{4} p_1 \left( p_2 + \frac{q^2(1 + q)}{4(1 + 2q) p_1^2} \right) \right\}.
Taking the absolute value and applying Lemmas 1–3, we obtain
\[ |b_4| \leq \frac{1 + q}{2(1 + q + q^2)} \left\{ 2 \left( \frac{q(1 + 2q)}{2} + 1 \right) + q(1 + 2q) \left( \frac{2q^2(1 + q)}{1 + 2q} + 1 \right) \right\}, \]
which yields the inequality (12).

The series expansion of \( f(z) \) given by (13) is
\[ f(z) = z + \frac{1 + q}{\psi_{2,\lambda,q}} z^2 + \frac{1 + q + q^2}{\psi_{3,\lambda,q}} z^3 + \frac{(1 + q)(1 + q^2)}{\psi_{4,\lambda,q}} z^4 + \cdots, \] (26)
which clearly demonstrates the sharpness of the inequalities (10)–(12).

Theorem 3. Let \( u \in S(\lambda, q) \) be given by Equation (1); then, for any \( \mu \in \mathbb{C} \),
\[ |b_3 - \mu b_2^2| \leq \frac{1}{\psi_{3,\lambda,q}} \max \left\{ 1, \left| \frac{\mu(1 + q)^2 \psi_{3,\lambda,q}^2}{\psi_{2,\lambda,q}^2} - q(1 + q) - 1 \right| \right\}. \] (27)
The sharp bound occurs for the function \( f(z) \) described by Equation (13) under the condition that
\[ \left| \frac{\mu(1 + q)^2 \psi_{3,\lambda,q}^2}{\psi_{2,\lambda,q}^2} - q(1 + q) - 1 \right| \geq 1. \] Alternatively, if this condition is not met, the sharp bound pertains to the function \( g(z) \) defined by the equation
\[ \frac{z D_q (I_q g(z))}{I_q g(z)} = 1 + q z^2 \] (28)

Proof. Using Equations (23) and (24), we can write
\[ b_3 - \mu b_2^2 = \frac{1}{2 \psi_{3,\lambda,q}} \left( p_2 + \frac{q + q^2}{2} p_1^2 - \mu \left( \frac{1 + q}{2 \psi_{2,\lambda,q}} \right)^2 \right) \]
\[ = \frac{1}{2 \psi_{3,\lambda,q}} \left( p_2 - \left( \frac{\mu(1 + q)^2 \psi_{3,\lambda,q}^2}{2 \psi_{2,\lambda,q}^2} - q(1 + q) - 1 \right) p_1^2 \right). \]
Applying modulus and using Lemma 1, we obtain the required inequality.

Corollary 1. Let \( u \in S(\lambda, q) \) be given by Equation (1); then, for any \( v \in \mathbb{R} \), we have
\[ |b_3 - v b_2^2| \leq \begin{cases} \frac{1}{\psi_{3,\lambda,q}} \left( 1 + q(1 + q) - \frac{v(1 + q)^2 \psi_{3,\lambda,q}}{\psi_{2,\lambda,q}^2} \right), & \text{if } v \leq \frac{q \psi_{2,\lambda,q}}{(1 + q) \psi_{3,\lambda,q}}, \\
\frac{1}{\psi_{3,\lambda,q}}, & \text{if } \frac{q \psi_{2,\lambda,q}}{(1 + q) \psi_{3,\lambda,q}} \leq v \leq \frac{(2 + q + q^2) \psi_{2,\lambda,q}^2}{(1 + q)^2 \psi_{3,\lambda,q}}, \\
\frac{1}{\psi_{3,\lambda,q}} \left( \frac{v(1 + q)^2 \psi_{3,\lambda,q}}{\psi_{2,\lambda,q}^2} - q(1 + q) - 1 \right), & \text{if } v \geq \frac{(2 + q + q^2) \psi_{2,\lambda,q}^2}{(1 + q)^2 \psi_{3,\lambda,q}}. \end{cases} \] (29)
Proof.  
\[
\left| \frac{\nu (q+1)^2 \psi_{2,\lambda,q}^2 - q(q+1) - 1}{\psi_{2,\lambda,q}^2} \right| \leq 1, \\
\iff -1 \leq \frac{\nu (q+1)^2 \psi_{3,\lambda,q}^2 - q(q+1) - 1}{\psi_{2,\lambda,q}^2} \leq 1, \\
\iff \frac{q \psi_{2,\lambda,q}^2}{(q+1) \psi_{3,\lambda,q}} \leq \nu \leq \frac{(2 + q + q^2) \psi_{2,\lambda,q}^2}{(q+1)^2 \psi_{3,\lambda,q}},
\]
therefore, the result follows. \( \Box \)

