

Article

Properties of a Special Holomorphic Function Linked with a Generalized Multiplier Transformation

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Abstract: In the present paper, we introduce a special holomorphic function in $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ which is associated with new generalized multiplier transformations. We investigate several properties of the defined function using the concept of subordination, then highlight a number of cases with interesting results.

Keywords: Holomorphic function; generalized multiplier transformation; subordination

MSC: 30C45



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1. Introduction

Numerous geometric function theory (GFT) domains have found considerable applicability for a number of operators, with the differential and integral operators being the most fascinating of these. From the beginning of GFT, many mathematicians have worked on integral operators, especially J.W. Alexander [1], S.D. Bernardi [2], and R.J. Libera [3]. The Ruscheweyh differential operator [4] and the Sălăgean differential operator [5] are frequently used by researchers in their findings. Interested readers can refer to [6] for an application of the Sălăgean operator. These operators have been generalized in GFT by adding new ones. Al-Oboudi introduced and researched an operator [7] which generalizes the Sălăgean differential operator. Alb Lupas generalized the Al-Oboudi operator by exploring a multiplier transformation [8]. Cătaș in [9] examined a new multiplier transformation in order to take p-valent functions into account. Recently, Swamy investigated a new generalized multiplier transformation [10,11]. Readers may refer to [12–18] to learn more about other operators in GFT. Numerous novel operators have been defined, their properties have been studied, and new families of exceptional univalent functions have been defined using these novel operators, leading to the discovery of remarkable results. A number of mathematicians have simultaneously produced exciting results in a range of research fields involving GFT using linear combinations of two operators, as demonstrated in several very recent papers [19–23].

Let the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ be denoted by $\mathcal{U}$, where $\mathbb{C}$ is the set of all complex numbers, let $\mathbb{N}_0 = \{0, 1, 2, \dots\} = \mathbb{N} \cup \{0\}$, and let $\mathbb{R} = (-\infty, +\infty)$. The family of all holomorphic functions in $\mathcal{U}$ that have the form $f(z) = z^p + \sum_{j=p+n}^{\infty} a_j z^j$, are indicated by $\mathfrak{A}(n, p)$, $n, p \in \mathbb{N}$. We set $\mathfrak{A}(1, 1) = \mathfrak{A}$, $\mathfrak{A}(1, p) = \mathfrak{A}_p$, and $\mathfrak{A}(n, 1) = \mathfrak{A}(n)$, which are known families of holomorphic functions in the unit disc $\mathcal{U}$.

In order to complete our research in this paper, we must first review the definitions of the new multiplier transformation and other required operators.

Definition 1 ([11,24]). Let $f \in \mathfrak{A}(n, p)$. The new generalized multiplier transformation $\mathfrak{I}_{p,\alpha,\beta}^m : \mathfrak{A}(n, p) \rightarrow \mathfrak{A}(n, p)$ is defined by

$$\mathfrak{I}_{p,\alpha,\beta}^m f(z) = z^p + \sum_{j=p+n}^{\infty} \left(\frac{j\beta + \alpha}{p\beta + \alpha} \right)^m a_j z^j, \quad (1)$$

where $\alpha \in \mathbb{R}, \beta \geq 0$ such that $p\beta + \alpha > 0, n, p \in \mathbb{N}, m \in \mathbb{N}_0$, and $z \in \mathfrak{U}$.

Definition 2 ([9]). Let $f \in \mathfrak{A}(n, p)$. The Cătaş operator $L_p^m(l, \beta) : \mathfrak{A}(n, p) \rightarrow \mathfrak{A}(n, p)$ is defined by

$$L_p^m(l, \beta) f(z) = z^p + \sum_{j=p+n}^{\infty} \left(\frac{l + p + (j - p)\beta}{l + p} \right)^m a_j z^j, \quad (2)$$

where $l \geq 0, \beta \geq 0, n, p \in \mathbb{N}, m \in \mathbb{N}_0$, and $z \in \mathfrak{U}$.

Definition 3 ([25]). Let $f \in \mathfrak{A}(1, p)$. The operator $\mathcal{AD}_{p,\beta}^m : \mathfrak{A}(1, p) \rightarrow \mathfrak{A}(1, p)$ is defined by

$$\mathcal{AD}_{p,\beta}^m f(z) = z^p + \sum_{j=p+1}^{\infty} \left(\frac{p + (j - p)\beta}{p} \right)^m a_j z^j, \quad (3)$$

where $\beta \geq 0, n, p \in \mathbb{N}, m \in \mathbb{N}_0$, and $z \in \mathfrak{U}$.

