Article

Decay of a Thermoelastic Laminated Beam with Microtemperature Effects, Nonlinear Delay, and Nonlinear Structural Damping

Hicham Saber 1, Fares Yazid 2, Djamel Ouchenane 2, Fatima Siham Djeradi 2, Keltoum Bouhali 3,*, Abdelkader Moumen 1, Yousef Jawarneh 1 and Tariq Alraqad 1

1 Department of Mathematics, Faculty of Sciences, University of Ha’il, Ha’il 55425, Saudi Arabia; hi.saber@uoh.edu.sa (H.S.); mo.abdelkader@uoh.edu.sa (A.M.); y.jawarneh@uoh.edu.sa (Y.J.); t.alraqad@uoh.edu.sa (T.A.)
2 Laboratory of Pure and Applied Mathematics, Amar Teledji University, Laghouat 03000, Algeria; f.yazid@lagh-univ.dz (F.Y.); d.ouchenane@lagh-univ.dz (D.O.); fs.djeradi@lagh-univ.dz (F.S.D.)
3 Department of Mathematics, College of Sciences and Arts, Qassim University, Ar-Rass 51921, Saudi Arabia
* Correspondence: k.bouhali@qu.edu.sa

Abstract: This article deals with a non-classical model, namely a thermoelastic laminated beam along with microtemperature effects, nonlinear delay, and nonlinear structural damping, where the last two terms both affect the equation which depicts the dynamics of slip. With the help of convenient conditions in both weight delay and wave speeds, we demonstrate explicit and general energy decay rates of the solution. To attain our interests, we highlight useful properties regarding convex functions and apply a specific approach known as the multiplier technique, which enables us to prove the stability results. Our results here aim to show the impact of different types of damping by taking into account the interaction between them, which extends recent publications in the literature.

Keywords: Laminated beam; Lyapunov functions; nonlinear damping; microtemperature effects; general decay; nonlinear delay; Partial differential equations

MSC: 35B40; 35L56; 74F05; 93D15; 93D20

1. Introduction

Nowadays, both scientists and engineers are becoming quite interested in the laminated beam model, since it gained their attention owing to the wide-ranging applications of this sort of material in various industries. Hansen and Spies [1] were the pioneers in this field and the first to introduce the following beam with two layers by developing the ensuing mathematical model, which has emerged as a crucial research topic

\[
\begin{align*}
\rho_s \ddot{s} + G(\bar{\omega} - s_x)x &= 0, \\
I_3(3\dot{\varphi}_H - \omega_H) - D(3\varphi_{xx} - \omega_{xx}) - G(\bar{\omega} - s_x) &= 0, \\
3I_4\dot{\varphi}_H - 3D\varphi_{xx} + 3G(\bar{\omega} - s_x) + 4\delta\varphi + 4\beta\varphi_t &= 0.
\end{align*}
\]

The equations of movement of the system were formulated using the principles of the Timoshenko beam theory, giving it a similar nature to the established classical Timoshenko system. The first two equations interlock with a third one that captures the effects of structural damping and depicts the interfacial slip dynamics. Such problems have grown greatly within the engineering community because of their significance.

When it comes to the examination of the asymptotic behavior of solutions to various laminated beam problems, there has been a surge in curiosity surrounding its behavior especially in the few past years, such as in [2,3].
The characteristic of a physical system in which there is a delay in the response to an applied force is known as time delay. Transmission of material or energy from one location to another is associated with a delay. Time delay frequently emerges in various phenomena, especially in physics and the economy. Furthermore, in the past few years the control of partial differential equations with time delay has been an active research field.

In the research of Mpungu and Apalara in [4], system (1) was taken into consideration. By including both nonlinear delay and nonlinear structural damping in the third equation, the authors managed to provide the general energy decay rates of the solutions, assuming that certain assumptions regarding the weight delay and wave speeds hold.

Concerning nonlinear structural damping, Djilali et al. in [5] included a nonlinear delay term in a viscoelastic Timoshenko beam, provided that certain conditions among the weight of the term with no delay and the weight of delay hold. The authors proved that they were able to obtain a global existence result and asymptotic behavior of the solutions.

For boundary requirements, Wang et al. in [6], were the first to provide results when the authors considered system (1) along with mixed homogeneous boundary conditions and unequal wave speeds to prove the exponential decay result. After that, many results were obtained, taking the initiative to ameliorate the work of [6], and were able to establish a similar exponential decay result assuming that $\varrho G < I_\varrho$.

The stabilization of laminated beams can also be achieved through the thermal effect in [7]; Apalara demonstrated that the thermal effect alone can lead to exponential stabilization of laminated beams without the need for additional damping terms, under the condition that (2) is satisfied.

Lately, Fayssal, in [8], revealed that the thermoelastic laminated beam problem with structural damping is exponentially stable if

$$\varrho = \frac{I_\varrho}{D},$$

holds.

The remainder of this paper follows this order. In Section 2, useful assumptions and resources are considered, followed by stating the major results. Additionally, we present our problem. In Section 3, some useful and needed Lemmas are proved to facilitate the proof to our main results. In Section 4, with the use of the multiplier technique, our stability results are established.

2. Preliminaries and Position of Problem

This section focuses on providing, after the introduction of the main system, the necessary materials and then stating the main results.

We are interested in the following thermoelastic laminated beam along with micro-temperature effects, nonlinear structural damping, and nonlinear delay

$$\begin{cases}
\varrho s_{tt} + G(\varrho - s_x)_x = 0, \\
I_\varrho(3\varrho - \varrho)_{tt} - D(3\varrho - \varrho)_{xx} - G(\varrho - s_x) = 0, \\
3I_\varrho \varphi_{tt} - 3D \varphi_{xx} + 3G(\varrho - s_x) + \gamma \theta_x + m r_x + 4\delta \varphi \\
+ \beta f_1(\varphi_t(x, t)) + \mu f_2(\varphi_t(x, t - \varphi)) = 0, \\
c\theta_t - k_0 \theta_{xx} + \gamma \varphi_{tx} + k_3 r_x = 0, \\
k_3 r_t - k_2 r_{xx} + k_3 r + m \varphi_{tx} + k_1 \theta_x = 0,
\end{cases}$$

where

$$(x, t) \in (0, 1) \times (0, \infty),$$
with initial and boundary conditions
\[
\begin{align*}
\{ & s(x, 0) = s_0, \, \varphi(x, 0) = \varphi_0, \, \omega(x, 0) = \omega_0, \, \theta(x, 0) = \theta_0, \, r(x, 0) = r_0, \, x \in (0, 1), \\
& \alpha_1(x, 0) = \alpha_0, \, \varphi_1(x, 0) = \varphi_1, \, \omega_1(x, 0) = \omega_1, \, x \in (0, 1), \\
& s_1(0, t) = \varphi(0, t) = \omega(0, t) = \theta(0, t) = r(0, t) = 0, \quad t > 0, \\
& \varphi_x(1, t) = \omega_x(1, t) = \theta_x(1, t) = r(1, t) = 0, \quad t > 0, \\
& \varphi_t(x, t - \varsigma) = f_0(x, t - \varsigma), \quad (x, t) \in (0, 1) \times (0, \varsigma).
\end{align*}
\] (4)