Example 1. For \( \lambda = 1 \), the function  
\[
f(z) = \frac{z}{1 - z} = z + z^2 + z^3 + z^4 + \ldots
\]
is such that  
\[
\begin{align*}
\frac{z \partial_q (L_1 f(z))}{L_1 f(z)} &= \frac{z \partial_q (z \partial_q f(z))}{z \partial_q f(z)} = 1 + qz, \\
\frac{1 - q^2}{1 - q^2},
\end{align*}
\]
which shows that \( f \in S(1,q) \), for all \( 0 < q < 1 \). Since for \( \lambda = 1 \), \( \psi_{n,\lambda,q} = \psi_{n,1,q} = [n]_q \), (29) can be written as  
\[
|b_3 - \nu b_2^2| \leq \begin{cases} 
1 - v, & \text{if } v \leq 1 - \frac{1}{[3]_q}, \\
\frac{1}{[3]_q}, & \text{if } 1 - \frac{1}{[3]_q} \leq v \leq 1 + \frac{1}{[3]_q}, \\
v - 1, & \text{if } v \geq 1 + \frac{1}{[3]_q}.
\end{cases}
\]
For \( f(z) \) represented by Equation (30),  
\[
\begin{align*}
\nu &= 1 - \frac{1}{[4]_q}, \quad |b_3 - \nu b_2^2| = \left| 1 - \left(1 - \frac{1}{[4]_q}\right) (1)^2 \right| = 1 - v, \\
\nu &= 1, \quad |b_3 - \nu b_2^2| = \left| 1 - (1)(1)^2 \right| = 0 < \frac{1}{[3]_q}, \\
\nu &= 1 + \frac{1}{[4]_q}, \quad |b_3 - \nu b_2^2| = \left| 1 - \left(1 + \frac{1}{[4]_q}\right) (1)^2 \right| = v - 1.
\end{align*}
\]
This confirms the validity of Corollary 1, for the function \( f(z) \) represented by Equation (30), considering certain particular values of \( v \).

4. Toeplitz Determinant

In this section, we will find sharp bounds for the Toeplitz determinants \( T_2(1), T_2(2), T_2(3), T_3(1) \) and \( T_3(2) \) for the functions belonging to the class \( S(\lambda, q) \).
Theorem 4. If \( u \in S(\lambda, q) \), then
\[
|T_2(1)| \leq 1 + \frac{(1+q)^2}{\psi^2_{2, \lambda, q}},
\]
\[
|T_2(2)| \leq \frac{(1+q)^2}{\psi^2_{2, \lambda, q}} + \frac{(1+q+q^2)^2}{\psi^2_{3, \lambda, q}},
\]
\[
|T_2(3)| \leq \frac{(1+q+q^2)^2}{\psi^2_{3, \lambda, q}} + \frac{(1+q)(1+q^2)^2}{\psi^2_{4, \lambda, q}}.
\]
All the inequalities are sharp for the function
\[
g(z) = z + i(1+q)z^2 - \frac{1+q+q^2}{\psi_{3, \lambda, q}}z^3 - \frac{i(1+q)(1+q^2)^2}{\psi_{4, \lambda, q}}z^4 + \cdots,
\] (32)
which is the solution of
\[
\frac{zD_q(L^1_q g(z))}{L^1_q g(z)} = \frac{1+iqz}{1-iq^2z}.
\] (33)

Proof. Let \( u \in S(\lambda, q) \) be given by Equation (1); then,
\[
T_2(n) = \begin{vmatrix} b_n & b_{n+1} \\ b_{n+1} & b_n \end{vmatrix} = b_n^2 - b_{n+1}^2.
\]
Taking absolute and applying triangle inequality, we have
\[
|T_2(n)| = |b_n^2 - b_{n+1}^2| \leq |b_n^2| + |b_{n+1}^2|.
\]
Taking \( n = 1, n = 2, \) and \( n = 3 \) and using inequalities (10)–(12), we obtain the required inequalities. □