Definition 4 ([26]). Let $f \in \mathfrak{A}(1, p)$. The operator $\mathcal{D}_p^m : \mathfrak{A}(1, p) \rightarrow \mathfrak{A}(1, p)$ is defined by

$$\mathcal{D}_p^m f(z) = z^p + \sum_{j=p+1}^{\infty} \left(\frac{j}{p} \right)^m a_j z^j, \quad (4)$$

where $n, p \in \mathbb{N}, m \in \mathbb{N}_0$, and $z \in \mathfrak{U}$.

It follows from (1) that $\mathfrak{I}_{p,\alpha,0}^m f(z) = f(z)$, $\mathfrak{I}_{p,0,\beta}^m f(z) = z f'(z) / p$, and

$$(p\beta + \alpha) \mathfrak{I}_{p,\alpha,\beta}^{m+1} f(z) = \alpha \mathfrak{I}_{p,\alpha,\beta}^m f(z) + \beta z (\mathfrak{I}_{p,\alpha,\beta}^m f(z))' \quad (5)$$

The operator $\mathfrak{I}_{p,\alpha,\beta}^m$ on $f \in \mathfrak{A}(n, p)$ was explored in [24], $\mathfrak{I}_{p,\alpha,\beta}^m$ on $f \in \mathfrak{A}_p = \mathfrak{A}(1, p)$ was examined in [11], $\mathfrak{I}_{p,\alpha,1}^m = \mathfrak{I}_p^m(\alpha)$ on $f \in \mathfrak{A}_p$, $\alpha > -p$ was investigated by Shivaprasad Kumar et al. [21] and by Srivastava et al. [27], and $\mathfrak{I}_{p,l+p-p\beta,\beta}^m = L_p^m(l, \beta)$, $l > -p, \beta \geq 0$ on $f \in \mathfrak{A}(n, p)$ was invented by Cătaş in [9]. Moreover, $\mathfrak{I}_{p,p-p\beta,\beta}^m = \mathcal{AD}_{p,\beta}^m$, the operator on $f \in \mathfrak{A}_p$ was established by Aouf et al. [25], and $\mathfrak{I}_{p,0,\beta}^m = \mathcal{D}_p^m$ on $f \in \mathfrak{A}_p$ by Aouf and Mustafa [26], Kamali and Orhan [28], and Orhan and Kiziltunc [29]. Finally, we mention that the operators $\mathfrak{I}_p^m(\alpha)$ and $L_p^m(l, \beta)$ have been investigated for $\alpha \geq 0$ and $l \geq 0$, respectively.

Setting $p = n = 1$ in (1) and choosing suitable values for α and β , we obtain:

(i) The operator $\mathfrak{I}_{1,\alpha,\beta}^m$ on $f \in \mathfrak{A}$, examined in [10]; (ii) $\mathfrak{I}_{1,1-\beta,\beta}$ on $f \in \mathfrak{A}$, explored by Al-Oboudi [7]; (iii) $\mathfrak{I}_{1,0,1}$ on $f \in \mathfrak{A}$, due to Sălăgean [5] and considered for $m \geq 0$ in [30]; (iv) $\mathfrak{I}_1^m(\alpha)$, $\alpha \geq 0$, invented in [31] and [32]; and finally (v) $\mathfrak{I}_1^m(1)$, experimented with by Uralagaddi and Somanatha [33].

The results of studying the functions connected to any of the aforementioned operators are current research trends. We define the following in response to these trends utilizing the newly generated multiplier transformation $\mathfrak{I}_{p,\alpha,\beta}^m$ provided by

$$\mathfrak{S}_{p,\alpha,\beta}^{m,\delta,\gamma}(z) = \delta \mathfrak{I}_{p,\alpha,\beta}^m f(z) + \gamma \mathfrak{I}_{p,\alpha,\beta}^{m+1} f(z), \quad z \in \mathfrak{U}, \quad (6)$$

where $f \in \mathfrak{A}(n, p)$, $\delta, \gamma \in \mathbb{C}$, $\beta \geq 0$, $n, p \in \mathbb{N}$, $m \in \mathbb{N}_0$, and $\alpha \in \mathbb{R}$ such that $\Re(\gamma) \geq 0$, $\gamma + \delta \in \mathbb{R}$, $p\beta + \alpha > 0$, allowing us to demonstrate its interesting properties.