Here, \( s, \varphi, \omega, \theta, \) and \( r \) stand for the transverse displacement, the rotation angle, the amount of slip along the interface, the difference temperature, and the microtemperature vector, respectively. The \( \delta \) coefficients as well as in what comes, Remark 1. Once we exploit \((\varphi, \omega, \theta, r)\) and satisfies \( F \) between the various constituents of the materials. Herein, \( \varsigma \) is the time delay and the positive parameter \( \mu \) is considered as a delay weight.

We shall advance by making the following needed assumptions \([9]\):

- \((A_1)\) The function \( f_1 : \mathbb{R} \rightarrow \mathbb{R} \) is increasing and of class \( C^0 \). Moreover, there exist constants \( \beta_1, \beta_2, \epsilon > 0 \), and a function \( \varepsilon \in C^1([\epsilon, +\infty)) \), which are convex increasing, satisfying \( \varepsilon(0) = 0 \), and the latter is linear on \([0, \epsilon]\) or strictly convex of class \( C^2 \) on \([0, \epsilon] \), in a way that we have

\[
\begin{align*}
\{ & a^2 + f_1^2(a) \leq \varepsilon^{-1}(af_1(a)), \quad \text{for all } |a| \leq \epsilon, \\
& \beta_1 |a| \leq |f_1(a)| \leq \beta_2 |a|, \quad \text{for all } |a| \geq \epsilon.
\end{align*}
\] (5)

- \((A_2)\) The function \( f_2 : \mathbb{R} \rightarrow \mathbb{R} \) is odd and increasing, with \( f_2 \in C^1(\mathbb{R}) \); in addition, there exist \( \tau_1, \tau_2 > 0 \), and \( \lambda \) such that

\[
|f_2^2(a)| \leq \lambda,
\] (6)

and

\[
\tau_1 af_2(a) \leq K(a) \leq \tau_2 af_1(a),
\] (7)

where

\[
K(a) = \int_0^a f_2(y)dy,
\]

and

\[
\tau_2 \mu < \tau_1 \beta.
\] (8)

Remark 1. Once we exploit \((A_1)\), we find

\[
a f_1(a) > 0, \quad \forall a \neq 0.
\]

We employ both the monotonicity of \( f_2 \) and the mean value theorem (for integrals) to obtain

\[
K(a) \leq af_2(a),
\] (9)

as well as in what comes,

\[
\tau_1 \leq \tau_2 \leq 1.
\]

In order to address the nonlinearity of the delay, we consider a constant \( \nu \), which is positive and satisfies

\[
\frac{\mu(1 - \tau_1)}{\tau_1} < \nu < \frac{\beta - \tau_2 \mu}{\tau_2}.
\] (10)
To start, like in [10], we introduce
\[ \mathcal{L}(x, p, t) = q_t(x, t - cp) \quad \text{in} \quad (0, 1) \times (0, 1) \times (0, \infty). \] (11)

We then obtain
\[ \zeta \mathcal{L}(x, p, t) + \mathcal{L}_p(x, p, t) = 0. \] (12)

Therefore, we can rewrite system (3) as
\[
\begin{aligned}
&\varphi_{tt} + G(\omega - s_x)_x = 0, \\
&I_q(3\varphi - \omega)_t - D(3\varphi - \omega)_{xx} - G(\omega - s_x)_x = 0, \\
&3I_q\varphi_t - 3D\varphi_{xx} + 3G(\omega - s_x) + \gamma \theta + mr + 4\delta \varphi \\
&\quad + \beta f_1(\varphi_t(x, t)) + \mu f_2(\mathcal{L}(x, 1, t)) = 0, \\
&c\theta_t - k_0 \theta_{xx} + \gamma \varphi_{tx} + k_1 r_x = 0, \\
&k r_t - k_2 r_{xx} + k_3 r + m \varphi_{tx} + k_1 \theta_x = 0, \\
&\zeta \mathcal{L}(x, p, t) + \mathcal{L}_p(x, p, t) = 0.
\end{aligned}
\] (13)

The initial and boundary conditions take the form below
\[
\begin{aligned}
&s(x, 0) = s_0, \quad \varphi(x, 0) = \varphi_0, \quad \omega(x, 0) = \omega_0, \quad \theta(x, 0) = \theta_0, \quad r(x, 0) = r_0, \quad x \in (0, 1), \\
s(t, x) = s_1, \quad \varphi_t(x, 0) = \varphi_1, \quad \omega_t(x, 0) = \omega_1, \quad x \in (0, 1), \\
s_x(0, t) = \varphi(0, t) = \omega(0, t) = \theta(0, t) = r(0, t) = 0, \quad t > 0, \\
\varphi_x(1, t) = \omega_x(1, t) = \theta(1, t) = r(1, t) = 0, \quad t > 0, \\
\mathcal{L}(x, 0, t) = \varphi_t(x, t), \quad \mathcal{L}(x, p, 0) = f_0(x, -c p), \quad (x, p) \in ((0, 1))^2, \quad t > 0.
\end{aligned}
\] (14)

To prove the existence and uniqueness results, we should use the Faedo Galerkin approach, as in [11]. Herein, \( \mathcal{L}(p) \) will be used to represent \( \mathcal{L}(x, p, t) \).

The total energy of systems (13) and (14) is introduced as
\[
\mathcal{E}(t) = \frac{1}{2} \int_0^1 \left\{ \varphi_t^2 + I_q(3\varphi_t - \omega_t)^2 + D(3\varphi_x - \omega_x)^2 + 3I_q\varphi_t^2 + 3D\varphi_x^2 \right\} dx \\
+ \frac{1}{2} \int_0^1 \left\{ G(\omega - s_x)^2 + 4\delta \varphi^2 + c\theta^2 + k r^2 \right\} dx \\
+ \int_0^1 \int_0^1 \zeta \mathcal{K}(\mathcal{L}(p)) \, dp \, dx.
\] (15)

Then, we can state our stability results.

