Article

Delivery Times Scheduling with Deterioration Effects in Due Window Assignment Environments

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Abstract: In practical problems, in addition to the processing time of the job, the impact of the time required for delivering the service to customers on the cost is also considered, i.e., delivery time, where the job processing time is a simple linear function of its starting time. This paper considers the impact of past-sequence-dependent delivery times (which can be referred to as \textit{psddt}) on the studied objectives in three types of due windows (common, slack and different due windows). This serves to minimize the weighted sum of earliness, tardiness, starting time and size of due window, where the weights (coefficients) are related to the location. Through the theoretical analysis of the optimal solution, it is found that these three problems can be solved in time \(O(N\log N)\), respectively, where \(N\) is the number of jobs.

Keywords: scheduling; single-machine; due window; delivery time; deterioration effect

MSC: 90B35

1. Introduction

In the actual production scheduling process, it is often overlooked that the processing time of a job increases with the wear and tear of the machine. The scheduling in which the processing time of a job is an increasing function of its starting time, which is called deterioration effect (time-dependent) scheduling, represented by \textit{DES} (Yin et al. [1], Sun and Geng [2], Gawiejnowicz [3], Miao et al. [4], Sun et al. [5]). Zhao and Hsu [6] studied the problem of minimizing the number of weighted tardy jobs in a single machine using a general linear deterioration model. They proposed a fully polynomial-time approximation scheme to solve the problem. Li et al. [7] discussed online batch scheduling with simple linear deterioration. For the makespan minimization, they proposed the best online algorithms for incompatible families. Chen and Yuan [8] also proposed two unified methods for general linear deterioration models in environments with deadlines. Liang et al. [9] considered single-machine scheduling with the linear combination of the convex resource allocation and deterioration effect. For the total completion time minimization, they proposed a heuristic and a branch-and-bound for solving the group scheduling problem. Cheng et al. [10] explored single-machine scheduling with step-deteriorating jobs. For the total completion time minimization, they developed a polynomial-time algorithm for this NP-hard problem. And, Pei et al. [11] provided a review of the application of deterioration effects in practical production in recent years.

In addition, the due window assignments will be established based on customer requirements (Janiak et al. [12]). There are generally three kinds of due windows, one of which is a common due window (denoted as \textit{CONDW}), that is, the \textit{CONDW} is shared by all jobs and can be recorded as \([d_1, d_2]\), where \(d_1\) (resp. \(d_2\)) represents the starting...
Wang [18] researched the two-machine flowshop scheduling with the weight not related to the job but to the position, as can be seen in Sun et al. [5] and Liu [6] within the interval (i.e., \( CONDW \)) and it incurs an earliness cost; if completed after \( d_k^2 \), there will be a tardiness cost; if a job is completed within the interval \( [d_k^1, d_k^2] \), no additional costs will be incurred. Among them, Huang et al. [13] and Xu et al. [14] considered single-machine scheduling with the \( CONDW \). Shabtay et al. [15] proposed a pseudo-polynomial-time algorithm for the NP-hard problem of a common due window assignment with a bounded location. Shabtay et al. [15] proposed a pseudo-polynomial-time algorithm for the common due window assignment problem with a bounded location. Zhao [16] investigated the two-machine flowshop scheduling with both the \( SLKDW \) and resource allocation. For the three versions of scheduling cost and resource consumption cost, they proved that the problem can be solved in polynomial time. Jia et al. [17] discussed the combination of the \( SLKDW \) and deterioration effects. Lv and Wang [18] researched the two-machine flowshop scheduling with the \( DIFDW \) and resource allocation. Lin [19] worked on single-machine scheduling with the \( CONDW, SLKDW \) and \( DIFDW \). Under position-dependent weights, learning and deterioration effects, they demonstrated that some problems are polynomially solvable. Teng et al. [20] investigated single-machine scheduling with deterioration effects under the \( SLKDW \) and \( DIFDW \).

Besides the due window, there is another important factor to consider in scheduling problems, namely the past-sequence-dependent delivery time (denoted as \( psddt \)), as can be seen in Koulamas and Kyparisis [21]). Toksari et al. [22] and Ren et al. [23] examined the scheduling problems with an exponential \( psddt \) and learning effects. Qian and Han [24] studied the single-machine problem with simple linear deterioration and \( psddt \). Under three due date assignments, they proposed a polynomial time algorithm to solve the problem, respectively. Recently, Qian and Zhan [25] integrated two types of due windows (i.e., \( CONDW \) and \( SLKDW \)), deterioration effects and \( psddt \). They proposed polynomial-time algorithms for minimizing total weighted earliness, tardiness, starting time, and size of due window, where the weights of all jobs (corresponding to earliness, tardiness, starting time and size of due window) are equal. Pan et al. [26] considered the single-machine scheduling with deterioration effects and \( psddt \). Under the common, slack, different due dates and position-dependent weights, they proved that the problem can be solved in polynomial time.

Based on the above description, in order to investigate the deterioration effects and past-sequence-dependent delivery time on the job processing under the conditions of the specified due windows, then building upon the work of Qian and Zhan [25] and Pan et al. [26], this paper minimizes the weighted sum of earliness, tardiness, starting time, and the size of due window, where the weights are position-dependent coefficients (i.e., the weight is not related to the job but to the position, as can be seen in Sun et al. [5] and Liu et al. [27]), i.e., the work of Qian and Zhan [25] and Pan et al. [26] is a special case of this paper. After theoretical analysis, the positions of two jobs can be determined by comparing the difference between them (see Sections 3 and 4 in details), and this method can be solved in a simple polynomial time with a complexity of \( O(N \log N) \), where \( N \) is the number of jobs. The structure of this paper is as follows: Section 2 describes the problem; Sections 3–5 provide specific algorithms for solving the three kinds of due windows. Section 6 gives a numerical example. Section 7 presents the conclusion.

The aforementioned literature and the specific problems studied in this paper are given in the following table (Table 1).
Table 1. Literature content and achievements of this paper.

<table>
<thead>
<tr>
<th>Scheduling Problems</th>
<th>Time Complexity</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>CONDW, $d^1 \leq d^2 \sum (\epsilon_k Q_k + \omega_k Y_k + \Omega_d^2 + \omega D)$</td>
<td>NP-hard</td>
</tr>
<tr>
<td>1</td>
<td>CONDW, DES $\sum (\mu_k + \tau_k T_k + \Omega d^2 + \omega D)$</td>
<td>O(N log N)</td>
</tr>
<tr>
<td>1</td>
<td>CONDW, DES $\sum (\theta_k U_k + \delta V_k + \Omega d^2 + \omega D)$</td>
<td>O(N^3)</td>
</tr>
<tr>
<td>1</td>
<td>SLKDW, DES, DMA $\sum (\mu_k + \tau_k T_k + \theta_k U_k + \delta V_k + \Omega d^2 + \omega D)$</td>
<td>O(N^6)</td>
</tr>
<tr>
<td>1</td>
<td>SLKDW/DIFDW, DES $\sum (\mu_k + \tau_k T_k + \theta_k U_k + \delta V_k + \Omega d^2 + \omega D)$</td>
<td>O(N^4) / O(N log N)</td>
</tr>
<tr>
<td>1</td>
<td>CON / SLK / DIF, DES, psddt $\sum (\mu_k + \tau_k T_k + \Omega d^2 + \omega D)$</td>
<td>O(N log N)</td>
</tr>
<tr>
<td>1</td>
<td>CON / SLK / DIF, DES, psddt $\sum (\mu_k + \tau_k T_k + \Omega d^2 + \omega D)$</td>
<td>O(N log N)</td>
</tr>
<tr>
<td>1</td>
<td>CON / SLK / DIF, DES, psddt $\sum (\mu_k + \tau_k T_k + \Omega d^2 + \omega D)$</td>
<td>O(N log N)</td>
</tr>
<tr>
<td>1</td>
<td>DIFDW, DES, psddt $\sum (\mu_k + \tau_k T_k + \Omega d^2 + \omega D)$</td>
<td>O(N log N)</td>
</tr>
</tbody>
</table>

Notes: $d$ is a bound of $d^1$, $\epsilon_k$, $\delta_k$, $\mu_k$ (d), $\theta_k$ (\theta) are given constants, $Q_k = \min\{P_k, \max\{0, d^1 - C_k + P_k\}\}$, $Y_k = \min\{P_k, \max\{0, C_k - d^1\}\}$, $U_k = 1$ (resp. $U_k = 0$) if $d_k^1 > C_k$ (resp. $d_k^1 \leq C_k$), $V_k = 1$ (resp. $V_k = 0$) if $d_k^2 < C_k$ (resp. $d_k^2 \geq C_k$), DMA is the deteriorating maintenance activity, $d_k$ is the due date of job $j_k$, and CON / SLK / DIF is the common / slack / different due date assignment, respectively.