Throughout this paper, unless otherwise mentioned we assume that $n, p \in \mathbb{N}$, $m \in \mathbb{N}_0$, $f \in \mathfrak{A}(n, p)$, $\delta, \gamma \in \mathbb{C}$, $\beta \geq 0$, and $\alpha \in \mathbb{R}$ such that $\Re(\gamma) \geq 0$, $\gamma + \delta \in \mathbb{R}$ and $p\beta + \alpha > 0$.

We notice the following:

(i) If $\delta = 1 - \gamma$ in (6), then we obtain

$$\mathfrak{M}_{p,\alpha,\beta}^{m,1-\gamma,\gamma}(z) = (1 - \gamma)\mathfrak{I}_{p,\alpha,\beta}^m f(z) + \gamma\mathfrak{I}_{p,\alpha,\beta}^{m+1} f(z), z \in \mathfrak{U}. \quad (7)$$

(ii) If $\alpha = l + p - p\beta$, $l > -p$ in (6), then we obtain

$$\mathfrak{R}_{p,l,\beta}^{m,\delta,\gamma}(z) = \delta L_p^m(l, \beta)f(z) + \gamma L_p^{m+1}(l, \beta)f(z), z \in \mathfrak{U}. \quad (8)$$

(iii) If $\alpha = p - p\beta$ in (6), then we have

$$\mathfrak{T}_{p,\beta}^{m,\delta,\gamma}(z) = \delta \mathcal{A}D_{p,\beta}^m f(z) + \gamma \mathcal{A}D_{p,\beta}^{m+1} f(z), z \in \mathfrak{U}. \quad (9)$$

(iv) If $\alpha = 0$ in (6), then we have

$$\mathfrak{Q}_p^{m,\delta,\gamma}(z) = \delta D_p^m f(z) + \gamma D_p^{m+1} f(z), z \in \mathfrak{U}. \quad (10)$$

(v) If $\alpha = m = 0$ in (6), then we obtain

$$\mathfrak{K}_p^{\delta,\gamma}(z) = \delta f(z) + \gamma \frac{zf'(z)}{p}, z \in \mathfrak{U}. \quad (11)$$

Remark 1. $\mathfrak{M}_{p,\alpha,\beta}^{m,1-\gamma,\gamma}$ was defined by Aouf et al. in [25] when $n = 1$ and $\alpha = l + p - p\beta$, while $\mathfrak{K}_p^{\delta,\gamma}$ was taken into consideration by Aouf et al. in [25] when $n = 1$ and $\delta = l - \gamma$.

We require the lemma stated below in order to derive our results.

Lemma 1 ([34,35]). Let $x = x_1 + ix_2$, $y = y_1 + iy_2$ and $F(x, y)$ be a complex valued function $F : \Lambda \rightarrow \mathbb{C}$, $\Lambda \subset \mathbb{C} \times \mathbb{C}$, and let $F(x, y)$ satisfy the following: (i) $F(x, y)$ is continuous in Λ ; (ii) $\Re\{F(1, 0)\} > 0$ and $(1, 0) \in \Lambda$; and (iii) $\Re\{F(ix_2, y_1)\} \leq 0$ and $(ix_2, y_1) \in \Lambda$ such that $y_1 \leq -\frac{n(1+x_2^2)}{2}$. Let $p(z) = 1 + \vartheta_n z^n + \vartheta_{n+1} z^{n+1} + \dots$ be holomorphic in $\mathfrak{U}$ such that $(p(z), zp'(z)) \in \Lambda$; then, $\Re(p(z)) > 0$ whenever $\Re\{F(p(z), zp'(z))\} > 0$, $z \in \mathfrak{U}$.