**Theorem 1.** Consider \((s, \omega, \varphi, \theta, r, \mathcal{L})\) the solution of (13) and (14) and suppose that \((A_1), (A_2), \) and (2) hold. Then, there exist positive constants \(\kappa_0, \kappa_1, \kappa_2, \) and \(\varepsilon_0, \) such that
\[
\mathcal{E}(t) \leq \kappa_0 \mathcal{E}_1^{-1}(\kappa_1 t + \kappa_2), \quad t \geq 0,
\] (16)

where
\[
\mathcal{E}_1(t) = \int_t^1 \frac{1}{\mathcal{E}(a)} da,
\]
and
\[
\mathcal{E}_0(t) = \begin{cases} 
\varepsilon \mathcal{E}(\varepsilon t) & \text{if } \mathcal{E} \text{ is linear on } [0, \varepsilon], \\
2\varepsilon \mathcal{E}(\varepsilon t) & \text{if } \mathcal{E}'(0) = 0 \text{ and } \mathcal{E}'' > 0 \text{ on } (0, \varepsilon).
\end{cases}
\]
Some previous research projects have provided examples related to our established assumptions and our stability results; see [4] for more details.

3. Technical Lemmas

In this section, we shall establish the required Lemmas to support our proof of stability results. The demonstration of the stability result of problem (13) will be attained by employing a particular method called the multiplier technique. For the sake of simplicity, we will use $m_\ast > 0$ to represent a constant. The value of $m_\ast$ may differ from line to line and even within the same line.

**Lemma 1.** Consider $(s, \omega, \varphi, \theta, r, \mathcal{X})$ the solution of (13) and (14) and suppose that $(A_1)$ and $(A_2)$ hold. Then, the energy functional satisfies

\[
E'(t) \leq -k_0 \int_0^1 \theta^2_x dx - k_2 \int_0^1 r^2_x dx - k_3 \int_0^1 r^2 dx - W_0 \int_0^1 \varphi f_1(\varphi_1) dx \tag{17}
- W_1 \int_0^1 \mathcal{X}(1)f_2(\mathcal{X}(1)) \, dx, \quad \forall t \geq 0,
\]

where $W_0$ and $W_1$ are positive constants.

**Proof.** To begin, we multiply (13), (13)$_2$, (13)$_4$, (13)$_6$, and (13)$_8$ by $s_t$, $(3\varphi_t - \varphi_3)$, $\varphi_t$, $\theta_t$, and $r$, respectively. We then continue by integrating over $(0, 1)$, and considering integration by parts, to find

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 \left\{ \varphi^2_t + I_2(3\varphi_t - \varphi_3)^2 + D(3\varphi_t - \varphi_3)^2 + 3I_2 \varphi^2_t + 3D \varphi^2_t + 4E \varphi^2 \right\} \, dx \tag{18}
+ \frac{1}{2} \frac{d}{dt} \int_0^1 \left\{ G(\omega - s_x)^2 + c\theta^2 + \kappa \theta^2 \right\} \, dx = -k_0 \int_0^1 \theta^2_x dx - k_2 \int_0^1 r^2_x dx - k_3 \int_0^1 r^2 dx
- \beta \int_0^1 \varphi f_1(\varphi_1) \, dx - \mu \int_0^1 \varphi f_2(\mathcal{X}(1)) \, dx.
\]

Thus, we take Equation (13) and multiply it by $\nu f_2(\mathcal{X}(p))$. Then, by integration over $(0, 1) \times (0, 1)$, we obtain

\[
\nu \int_0^1 \int_0^1 f_2(\mathcal{X}(p)) \mathcal{X}_1(p) \, dp \, dx = -\nu \int_0^1 \int_0^1 \partial p K(\mathcal{X}(p)) \, dp \, dx
= \nu \int_0^1 K(\mathcal{X}(0)) \, dx - \nu \int_0^1 K(\mathcal{X}(1)) \, dx \tag{19}
= \nu \int_0^1 K(\varphi_t) \, dx - \nu \int_0^1 K(\mathcal{X}(1)) \, dx,
\]

then,

\[
\nu \int_0^1 \int_0^1 K(\mathcal{X}(p)) \, dp \, dx = \nu \int_0^1 K(\varphi_t) \, dx - \nu \int_0^1 K(\mathcal{X}(1)) \, dx, \tag{20}
\]

When we combine (18) and (20) using (7), we obtain

\[
E'(t) \leq -k_0 \int_0^1 \theta^2_x dx - k_2 \int_0^1 r^2_x dx - k_3 \int_0^1 r^2 dx - (\beta - \tau_2 \nu) \int_0^1 \varphi f_1(\varphi_1) \, dx \tag{21}
- \nu \int_0^1 K(\mathcal{X}(1)) \, dx - \mu \int_0^1 \varphi f_2(\mathcal{X}(1)) \, dx.
\]

Let $K^*$ be the conjugate function of $K$

\[
K^*(a) = \sup_{\nu \in \mathbb{R}^+} (\nu a - K(\nu)).
\]
Thus, the Legendre transformation of $K$ is noted $K^*$, which is given by

$$K^*(a) = a(K')^{-1}(a) - K[(K')^{-1}(a)], \quad \forall a \geq 0.$$  \hspace{1cm} (22)

This allows us to write (see [9,12])

$$a\varpi \leq K^*(a) + K(\varpi), \quad \forall a, \varpi \geq 0.$$  \hspace{1cm} (23)

The next step would be to consider the definition of $K$, while exploiting (22), to achieve

$$K^*(a) = a f_2^{-1}(a) - K\left(f_2^{-1}(a)\right).$$  \hspace{1cm} (24)

Thanks to (24), along with (7), we write

$$K^*(f_2(\mathcal{Z}(1))) = \mathcal{Z}(1)f_2(\mathcal{Z}(1)) - K(\mathcal{Z}(1)) \leq (1 - \tau) \mathcal{Z}(1)f_2(\mathcal{Z}(1)).$$  \hspace{1cm} (25)

After that, by (23) and (25) as well as (7), we have

$$-\mu \int_0^1 \varphi_1 f_2(\mathcal{Z}(1)) dx \leq \mu \int_0^1 K(\varphi_1) dx + \mu \int_0^1 K^*(f_2(\mathcal{Z}(1))) dx$$
$$\leq \mu \int_0^1 K(\varphi_1) dx + \mu(1 - \tau) \int_0^1 \mathcal{Z}(1)f_2(\mathcal{Z}(1)) dx$$
$$\leq \tau_2 \mu \int_0^1 \varphi_1 f_1(\varphi_1) dx + \mu(1 - \tau) \int_0^1 \mathcal{Z}(1)f_2(\mathcal{Z}(1)) dx.$$  \hspace{1cm} (26)