Corollary 2 ([39]). If \( u \in S^* \), then
\[
|T_2(1)| \leq 5,
|T_2(2)| \leq 13,
|T_2(3)| \leq 25.
\]
The inequalities are sharp for the function
\[
g(z) = \frac{z}{(1-iz)^2} = z + i2z^2 - 3z^3 - 4iz^4 + \cdots.
\] (34)

Proof. Using \( \lambda = 0 \) and taking limit \( q \to 1^- \), we obtain the desired inequalities. □

Corollary 3 ([39]). If \( u \in C \), then
\[
|T_2(n)| \leq 2, \quad \text{for all} \quad n \in \mathbb{N}.
\]
The inequality is sharp for the function
\[
g(z) = \frac{z}{1-iz} = z + iz^2 - z^3 - iz^4 + \cdots.
\] (35)
Proof. Using \( \lambda = 1 \) and taking limit \( q \to 1^- \), we obtain the desired inequality. \( \square \)

Example 2. The function
\[
v(z) = \frac{4z}{(2-z)^2} = z + z^2 + \frac{3}{4}z^3 + \frac{1}{2}z^4 + \cdots
\]
is a starlike function because
\[
\text{Re}\left( \frac{z v'(z)}{v(z)} \right) = \text{Re}\left( \frac{2 + z}{2 - z} \right) = \frac{4 - |z|^2}{4 + |z|^2} > 0.
\]
For the function \( v(z) \) given by Equation (36), we have
\[
|T_2(1)| = 0 < 5, \quad |T_2(2)| = \frac{7}{16} < 13, \quad |T_2(3)| = \frac{5}{16} < 25,
\]
which validates Corollary 2 for the function \( v \in \mathcal{S}^* \).

Theorem 5. If \( u \in \mathcal{S}(\lambda, q) \), then
\[
|T_3(1)| \leq 1 + \frac{2(1 + q)^2}{\psi_{2,\lambda,q}^2} + \frac{(1 + q + q^2)}{\psi_{3,\lambda,q}^2} \left| \frac{2(1 + q)^2}{\psi_{2,\lambda,q}^2} - q(1 + q) - 1 \right|.
\]
The inequality is sharp for the function \( g(z) \) defined in Equation (32).

Proof. Let \( u \in \mathcal{S}(\lambda, q) \) be given by Equation (1), then
\[
T_3(1) = \begin{vmatrix} 1 & b_2 & b_3 \\ b_2 & 1 & b_2 \\ b_3 & b_2 & 1 \end{vmatrix} = 1 - 2b_2^2 + 2b_2b_3 - b_3^2.
\]
Taking the absolute and applying triangle inequality, we can write
\[
|T_3(1)| = |1 - 2b_2^2 + 2b_2b_3 - b_3^2| \leq 1 + 2|b_2|^2 + |b_3||b_3 - 2b_2^2|.
\]
Using inequalities (10) and (11) and applying Theorem 3, we obtain the required result. \( \square \)

Corollary 4 ([39]). If \( u \in \mathcal{S}^* \), then
\[
|T_3(1)| \leq 24.
\]
Proof. Using \( \lambda = 0 \) and taking limit \( q \to 1^- \), we obtain the desired inequalities. \( \square \)

Corollary 5 ([39]). If \( u \in \mathcal{C} \), then
\[
|T_3(1)| \leq 4.
\]
The inequality is sharp for the function \( g(z) \) defined by Equation (35).

Proof. Using \( \lambda = 1 \) and taking limit \( q \to 1^- \), we obtain the desired inequalities. \( \square \)
Theorem 6. If \( u \in S(\lambda, q) \), then
\[
|T_3(2)| \leq 4 \left( \frac{1 + q}{\psi_{2,\lambda,q}} + \frac{(1 + q)(1 + q^2)}{\psi_{4,\lambda,q}} \right) \left( a_1 + 4|a_2| + 2a_3 + a_4|1 - 2a_5| \right),
\]
where
\[
\begin{align*}
a_1 & = \frac{(1 + q)^2}{4\psi_{2,\lambda,q}}, \\
a_2 & = \frac{q^3(1 + q)^3}{16(1 + q + q^2)\psi_{2,\lambda,q}\psi_{4,\lambda,q}} - \frac{q^2(1 + q)^2}{8\psi_{4,\lambda,q}^2}, \\
a_3 & = \frac{1}{2\psi_{3,\lambda,q}^2}, \\
a_4 & = \frac{(1 + q)^2}{4(1 + q + q^2)\psi_{2,\lambda,q}\psi_{4,\lambda,q}}, \\
a_5 & = \frac{2q(1 + q + q^2)\psi_{2,\lambda,q}\psi_{4,\lambda,q}}{(1 + q)\psi_{4,\lambda,q}^2} - \frac{q(1 + 2q)}{2}.
\end{align*}
\]
The inequality is sharp for the function \( g(z) \) defined in Equation (32).