2. Main Results.

Theorem 1. Let $\mathfrak{S}_{p,\alpha,\beta}^{m,\delta,\gamma}$ be as detailed in (6). If

$$\Re\left\{\frac{\mathfrak{S}_{p,\alpha,\beta}^{m,\delta,\gamma}(z)}{z^p}\right\} > \rho, z \in \mathfrak{U} \quad (12)$$

and $\rho < \gamma + \delta$, then

$$\Re\left\{\frac{\mathfrak{I}_{p,\alpha,\beta}^m f(z)}{z^p}\right\} > \frac{n\beta \Re(\gamma) + 2\rho(p\beta + \alpha)}{n\beta \Re(\gamma) + 2(p\beta + \alpha)}, z \in \mathfrak{U}. \quad (13)$$

Proof. Let $\Omega = \frac{n\beta \Re(\gamma) + 2\rho(p\beta + \alpha)}{n\beta \Re(\gamma) + 2(p\beta + \alpha)}$. Let $p(z)$ be defined by

$$\frac{\mathfrak{I}_{p,\alpha,\beta}^m f(z)}{z^p} = \Omega + (1 - \Omega)p(z). \quad (14)$$

It can be seen that $p(z) = 1 + \vartheta_n z^n + \vartheta_{n+1} z^{n+1} + \dots$ is holomorphic in the open unit disc $\mathfrak{U}$. Using (2), we can deduce from (14) that

$$\frac{\mathfrak{I}_{p,\alpha,\beta}^{m+1} f(z)}{z^p} = \Omega + (1 - \Omega)p(z) + \frac{\beta(1 - \Omega)}{(p\beta + \alpha)} z p'(z). \quad (15)$$

From (6), (14), and (15) it is evident that

$$\frac{\mathfrak{S}_{p,\alpha,\beta}^{m,\delta,\gamma}(z)}{z^p} = (\gamma + \delta)\Omega + (\gamma + \delta)(1 - \Omega)p(z) + \frac{\beta\gamma(1 - \Omega)}{(p\beta + \alpha)} z p'(z). \quad (16)$$

From (12) and (16), we obtain

$$\begin{aligned} & \Re \left\{ \frac{\mathfrak{S}_{p,\alpha,\beta}^{m,\delta,\gamma}(z)}{z^p} - \rho \right\} \\ &= \Re \left\{ (\gamma + \delta)\Omega - \rho + (\gamma + \delta)(1 - \Omega)p(z) + \frac{\beta\gamma(1 - \Omega)}{(p\beta + \alpha)} z p'(z) \right\} > 0. \end{aligned}$$

We can then define $F(x, y)$ by

$$F(x, y) = (\gamma + \delta)\Omega - \rho + (\gamma + \delta)(1 - \Omega)x + \frac{\beta\gamma(1 - \Omega)}{(p\beta + \alpha)} y$$

with $x_1 + ix_2 = x = p(z)$ and $y_1 + iy_2 = y = z p'(z)$. Now, we have the following:

(i) $F(x, y)$ is continuous in Λ ; (ii) $(1, 0) \in \Lambda$, $\Re\{F(1, 0)\} = (\gamma + \delta) - \rho > 0$; and (iii) $\Re\{F(ix_2, y_1)\} = \rho - (\gamma + \delta)\Omega - \frac{\beta(1 - \Omega)\Re(\gamma)}{p\beta + \alpha} y_1 \leq \Omega - \rho + \frac{n\beta(1 - \Omega)(1 + x_2^2)\Re(\gamma)}{2(p\beta + \alpha)} \leq 0$, $(ix_2, y_1) \in \Lambda$ with $y_1 \leq -\frac{n(1 + x_2^2)}{2}$.

Thus, every requirement of Lemma 1 is satisfied by the function $F(x, y)$; hence, $\Re\{p(z)\} > 0$ ($z \in \mathfrak{U}$), leading to (13). This establishes our theorem. $\square$

Theorem 2. Let $\mathfrak{S}_{p,\alpha,\beta}^{m,\delta,\gamma}$ be as provided by (6). If

$$\Re \left\{ \frac{\mathfrak{S}_{p,\alpha,\beta}^{m,\delta,\gamma}(z)}{z^p} \right\} < \rho, \quad z \in \mathfrak{U}$$

and $\rho > \gamma + \delta$, then

$$\Re \left\{ \frac{I_{p,\alpha,\beta}^m f(z)}{z^p} \right\} < \frac{n\beta\Re(\gamma) + 2\rho(p\beta + \alpha)}{n\beta\Re(\gamma) + 2(\gamma + \delta)(p\beta + \alpha)}, \quad z \in \mathfrak{U}. \quad (17)$$

Proof. Let $\Omega = \frac{n\beta\Re(\gamma) + 2\rho(p\beta + \alpha)}{n\beta\Re(\gamma) + 2(\gamma + \delta)(p\beta + \alpha)} > 1$. Let $p(z)$ be defined by

$$\frac{\mathfrak{I}_{p,\alpha,\beta}^m f(z)}{z^p} = \Omega + (1 - \Omega)p(z).$$