2. Problem Definition

This problem can be described as follows: $N$ jobs (represented by the set $J = \{1, \ldots, N\}$) are processed through a single-machine, and all jobs starting from $r_0$ ($r_0 > 0$). The starting time and finishing time of the CONDW can be represented by the interval $[d^1, d^2]$, and $D = d^2 - d^1$ is the size of a common due window. Let $k$ be the $k$th position, the earliness and tardiness are denoted by $E_k = \max\{d^1 - C_k, 0\}$ and $T_k = \max\{C_k - d^2, 0\}$, respectively, where $C_k$ is the completion time of job $j_k$. For the interval of SLKDW, it can be expressed as $[P_k + q^1, P_k + q^2]$ with the due window size is $D = q^2 - q^1$. The earliness and tardiness in this case are $E_k = \max\{0, P_k + q^1 - C_k\}$ and $T_k = \max\{0, C_k - P_k - q^2\}$, where $d^1, d^2, q^1, q^2$ are all decision variables; for DIFDW, the interval is $[d_k^1, d_k^2]$ and the size of the due window $D_k = d_k^2 - d_k^1$. The problems under CONDW, SLKDW, and DIFDW due to window conditions can be described as follows:

1. CONDW, DES, psddt $\sum_{k=1}^{N} (\mu_k E_k + \tau_k T_k + \Omega_k d^1 + \omega_k D)$ (1)

2. SLKDW, DES, psddt $\sum_{k=1}^{N} (\mu_k E_k + \tau_k T_k + \Omega_k q^1 + \omega_k D)$ (2)

and

3. DIFDW, DES, psddt $\sum_{k=1}^{N} (\mu_k E_k + \tau_k T_k + \Omega_k d_k^1 + \omega_k D_k)$ (3)

where $\mu_k, \tau_k, \Omega_k$, and $\omega_k$ ($k = 1, \ldots, N$) are position-dependent weights. The actual processing time $P_k$ of job $j_k$ (which is scheduled at the $k$-th position in a sequence) can be expressed as:

$$P_k = \lambda_k r_k$$ (4)

where $\lambda_k$ (respectively, $r_k$) is deterioration rate (respectively, starting time) of $j_k$. In addition, the past-sequence-dependent delivery time (psddt) of $j_k$ is:

$$q_k = YW_k = Y \sum_{m=0}^{k-1} P_m$$ (5)
where \( Y > 0 \) is a delivery rate, and \( W[k] = \sum_{m=0}^{k-1} P[m] = r_0 \prod_{m=1}^{k-1} (1 + \lambda[m]) \) with \( P[0] = r_0 \).

The corresponding completion time \( C[k] \) is:

\[
C[k] = \begin{cases} 
  W[k] + P[k] + q[k] & \text{Case I. When} \\
  r_0(1 + \lambda[k] + Y) \prod_{m=1}^{k-1} (1 + \lambda[m]) & \text{otherwise}
\end{cases}
\]

(6)

3. Solution of \( CONDW \)

In order to solve \( |CONDW, DES, psddt| \sum_{k=1}^{N} (\mu_k E[k] + \tau_k T[k] + \Omega_k d^1 + \omega_k D) \), the following optimal properties are given.

**Lemma 1.** For any given job sequence, the optimal \( d^1 \) and \( d^2 \) are equal to the completion times of two certain jobs.

**Proof.** Case I. When \( C[v] < d^1 < C[v+1] \) and \( d^2 = C[u] \) with \( 0 \leq v < u \leq N \), \( C[0] = P[0] = r_0 \), and the objective is

\[
F = \sum_{k=1}^{v} \mu_k (d^1 - C[k]) + \sum_{k=u+1}^{N} \tau_k (C[v] - C[u]) + \sum_{k=1}^{N} \Omega_k d^1 + \sum_{k=1}^{N} \omega_k (C[u] - d^1) \\
= -\sum_{k=1}^{v} \mu_k C[k] + \sum_{k=u+1}^{N} \tau_k C[k] + \left[ \sum_{k=1}^{v} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \omega_k \right] d^1 \\
+ \left[ \sum_{k=1}^{N} \omega_k - \sum_{k=u+1}^{N} \tau_k \right] C[u] 
\]

(7)

For \( d^1 = C[u] \), the objective function is:

\[
F' = -\sum_{k=1}^{v} \mu_k C[k] + \sum_{k=u+1}^{N} \tau_k C[k] \\
+ \left[ \sum_{k=1}^{v} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \omega_k \right] C[v] + \left[ \sum_{k=1}^{N} \omega_k - \sum_{k=u+1}^{N} \tau_k \right] C[u] 
\]

(8)

\[
F - F' = \left[ \sum_{k=1}^{v} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \omega_k \right] (d^1 - C[v]) 
\]

For \( d^1 = C[v+1] \), we have

\[
F'' = -\sum_{k=1}^{v} \mu_k C[k] + \sum_{k=u+1}^{N} \tau_k C[k] \\
+ \left[ \sum_{k=1}^{v} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \omega_k \right] C[v+1] + \left[ \sum_{k=1}^{N} \omega_k - \sum_{k=u+1}^{N} \tau_k \right] C[u] 
\]

(9)

\[
F - F'' = \left[ \sum_{k=1}^{v} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \omega_k \right] (d^1 - C[v+1]) 
\]

(10)

If \( \sum_{k=1}^{v} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \omega_k < 0 \), \( F > F'' \); otherwise \( F \geq F' \).
I. When \( d^1 = C[v] \) and \( C[u] < d^2 < C[u+1] \) with \( 0 \leq v \leq u < N \), we have

\[
F = -\sum_{k=1}^{v-1} \mu_k C[k] + \sum_{k=u+1}^N \tau_k C[k] \\
+ \left[ \sum_{k=1}^{v-1} \mu_k + \sum_{k=1}^N \Omega_k - \sum_{k=1}^N \omega_k \right] C[v] + \left[ \sum_{k=1}^N \omega_k - \sum_{k=u+1}^N \tau_k \right] d^2
\]

(11)

For \( d^2 = C[u] \), we have

\[
F' = -\sum_{k=1}^{v-1} \mu_k C[k] + \sum_{k=u+1}^N \tau_k C[k] \\
+ \left[ \sum_{k=1}^{v-1} \mu_k + \sum_{k=1}^N \Omega_k - \sum_{k=1}^N \omega_k \right] C[u] + \left[ \sum_{k=1}^N \omega_k - \sum_{k=u+1}^N \tau_k \right] C[u]
\]

(12)

\[
F - F' = \left[ \sum_{k=1}^N \omega_k - \sum_{k=u+1}^N \tau_k \right] (d^2 - C[u])
\]

(13)

For \( d^2 = C[u+1] \), we have

\[
F'' = -\sum_{k=1}^{v-1} \mu_k C[k] + \sum_{k=u+2}^N \tau_k C[k] \\
+ \left[ \sum_{k=1}^{v-1} \mu_k + \sum_{k=1}^N \Omega_k - \sum_{k=1}^N \omega_k \right] C[u+1] + \left[ \sum_{k=1}^N \omega_k - \sum_{k=u+2}^N \tau_k \right] C[u+1]
\]

(14)

\[
F - F'' = \left[ \sum_{k=1}^N \omega_k - \sum_{k=u+1}^N \tau_k \right] (d^2 - C[u+1])
\]

(15)

If \( \sum_{k=1}^N \omega_k - \sum_{k=u+1}^N \tau_k < 0 \), \( F > F'' \); otherwise \( F \geq F' \).

