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Trace Formulae for Second-Order Differential Pencils with a Frozen Argument

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Abstract: This paper deals with second-order differential pencils with a fixed frozen argument on a finite interval. We obtain the trace formulae under four boundary conditions: Dirichlet–Dirichlet, Neumann–Neumann, Dirichlet–Neumann, Neumann–Dirichlet. Although the boundary conditions and the corresponding asymptotic behaviour of the eigenvalues are different, the trace formulae have the same form which reveals the impact of the frozen argument.

Keywords: differential pencils; regularized trace formulae; frozen argument

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1. Introduction

In this paper, we consider boundary value problem generated by

$$-y''(x) + [q_0(x) + \rho q_1(x)]y(a) = \rho^2 y(x), \quad 0 < x < 1, \tag{1}$$

and boundary conditions

$$y^{(\alpha)}(0) = y^{(\beta)}(1) = 0, \tag{2}$$

where $a \in (0,1)$, $q_j(x), j = 0,1$ are complex-valued functions in Sobolev space $W_2^j[0,1]$, ρ is the spectral parameter and $\alpha, \beta \in \{0,1\}$. We note that $(\alpha,\beta) = (0,0), (1,1), (0,1)$ or $(1,0)$ represent Dirichlet–Dirichlet, Neumann–Neumann, Dirichlet–Neumann, Neumann–Dirichlet boundary conditions correspondingly. We denote the corresponding operator by $L_{\alpha,\beta} = L(q_0, q_1, \alpha, \beta, a)$. We call $L_{\alpha,\beta}$ the second-order differential pencils with frozen argument. Specifically, we deduce the trace formulae for $L_{\alpha,\beta}$.

Trace is an important conserved quantity in matrix theory. In finite dimensional space, the sum of principal diagonal elements of a matrix equals to the sum of eigenvalues which we call the trace. While considering the differential operators in the Hilbert space, however, a sum of infinitely many eigenvalues leads to a divergence series. In 1953, for the first time, Gelfand and Levitan [1] introduced an interesting formula for the Sturm–Liouville operator:

$$\sum_{n=0}^{\infty}\left[\lambda_n - n^2 - \frac{1}{\pi}\int_0^{\pi} q(t)dt\right] = \frac{1}{4}[q(0) + q(\pi)] - \frac{1}{2\pi}\int_0^{\pi} q(t)dt, \tag{3}$$

where the operator was generated by the Neumann–Neumann-type boundary problem

$$-y''(x) + q(x)y(x) = \lambda y(x), \quad y'(0) = y'(\pi) = 0,$$

and $q(x) \in C^1[0,\pi]$, λ_n are the corresponding eigenvalues. After that, many scholars put attention to this quantity, which has many applications in integrable system theory and the inverse problem [2–6]. Also, it turned out that regularized trace formulae had physical meaning, as discussed by Sadovnichii and Podol'skii [7]: "*The meaning of the regularized trace as a measure of the defect of the total energy of the system when it is perturbed in the case when the total energy itself of the system (more precisely, of the model under consideration) is infinite* $\cdots$".

In physics, the interactions between colliding particles is of fundamental significance. For example, Jaulent and Jean [8] describe this phenomenon by s-wave Schrödinger equation with a radial static potential $V(x)$:

$$-y''(x) + V(E,x)y(x) = Ey(x),$$

where $V(E,x)$ is the following form for the energy dependence:

$$V(E,x) = Q(x) + 2\sqrt{E}P(x).$$

With an additional condition $Q(x) = -P^2(x)$, the above Schrödinger equation reduces to the Klein–Gordon equation for a particle of zero mass and energy E, which could serve as part of Lax pair in a two-component Camassa–Holm Equation [9]. Due to the nonlinear dependence on the spectral parameter, the corresponding inverse spectral problem and inverse scattering theory are difficult; we refer to papers [10–16]. With the method of asymptotic estimation on the family of contours [17], Cao and Zhuang [18] studied regularized traces of the Schrödinger equation with energy-dependent potential, in which the final quantity only contains the term with $P(x)$. Further, Yang [19–21] obtained some new formulae related to both $P(x)$ and $Q(x)$.