Proof. Let \( u \in S(\lambda, q) \) be given by Equation (1), then
\[
T_3(1) = \begin{vmatrix} b_2 & b_3 & b_4 \\ b_3 & b_2 & b_3 \\ b_4 & b_5 & b_2 \end{vmatrix} = (b_2 - b_4)(b_2^2 - 2b_3^2 + b_2b_4).
\]
Taking the absolute value, we can write
\[
|T_3(2)| = |b_2 - b_4||b_2^2 - 2b_3^2 + b_2b_4|.
\] Using inequalities (10) and (12), we obtain
\[
|b_2 - b_4| \leq |b_2| + |b_4| \leq \frac{1 + q}{\psi_{2,\lambda,q}} + \frac{(1 + q)(1 + q^2)}{\psi_{4,\lambda,q}}.
\]
Using Equations (23)–(25), we can write
\[
\begin{align*}
b_2^2 - 2b_3^2 + b_2b_4 & = \left( \frac{1 + q}{2\psi_{2,\lambda,q}} p_1 \right)^2 - 2 \left( \frac{1}{2\psi_{3,\lambda,q}} \left( p_2 + \frac{q + q^2}{2} \right) p_1^2 \right) \\
& \quad + \left( \frac{1 + q}{2\psi_{2,\lambda,q}} p_1 \right) \left( \frac{1 + q}{2(1 + q + q^2)\psi_{4,\lambda,q}} \left( p_3 + \frac{q(1 + 2q)}{2} p_1 p_2 + \frac{q^3(1 + q)}{4} p_3^3 \right) \right) \\
& = \frac{(1 + q)^2}{4\psi_{2,\lambda,q}^2} p_1^2 + \left( \frac{q^3(1 + q)^3}{16(1 + q + q^2)\psi_{2,\lambda,q}\psi_{4,\lambda,q}} - \frac{q^2(1 + q)^2}{8\psi_{4,\lambda,q}^2} \right) p_1^3 - \frac{1}{2\psi_{3,\lambda,q}^2} p_2^2 \\
& \quad + \left( \frac{(1 + q)^2}{4(1 + q + q^2)\psi_{2,\lambda,q}\psi_{4,\lambda,q}} \right) p_1 \left( p_3 - \frac{2q(1 + q + q^2)\psi_{2,\lambda,q}\psi_{4,\lambda,q}}{(1 + q)\psi_{4,\lambda,q}^2} - \frac{q(1 + 2q)}{2} \right) p_2 p_1 \\
& = a_1 p_1^2 + a_2 p_1^4 - a_3 p_2^2 + a_4 p_1(p_3 - a_5 p_2).
\end{align*}
\]
Taking the absolute value, applying the triangle inequality and using Lemmas 2 and 3, we have

\[ |b_2^2 - 2b_2^2 + b_2b_4| \leq 4(\alpha_1 + 4|\alpha_2| + \alpha_3 + \alpha_4|1 - 2\alpha_5|). \] (39)

Using inequalities (38) and (39) in (37), we obtain the required result. \( \square \)

**Corollary 6 ([39]).** If \( u \in S^* \), then

\[ |T_3(2)| \leq 84. \]

**Proof.** Using \( \lambda = 0 \) and taking limit \( q \to 1^- \), we obtain the desired inequalities. \( \square \)

**Corollary 7 ([39]).** If \( u \in C \), then

\[ |T_3(2)| \leq 4. \]

The inequality is sharp for the function \( g(z) \) defined by Equation (35).

**Proof.** Using \( \lambda = 1 \) and taking limit \( q \to 1^- \), we obtain the required inequality. \( \square \)

5. Conclusions

In this article, we have defined a new subclass of analytic functions associated with the \( q \)-Ruscheweyh operator. After finding sufficient conditions for the analytic functions belonging to this class, we establish sharp bounds for the initial coefficients, second- and third-order Toeplitz determinants and the Fekete–Szegö functional for the functions belonging to the newly defined class. We also demonstrate several established corollaries of our primary findings to highlight the interrelation between existing and novel research.

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