Case III. When \( C[v] < d^1 < C[v+1] \) and \( C[u] < d^2 < C[u+1] \), in which \( 0 \leq v \leq u < N \), it is considered the general case. When \( d^1 \) moves left or right so that \( d^1 = C[v]/C[v+1] \), it becomes Case II; when \( d^2 \) moves left or right, so that \( d^2 = C[u]/C[u+1] \), it becomes Case I. \( \square \)

**Lemma 2.** For the optimal sequence of jobs, \( d^1 = C[v] \) and \( d^2 = C[u] \), where \( v \) satisfies \( \sum_{k=1}^{v-1} \mu_k + \sum_{k=1}^N \Omega_k - \sum_{k=1}^N \omega_k \leq 0 \) and \( \sum_{k=1}^N \mu_k + \sum_{k=1}^N \omega_k - \sum_{k=1}^N \tau_k \geq 0 \); and \( u \) satisfies \( \sum_{k=1}^N \omega_k - \sum_{k=u+1}^N \tau_k \geq 0 \).

**Proof.** When \( d^1 = C[v] \) and \( d^2 = C[u] \), the objective function can be expressed as

\[
F = -\sum_{k=1}^{v-1} \mu_k C[k] + \sum_{k=u+1}^N \tau_k C[k] + \left[ \sum_{k=1}^{v-1} \mu_k + \sum_{k=1}^N \omega_k - \sum_{k=1}^N \omega_k \right] C[v]
\]

\[
+ \left[ \sum_{k=1}^N \omega_k - \sum_{k=u+1}^N \tau_k \right] C[u]
\]

(16)
Case I. When \( d^1 \) moves \( \delta(n \gg 0) \) units to the left and \( d^2 = C_{[u]} \), the objective function becomes

\[
F' = -\sum_{k=1}^{v-2} \mu_k C_{[k]} + \sum_{k=u+1}^{N} \tau_k C_{[k]}
+ \left[ \sum_{k=1}^{v-2} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \alpha_k \right] (C_{[v]} - \delta) + \left[ \sum_{k=1}^{N} \alpha_k - \sum_{k=u+1}^{N} \tau_k \right] C_{[u]}
\]  
(17)

\[
F - F' = \left[ \sum_{k=1}^{v-1} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \alpha_k \right] \delta \leq 0
\]  
(18)

that is, \( \sum_{k=1}^{v-1} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \alpha_k \leq 0. \)

When \( d^1 \) moves \( \delta \) units to the right with \( d^2 = C_{[u]} \), the objective function is

\[
F'' = -\sum_{k=1}^{v} \mu_k C_{[k]} + \sum_{k=u+1}^{N} \tau_k C_{[k]}
+ \left[ \sum_{k=1}^{v} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \alpha_k \right] (C_{[v]} + \delta) + \left[ \sum_{k=1}^{N} \alpha_k - \sum_{k=u+1}^{N} \tau_k \right] C_{[u]}
\]  
(19)

\[
F - F'' = -\left[ \sum_{k=1}^{v} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \alpha_k \right] \delta \leq 0
\]  
(20)

namely, \( \sum_{k=1}^{v} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \alpha_k \geq 0. \)

Case II. When \( d^1 = C_{[v]} \) and \( d^2 \) moves \( \delta \) units to the left, such that the objective function is

\[
\hat{F} = -\sum_{k=1}^{v-1} \mu_k C_{[k]} + \sum_{k=u+1}^{N} \tau_k C_{[k]}
+ \left[ \sum_{k=1}^{v-1} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \alpha_k \right] C_{[v]} + \left[ \sum_{k=1}^{N} \alpha_k - \sum_{k=u+1}^{N} \tau_k \right] (C_{[u]} - \delta)
\]  
(21)

\[
F - \hat{F} = \left[ \sum_{k=1}^{N} \alpha_k - \sum_{k=u+1}^{N} \tau_k \right] \delta \leq 0
\]  
(22)

that is, \( \sum_{k=1}^{N} \alpha_k - \sum_{k=u+1}^{N} \tau_k \leq 0. \)

When \( d^1 = C_{[u]} \) and \( d^2 \) moves \( \delta \) units to the right, and the objective function is

\[
F = -\sum_{k=1}^{v-1} \mu_k C_{[k]} + \sum_{k=u+1}^{N} \tau_k C_{[k]}
+ \left[ \sum_{k=1}^{v-1} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \alpha_k \right] C_{[v]} + \left[ \sum_{k=1}^{N} \alpha_k - \sum_{k=u+1}^{N} \tau_k \right] (C_{[u]} + \delta)
\]  
(23)

\[
F - F = -\left[ \sum_{k=1}^{N} \alpha_k - \sum_{k=u+1}^{N} \tau_k \right] \delta \leq 0
\]  
(24)

namely \( \sum_{k=1}^{N} \alpha_k - \sum_{k=u+1}^{N} \tau_k \geq 0. \) □

Based on the above lemmas, it can be assumed that, in a given optimal sequence, \( d^1 = C_{[v]} \) and \( d^2 = C_{[u]} \). Define the sets \( \pi_1 = \{ j_k \in \pi \mid k \leq v \} \), \( \pi_2 = \{ j_k \in \pi \mid k = v \} \), \( \pi_3 = \{ j_k \in \pi \mid v+1 \leq k \leq u \} \), \( \pi_4 = \{ j_k \in \pi \mid k = u \} \), and \( \pi_5 = \{ j_k \in \pi \mid u+1 \leq k \leq N \} \), where \( \pi \) is the sequence of jobs.
Lemma 3. In the optimal sequence, the jobs in \( \pi_1 \) are sorted in descending order of \( \lambda_k \).

Proof. Let \( I_g \) and \( I_h \), respectively, be in the \( x \)-th and \((x + 1)\)-th positions in \( \pi_1 \); thus, the sequence can be written as \( \theta = \{ I_1, \ldots, I_g, I_h, \ldots, I_N \} \), and the sequence \( \theta' = \{ I_1, \ldots, I_h, I_g, \ldots, I_N \} \) is obtained by exchanging the two jobs. Then, the subtraction of the objective function \( F \) corresponding to the sequence \( \theta \) and \( F' \) corresponding to \( \theta' \) is

\[
F - F' = (\mu_{x+1}Y + \mu_x)r_0(\lambda_h - \lambda_g) \prod_{m=1}^{x-1} (1 + \lambda_{[m]})
\]  

(25)

Note that \((\mu_{x+1}Y + \mu_x)r_0\) is constant, and if \( \lambda_g > \lambda_h \), then \( F < F' \). \( \Box \)

Lemma 4. The deterioration rates of the jobs in \( \pi_2 \) are lower than those of any job in \( \pi_1 \).

Proof. Assuming that \( I_g \) and \( I_h \) are at the \( v \)-th and \((v + 1)\)-th positions, and the sequences before and after exchange are \( \theta = \{ I_1, \ldots, I_g, I_h, \ldots, I_N \} \) and \( \theta' = \{ I_1, \ldots, I_h, I_g, \ldots, I_N \} \). The difference between the objective functions of the two is

\[
F - F' = \left[ (\sum_{k=1}^{v-1} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \omega_k)Y - \mu_{v-1} \right] r_0(\lambda_g - \lambda_h) \prod_{m=1}^{v-2} (1 + \lambda_{[m]})
\]  

(26)

It follows from Lemma 2 that \( v \) satisfies the inequality \( \sum_{k=1}^{v-1} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \omega_k \leq 0 \), then the term \( \left[ (\sum_{k=1}^{v-1} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \omega_k)Y - \mu_{v-1} \right] < 0 \) can be obtained. To make \( F < F' \), there is \( \lambda_g > \lambda_h \). \( \Box \)

Lemma 5. For the jobs in \( \pi_3 \), the optimal sequence is sorted according to any ordering of \( \lambda_k \).

Proof. \( I_g \) and \( I_h \) are the jobs at the \( x \)-th and \((x + 1)\)-th positions in \( \pi_3 \), in which \( v + 1 \leq x < x + 1 \leq u - 1 \). Let \( \theta = \{ I_1, \ldots, I_g, I_h, \ldots, I_N \} \) and \( \theta' = \{ I_1, \ldots, I_h, I_g, \ldots, I_N \} \) be the sequences before and after the exchange, respectively. Since \( d^1 \) and \( d^2 \) do not change, there is \( F = F' \). \( \Box \)

Lemma 6. The deterioration rates of any job in \( \pi_3 \) are less than those in \( \pi_4 \).