Recently, the Sturm–Liouville equation with the frozen argument of the form

$$-y''(x) + q(x)y(a) = \lambda y(x), \quad x \in (0,1) \tag{4}$$

has attracted much attention. This equation can be classified as a special case of a functional differential equation with a deviating argument. Especially Equation (4) belongs to the class of loaded equations [22] which arise in mathematical physics, such as groundwater dynamics [23,24], heat conduction [25,26], system with energy feedback [27].

For the inverse spectral problem of differential operators with frozen argument, the classical approaches like the method of spectral mappings and the Gelfand–Levian–Marchenko method do not work. Albeverio et al. [28] and Nizhnik [29,30] studied some special cases where the nonlocal boundary condition guarantees the self-adjointness of the corresponding operator. Bondarenko et al. [31] studied Equation (4) with Boundary conditions (2) where $1/a \in \mathbb{N}$ and $\alpha, \beta \in \{0,1\}$. They classified two cases: degenerate and non-degenerate, depending on the values of α, β and on the parity of $k = 1/a$. Moreover, Bondarenko et al. established the unique solvability of the inverse problem. For the study of different aspects of this operator, such as arbitrary $a \in (0,1)$, non-separated boundary conditions, etc., we refer to [32–40]. Namely, Kuznetsova [41,42] proved the well-posedness of the inverse spectral problem generated by (4) and (2) by a new approach, which is effective in both the rational and irrational cases. Bondarenko [43] explained the relation between the Sturm–Liouville operators with frozen argument and the Laplace operator with integral matching conditions on a star-shaped graph. Also, as pointed out by Buterin [44], the frozen argument term appeared naturally in the study of a Sturm–Liouville operator with constant delay.

However, there are few works on differential pencils with frozen argument. Equation (1) appears, for example, after applying the Fourier method of separation of variables to the following loaded hyperbolic equation:

$$\frac{\partial^2}{\partial t^2}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t) - (\lambda r(x) + q(x))u(a,t),\ 0 < x < 1,\ t > 0,$$

where $a \in (0,1)$, λ is a spectral parameter, $r(x)$ is called the loss function and $q(x)$ the impedance function. This model arises in the study of inverse scattering in lossy layered media [45]; moreover, we assume that the model is affected by a magnetic field exerting a force-per-unit mass represented by $-(\lambda r(x) + q(x))u(a,t)$, i.e., depending on the lateral displacement $u(a,t)$ at point a at time t. The author of [46] studied the inverse spectral problem for $L_{\alpha,\beta}$ by using the approach suggested in [31]. Namely, $L_{\alpha,\beta}$ share the same degenerate and non-degenerate conditions with boundary value problem (4) and (2).

Motivated by the works of Cao and Zhuang [18], we focus on regularized traces of $L_{\alpha,\beta}$. For the next section, we recall some basic facts from [46], i.e., the integral equation for the characteristic functions of $L_{\alpha,\beta}$ and asymptotic behaviour of the corresponding eigenvalues; then, we provide the main results. Finally, we offer a conclusion.

2. Preliminaries and Main Results

We let $C(x,\rho),\ S(x,\rho)$ be the solutions of Equation (1) under the initial conditions

$$C(a,\rho) = S'(a,\rho) = 1, \quad S(a,\rho) = C'(a,\rho) = 0.$$

It is easy to verify that

$$C(x,\rho) = \cos\rho(x-a) + \int_a^x q_1(t)\sin\rho(x-t)dt + \int_a^x q_0(t)\frac{\sin\rho(x-t)}{\rho}dt, \tag{5}$$

$$S(x,\rho) = \frac{\sin\rho(x-a)}{\rho}. \tag{6}$$

Integrating by parts the second term of (5), we obtain

$$\begin{aligned} C(x,\rho) =& \cos\rho(x-a) + \frac{1}{\rho}(q_1(x) - q_1(a)\cos\rho(x-a)) \\ &- \frac{1}{\rho}\int_a^x q_1'(t)\cos\rho(x-t)dt + \frac{1}{\rho}\int_a^x q_0(t)\sin\rho(x-t)dt. \end{aligned} \tag{7}$$