By using (10) and (8) in addition to the combination of (26) with (21), (17) is proved. \hfill \Box

**Lemma 2.** Consider the functional

$$\mathcal{J}_1(t) := 3 \mu G \int_0^1 (3\varphi - \omega) \varphi_1 dx - \varphi D \int_0^1 (3\varphi_1 - \omega s_1) dx - I_\theta G \int_0^1 (3\varphi_1 - \omega s_1) dx,$$  \hspace{1cm} (27)

then, it satisfies

$$\mathcal{J}_1'(t) \leq -\frac{GD}{2} \int_0^1 (3\varphi_1 - \omega s_1)^2 dx + \epsilon_1 \int_0^1 (3\varphi_1 - \omega s_1)^2 dx + m_0 \int_0^1 \varphi_1^2 dx$$
$$+ m_0 \int_0^1 (\omega - \eta_1)^2 dx + m_0 \int_0^1 \theta_1^2 dx + m_0 \int_0^1 r_1^2 dx, \forall \epsilon_1 > 0.$$  \hspace{1cm} (28)

**Proof.** We exploit $\mathcal{J}_1'$ while considering Equations (13)$_{1,2,3}$, the integration by parts, and let $s_1 = \omega - (\omega - s_1)$, to arrive at

$$\mathcal{J}_1'(t) = -GD \int_0^1 (3\varphi_1 - \omega s_1)^2 dx + 3 \mu G \int_0^1 \varphi_1 (3\varphi_1 - \omega s_1) dx - 3G^2 \int_0^1 (\omega - s_1)(3\varphi - \omega) dx$$
$$- 4G \int_0^1 (3\varphi - \omega) \varphi_1 dx - \gamma G \int_0^1 (3\varphi - \omega) \theta_1 dx - m G \int_0^1 (3\varphi - \omega) r_1 dx$$
$$- G \int_0^1 (3\varphi - \omega) f_1(\varphi_1) dx - \mu G \int_0^1 (3\varphi - \omega) f_2(\mathcal{Z}(1)) dx$$
$$- G^2 \int_0^1 s_1(3\varphi - \omega) dx + (I_\theta G - \varphi D) \int_0^1 s_1(3\varphi - \omega) dx.$$
With the simple substitution \( s_x = 3 \varphi - (\varpi - s_x) - (3 \varphi - \omega) \) and (2), we find that
\[
\mathcal{J}_1'(t) = -GD \int_0^1 (3 \varphi_x - \varpi_x)^2 dx + 3 \varphi \int_0^1 \varphi_t (3 \varphi_t - \varpi_t) dx - \frac{1}{2} \varphi \int_0^1 (\varpi - s_x) (3 \varphi - \omega) dx
\]
\[
- 4 \delta G \int_0^1 (3 \varphi - \omega) \varphi dx - \gamma G \int_0^1 (3 \varphi - \omega) \theta_x dx - mG \int_0^1 (3 \varphi - \omega) r_x dx
\]
\[
- \beta G \int_0^1 (3 \varphi - \omega) f_1 (\varphi_t) dx - \mu G \int_0^1 (3 \varphi - \omega) (f_2 (\varphi))(1) dx
\]
\[
+ G^2 \int_0^1 (\varpi - s_x)^2 dx - 3 G \int_0^1 (\varpi - s_x) \varphi dx.
\]

On account of (9), (23), and (25), we achieve
\[
f_2^2 (\varphi)(1) \leq 2 \varphi (1) f_2 (\varphi)(1).
\]

We finally, owing to Young and Poincaré’s inequalities with (30), establish (28).

\[\square\]

Lemma 3. Consider the functional
\[
\mathcal{J}_2(t) := 3 \varphi \int_0^1 \varphi_t (\varpi - s_x) dx - 3 \varphi D \int_0^1 s_t q_x dx,
\]
then, it satisfies for any \( \epsilon_2 > 0 \)
\[
\mathcal{J}_2(t) \leq -G^2 \int_0^1 (\varpi - s_x)^2 dx + \epsilon_2 \int_0^1 (3 \varphi_t - \varpi_t)^2 dx + m_s \left(1 + \frac{1}{\epsilon_2}\right) \int_0^1 \varphi_t^2 dx
\]
\[
+ m_s \int_0^1 q_t^2 dx + m_s \int_0^1 \theta_t^2 dx + m_s \int_0^1 r_t^2 dx + m_s \int_0^1 f_1^2 (\varphi_t) dx + m_s \int_0^1 \varphi (1) f_2 (\varphi)(1) dx.
\]

Proof. With easy calculations, taking (13)_1 and (13)_3 while integrating by parts, we achieve that
\[
\mathcal{J}_2'(t) = -3G^2 \int_0^1 (\varpi - s_x)^2 dx + 3 \varphi D - 3 \varphi \int_0^1 \varphi_t s_t dx + 3 \varphi \int_0^1 \varphi_t \varphi_t dx
\]
\[
- 4 \delta G \int_0^1 (\varpi - s_x) \varphi dx - \gamma G \int_0^1 (\varpi - s_x) \theta_x dx - mG \int_0^1 (\varpi - s_x) r_x dx
\]
\[
- \beta G \int_0^1 (\varpi - s_x) f_1 (\varphi_t) dx - \mu G \int_0^1 (\varpi - s_x) (f_2 (\varphi))(1) dx,
\]
and employing \( \varpi_t = 3 \varphi_t - (3 \varphi - \omega_t) \) and (2) yields
\[
\mathcal{J}_2'(t) = -3G^2 \int_0^1 (\varpi - s_x)^2 dx - 3 \varphi \int_0^1 \varphi_t (3 \varphi_t - \varpi_t) dx - 4 \delta G \int_0^1 (\varpi - s_x) \varphi dx
\]
\[
+ 9 \varphi \int_0^1 \varphi_t^2 dx - \gamma G \int_0^1 (\varpi - s_x) \theta_x dx - mG \int_0^1 (\varpi - s_x) r_x dx
\]
\[
- \beta G \int_0^1 (\varpi - s_x) f_1 (\varphi_t) dx - \mu G \int_0^1 (\varpi - s_x) (f_2 (\varphi))(1) dx.
\]

By (30) and thanks to Young and Poincaré’s inequalities, one concludes the proof.

\[\square\]

Lemma 4. Consider the functional
\[
\mathcal{J}_3(t) := 3 \varphi \int_0^1 \varphi \varphi_t dx - 3 \varphi \int_0^1 \varphi \int_0^x s_t (y) dy dx,
\]
then, it satisfies
\[
\mathcal{J}_3(t) \leq -\delta \int_0^1 \varphi'^2 dx - 3D \int_0^1 \varphi'^2 dx + \varepsilon_3 \int_0^1 s_1^2 dx + m_s \int_0^1 \theta'^2 dx + m_s \left( 1 + \frac{1}{\varepsilon_3} \right) \int_0^1 \varphi'^2 dx + m_s \int_0^1 \varphi'^2 dx + m_s \int_0^1 f_1'(\varphi') dx \\
+ m_s \int_0^1 \mathcal{X}(1)f_2(\mathcal{X}(1))dx,
\]
for any \( \varepsilon_3 > 0 \).