Proof. \( I_g \) is in the \( x \)-th position, where \( v + 1 \leq x \leq u - 1 \), and \( I_h \) is in the \( u \)-th position where the corresponding sequence is \( \theta = \{ I_1, \ldots, I_g, I_h, \ldots, I_N \} \). The exchange of the two jobs yields \( \theta' = \{ I_1, \ldots, I_h, I_g, \ldots, I_N \} \). Then, the objective function \( F \) corresponding to the sequence \( \theta = \{ I_1, \ldots, I_g, I_h, \ldots, I_N \} \) is subtracted from the \( F' \) corresponding to \( \theta' = \{ I_1, \ldots, I_h, I_g, \ldots, I_N \} \) to obtain

\[
F - F' = \left( \sum_{k=1}^{N} \omega_k - \sum_{k=u+1}^{N} \tau_k \right) Y r_0(\lambda_g - \lambda_h) \prod_{m=1,m\neq x}^{u-1} (1 + b_{[m]})
\]  

(27)

It follows from Lemma 2 that \( \sum_{k=1}^{N} \omega_k - \sum_{k=u+1}^{N} \tau_k \geq 0 \). When \( \lambda_g < \lambda_h \), we have \( F < F' \). \( \Box \)

Lemma 7. In a given optimal sequence, the jobs in \( \pi_5 \) are ordered in the ascending order of \( \lambda_k \).
Proof. Suppose that \( J_g \) and \( J_h \) are in the \( x \)-th and \( (x + 1) \)-th positions, separately. The original sequence and the sequence after swapping are \( \theta = \{ J_1, ..., J_g, J_h, ..., J_N \} \) and \( \theta' = \{ J_1, ..., J_h, J_g, ..., J_N \} \). Then, the difference between the two objective functions is

\[
F - F' = (\tau_x + \tau_{x+1} Y) r_0 (\lambda_g - \lambda_h) \prod_{m=1}^{x-1} (1 + \lambda_{[m]})
\]

(28)

Obviously, \((\tau_x + \tau_{x+1} Y) r_0 > 0\), then \( F < F' \) when \( \lambda_g < \lambda_h \). \( \square \)

Lemma 8. The deterioration rates of any job in \( \pi_q \) are smaller than those of the jobs in \( \pi_s \).

Proof. Assuming that \( J_g \) and \( J_h \) are in the \( u \)-th and \( (u + 1) \)-th positions, the sequences before and after the exchange are \( \theta = \{ J_1, ..., J_g, J_h, ..., J_N \} \) and \( \theta' = \{ J_1, ..., J_h, J_g, ..., J_N \} \). And, the difference between the two is

\[
F - F' = [\tau_{u+1} Y + \left( \sum_{k=1}^{N} \omega_k - \sum_{k=u+1}^{N} \tau_k \right)] r_0 (\lambda_g - \lambda_h) \prod_{m=1}^{u-1} (1 + \lambda_{[m]})
\]

(29)

According to Lemma 2, when \( \lambda_g < \lambda_h \) there is \( F < F' \). \( \square \)

Suppose that \( J_g \) and \( J_h \) are in the \( x \)-th and \( y \)-th positions, respectively. That is, the sequence \( \theta = \{ J_1, ..., J_g, ..., J_h, ..., J_N \} \), where \( 1 \leq x \leq u \) and \( u \leq y \leq N \). The exchange of the two jobs yields \( \theta' = \{ J_1, ..., J_h, ..., J_g, ..., J_N \} \). Then, the objective function difference between the two is

\[
F - F' = r_0 (\lambda_g - \lambda_h) \prod_{m=1}^{x-1} (1 + \lambda_{[m]}) \begin{bmatrix}
- \mu_x & - \sum_{k=x+1}^{v} \mu_k (1 + \lambda_{[k]} + Y) \prod_{m=x+1}^{k-1} (1 + \lambda_{[m]}) \\
+ \sum_{k=u}^{y-1} \tau_k (1 + \lambda_{[k]} + Y) \prod_{m=x+1}^{k-1} (1 + \lambda_{[m]}) & + \tau_y Y \prod_{m=x+1}^{y-1} (1 + \lambda_{[m]}) \\
+ \left( \sum_{k=1}^{v} \omega_k - \sum_{k=x+1}^{N} \tau_k \right) (1 + \lambda_{[u]} + Y) & \times \prod_{m=x+1}^{u-1} (1 + \lambda_{[m]}) \\
+ \left( \sum_{k=1}^{N} \omega_k - \sum_{k=u+1}^{N} \tau_k \right) (1 + \lambda_{[y]} + Y) & \times \prod_{m=x+1}^{u-1} (1 + \lambda_{[m]})
\end{bmatrix}
\]

(30)

Define

\[
S_{xy} = - \mu_x - \sum_{k=x+1}^{v} \mu_k (1 + \lambda_{[k]} + Y) \prod_{m=x+1}^{k-1} (1 + \lambda_{[m]}) \\
+ \sum_{k=u}^{y-1} \tau_k (1 + \lambda_{[k]} + Y) \prod_{m=x+1}^{k-1} (1 + \lambda_{[m]}) + \tau_y Y \prod_{m=x+1}^{y-1} (1 + \lambda_{[m]}) \\
+ \left( \sum_{k=1}^{v} \mu_k + \sum_{k=1}^{N} \omega_k - \sum_{k=x+1}^{N} \omega_k \right) (1 + \lambda_{[u]} + Y) \prod_{m=x+1}^{v-1} (1 + \lambda_{[m]}) \\
+ \left( \sum_{k=1}^{N} \omega_k - \sum_{k=u+1}^{N} \tau_k \right) (1 + \lambda_{[y]} + Y) \prod_{m=x+1}^{u-1} (1 + \lambda_{[m]})
\]

(31)

If \( S_{xy} > 0 \), \( J_g \) should be placed at the \( x \)-th position; otherwise, \( J_g \) should be placed at the \( y \)-th position.
From the analysis above, we can propose the algorithm to solve as follows

$$1|\text{CONDW, DES, psddt}| \sum_{k=1}^{N} (\mu_k E_k[k] + \tau_k T_k[k] + \Omega_k d^1 + \omega_k D).$$

**Theorem 1.** The problem $1|\text{CONDW, DES, psddt}| \sum_{k=1}^{N} (\mu_k E_k[k] + \tau_k T_k[k] + \Omega_k d^1 + \omega_k D)$ can be solved in $O(N \log N)$.

**Proof.** The required time to calculate $S_1$ is $O(N \log N)$, the required time to calculate $S_2$ and $S_3$ is constant, and $S_4$ required $O(N)$ time, so the time required for Algorithm 1 is $O(N \log N)$. □

### Algorithm 1: Solution based on CONDW

**Input:** $N, r_0, Y, \lambda_k, \mu_k, \tau_k, \Omega_k$ and $\omega_k$.

1. Sort by increasing order of $\lambda_k$, i.e., $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$.
2. According to Lemma 2, the optimal starting time $d^1$ and finishing time $d^2$ of due window are determined.
3. Determine the set $\pi_3$, which contains $u - v - 1$ jobs, namely, $\lambda_1 \ldots \lambda_{u-v-1}$.
4. Identify the jobs in $\pi_1 \cup \pi_2$ and $\pi_4 \cup \pi_5$ through (31).

**Output:** The optimal sequence $\pi_1, d^1$ and $d^2$.

### 4. Solution of SLKDW

For the problem $1|\text{SLKDW, DES, psddt}| \sum_{k=1}^{N} (\mu_k E_k[k] + \tau_k T_k[k] + \Omega_k q^1 + \omega_k D)$, the following properties can be given:

**Lemma 9.** For any given sequence of jobs, $q^1$ and $q^2$ in the optimal sequence are computed as $(1 + Y)$ times either the sum of the actual processing time of certain jobs or $r_0$.