We define

$$\Delta_{\alpha,\beta}(\rho) = \begin{vmatrix} C^{(\alpha)}(0,\rho) & S^{(\alpha)}(0,\rho) \\ C^{(\beta)}(1,\rho) & S^{(\beta)}(1,\rho) \end{vmatrix}; \tag{8}$$

then, it is easy to verify that the eigenvalues of $L_{\alpha,\beta}$ coincide with the zeros of $\Delta_{\alpha,\beta}(\rho)$.

We note that the spectrum of the operator $L(q_0(1-x), q_1(1-x), 1-a, \alpha, \beta)$ coincides with the one of $L(q_0(x), q_1(x), a, \alpha, \beta)$; without loss of generality, we assume $0 < a \le 1/2$ for definiteness.

Theorem 1 ([46])**.** *The characteristic functions $\Delta_{\alpha,\beta}(\rho)$ of the problem $L_{\alpha,\beta}$ have the form of*

$$\Delta_{\alpha,\alpha}(\rho) = \rho^{2\alpha-2}\left(\rho\sin\rho - W_{\alpha,\alpha}(a,\rho) + \int_0^1 (U_{\alpha,\alpha}(t)\cos\rho t + V_{\alpha,\alpha}(t)\sin\rho t)\,dt\right), \tag{9}$$

if $\alpha = \beta$, and

$$\Delta_{\alpha,\beta}(\rho) = \rho^{-1}\left((-1)^\alpha(\rho\cos\rho - W_{\alpha,\beta}(a,\rho)) + \int_0^1 (U_{\alpha,\beta}(t)\sin\rho t + V_{\alpha,\beta}(t)\cos\rho t)\,dt\right), \tag{10}$$

if $\alpha \neq \beta$*, where*

$$W_{\alpha,\beta}(a,\rho) = \begin{cases} q_1(a)\sin\rho - q_1(0)\sin\rho(1-a) - q_1(1)\sin\rho a, & (\alpha,\beta) = (0,0), \\ q_1(a)\sin\rho, & (\alpha,\beta) = (1,1), \\ q_1(a)\cos\rho - q_1(0)\cos\rho(1-a), & (\alpha,\beta) = (0,1), \\ q_1(a)\cos\rho - q_1(1)\cos\rho a, & (\alpha,\beta) = (1,0). \end{cases}$$

Moreover, the functions $U_{\alpha,\beta}(t)$ *and* $V_{\alpha,\beta}(t)$ *have the following form:*

$$U_{\alpha,\beta}(t) = \frac{(-1)^{\alpha\beta}}{2} \begin{cases} q_0(1-a+t) + dq_0(1-a-t), & t \in (0,a), \\ cq_0(1+a-t) + dq_0(1-a-t), & t \in (a,1-a), \\ c\big(q_0(1+a-t) + q_0(t-1+a)\big), & t \in (1-a,1) \end{cases} \tag{11}$$

and

$$V_{\alpha,\beta}(t) = \frac{(-1)^{\gamma}}{2} \begin{cases} -q_1'(1-a+t) + dq_1'(1-a-t), & t \in (0,a), \\ cq_1'(1+a-t) + dq_1'(1-a-t), & t \in (a,1-a), \\ c\big(q_1'(1+a-t) - q_1'(t-1+a)\big), & t \in (1-a,1), \end{cases} \tag{12}$$

where $c = (-1)^{1+\beta}$*,* $d = (-1)^{\alpha+\beta}$ *and* $\gamma = \max\{\alpha,\beta\}$*.*

We let $\mathbb{Z}_0 := \mathbb{Z} \setminus \{0\}$, $\mathbb{Z}_1 := \{\pm 0, \pm 1, \pm 2, \cdots\}$ and $\mathbb{Z}_2 := \mathbb{Z}$. From this, we stipulate that if n denotes an index for eigenvalues, then $n \in \mathbb{Z}_0$ for $(\alpha,\beta) = (0,0)$, $n \in \mathbb{Z}_1$ for $(\alpha,\beta) = (1,1)$ and $n \in \mathbb{Z}_2$ for $(\alpha,\beta) = (0,1)$ or $(\alpha,\beta) = (1,0)$.