\textbf{Proof.} The derivative of \( \mathcal{J}_3 \), along with Equation (13)_1, and integration by parts, give
\[
\mathcal{J}'_3(t) = -4\delta \int_0^1 \varphi'^2 dx + 3I_0 \int_0^1 \varphi'^2 dx - 3D \int_0^1 \varphi'^2 dx - \gamma \int_0^1 \varphi \theta_x dx - m \int_0^1 \varphi \rho_x dx \\
- \beta \int_0^1 f_1(\varphi) \varphi dx - \mu \int_0^1 f_2(\mathcal{X}(1)) \varphi dx - 3\varepsilon \int_0^1 \varphi_1 \int_0^x s_1(y) dy dx.
\]

We apply Young and Poincaré’s inequalities and use (30) to complete this proof.

\textbf{Lemma 5.} Consider the functional
\[
\mathcal{J}_4(t) := -\varepsilon \int_0^1 s_1 s dx,
\]
then, it satisfies
\[
\mathcal{J}'_4(t) \leq -\varepsilon \int_0^1 s_1^2 dx + m_s \int_0^1 \varphi'^2 dx + D \int_0^1 (3\varphi - \omega_s)^2 dx + m_s \int_0^1 (\omega - s_x)^2 dx.
\]

\textbf{Proof.} An easy calculation, involving the derivative of \( \mathcal{J}_4 \), Equation (13)_1, and integration by parts implies that
\[
\mathcal{J}'_4(t) = -\varepsilon \int_0^1 s_1^2 dx - G \int_0^1 s_1(\omega - s_x) dx.
\]

Then, we rewrite \( s_x \) as \( s_x = 3\varphi - (\omega - s_x) - (3\varphi - \omega) \) to obtain
\[
\mathcal{J}'_4(t) = -\varepsilon \int_0^1 s_1^2 dx + G^2 \int_0^1 (\omega - s_x)^2 dx - 3G \int_0^1 (\omega - s_x) \varphi dx \\
+ G \int_0^1 (3\varphi - \omega)(\omega - s_x) dx.
\]

By Young and Poincaré’s inequalities, we obtain
\[
-3G \int_0^1 (\omega - s_x) \varphi dx \leq \frac{3G}{2} \int_0^1 \varphi_x^2 dx + \frac{G}{2} \int_0^1 (\omega - s_x)^2 dx,
\]

\[
G \int_0^1 (3\varphi - \omega)(\omega - s_x) dx \leq D \int_0^1 (3\varphi_x - \omega_s)^2 dx + \frac{G^2}{4D} \int_0^1 (\omega - s_x)^2 dx.
\]
We finally combine (38)–(37) to complete the proof of (35).

\textbf{Lemma 6.} Consider functional
\[
\mathcal{J}_5(t) := -I_0 \int_0^1 (3\varphi - \omega)_1(3\varphi - \omega) \, dx,
\]

(40)
it satisfies,
\[ \mathcal{H}_5'(t) \leq -I_\varepsilon \int_0^1 (3\varphi_t - \omega_t)^2 \, dx + 2D \int_0^1 (3\varphi_x - \omega_x)^2 \, dx + m_\varepsilon \int_0^1 (\omega - s_x)^2 \, dx. \tag{41} \]

**Proof.** The derivative of \( \mathcal{H}_5 \), along with Equation (13) and integration by parts, yields
\[ \mathcal{H}_5'(t) = -I_\varepsilon \int_0^1 (3\varphi - \omega)_t (3\varphi - \omega) \, dx - I_\varepsilon \int_0^1 (3\varphi_t - \omega_t)^2 \, dx \]
\[ = -I_\varepsilon \int_0^1 (3\varphi_t - \omega_t)^2 \, dx + D \int_0^1 (3\varphi_x - \omega_x)^2 \, dx - G \int_0^1 (\omega - s_x)(3\varphi - \omega) \, dx. \tag{42} \]

Owing to Young and Poincaré’s inequalities, we have
\[ -G \int_0^1 (\omega - s_x)(3\varphi - \omega) \, dx \leq D \int_0^1 (3\varphi_x - \omega_x)^2 \, dx + \frac{G^2}{4D} \int_0^1 (\omega - s_x)^2 \, dx, \tag{43} \]
and then, by (43) and (42), we conclude the proof. \( \square \)

**Lemma 7.** Consider the functional
\[ \mathcal{H}_6(t) := \xi \int_0^1 \int_0^1 e^{-pK(\mathcal{I}(p))} dp \, dx, \tag{44} \]
it satisfies, then
\[ \mathcal{H}_6'(t) \leq -\tau_1 e^{-\xi} \int_0^1 \mathcal{I}(1)f_2(\mathcal{I}(1)) \, dx + \tau_2 \int_0^1 \varphi_t f_1(\varphi_t) \, dx - \xi e^{-\xi} \int_0^1 \int_0^1 K(\mathcal{I}(p)) dp \, dx. \tag{45} \]

**Proof.** The calculations, involving the derivative of \( \mathcal{H}_6 \), Equation (13), and \( \mathcal{I}(0) = \varphi_t \), imply that
\[ \mathcal{H}_6'(t) = \xi \int_0^1 \int_0^1 e^{-sp} \mathcal{I}(p)f_2(\mathcal{I}(p)) \, dp \, dx \]
\[ = -\xi \int_0^1 \int_0^1 e^{-sp} \mathcal{I}(p)f_2(\mathcal{I}(p)) \, dp \, dx \]
\[ = -\xi \int_0^1 \int_0^1 e^{-sp} \partial_p K(\mathcal{I}(p)) \, dp \, dx \]
\[ = -\xi \int_0^1 \int_0^1 e^{-sp} K(\mathcal{I}(p)) \, dp \, dx - \int_0^1 \int_0^1 \partial_p [e^{-sp} K(\mathcal{I}(p))] \, dp \, dx \]
\[ = -\xi \int_0^1 \int_0^1 e^{-sp} K(\mathcal{I}(p)) \, dp \, dx - e^{-\xi} \int_0^1 K(\mathcal{I}(1)) \, dx + \int_0^1 K(\varphi_t) \, dx. \]

Exploiting (7) and \( e^{-\xi} \leq e^{-p^\xi} \leq 1, p \in (0,1) \), we establish (45). \( \square \)

4. Stability Result

We utilize the Lemmas presented in Section 3 to prove our stability result.