We note that $p(z) = 1 + \vartheta_n z^n + \vartheta_{n+1} z^{n+1} + \dots$ is holomorphic in the open unit disc $\mathfrak{U}$. The methodology from Theorem 1 yields the following:

$$\Re \left\{ \rho - \frac{\mathfrak{S}_{p,\alpha,\beta}^{m,\delta,\gamma}(z)}{z^p} \right\} = \Re \left\{ \rho - (\gamma + \delta)\Omega - (\gamma + \delta)(1 - \Omega)p(z) - \frac{\beta\gamma(1 - \Omega)}{p\beta + \alpha}zp'(z) \right\} > 0.$$

Let

$$F(x, y) = \rho - (\gamma + \delta)\Omega - (\gamma + \delta)(1 - \Omega)x + \frac{\beta\gamma(1 - \Omega)}{p\beta + \alpha}y \quad (18)$$

with $x = x_1 + ix_2 = p(z)$ and $y = y_1 + iy_2 = zp'(z)$. Then, from (18) it follows that: (i) $F(x, y)$ is continuous in Λ ; (ii) $(1, 0) \in \Lambda$, $\Re\{F(1, 0)\} = \rho - (\gamma + \delta) > 0$; and (iii) $\Re\{F(ix_2, y_1)\} = \rho - (\gamma + \delta)\Omega - \frac{\beta(1-\Omega)\Re(\gamma)}{p\beta+\alpha}y_1 \leq \Omega - \rho + \frac{n\beta(1-\Omega)(1+x_2^2)\Re(\gamma)}{2(p\beta+\alpha)} \leq 0$, for all $(ix_2, y_1) \in \Lambda$ such that $y_1 \leq -\frac{n(1+x_2^2)}{2}$.

Thus, every requirement of Lemma 1 is satisfied by the function $F(x, y)$; hence, $\Re\{p(z)\} > 0$ ($z \in \mathfrak{U}$), from which we obtain (17). This establishes our theorem. $\square$

Analogous to above, we have the following theorems.

Theorem 3. Let $\mathfrak{S}_{p,\alpha,\beta}^{m,\delta,\gamma}$ be as detailed in (6). Then,

$$\Re \left\{ \frac{(\mathfrak{I}_{p,\alpha,\beta}^m f(z))'}{z^p} \right\} > \frac{n\beta\Re(\gamma) + 2\rho(p\beta + \alpha)}{n\beta\Re(\gamma) + 2(\gamma + \delta)(p\beta + \alpha)}, \quad z \in \mathfrak{U}$$

whenever

$$\Re \left\{ \frac{(\mathfrak{S}_{p,\alpha,\beta}^{m,\delta,\gamma}(z))'}{pz^{p-1}} \right\} > \rho, \quad z \in \mathfrak{U}$$

and $\rho < \gamma + \delta$.

Theorem 4. Let $\mathfrak{S}_{p,\alpha,\beta}^{m,\delta,\gamma}$ be as detailed in (6). Then,

$$\Re \left\{ \frac{(\mathfrak{I}_{p,\alpha,\beta}^m f(z))'}{z^p} \right\} < \frac{n\beta\Re(\gamma) + 2\rho(p\beta + \alpha)}{n\beta\Re(\gamma) + 2(\gamma + \delta)(p\beta + \alpha)}, \quad z \in \mathfrak{U}$$

whenever

$$\Re \left\{ \frac{(\mathfrak{S}_{p,\alpha,\beta}^{m,\delta,\gamma}(z))'}{pz^{p-1}} \right\} < \rho, \quad z \in \mathfrak{U}$$

and $\rho > \gamma + \delta$.

Remark 2. Analogous results can be obtained by taking (i) $\alpha = p - p\beta + l$, $l > -p$, (ii) $\alpha = p - p\beta$, and (iii) $\alpha = 0$ in Theorems 1, 2, 3, and 4 for $\mathfrak{R}_{p,l,\beta}^{m,\delta,\gamma}(z)$, $\mathfrak{T}_{p,\beta}^{m,\delta,\gamma}(z)$, and $\mathfrak{Q}_p^{m,\delta,\gamma}(z)$, which are detailed in (8), (9), and (10), respectively.