Theorem 2 ([46])**.** *The eigenvalues of* $L_{\alpha,\beta}$ *can be numbered as* $\{\rho_{n,\alpha,\beta}\}$*, counting with their multiplicities, such that the following asymptotics hold:*

(i) For $(\alpha,\beta) = (0,0)$*,*

$$\rho_{n,0,0} = n\pi + \frac{q_1(0) + (-1)^{n+1} q_1(1)}{n\pi} \sin n\pi a + \frac{\kappa_{0,0,n}}{n}, \{\kappa_{n,0,0}\} \in l_2; \tag{13}$$

(ii) For $(\alpha,\beta) = (1,1)$*,*

$$\rho_{n,1,1} = n\pi + \frac{\kappa_{1,1,n}}{n}, \ \{\kappa_{n,1,1}\} \in l_2; \tag{14}$$

(iii) For $(\alpha,\beta) = (0,1)$*,*

$$\rho_{n,0,1} = \left(n - \frac{1}{2}\right)\pi + \frac{q_1(0)}{n\pi} \sin\left(n - \frac{1}{2}\right)\pi a + \frac{\kappa_{0,1,n}}{n}, \ \{\kappa_{n,0,1}\} \in l_2; \tag{15}$$

(iv) For $(\alpha,\beta) = (1,0)$*,*

$$\rho_{n,1,0} = \left(n - \frac{1}{2}\right)\pi + \frac{(-1)^{n+1} q_1(1)}{n\pi} \cos\left(n - \frac{1}{2}\right)\pi a + \frac{\kappa_{1,0,n}}{n}, \ \{\kappa_{n,1,0}\} \in l_2. \tag{16}$$

In order to obtain the trace formulae of $L_{\alpha,\beta}$, we need the following lemma.

Lemma 1 ([17])**.** *Let* $\omega(z)$ *and* $\omega_0(z)$ *be two entire functions on a* $z-$*plane and have no zeros on some closed contour* Γ*. Suppose that* $\omega(z) \backslash \omega_0(z) = 1 + \theta(z)$*, where* $|\theta(z)| \leq \delta$ *on* Γ*,* $0 < \delta < 1$*; then,*

$$\sum_{\Gamma} (\lambda_n^\sigma - \mu_n^\sigma) = -\frac{1}{2\pi i} \oint_{\Gamma} \sigma z^{\sigma-1} \ln \frac{\omega(z)}{\omega_0(z)} dz, \tag{17}$$

where λ_n *and* μ_n *are zeros of* $\omega(z)$ *and* $\omega_0(z)$ *inside* Γ *correspondingly, and* σ *is a positive integer.*

We let $\{\tau_{n,\alpha,\beta}\}$ be the spectrum of $L(0,0,\alpha,\beta,a)$, $\alpha,\beta\in\{0,1\}$.

Theorem 3. *The following formulae hold:*

$$\sum_{n\in\mathbb{Z}_j}(\rho_{n,\alpha,\beta}-\tau_{n,\alpha,\beta})=q_1(a), \tag{18}$$

where $j=0,1,2$.

Proof. We let Γ_N, $N=1,2,\cdots$ be the counterclockwise square contours $A_NB_NC_ND_N$ with

$$A_N=\left(N+\frac{3}{4}\right)(1-i),\ \ B_N=\left(N+\frac{3}{4}\right)(1+i),$$
$$C_N=\left(N+\frac{3}{4}\right)(-1+i),\ \ D_N=\left(N+\frac{3}{4}\right)(-1-i).$$

Formulae (13)–(16) imply that, for sufficiently large N, the eigenvalues $\rho_{n,\alpha,\beta}$, $|n|\le N$ are inside Γ_N, and the eigenvalues $\rho_{n,\alpha,\beta}$ with $|n|>N$ are outside Γ_N. Also, since $\{\tau_{n,\alpha,\beta}\}$ is the the spectrum of $L(0,0,\alpha,\beta,a)$, we have $\{\tau_{n,\alpha,\beta}\}\cap\Gamma_N=\varnothing$.