**Proof of Theorem 1.** To start, we introduce a Lyapunov functional
\[ \mathcal{R}(t) = N\mathcal{E}(t) + \sum_{i=1}^{6} N_i \mathcal{F}_i(t), \quad \forall t \geq 0, \tag{46} \]
where constants \( N, N_i > 0 \) and \( i = 1 \cdots 6 \) will be fixed later.
From (46), we can write

\[ |\mathcal{R}(t) - N\dot{\mathcal{R}}(t)| \leq \epsilon DN_1 \int_0^1 s_i(3\varphi_x - \alpha_x)dx + 3I_0GN_1 \int_0^1 \varphi_i(3\varphi - \omega)dx \]
\[ + I_0GN_1 \int_0^1 |s_x(3\varphi_i - \alpha_i)|dx + 3qDN_2 \int_0^1 |s_i\varphi_x|dx \]
\[ + 3I_0GN_2 \int_0^1 (\omega - s_x)\varphi_i|dx + 3I_0N_3 \int_0^1 |\varphi\varphi_i|dx \]
\[ + 3qN_3 \int_0^1 \varphi \int_0^x s_i(y)dy|dx + qN_4 \int_0^1 |s_i\varphi|dx \]
\[ + I_0N_5 \int_0^1 |(3\varphi - \omega)(3\varphi - \omega_i)|dx \]
\[ + \psi N_6 \int_0^1 e^{-\nu} |K(\mathcal{X}(p))|dpdx. \]

Exploiting the energy definition and by Young, Cauchy–Schwarz, and Poincaré’s inequalities, we obtain

\[ |\mathcal{R}(t) - \delta'(t)| \leq \epsilon \delta'(t), \quad \text{where } \epsilon > 0, \]

then,

\[ (N - \epsilon)\delta'(t) \leq \mathcal{R}(t) \leq (N + \epsilon)\delta'(t). \quad (47) \]

The derivative of \( \mathcal{R} \), along with (17), (28), (31), (33), (35), (41), and (45), gives us after setting

\[ N_1 = \frac{8}{G}, \quad N_4 = N_5 = N_6 = 1, \quad \epsilon_1 = \frac{l_0}{4N_1}, \quad \epsilon_2 = \frac{l_0}{4N_2}, \quad \epsilon_3 = \frac{\theta}{2N_3}, \]

the estimate below

\[ \mathcal{R}'(t) \leq -[3DN_3 - m_*N_2 - m_\ast] \int_0^1 \varphi_\ast^2dx - \frac{\theta}{2} \int_0^1 s_\ast^2dx - \delta N_3 \int_0^1 \varphi_\ast^2dx \]
\[ - \frac{l_0}{2} \int_0^1 (3\varphi_i - \alpha_i)^2dx - D \int_0^1 (3\varphi_x - \alpha_x)^2dx \left[ G^2N_2 - m_* \right] \int_0^1 (\omega - s_x)^2dx \]
\[ - [k_2N - m_*N_2 - m_*N_3 - m_*] \int_0^1 \vartheta_\ast^2dx - [k_0N - m_*N_2 - m_*N_3 - m_*] \int_0^1 \theta_\ast^2dx \]
\[ - k_3N \int_0^1 \vartheta^2dx - [W_1N - m_*N_2 - m_*N_3 - m_* + e^{-\nu_1}] \int_0^1 \mathcal{X}(1)f_2(\mathcal{X}(1))dx \]
\[ - [W_0N - \nu_2] \int_0^1 \varphi_\ast f_1(\varphi_\ast)dx - \epsilon e^{-\nu} \int_0^1 \int_0^1 K(\mathcal{X}(p))dpdx \]
\[ + [N_2m_\ast(1 + N_2) + N_3m_\ast(1 + N_3) + m_\ast] \int_0^1 \varphi_\ast^2dx \]
\[ + [m_*N_2 + m_*N_3 + m_*] \int_0^1 f_1^2(\varphi_i)dx. \quad (48) \]

Now, we choose the coefficients in (48) such that the last two terms are negative. Choosing an \( N_2 \) large enough so that

\[ G^2N_2 - m_* > 0, \]

allows us to proceed by taking a sufficiently large \( N_3 \), such that

\[ 3DN_3 - m_*N_2 - m_* > 0, \]
and to conclude by selecting $N$ to have (47) and
\[
\begin{aligned}
&k_2 N - m_3 N_2 + m_4 N_3 - m_5 > 0, \\
k_0 N - m_3 N_2 - m_4 N_3 - m_5 > 0, \\
W_1 N - m_3 N_2 - m_4 N_3 + e^{-\varsigma} \tau_1 > 0, \\
W_0 N - \tau_2 > 0.
\end{aligned}
\]

The above choices and Poincaré’s inequality yield
\[
\mathfrak{R}'(t) \leq -\tau_3 \mathcal{E}'(t) + \tau_4 \int_0^1 (\varphi_1^2 + f_1(\varphi_1)) dx, \quad \tau_3, \tau_4 > 0. \tag{49}
\]

We continue by dividing the proof in two cases:

**Case 1:** Assume that $\mathcal{E}$ is linear on $[0, \varepsilon]$. By \((A_1)\), we have
\[
\begin{aligned}
&\beta_1 a^2 \leq f_1(a) \leq \beta_2 a^2, \\
&a \beta_1 f_1(a) \leq f_1^2(a) \leq a \beta_2 f_1(a), \quad \forall a \in \mathbb{R},
\end{aligned}
\]
which, along with (49), leads to
\[
\mathfrak{R}'(t) \leq -\tau_3 \mathcal{E}'(t) + \tau_4 \int_0^1 \varphi_1 f_1(\varphi_1) dx, \quad \tau_3, \tau_4 > 0. \tag{50}
\]
Combining (17) and (50), we easily come to
\[
\mathfrak{R}'(t) \leq -\tau_3 \mathcal{E}'(t) - \tau_3 \mathcal{E}'(t), \quad \tau_3 > 0. \tag{51}
\]
Then, introducing the functional
\[
\mathfrak{R}_*(t) := \mathfrak{R}(t) + \tau_3 \mathcal{E}(t), \tag{52}
\]
and by (47), we obtain
\[
\mathcal{E}'(t) \leq \mathfrak{R}_*(t) \leq \mathfrak{R}_*(t), \quad \mathfrak{R}_*, \mathfrak{R}_* > 0. \tag{53}
\]