3. Consequences

By inserting $\delta = 1 - \gamma$ into the first theorem, we arrive at the following conclusion.

Corollary 1. Let $\mathfrak{M}_{p,\alpha,\beta}^{m,1-\gamma,\gamma}$ be as detailed in (7). If $\rho < 1$ and

$$\Re \left\{ \frac{\mathfrak{M}_{p,\alpha,\beta}^{m,1-\gamma,\gamma}(z)}{z^p} \right\} > \rho, \quad z \in \mathfrak{U},$$

then

$$\Re \left\{ \frac{\mathfrak{I}_{p,\alpha,\beta}^m f(z)}{z^p} \right\} > \frac{n\beta\Re(\gamma) + 2\rho(p\beta + \alpha)}{n\beta\Re(\gamma) + 2(p\beta + \alpha)}, \quad z \in \mathfrak{U}.$$

When $\delta = \bar{\gamma}$, Corollary 2 claims the following conclusion of the first theorem.

Corollary 2. Let $\mathfrak{S}_{p,\alpha,\beta}^{m,\bar{\gamma},\gamma}$ be as detailed in (6). If

$$\Re \left\{ \frac{\mathfrak{S}_{p,\alpha,\beta}^{m,\bar{\gamma},\gamma}(z)}{z^p} \right\} > \rho, \quad z \in \mathfrak{U}$$

and $\rho < 2\Re(\gamma)$, then

$$\Re \left\{ \frac{\mathfrak{I}_{p,\alpha,\beta}^m f(z)}{z^p} \right\} > \frac{n\beta\Re(\gamma) + 2\rho(p\beta + \alpha)}{4\Re(\gamma)[(p\beta + \alpha) + n\beta]}, \quad z \in \mathfrak{U}.$$

Further, $\Re \left\{ \frac{\mathfrak{I}_{p,\alpha,\beta}^m f(z)}{z^p} \right\} > \frac{n\beta + 3\rho(p\beta + \alpha)}{n\beta + 4(p\beta + \alpha)}, \quad z \in \mathfrak{U}$ whenever $\Re \left\{ \frac{\mathfrak{S}_{p,\alpha,\beta}^{m,\bar{\gamma},\gamma}(z)}{z^p} \right\} > \frac{3\Re(\gamma)}{2}, \quad z \in \mathfrak{U}.$

By allowing $\delta = 1 - \gamma$ in the second theorem, we arrive at the corollary stated below.

Corollary 3. Let $\mathfrak{M}_{p,\alpha,\beta}^{m,1-\gamma,\gamma}$ be as detailed in (7). If $\rho < 1$ and

$$\Re \left\{ \frac{\mathfrak{M}_{p,\alpha,\beta}^{m,1-\gamma,\gamma}(z)}{z^p} \right\} > \rho, \quad z \in \mathfrak{U},$$

then

$$\Re \left\{ \frac{\mathfrak{I}_{p,\alpha,\beta}^m f(z)}{z^p} \right\} > \frac{n\beta\Re(\gamma) + 2\rho(p\beta + \alpha)}{n\beta\Re(\gamma) + 2(p\beta + \alpha)}, \quad z \in \mathfrak{U}.$$

When $\delta = \bar{\gamma}$, Corollary 4 affirms an outcome of the second theorem.

Corollary 4. Let $\mathfrak{S}_{p,\alpha,\beta}^{m,\bar{\gamma},\gamma}$ be as indicated (6). If $\rho < 2\Re(\gamma)$ and

$$\Re \left\{ \frac{\mathfrak{S}_{p,\alpha,\beta}^{m,\bar{\gamma},\gamma}(z)}{z^p} \right\} > \rho, \quad z \in \mathfrak{U},$$

then

$$\Re \left\{ \frac{\mathfrak{I}_{p,\alpha,\beta}^m f(z)}{z^p} \right\} > \frac{n\beta\Re(\gamma) + 2\rho(p\beta + \alpha)}{4\Re(\gamma)[(p\beta + \alpha) + n\beta]}, \quad z \in \mathfrak{U}.$$

Further, $\Re \left\{ \frac{\mathfrak{I}_{p,\alpha,\beta}^m f(z)}{z^p} \right\} > \frac{n\beta + 3\rho(p\beta + \alpha)}{n\beta + 4(p\beta + \alpha)}, \quad z \in \mathfrak{U}$ whenever $\Re \left\{ \frac{\mathfrak{S}_{p,\alpha,\beta}^{m,\bar{\gamma},\gamma}(z)}{z^p} \right\} > \frac{3\Re(\gamma)}{2}, \quad z \in \mathfrak{U}.$

By allowing $\delta = 1 - \gamma$ in the third theorem, we arrive at the following conclusion.