**Proof.** Case I. When $(1 + Y) \sum_{m=0}^{u-1} P_{[m]} < q^1 < (1 + Y) \sum_{m=0}^{v} P_{[m]}$ and $q^2 = (1 + Y) \sum_{m=0}^{u-1} P_{[m]}$, where $1 \leq v < u \leq N$. There is

$$F = - (1 + Y) \sum_{k=1}^{v} \mu_k W_k[k] + (1 + Y) \sum_{k=u+1}^{N} \tau_k W_k[k] + \left[ \sum_{k=1}^{v} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \omega_k \right] q^1 + \left[ \sum_{k=1}^{N} \omega_k - \sum_{k=u+1}^{N} \tau_k \right] (1 + Y) \sum_{m=0}^{u-1} P_{[m]}$$

(32)

When $q^1 = (1 + Y) \sum_{m=0}^{v-1} P_{[m]}$, the objective function is

$$F' = - (1 + Y) \sum_{k=1}^{v} \mu_k W_k[k] + (1 + Y) \sum_{k=u+1}^{N} \tau_k W_k[k] + \left[ \sum_{k=1}^{v} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \omega_k \right] (1 + Y) \sum_{m=0}^{v-1} P_{[m]} + \left[ \sum_{k=1}^{N} \omega_k - \sum_{k=u+1}^{N} \tau_k \right] (1 + Y) \sum_{m=0}^{u-1} P_{[m]}$$

(33)

and

$$F - F' = \left[ \sum_{k=1}^{v} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \omega_k \right] \left( q^1 - (1 + Y) \sum_{m=0}^{v-1} P_{[m]} \right)$$

(34)
When \( q^1 = (1 + Y) \sum_{m=0}^{u} P_{[m]} \), the objective function can be written as

\[
F'' = - (1 + Y) \sum_{k=1}^{v+1} \mu_k W_{[k]} + (1 + Y) \sum_{k=u+1}^{N} \tau_k W_{[k]}
\]

\[
+ \left[ \sum_{k=1}^{v+1} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \omega_k \right] (1 + Y) \sum_{m=0}^{v} P_{[m]}
\]

\[
+ \left[ \sum_{k=1}^{N} \omega_k - \sum_{k=u+1}^{N} \tau_k \right] (1 + Y) \sum_{m=0}^{u-1} P_{[m]}
\]

(35)

and

\[
F - F'' = \left[ \sum_{k=1}^{v} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \omega_k \right] \left( q^1 - (1 + Y) \sum_{m=0}^{v} P_{[m]} \right)
\]

(36)

Obviously, when \( \sum_{k=1}^{v+1} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \omega_k < 0 \), there is \( F > F'' \); otherwise \( F \geq F' \).

Case II. When \( q^1 = (1 + Y) \sum_{m=0}^{u-1} P_{[m]} \), and \( (1 + Y) \sum_{m=0}^{u-1} P_{[m]} < q^2 < (1 + Y) \sum_{m=0}^{u} P_{[m]} \) with \( 1 \leq v < u \leq N \), the function is

\[
F = -(1 + Y) \sum_{k=1}^{v-1} \mu_k W_{[k]} + (1 + Y) \sum_{k=u+1}^{N} \tau_k W_{[k]}
\]

\[
+ \left[ \sum_{k=1}^{v-1} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \omega_k \right] (1 + Y) \sum_{m=0}^{v-1} P_{[m]}
\]

\[
+ \left[ \sum_{k=1}^{N} \omega_k - \sum_{k=u+1}^{N} \tau_k \right] (1 + Y) \sum_{m=0}^{u-1} P_{[m]}
\]

(37)

When \( q^2 = (1 + Y) \sum_{m=0}^{u-1} P_{[m]} \), the function is

\[
F' = -(1 + Y) \sum_{k=1}^{v-1} \mu_k W_{[k]} + (1 + Y) \sum_{k=u+1}^{N} \tau_k W_{[k]}
\]

\[
+ \left[ \sum_{k=1}^{v-1} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \omega_k \right] (1 + Y) \sum_{m=0}^{v-1} P_{[m]}
\]

\[
+ \left[ \sum_{k=1}^{N} \omega_k - \sum_{k=u+1}^{N} \tau_k \right] (1 + Y) \sum_{m=0}^{u-1} P_{[m]}
\]

(38)

and

\[
F - F' = \left[ \sum_{k=1}^{N} \omega_k - \sum_{k=u+1}^{N} \tau_k \right] \left( q^2 - (1 + Y) \sum_{m=0}^{u-1} P_{[m]} \right)
\]

(39)

When \( q^2 = (1 + Y) \sum_{m=0}^{u} P_{[m]} \), there is

\[
F'' = -(1 + Y) \sum_{k=1}^{v-1} \mu_k W_{[k]} + (1 + Y) \sum_{k=u+2}^{N} \tau_k W_{[k]}
\]

\[
+ \left[ \sum_{k=1}^{v-1} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \omega_k \right] (1 + Y) \sum_{m=0}^{v-1} P_{[m]}
\]

\[
+ \left[ \sum_{k=1}^{N} \omega_k - \sum_{k=u+2}^{N} \tau_k \right] (1 + Y) \sum_{m=0}^{u} P_{[m]}
\]

(40)

and

\[
F - F' = \left[ \sum_{k=1}^{N} \omega_k - \sum_{k=u+2}^{N} \tau_k \right] \left( q^2 - (1 + Y) \sum_{m=0}^{u} P_{[m]} \right)
\]

(41)

If \( \sum_{k=1}^{N} \omega_k - \sum_{k=u+1}^{N} \tau_k < 0 \), \( F > F'' \); otherwise \( F \geq F' \).
Case III. When $(1 + Y) \sum_{m=0}^{v-1} P_{[m]} < q^3 < (1 + Y) \sum_{m=0}^{u} P_{[m]}$ and $(1 + Y) \sum_{m=0}^{u-1} P_{[m]} < q^2 < (1 + Y) \sum_{m=0}^{u} P_{[m]}$ with $1 \leq v, u \leq N$. It becomes Case II when either $q^1 = (1 + Y) \sum_{m=0}^{v-1} P_{[m]}$ or $q^4 = (1 + Y) \sum_{m=0}^{u-1} P_{[m]}$; it becomes Case I when either $q^2 = (1 + Y) \sum_{m=0}^{u} P_{[m]}$ or $q^2 = (1 + Y) \sum_{m=0}^{u} P_{[m]}$. \(\square\)

**Lemma 10.** For the optimal sequence, let $q^1 = (1 + Y) \sum_{m=0}^{v-1} P_{[m]}$ and $q^2 = (1 + Y) \sum_{m=0}^{u-1} P_{[m]}$, where $v$ and $u$, respectively, satisfy $\sum_{k=1}^{v-1} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \omega_k \leq 0$ and $\sum_{k=1}^{u} \mu_k + \sum_{k=1}^{N} \omega_k - \sum_{k=1}^{N} \Omega_k \geq 0$; $\sum_{k=1}^{N} \omega_k - \sum_{k=1}^{N} \Omega_k \leq 0$ and $\sum_{k=1}^{N} \omega_k - \sum_{k=1}^{N} \Omega_k \geq 0$.

**Proof.** The objective function can be written as follows with $q^1 = (1 + Y) \sum_{m=0}^{v-1} P_{[m]}$ and $q^2 = (1 + Y) \sum_{m=0}^{u-1} P_{[m]}$:

$$F = -(1 + Y) \sum_{k=1}^{v-1} \mu_k W_{[k]} + (1 + Y) \sum_{k=u+1}^{N} \tau_k W_{[k]} + \left[ \sum_{k=1}^{v-1} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \omega_k \right] (1 + Y) \sum_{m=0}^{v-1} P_{[m]} + \sum_{k=1}^{N} \omega_k - \sum_{k=u+1}^{N} \tau_k (1 + Y) \sum_{m=0}^{u-1} P_{[m]}$$

(42)

Case I. When $q^1$ moves $(1 + Y)\delta$ units to the left with $\delta > 0$, and $q^2 = (1 + Y) \sum_{m=0}^{u-1} P_{[m]}$. Then, the function is

$$F' = -(1 + Y) \sum_{k=1}^{v-2} \mu_k W_{[k]} + (1 + Y) \sum_{k=u+1}^{N} \tau_k W_{[k]} + \left[ \sum_{k=1}^{v-2} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \omega_k \right] (1 + Y) \left( \sum_{m=0}^{v-1} P_{[m]} - \delta \right) + \sum_{k=1}^{N} \omega_k - \sum_{k=u+1}^{N} \tau_k (1 + Y) \sum_{m=0}^{u-1} P_{[m]}$$

(43)

and

$$F - F' = \left[ \sum_{k=1}^{v-1} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \omega_k \right] (1 + Y) \delta \leq 0$$

(44)

Obviously, $\sum_{k=1}^{v-1} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \omega_k \leq 0$.