Once we employ (52) together with (53), we obtain
\[
\mathfrak{R}_*(t) \leq -\kappa_1 \mathfrak{R}_*(t), \quad \kappa_1 = \frac{\tau_3}{\mathfrak{R}_*}, \quad \forall t \geq 0. \tag{54}
\]

We integrate (54) and exploit (53); we achieve
\[
\mathcal{E}(t) \leq \kappa_0 e^{-\kappa_1 t}, \quad \text{where} \quad \kappa_0 = \frac{\mathfrak{R}_*(0)}{\mathfrak{R}_*}, \quad t \geq 0. \tag{55}
\]

**Case 2:** Assume that $\mathcal{E}$ is nonlinear on $(0, \varepsilon]$.
As in \cite{13}, we pick $0 < \varepsilon_1 \leq \varepsilon$, so that
\[
a f_1(a) \leq \min\{\varepsilon, \mathcal{E}(\varepsilon)\}, \quad \forall |a| \leq \varepsilon_1.
\]

Then, once employing \((A_1)\) and function $f_1$, which is continuous, along with $|f_1(a)| > 0$ and $a \neq 0$, it results in
\[
\begin{cases}
a^2 + f_1^2(a) \leq \mathcal{E}^{-1}(a f_1(a)), \quad |a| \leq \varepsilon_1, \\
|\beta_1| a \leq |f_1(a)| \leq |\beta_2| a, \quad |a| \geq \varepsilon_1.
\end{cases} \tag{56}
\]
Let us now deal with
\[ \int_0^1 \left( \varphi_i^2 + f_i^2(\varphi_i) \right) dx, \]
where, to be able to estimate this term, we need to proceed as in [14] and present the ensuing partitions
\[ A_1 = \{ x \in (0, 1) : |\varphi_i| \leq \varepsilon_1 \}, \quad A_2 = \{ x \in (0, 1) : |\varphi_i| > \varepsilon_1 \}, \]
which, once used with Jensen’s inequality and with \( \mathcal{E}^{-1} \) being a concave function, we obtain
\[ \mathcal{E}^{-1}(A(t)) \geq \beta_5 \int_{A_1} \mathcal{E}^{-1}(\varphi_i f_i(\varphi_i)) \, dx, \quad (57) \]
where
\[ A(t) = \int_{A_1} \varphi_i f_i(\varphi_i) \, dx, \quad \text{and} \quad \beta_5 > 0. \]

Considering (56), (57), and (17) yields
\[ \int_0^1 \left( \varphi_i^2 + f_i^2(\varphi_i) \right) dx = \int_{A_1} \left( \varphi_i^2 + f_i^2(\varphi_i) \right) dx + \int_{A_2} \left( \varphi_i^2 + f_i^2(\varphi_i) \right) dx \leq \int_{A_1} \mathcal{E}^{-1}(\varphi_i f_i(\varphi_i)) \, dx + \beta_6 \int_{A_2} (\varphi_i f_i(\varphi_i)) \, dx \leq \beta_6 \mathcal{E}^{-1}(A(t)) - \beta_6 \mathcal{E}'(t), \quad \beta_6 > 0. \]

We continue by introducing
\[ \mathcal{R}_0(t) := \mathcal{R}(t) + \beta \mathcal{E}'(t), \quad \text{where} \quad \beta > 0, \quad (59) \]
then, with relation (47) being taken into consideration, we easily derive
\[ d_1 \mathcal{E}'(t) \leq \mathcal{R}_0(t) \leq d_2 \mathcal{E}'(t), \quad d_1, d_2 > 0. \quad (60) \]

By (59) together with (58) and (49), we have
\[ \mathcal{R}_0'(t) \leq -\gamma_3 \mathcal{E}'(t) + \beta \mathcal{E}^{-1}(A(t)). \quad (61) \]

Taking now the functional
\[ \mathcal{R}_1(t) := \mathcal{E}' \left( \frac{\mathcal{E}'(t)}{\mathcal{E}'(0)} \varepsilon_0 \right) \mathcal{R}_0(t) + \gamma_0 \mathcal{E}'(t), \quad \varepsilon_0 < \varepsilon, \quad \gamma_0 > 0, \quad (62) \]

we have, together with (60) and \( \mathcal{E}' \leq 0, \quad \mathcal{E}' > 0, \quad \mathcal{E}'' > 0, \) on \( (0, \varepsilon] \), we find
\[ d_1 \mathcal{E}'(t) \leq \mathcal{R}_1(t) \leq d_2 \mathcal{E}'(t), \quad d_1, d_2 > 0. \quad (63) \]

In addition, (61) leads us to
\[ \mathcal{R}_1'(t) = \varepsilon_0 \frac{\mathcal{E}''(t)}{\mathcal{E}'(0)} \mathcal{E}' \left( \frac{\mathcal{E}'(t)}{\mathcal{E}'(0)} \varepsilon_0 \right) \mathcal{R}_0(t) + \mathcal{E}' \left( \frac{\mathcal{E}'(t)}{\mathcal{E}'(0)} \varepsilon_0 \right) \mathcal{R}_0'(t) + \gamma_0 \mathcal{E}'(t) \leq -\gamma_3 \mathcal{E}' \left( \frac{\mathcal{E}'(t)}{\mathcal{E}'(0)} \varepsilon_0 \right) \mathcal{E}'(t) + \beta \mathcal{E}' \left( \frac{\mathcal{E}'(t)}{\mathcal{E}'(0)} \varepsilon_0 \right) \mathcal{E}^{-1}(A(t)) + \gamma_0 \mathcal{E}'(t). \quad (64) \]

Let us set
\[ \mathcal{E}' = \beta \mathcal{E}' \left( \frac{\mathcal{E}'(t)}{\mathcal{E}'(0)} \varepsilon_0 \right) \mathcal{E}^{-1}(A(t)). \]

As in (22), we estimate \( \mathcal{E}' \); for this, we let \( \mathcal{E}'^* \) be the convex conjugate of \( \mathcal{E} \), given by
\[ \mathcal{E}'^*(a) = a (\mathcal{E}'^{-1}(a) - \mathcal{E} \left( (\mathcal{E}')^{-1}(a) \right) \leq a (\mathcal{E}')^{-1}(a), \quad \text{where} \quad a \in (0, \mathcal{E}'(\varepsilon)). \quad (65) \]
By general Young’s inequality, we achieve
\[ av \leq E^*(a) + E(v), \quad \text{where} \quad a \in (0, E'(\epsilon)), \quad v \in (0, \epsilon]. \tag{66} \]