Corollary 5. Let $\mathfrak{M}_{p,\alpha,\beta}^{m,1-\gamma,\gamma}$ be as defined in (7). Then, we have

$$\Re \left\{ \frac{\left(\mathfrak{I}_{p,\alpha,\beta}^m f(z) \right)'}{z^p} \right\} > \frac{n\beta\Re(\gamma) + 2\rho(p\beta + \alpha)}{n\beta\Re(\gamma) + 2(p\beta + \alpha)}, z \in \mathfrak{U}$$

whenever $\rho < 1$ and

$$\Re \left\{ \frac{\left(\mathfrak{M}_{p,\alpha,\beta}^{m,1-\gamma,\gamma}(z) \right)'}{z^p} \right\} > \rho, z \in \mathfrak{U}.$$

When $\delta = \bar{\gamma}$, Corollary 6 asserts a result of the third theorem.

Corollary 6. Let $\mathfrak{S}_{p,\alpha,\beta}^{m,\bar{\gamma},\gamma}$ be as indicated (7). Then, we have

$$\Re \left\{ \frac{\left(\mathfrak{I}_{p,\alpha,\beta}^m f(z) \right)'}{z^p} \right\} > \frac{n\beta\Re(\gamma) + 2\rho(p\beta + \alpha)}{4\Re(\gamma)[(p\beta + \alpha) + \beta n]}, z \in \mathfrak{U}$$

whenever $\rho < 2\Re(\gamma)$ and

$$\Re \left\{ \frac{\left(\mathfrak{S}_{p,\alpha,\beta}^{m,\bar{\gamma},\gamma}(z) \right)'}{z^p} \right\} > \rho, z \in \mathfrak{U}.$$

Further, $\Re \left\{ \frac{\mathfrak{I}_{p,\alpha,\beta}^m f(z)}{z^p} \right\} > \frac{n\beta + 3\rho(p\beta + \alpha)}{n\beta + 4(p\beta + \alpha)}, z \in \mathfrak{U}$ whenever $\Re \left\{ \frac{\mathfrak{S}_{p,\alpha,\beta}^{m,\bar{\gamma},\gamma}(z)}{z^p} \right\} > \frac{3\Re(\gamma)}{2}, z \in \mathfrak{U}$.

By allowing $\delta = 1 - \text{gamma}$ in the fourth theorem, we arrive at the following conclusion.

Corollary 7. Let $\mathfrak{M}_{p,\alpha,\beta}^{m,1-\gamma,\gamma}$ be as detailed in (7). Then, we have

$$\Re \left\{ \frac{\left(\mathfrak{I}_{p,\alpha,\beta}^m f(z) \right)'}{z^p} \right\} < \frac{n\beta\Re(\gamma) + 2\rho(p\beta + \alpha)}{\beta n\Re(\gamma) + 2(p\beta + \alpha)}, z \in \mathfrak{U}$$

whenever $\rho > 1$ and

$$\Re \left\{ \frac{\left(\mathfrak{M}_{p,\alpha,\beta}^{m,1-\gamma,\gamma}(z) \right)'}{pz^{p-1}} \right\} < \rho, z \in \mathfrak{U}.$$

Corollary 8 asserts an outcome of the fourth theorem when $\delta = \bar{\gamma}$.

Corollary 8. Let $\mathfrak{S}_{p,\alpha,\beta}^{m,\bar{\gamma},\gamma}$ be as provided in (7); then, we have

$$\Re \left\{ \frac{\left(\mathfrak{I}_{p,\alpha,\beta}^m f(z) \right)'}{z^p} \right\} < \frac{n\beta\Re(\gamma) + 2\rho(p\beta + \alpha)}{4\Re(\gamma)[(p\beta + \alpha) + \beta n]}, z \in \mathfrak{U}$$

whenever $\rho > 2\Re(\gamma)$ and

$$\Re \left\{ \frac{\left(\mathfrak{S}_{p,\alpha,\beta}^{m,\bar{\gamma},\gamma}(z) \right)'}{z^p} \right\} < \rho, z \in \mathfrak{U}.$$

Further, $\Re\left\{\frac{\mathfrak{I}_{p,\alpha,\beta}^m f(z)}{z^p}\right\} < \frac{n\beta+3\rho(p\beta+\alpha)}{n\beta+4(p\beta+\alpha)}, z \in \mathfrak{U}$ whenever $\Re\left\{\frac{\mathfrak{S}_{p,\alpha,\beta}^{m,\gamma,\gamma}(z)}{z^p}\right\} < \frac{3\Re(\gamma)}{2}, z \in \mathfrak{U}$.