When $q^1$ moves $(1 + Y)\delta$ units to the right and $q^2 = (1 + Y) \sum_{m=0}^{u-1} P_{[m]}$. Then, the function can be expressed as

$$F'' = -(1 + Y) \sum_{k=1}^{v} \mu_k W_{[k]} + (1 + Y) \sum_{k=u+1}^{N} \tau_k W_{[k]} + \left[ \sum_{k=1}^{v} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \omega_k \right] (1 + Y) \left( \sum_{m=0}^{v-1} P_{[m]} + \delta \right) + \sum_{k=1}^{N} \omega_k - \sum_{k=u+1}^{N} \tau_k (1 + Y) \sum_{m=0}^{u-1} P_{[m]}$$

(45)

$$F - F'' = - \left[ \sum_{k=1}^{v} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \omega_k \right] (1 + Y) \delta \leq 0$$

(46)

It can be known that $\sum_{k=1}^{v} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \omega_k \geq 0$.  


Case II. When \( q^1 = (1 + Y) \sum_{m=0}^{v-1} P_m \) and \( q^2 \) moves \((1 + Y)\delta\) units to the left, then

\[
F' = -(1 + Y) \sum_{k=1}^{v-1} \mu_k W_k + (1 + Y) \sum_{k=u+1}^{N} \tau_k W_k \\
+ \left[ \sum_{k=1}^{v-1} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \alpha_k \right] \left( 1 + Y \right) \sum_{m=0}^{v-1} P_m \\
+ \left[ \sum_{k=1}^{N} \alpha_k - \sum_{k=u+1}^{N} \tau_k \right] \left( 1 + Y \right) \left( \sum_{m=0}^{u-1} P_m + \delta \right)
\]

(47)

and

\[
F - F' = \left[ \sum_{k=1}^{N} \alpha_k - \sum_{k=u+1}^{N} \tau_k \right] (1 + Y) \delta \leq 0
\]

(48)

That is, \( \sum_{k=1}^{N} \alpha_k - \sum_{k=u+1}^{N} \tau_k \leq 0 \).

When \( q^1 = (1 + Y) \sum_{m=0}^{v-1} P_m \) and \( q^2 \) moves \((1 + Y)\delta\) units to the right, then the function can be written as

\[
F'' = -(1 + Y) \sum_{k=1}^{v-1} \mu_k W_k + (1 + Y) \sum_{k=u+1}^{N} \tau_k W_k \\
+ \left[ \sum_{k=1}^{v-1} \mu_k + \sum_{k=1}^{N} \Omega_k - \sum_{k=1}^{N} \alpha_k \right] \left( 1 + Y \right) \sum_{m=0}^{v-1} P_m \\
+ \left[ \sum_{k=1}^{N} \alpha_k - \sum_{k=u+1}^{N} \tau_k \right] \left( 1 + Y \right) \left( \sum_{m=0}^{u-1} P_m + \delta \right)
\]

(49)

and

\[
F - F'' = - \left[ \sum_{k=1}^{N} \alpha_k - \sum_{k=u+1}^{N} \tau_k \right] (1 + Y) \delta \leq 0
\]

(50)

Then, there is \( \sum_{k=1}^{N} \alpha_k - \sum_{k=u+1}^{N} \tau_k \geq 0 \). \( \square \)

According to Lemma 10, assuming that \( q^1 = (1 + Y) \sum_{m=0}^{v-1} P_m \) and \( q^2 = (1 + Y) \sum_{m=0}^{u-1} P_m \) in the optimal sequence. The following sets can be determined based on the due window: \( \pi_1 = \{ J_k \in \pi | k \leq v - 1 \} \), \( \pi_2 = \{ J_k \in \pi | v \leq k \leq u - 1 \} \) and \( \pi_3 = \{ J_k \in \pi | u \leq k \leq N \} \), in which \( \pi \) is the job sequence.

**Lemma 11.** In the optimal sequence, the jobs in \( \pi_1 \) can be arranged in descending order of \( \lambda_k \).

**Proof.** \( J_S \) and \( J_h \) are the jobs at \( x \)-th and \( (x + 1) \)-th positions in \( \pi_1 \), respectively, where the sequence can be written as \( \theta = \{ J_1, ..., J_S, J_h, ..., J_N \} \). Swapping the two jobs yields \( \theta' = \{ J_1, ..., J_h, J_S, ..., J_N \} \). Then, subtracting the function \( F \) corresponding to \( \theta \) forms the \( F' \) corresponding to \( \theta' \) to obtain

\[
F - F' = -\mu_{x+1}(1 + Y)\tau_0(\lambda_S - \lambda_h) \prod_{m=1}^{x-1} (1 + \lambda_m)
\]

(51)

If \( \lambda_S > \lambda_h \), \( F < F' \). \( \square \)

**Lemma 12.** In the optimal sequence, the jobs in \( \pi_2 \) can be arranged in any order of \( \lambda_k \).

**Proof.** \( J_S \) and \( J_h \) are at the \( x \)-th and \( (x + 1) \)-th positions in \( \pi_2 \), where \( v + 1 \leq x < x + 1 \leq u - 1 \). The sequences before and after the exchange are \( \theta = \{ J_1, ..., J_S, J_h, ..., J_N \} \) and \( \theta' = \{ J_1, ..., J_h, J_S, ..., J_N \} \). Since \( q^1 \) and \( q^2 \) do not change, we have \( F = F' \). \( \square \)
**Lemma 13.** In the optimal sequence, the jobs in π₃ can be sorted in the ascending order of λᵦ.

**Proof.** Jₛ and Jₜ are the jobs at the x-th and (x + 1)-th positions in π₃, namely, the sequence can be expressed as θ = {J₁, ..., Jₛ, Jₜ, ..., Jₙ}, and θ' = {J₁, ..., Jₜ, Jₛ, ..., Jₙ} can be obtained by swapping the two jobs. And, the difference between the two is

\[ F - F' = \tau_{x+1}(1 + Y)r_0(\lambda_g - \lambda_h) \prod_{m=1}^{x-1}(1 + \lambda_{[m]}) \]  (52)

When λₙ < λₜ, there is F < F'. □

**Lemma 14.** The deterioration rates of any job in π₂ are less than π₁.

**Proof.** Jₛ and Jₜ are at the (v - 1)-th and v-th positions, the original sequence and the exchanged sequence are θ = {J₁, ..., Jₛ, Jₜ, ..., Jₙ}, and θ' = {J₁, ..., Jₜ, Jₛ, ..., Jₙ}. And, the difference between the objective functions is

\[ F - F' = \left[ \sum_{k=1}^{v-1} \mu_k + \sum_{k=1}^{N} \omega_k - \sum_{k=1}^{N} \omega_k \right] (1 + Y)r_0(\lambda_g - \lambda_h) \prod_{m=1}^{v-2}(1 + \lambda_{[m]}) \]  (53)

From the above equation, it can be seen that there is F < F' when λₙ < λₜ. □

**Lemma 15.** The deterioration rates of all jobs in π₂ are less than π₃.

**Proof.** Jₛ and Jₜ are at the (u - 1)-th and u-th positions, the original and the exchanged sequence are θ = {J₁, ..., Jₛ, Jₜ, ..., Jₙ}, and θ' = {J₁, ..., Jₜ, Jₛ, ..., Jₙ}. And, the difference between the functions is

\[ F - F' = \left[ \sum_{k=1}^{N} \omega_k - \sum_{k=u+1}^{N} \omega_k \right] (1 + Y)r_0(\lambda_g - \lambda_h) \prod_{m=1}^{u-2}(1 + \lambda_{[m]}) \]  (54)