We then set
\[ a = E'\left(\frac{E(t)}{\delta(0)} \epsilon_0\right), \quad \text{and} \quad v = E^{-1}(A(t)). \]

Thanks to (65) and (66), as well as
\[ \mathcal{A}(t) = \int_{A_t} \varphi_t f_1(\varphi_t) dx \leq \int_0^1 \varphi_t f_1(\varphi_t) dx \leq -\frac{1}{W_0} \mathcal{E}'(t), \]

we obtain
\[ \mathcal{O} = \beta \mathcal{E}'\left(\frac{\mathcal{E}(t)}{\delta(0)} \epsilon_0\right) \mathcal{E}^{-1}(A(t)) \leq \beta \mathcal{E}_0 \mathcal{E}'\left(\frac{\mathcal{E}(t)}{\delta(0)} \epsilon_0\right) - \beta_8 \mathcal{E}'(t), \quad \beta_8 > 0. \tag{67} \]

Replacing (67) with (64), we find
\[ \mathcal{R}_1(t) \leq -[\tau_3 \mathcal{E}(0) - \beta \mathcal{E}_0] \mathcal{E}'\left(\frac{\mathcal{E}(t)}{\delta(0)} \epsilon_0\right) + (\gamma_0 - \beta_8) \mathcal{E}'(t). \tag{68} \]

By picking \( \epsilon_0 = \frac{\tau_3 \mathcal{E}(0)}{2\beta_8} \) and \( \gamma_0 = 2\beta_8 \), we reach
\[ \mathcal{R}_1(t) \leq -\tau_3 \mathcal{E}(0) \mathcal{E}'\left(\frac{\mathcal{E}(t)}{\delta(0)} \epsilon_0\right) + \beta_8 \mathcal{E}'(t); \quad \tau_3 = \frac{\tau_3 \mathcal{E}(0)}{2}, \]

and, since \( \mathcal{E}'(t) \leq 0 \), we obtain
\[ \mathcal{R}_1(t) \leq -\tau_3 \mathcal{E}(0) \mathcal{E}'\left(\frac{\mathcal{E}(t)}{\delta(0)} \epsilon_0\right) = -\tau_3 \mathcal{E}_0 \mathcal{E}'\left(\frac{\mathcal{E}(t)}{\delta(0)} \epsilon_0\right), \tag{69} \]

where \( \mathcal{E}_0(a) = a \mathcal{E}'(\epsilon_0 a) \).

Now, since \( \mathcal{E} \) is strictly convex on \( (0, \epsilon] \), we have \( \mathcal{E}_0(a) > 0 \) on \( (0, 1] \).

Therefore, considering
\[ \mathcal{R}_{1+}(t) := \frac{\bar{d}_1 \mathcal{E}_1(t)}{\delta(0)}, \tag{70} \]

then
\[ \bar{d}_1 \mathcal{E}'(t) \leq \mathcal{R}_{1+}(t) \leq \bar{d}_2 \mathcal{E}'(t), \quad \bar{d}_1, \bar{d}_2 > 0, \tag{71} \]

and exploiting (69) yields
\[ \mathcal{R}_{1+}'(t) \leq -\frac{\bar{d}_1 \tau_3}{\delta(0)} \mathcal{E}_0 \mathcal{E}'\left(\frac{\mathcal{E}(t)}{\delta(0)} \epsilon_0\right). \]

By (71), with \( \mathcal{E}_0 \) increasing, we have
\[ \mathcal{R}_{1+}'(t) \leq -\kappa_1 \mathcal{E}_0(\mathcal{R}_{1+}(t)), \quad \kappa_1 > 0. \tag{72} \]

By (72), we have
\[ [\mathcal{E}_1(\mathcal{R}_{1+}(t))]' \geq \kappa_1, \quad \forall t \geq 0, \tag{73} \]

where
\[ \mathcal{E}_1(t) = \int_t^1 \frac{1}{\mathcal{E}_0(a)} da. \]

We simply integrate (73) over \((0, t)\) to establish
\[ \mathcal{E}_1(\mathcal{R}_{1+}(t)) \geq \kappa_1 t + \kappa_2, \quad \kappa_2 = \mathcal{E}_1(\mathcal{R}_{1+}(0)), \quad \forall t \geq 0. \tag{74} \]
Because $\varepsilon_1^{-1}$ is a decreasing function, we achieve

$$\mathcal{R}_1(t) \leq \varepsilon_1^{-1}(\kappa_1 t + \kappa_2), \quad \forall t \geq 0. \quad (75)$$

By exploiting relation (71), we conclude

$$\mathcal{E}(t) \leq \kappa_0 \varepsilon_1^{-1}(\kappa_1 t + \kappa_2), \quad \forall t \geq 0, \quad (76)$$

where $\kappa_0 = \frac{1}{d_i}$. The proof is then concluded.

5. Conclusions and Discussion

One of the main subjects of our research is to add some physical processes (damping terms) associated with thermoelastic laminated beams and develop techniques to establish one of the most qualitative properties of the solution, which is the general decay rate. The standard requirement of the initial and boundary conditions was very complicated, and we tried to improve them. To date, a variety of techniques are known to achieve the desired result. These include techniques associated with the convex functions and multiplier method. For researchers working in this field who want to learn something new and not easy, it is an extremely interesting section of modern science and engineering on new physical principles. However, it is possible to formulate a number of extremely important problems from the point of view of practical applications, the solution of which requires new methods in the literature, namely: problems that contain fractional derivatives in the boundary conditions, with a variable time delay (see [15–19]).

We can conclude that the application of this type of problem is very rich. It is found in all areas of modern physics and in many branches of applied science. Our novelty is located in the following points:

1. We considered a new non-classical model on thermoelastic laminated beams with microtemperature effects, nonlinear delay, and nonlinear structural damping.
2. We have clearly outlined and minimized the impact of the weight of the different damping terms.
3. Our results can be seen as an extension of many recent related works by applying a convex function and a specific approach known as the multiplier technique.

Author Contributions: writing—original draft preparation H.S., F.Y., D.O., T.A., A.M. and F.S.D.; writing—review and editing, H.S., A.M., Y.J. and T.A., Supervision, K.B., Methodology, F.Y., D.O. and F.S.D. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The researchers would like to thank the Deanship of Scientific Research, Qassim University for the continuous support.

Conflicts of Interest: The authors declare no conflict of interest.

References


17. Zennir, K.H. Stabilization for Solutions of Plate Equation with Time-Varying Delay and Weak-Viscoelasticity in \( \mathbb{R}^n \). *Russ. Math.* 2020, 64, 21–33. [CrossRef]


**Disclaimer/Publisher's Note:** The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.