Now we prove the theorem for the case $(\alpha,\beta)=(0,0)$; the other cases are similar. We let $\Delta^{\circ}_{0,0}(\rho)=\sin\rho/\rho$ be the characteristic function of $L(0,0,0,0,a)$. By using (9), (11) and (12), we estimate the fraction $\Delta_{0,0}(\rho)/\Delta^{\circ}_{0,0}(\rho)$ on the contour Γ_N for sufficiently large N:

$$\frac{\Delta_{0,0}(\rho)}{\Delta^{\circ}_{0,0}(\rho)}=1+\frac{q_1(0)\sin\rho(1-a)+q_1(1)\sin\rho a}{\rho\sin\rho}-\frac{q_1(a)}{\rho}+o\left(\frac{1}{\rho}\right),\rho\in\Gamma_N.$$

Using the Taylor series expansion, we have

$$\ln\frac{\Delta_{0,0}(\rho)}{\Delta^{\circ}_{0,0}(\rho)}=\frac{q_1(0)\sin\rho(1-a)+q_1(1)\sin\rho a}{\rho\sin\rho}-\frac{q_1(a)}{\rho}+o\left(\frac{1}{\rho}\right),\rho\in\Gamma_N.$$

Calculating the contour integral by (17) and using residue calculation, we obtain that for sufficiently large N,

$$\begin{aligned}\sum_{n=-N}^{N}(\rho_{0,0,n}-\mu_{0,0,n})&=-\frac{1}{2\pi i}\oint_{\Gamma_N}\ln\frac{\Delta_{0,0}(\rho)}{\Delta^{\circ}_{0,0}(\rho)}d\rho\\&=q_1(a)+q_1(0)\left(2\sum_{n=1}^{N}\theta_n-(1-a)\right)+q_1(1)\left(2\sum_{n=1}^{N}\zeta_n-a\right)+o(1),\end{aligned}$$

where

$$\theta_n=(-1)^{n+1}\frac{\sin n\pi(1-a)}{n\pi},\qquad \zeta_n=(-1)^{n+1}\frac{\sin n\pi a}{n\pi}.$$

Together with the Fourier series

$$x=2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin n\pi x}{n\pi},\quad x\in(-1,1),$$

we arrive at (18) for $(\alpha,\beta)=(0,0)$ by taking $N\to\infty$. Note that for the cases $(\alpha,\beta)=(0,1)$ and $(\alpha,\beta)=(1,0)$, we need the Fourier series expansion

$$\frac{1}{2}=2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(n-\frac{1}{2})\pi x}{(n-\frac{1}{2})\pi},x\in(-1,1).$$

□

3. Conclusions

In this paper, we deduce the trace formulae of second-order differential pencils with frozen argument. By applying the methods in complex analysis, we calculate the regularized sum of infinite eigenvalues of $L_{\alpha,\beta}$ in the Gelfand–Levitan sense. Let us mention some advantages of our approach:

1. Operator $L_{\alpha,\beta}$ is non-selfadjoint which may have complex eigenvalues with multiplicity; however, the method we use allows us dealing with the regularized sum of eigenvalues in the whole meaning.
2. The regularized trace of $L_{\alpha,\beta}$ depends only on the value of $q_1(x)$ at the frozen point a, regardless of the boundary conditions and the potential $q_0(x)$.
3. In the study of inverse spectral problem of $L_{\alpha,\beta}$, the rationality of frozen argument a is important. Whether a is rational leads to different approachs of inverse spectral problem. However, we do not need this distinction while calculating the trace formulae.

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