Obviously, there is F < F' when λₙ < λₜ. □

Assuming that Jₛ and Jₜ are the jobs at the x-th and y-th positions, the sequence can be written as θ = {J₁, ..., Jₛ, Jₜ, ..., Jₙ}, where 1 ≤ x ≤ v - 1 and u ≤ y ≤ N. The two jobs are exchanged to obtain θ' = {J₁, ..., Jₜ, Jₛ, ..., Jₙ}. And, the function difference between the two sequences is

\[ F - F' = r_0(\lambda_g - \lambda_h) \prod_{m=1}^{x-1}(1 + \lambda_{[m]}) \]

\[ \times \left[ -\mu_x - \sum_{k=x+1}^{v-2} \mu_k \prod_{m=x+1}^{k}(1 + \lambda_{[m]}) + \sum_{k=x+1}^{v-1} \tau_k \prod_{m=x+1}^{k}(1 + \lambda_{[m]}) \right] \]

\[ + \left( \sum_{k=1}^{N} \omega_k - \sum_{k=u+1}^{N} \omega_k \right) \prod_{m=x+1}^{u-1}(1 + \lambda_{[m]}) \]  (55)

Define

\[ S_{xy} = -\mu_x - \sum_{k=x+1}^{v-2} \mu_k \prod_{m=x+1}^{k}(1 + \lambda_{[m]}) + \sum_{k=x+1}^{v-1} \tau_k \prod_{m=x+1}^{k}(1 + \lambda_{[m]}) \]

\[ + \left( \sum_{k=1}^{N} \omega_k - \sum_{k=u+1}^{N} \omega_k \right) \prod_{m=x+1}^{u-1}(1 + \lambda_{[m]}) \]  (56)
If \( S_{xy} > 0 \), \( J_y \) should be placed at the \( x \)-th position; otherwise, \( J_y \) should be placed at the \( y \)-th position.

**Theorem 2.** The problem \(|\text{SLKDW}, DES, psddt| \sum_{k=1}^{N}(\mu_k T_k + \Omega_k q^1 + \omega_k D)\) can be solved in \( O(N \log N) \) by Algorithm 2.

**Algorithm 2: Solution based on SLKDW**

**Input:** \( N, r_0, Y, \lambda_k, \mu_k, \tau_k, \Omega_k \) and \( \omega_k \).

1. Sort according to \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N \).
2. According to Lemma 10, the optimal \( q^1 \) and \( q^2 \) of due window can be determined.
3. Determine the set \( \pi_2 \), that is, \( \pi_2 = \{J_1, \ldots, J_{|U-V|}\} \).
4. Define the jobs in \( \pi_1 \) and \( \pi_3 \) through (56). 

**Output:** The optimal sequence \( \pi, q^1 \) and \( q^2 \).

**5. Solution of DIFDW**

For the different due window (DIFDW) problem: \(|\text{DIFDW}, DES, psddt| \sum_{k=1}^{N}(\mu_k T_k + \Omega_k d^1_k + \omega_k D_k)\), the following properties are proposed:

**Lemma 16.** For the job \( J_k \) in a given sequence, the starting time and finishing time satisfy \( 0 \leq d^1_k \leq d^2_k \leq C_k \).

**Proof.** Obviously, \( 0 \leq d^1_k \leq d^2_k \).

Case I. For \( d^1_k \leq C_k \leq d^2_k \). It can be seen that the job \( J_k \) is within the due window and does not incur any earliness/tardiness penalties. Then, the objective function of \( J_k \) is

\[
F_k = \Omega_k d^1_k + \omega_k (d^2_k - d^1_k)
\]

(57)

Shift \( d^2_k \) to the left so that \( d^2_k = C_k \), and the objective function is

\[
F'_k = \Omega_k d^1_k + \omega_k (C_k - d^1_k) < F_k
\]

(58)

Case II. For \( C_k \leq d^1_k \leq d^2_k \), it can be observed that the job \( J_k \) is an early job, and the function is

\[
F_k = \mu_k (d^1_k - C_k) + \Omega_k d^1_k + \omega_k (d^2_k - d^1_k)
\]

(59)

Shift \( d^1_k \) and \( d^2_k \) to the left so that \( d^1_k = d^2_k = C_k \), and the function is

\[
F'_k = \Omega_k C_k < F_k
\]

(60)

In summary, \( 0 \leq d^1_k \leq d^2_k \leq C_k \).  \( \square \)

**Lemma 17.** For the given sequence, the optimal starting time and finishing time are as follows:

(a) If \( \min \{\mu_k, \tau_k, \Omega_k\} = \mu_k, d^1_k = 0 \) and \( d^2_k = C_k \);

(b) If \( \min \{\mu_k, \tau_k, \Omega_k\} = \tau_k, d^1_k = d^2_k = 0 \);

(c) If \( \min \{\mu_k, \tau_k, \Omega_k\} = \Omega_k, d^1_k = d^2_k = C_k \).
Theorem 3. The problem parameters for the example.

Table 2.

<table>
<thead>
<tr>
<th>$J_k$</th>
<th>$J_1$</th>
<th>$J_2$</th>
<th>$J_3$</th>
<th>$J_4$</th>
<th>$J_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_k$</td>
<td>2</td>
<td>0.3</td>
<td>0.6</td>
<td>1</td>
<td>0.5</td>
</tr>
<tr>
<td>$\mu_k$</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\tau_k$</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>9</td>
</tr>
<tr>
<td>$\Omega_k$</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\omega_k$</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

Proof. It can be known that $0 \leq d^1_{[k]} \leq d^2_{[k]} \leq C_{[k]}$ form Lemma 16 so that the objective function of $I_{[k]}$ can be expressed as

$$F_{[k]} = \tau_k(C_{[k]} - d^2_{[k]}) + \Omega_k d^1_{[k]} + \omega_k(d^2_{[k]} - d^1_{[k]}) = \tau_k C_{[k]} + d^1_{[k]}(\Omega_k - \omega_k) + d^2_{[k]}(\omega_k - \tau_k)$$

According to Lemma 16, when $\omega_k - \tau_k \geq 0$, i.e., $\min\{\mu_k, \tau_k, \Omega_k\} = \tau_k$, there is $d^1_{[k]} = d^2_{[k]} = 0$ to minimize the objective function; for $\omega_k - \tau_k < 0$ and $\Omega_k - \omega_k \geq 0$, namely $\min\{\mu_k, \tau_k, \Omega_k\} = \mu_k$, then $d^1_{[k]} = 0$ and $d^2_{[k]} = C_{[k]}$; for $\omega_k - \tau_k < 0$ and $\Omega_k - \omega_k < 0$, that is, $\min\{\mu_k, \tau_k, \Omega_k\} = \Omega_k$, there is $d^1_{[k]} = d^2_{[k]} = C_{[k]}$ to minimize the function.

According to the three cases mentioned in the above lemma, the objective function is

$$F = \sum_{k=1}^{N} \min\{\mu_k, \tau_k, \Omega_k\} C_{[k]} = r_0 \sum_{k=1}^{N} \min\{\mu_k, \tau_k, \Omega_k\} (1 + \lambda_{[k]} + Y) \prod_{m=1}^{k-1}(1 + \lambda_{[m]})$$

Lemma 18. For an optimal sequence, the jobs are ranked in ascending order of deterioration rate $b_k$.

Proof. $I_k$ and $J_k$ are at the $x$-th and $(x + 1)$-th positions, the original and exchanged sequences are $\theta = \{I_1, \ldots, I_k, J_k, \ldots, I_N\}$ and $\theta' = \{I_1, \ldots, I_k, J_k, \ldots, I_N\}$. The difference between the two is

$$F - F' = (\min\{\mu_x, \tau_x, \Omega_x\} + Y \min\{\mu_{x+1}, \tau_{x+1}, \Omega_{x+1}\})(\lambda_x - \lambda_{x+1}) \prod_{m=1}^{x-1}(1 + \lambda_{[m]})$$

Obviously, when $\lambda_x < \lambda_{x+1}$, this satisfies $F < F'$.

Theorem 3. The problem 1|DIFDW, DES, psddt| $\sum_{k=1}^{N} (\mu_k E_{[k]} + \tau_k T_{[k]} + \Omega_k d^1_{[k]} + \omega_k D_{[k]})$ can be solved in $O(N \log N)$ according to Algorithm 3.