By allowing $m = 0 = \alpha$ in the first and third theorems, we come to the following conclusion.

Corollary 9. Let $\mathfrak{K}_p^{\delta,\gamma}$ be as indicated in (11). If $\rho < \gamma + \delta$, then

$$(i) \Re\left\{\frac{f(z)}{z^p}\right\} > \frac{n\Re(\gamma) + 2pp}{n\Re(\gamma) + 2p(\gamma + \delta)}, z \in \mathfrak{U} \text{ whenever } \Re\left\{\frac{\mathfrak{K}_p^{\delta,\gamma}(z)}{z^p}\right\} > \rho, z \in \mathfrak{U}.$$

and

$$(ii) \Re\left\{\frac{f'(z)}{pz^{p-1}}\right\} > \frac{n\Re(\gamma) + 2pp}{n\Re(\gamma) + 2p(\gamma + \delta)}, z \in \mathfrak{U} \text{ whenever } \Re\left\{\frac{\mathfrak{K}_p^{\delta,\gamma}(z)}{pz^{p-1}}\right\} > \rho, z \in \mathfrak{U}.$$

When both m and α are zero, the second and fourth theorems lead to the following result.

Corollary 10. Let $\mathfrak{K}_p^{\delta,\gamma}$ be as defined in (11). If $\rho > \gamma + \delta$, then

$$(i) \Re\left\{\frac{f(z)}{z^p}\right\} < \frac{n\Re(\gamma) + 2pp}{n\Re(\gamma) + 2p(\gamma + \delta)}, z \in \mathfrak{U} \text{ whenever } \Re\left\{\frac{\mathfrak{K}_p^{\delta,\gamma}(z)}{z^p}\right\} < \rho, z \in \mathfrak{U}.$$

and

$$(ii) \Re\left\{\frac{f'(z)}{pz^{p-1}}\right\} < \frac{n\Re(\gamma) + 2pp}{n\Re(\gamma) + 2p(\gamma + \delta)}, z \in \mathfrak{U} \text{ whenever } \Re\left\{\frac{\mathfrak{K}_p^{\delta,\gamma}(z)}{pz^{p-1}}\right\} < \rho, z \in \mathfrak{U}.$$

Remark 3. For $n = 1, \alpha = l + p - p\beta, l > -p$, Corollaries 1, 3, 5, and 7 agree with Theorems 1, 2, 3, and 4 of [25], respectively (which have been examined for $l \geq 0$).

Remark 4. For $n = 1$ and $\delta = 1 - \gamma$, the second part of Corollaries 9 and 10 agree with Corollary 1 and Corollary 2 of [25], respectively.

Remark 5. Analogous results for $\mathfrak{R}_{p,l,\beta}^{m,\delta,\gamma}(z)$, $\mathfrak{T}_{p,\beta}^{m,\delta,\gamma}(z)$ and $\mathfrak{Q}_p^{m,\delta,\gamma}(z)$, which are detailed in (8), (9), and (10), can be obtained by taking (i) $\alpha = p - p\beta + l, l > -p$, (ii) $\alpha = p - p\beta$ and (iii) $\alpha = 0$ in the above-mentioned corollaries.

4. Conclusions

In this paper, we have introduced a unique holomorphic function in the open unit disc $\mathfrak{U}$ by means of a generalized multiplier transformation. We have deduced several features of this function using of the idea of subordination. Furthermore, by assigning appropriate values to the relevant parameters we have highlighted a number of repercussions. Additionally, we have identified important connections between the new findings of this research and those of past investigations.

The new multiplier transformation in our formulation can be replaced with any other recent operator mentioned in the introductory section to produce fresh outcomes. Following the concepts of “strong differential subordination” [36] and “fuzzy differential subordination” [37], Equation (6) can be formulated appropriately, resulting in new findings.

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