Algorithm 3: Solution based on DIFDW

<table>
<thead>
<tr>
<th>Input: $N, r_0, Y, \lambda_k, \mu_k, \tau_k, \Omega_k$ and $\omega_k$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$. Determine the optimal sequence by $\lambda_{[1]} \leq \lambda_{[2]} \leq \ldots \leq \lambda_{[N]}$.</td>
</tr>
<tr>
<td>$S_2$. Calculate $d^1_{[k]}$ and $d^2_{[k]}$ according to Lemma 17.</td>
</tr>
<tr>
<td>Output: The optimal sequence $\pi$.</td>
</tr>
</tbody>
</table>

6. An Example

Consider an example with $N = 5$, $r_0 = 2$, and $Y = 0.2$, and other parameters are detailed in the table below (i.e., Table 2):

Table 2. Parameters for the example.
For $1|\text{CONDW, DES, psddt}| \sum_{k=1}^{N} (\mu_{k}E_{[k]} + \tau_{k}T_{[k]} + \Omega_{k}d_{1} + \omega_{k}D)$, the specific calculation steps are as follows:

$S_{1}$. Since $\lambda_{2} < \lambda_{3} < \lambda_{4} < \lambda_{1}$, $J_{2} \rightarrow J_{3} \rightarrow J_{4} \rightarrow J_{1}$.

$S_{2}$. According to Lemma 2, $\mu_{1} + \sum_{k=1}^{5} \Omega_{k} - \sum_{k=1}^{3} \omega_{k} = 3 + 6 - 10 = -1 < 0$ and $\mu_{1} + \mu_{2} + \sum_{k=1}^{5} \Omega_{k} - \sum_{k=1}^{3} \omega_{k} = 6 + 6 - 10 = 2 > 0$, $\sum_{k=1}^{7} \omega_{k} = 10 - 10 = 0$ and $\sum_{k=1}^{5} \omega_{k} = 10 - 0 = 0$ can be calculated, i.e., $d_{1}^{1} = C_{[2]}$ and $d_{1}^{2} = C_{[4]}$. Then, the value of $J_{2} \rightarrow J_{3} \rightarrow J_{4} \rightarrow J_{1}$ is $F = 359.732$.

$S_{3}$. It can be known that $J_{2}$ is in the $\pi_{3}$ and in the third position in the sequence.

$S_{4}$. For $x = 2, y = 4$. It can be calculated that $S_{[2][4]} = -3 + 1 \times 0.2 \times (1 + 0.3) + 2 \times (1 + 0.5 + 0.2) = 0.66 > 0$, then place $J_{3}$ in the second position.

For $x = 1, y = 4$. It can be calculated that $S_{[1][4]} = -3 - 3 \times (1 + 0.5 + 0.2) + 1 \times 0.2 \times (1 + 0.5) \times (1 + 0.3) + 2 \times (1 + 0.5 + 0.2) = -4.31 < 0$, then place $J_{3}$ in the fourth position.

For $x = 2, y = 5$. It can be calculated that $S_{[2][5]} = -3 + 1 \times (1 + 0.5 + 0.2) + 9 \times 0.2 \times (1 + 0.3) \times (1 + 0.6) + 2 \times (1 + 0.5 + 0.2) = 6.484 > 0$, then place $J_{4}$ in the first position.

Thus, the optimal job sequence is $J_{4} \rightarrow J_{3} \rightarrow J_{2} \rightarrow J_{3} \rightarrow J_{1}$, and $d_{1}^{1} = C_{[2]} = 6.8$ and $d_{1}^{2} = C_{[4]} = 28.08$. And, the corresponding value of the objective function is $F = 243.112 < 359.732$.

For $1|\text{SLKDW, DES, psddt}| \sum_{k=1}^{N} (\mu_{k}E_{[k]} + \tau_{k}T_{[k]} + \Omega_{k}q_{1}^{1} + \omega_{k}D)$, the calculation process is as follows:

$S_{1}$. Since $\lambda_{2} < \lambda_{5} < \lambda_{4} < \lambda_{1}$, $J_{2} \rightarrow J_{3} \rightarrow J_{4} \rightarrow J_{1}$.

$S_{2}$. According to Lemma 10, $q_{1}^{1} = (1 + Y) \sum_{m=0}^{L} P_{[m]}$ and $q_{1}^{2} = (1 + Y) \sum_{m=0}^{L} P_{[m]}$. That is, $\nu = 2$ and $n = 4$. And, the value of $J_{2} \rightarrow J_{3} \rightarrow J_{4} \rightarrow J_{1}$ is $F = 131.232 < 131.952$.

$S_{3}$. It can be known that $J_{2}$ and $J_{3}$ are in the $\pi_{2}$, and in the second and third positions in the sequence.

$S_{4}$. For $x = 1, y = 4$. It can be calculated that $S_{[1][4]} = -3 - 1 \times 1 \times (1 + 0.3) \times (1 + 0.5) = -2.05 < 0$, then $J_{3}$ is placed in the fourth position.

For $x = 1, y = 5$. It can be calculated that $S_{[1][5]} = -3 + 1 \times (1 + 0.3) \times (1 + 0.5) - 1 \times 1 + 1 \times (1 + 0.3) \times (1 + 0.5) = 1.07 > 0$, then $J_{4}$ is placed in the first position.

Thus, the optimal sequence is $J_{3} \rightarrow J_{2} \rightarrow J_{3} \rightarrow J_{4} \rightarrow J_{1}$, and $q_{1}^{1} = 3.84$ and $q_{1}^{2} = 7.488$.

And, the corresponding value of function is $F = 131.232 < 131.952$.

For $1|\text{DIFDW, DES, psddt}| \sum_{k=1}^{N} (\mu_{k}E_{[k]} + \tau_{k}T_{[k]} + \omega_{k}d_{1}^{1} + \omega_{k}D_{[k]}),$, the optimal sequence, and the starting times and finishing times are given as follows:

$S_{1}$. The optimal sequence is $J_{2} \rightarrow J_{3} \rightarrow J_{3} \rightarrow J_{4} \rightarrow J_{1}$, which is determined by $\lambda_{2} < \lambda_{5} < \lambda_{3} < \lambda_{4} < \lambda_{1}$.

$S_{2}$. The starting time and finishing time can be shown as the following table (i.e., Table 3):

<table>
<thead>
<tr>
<th>The Sequence</th>
<th>$J_{2}$</th>
<th>$J_{3}$</th>
<th>$J_{4}$</th>
<th>$J_{1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_{1}$</td>
<td>0.4</td>
<td>0.52</td>
<td>0.78</td>
<td>1.248</td>
</tr>
<tr>
<td>$\bar{q}_{1}$</td>
<td>3.42</td>
<td>4.42</td>
<td>7.02</td>
<td>13.728</td>
</tr>
<tr>
<td>$d_{1}^{1}$</td>
<td>0</td>
<td>0</td>
<td>7.02</td>
<td>13.728</td>
</tr>
<tr>
<td>$d_{1}^{2}$</td>
<td>0</td>
<td>0</td>
<td>7.02</td>
<td>13.728</td>
</tr>
</tbody>
</table>

7. Conclusions

This paper studied single-machine delivery time scheduling with DES and psddt. The objective is to find the optimal sequence, starting time and finishing time of the due window so that the position-dependent weighted sum of earliness, tardiness, starting time and size of the due window is to be minimized. Under common, slack, and different due windows, it is shown that the studied problems can all be solved within a time complexity of $O(N \log N)$. Future research could explore scheduling with linear deterioration functions in the context of flow shop setting, since, due to the complexity of the flow shop environment,
heuristic algorithms may be proposed, such as the application of the gravitational search algorithm with hierarchy and distributed framework to it, as mentioned in Wang et al. [28].

Author Contributions: Investigation, J.-B.W.; Writing—original draft, R.-R.M. and Y.-C.W.; Writing—review & editing, R.-R.M., Y.-C.W., D.-Y.L., J.-B.W. and Y.-Y.L. All authors have read and agreed to the published version of the manuscript.

Funding: This Work Was Supported by LiaoNing Revitalization Talents Program (XLYC2002017) and Science Research Foundation of Educational Department of Liaoning Province (LJJKMZ20220532).

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare there is no conflict of interest.

References


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