On Normalized Laplacian Spectra of the Weakly Zero-Divisor Graph of the Ring $\mathbb{Z}_n$

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Abstract: For a finite commutative ring $\mathcal{R}$ with identity $1 \neq 0$, the weakly zero-divisor graph of $\mathcal{R}$ denoted as $\Gamma(\mathcal{R})$ is a simple undirected graph having vertex set as a set of non-zero zero-divisors of $\mathcal{R}$ and two distinct vertices $a$ and $b$ are adjacent if and only if there exist elements $r \in \text{ann}(a)$ and $s \in \text{ann}(b)$ satisfying the condition $rs = 0$. The zero-divisor graph of a ring is a spanning sub-graph of the weakly zero-divisor graph. This article finds the normalized Laplacian spectra of the weakly zero-divisor graph $\Gamma(\mathcal{R})$ for various values of $n$.

Keywords: normalized Laplacian spectra; weakly zero-divisor graph; ring of integers modulo $n$; Euler totient function

MSC: 05C25; 05C50; 15A18

1. Introduction

Let $\mathcal{R}$ be a finite commutative ring with multiplicative identity $1 \neq 0$. A non-zero element denoted as $a \in \mathcal{R}$ is referred to as a zero-divisor of $\mathcal{R}$ if there exists an element $b \in \mathcal{R}$ distinct from zero ($0 \neq b \in \mathcal{R}$) such that $ab = 0$. The collection of these zero-divisors within the ring $\mathcal{R}$ is denoted as $Z(\mathcal{R})$, and $Z(\mathcal{R})^* = Z(\mathcal{R}) \setminus \{0\}$. The notation $\mathbb{Z}_n$ represents the ring of integers modulo $n$ for a given positive integer $n$. Many researchers have looked closely at zero-divisor graphs related to rings. This information can be found in past studies mentioned in the references [1–3].

We consider only connected, undirected, simple and finite graphs throughout this paper. We can represent the graph $G$ as $G(\nu, e)$, where $\nu$ represents the set of vertices and $e$ is the set of edges of $G$. If there is a connection between vertices $a$ and $b$ in the graph $G$, it is denoted as $a \sim b$. The set of vertices connected to a particular vertex $a$ is formally known as the neighborhood of $a$, denoted as $N_G(a)$. The number of edges adjacent to a specific vertex $a$ in the vertex set $V$ is called the degree of that vertex, denoted as $\text{deg}(a)$. A graph $G$ is called $r$-regular if every vertex $a$ in $V$ has a degree $r$. The complete graph with vertices $a$ is denoted as $K_n$, and the complete bipartite graph with two sets of vertices of sizes $(a, b)$ is written as $K_{a,b}$. Also, it is important to mention that the references [4–6] might have more symbols and words that we have not explained in this text.

Consider any square matrix $A$ having distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ with algebraic multiplicity $\eta_1, \eta_2, \ldots, \eta_k$, respectively. In this context, the collection of these eigenvalues is referred to as the spectrum of the matrix $A$, denoted as $\sigma(A)$, represented as follows:

$$\sigma(A) = \left\{ \frac{\lambda_1}{\eta_1}, \frac{\lambda_2}{\eta_2}, \ldots, \frac{\lambda_k}{\eta_k} \right\}.$$
Let \( n \) be a natural number. The \( n \)-dimensional square matrix, known as the adjacency matrix of the graph \( G \), is denoted as \( A(G) \). This matrix is defined as follows:

\[
A(G) = (a_{ij}) = \begin{cases} 
1, & v_i v_j \in E(G) \\
0, & \text{otherwise.}
\end{cases}
\]

The matrices \( L(G) = \text{Deg}(G) - A(G) \) and \( SL(G) = \text{Deg}(G) + A(G) \) are known as the Laplacian and signless Laplacian matrices of graph \( G \), respectively. Here, \( \text{Deg}(G) \) represents the diagonal matrix of vertex degrees, which is given by \( \text{Deg}(G) = \text{diag}(x_1, x_2, \ldots, x_n) \), where each \( x_i \) denotes the degree of vertex \( v_i \) for \( 1 \leq i \leq n \). We refer the reader to [7–9] for a more in-depth understanding of adjacency and the Laplacian spectra.

The normalized Laplacian spectra of zero-divisor graphs were discussed in [10,11]. Also, the normalized Laplacian spectra of power graphs associated with finite cyclic groups were discussed in [12]. Different spectra of zero-divisor graphs have been examined in prior works [13–15]. Also, some works on a space-time spectral order, a predictor-corrector compact difference, an implicit robust numerical scheme and an efficient ADI difference scheme are found in [16–19].

In this paper, we continue the study of spectral analysis of a weakly zero-divisor graph. The weakly zero-divisor graph of \( R \) was introduced by Nikmehr et al. [20] and denoted as \( \Gamma^W(R) \). It is a simple undirected graph having vertex set as set of non-zero zero-divisors of \( R \), and two distinct vertices \( a \) and \( b \) are adjacent if and only if there exist elements \( r \in \text{ann}(a) \) and \( s \in \text{ann}(b) \) satisfying the condition \( rs = 0 \). It is easy to observe that the zero-divisor graph of a ring is a spanning sub-graph of the weakly zero-divisor graph. More information about the weakly zero-divisor graphs can be found in [21,22].

This article begins with an introduction that outlines the motivation and objectives, followed by a review of the relevant literature. In Section 2, fundamental concepts of zero-divisors and commutative rings are established. We then explore the structural properties of weakly zero-divisor graphs and their connection to zero-divisor graphs. In Section 3, we detail the approach and techniques used in our analysis, including the computation of the normalized Laplacian spectra of the weakly zero-divisor graphs. In Section 4, we obtain the normalized Laplacian spectra of the weakly zero-divisor graph \( \Gamma^W(\mathbb{Z}_n) \) for some values of \( n \in \{p_1 p_2, p_1^2 p_2, p_1 p_2 p_3, p_1 p_2 p_3^2, p_1 p_2 p_3^2 p_2\} \), where \( p_1, p_2 \) and \( p_3 \) are prime numbers with \( p_1 < p_2 < p_3 \), and \( m \geq 2 \) is a positive integer. Also, we find the normalized Laplacian spectra of \( \Gamma^W(\mathbb{Z}_n) \) for \( n = p_1^m p_2 \cdots p_m q_1^{k_1} q_2^{k_2} \cdots q_l^{k_l} \), where \( k_i \geq 2, m \geq 1, r \geq 0 \), and \( q_i \)'s are distinct primes. We conclude this work with a conclusion in Section 5.

We have used the computational software Wolfram Mathematica version 13.2 to determine approximations of eigenvalues and characteristic polynomials for different matrices.

### 2. Preliminaries

We initiate by presenting the essential definitions and certain pre-existing findings that will be employed in establishing the principal conclusions.

**Definition 1.** Consider the graph \( G(v, e) \) of order \( n \) having vertex set \( \{1, 2, \ldots, n\} \) and \( G_i(v, e_i) \) be disjoint graphs of order \( n_i \), \( 1 \leq i \leq n \). The graph \( G[G_1, G_2, \ldots, G_n] \) is formed by taking the graphs \( G_1, G_2, \ldots, G_n \) and joining each vertex of \( G_i \) to every vertex of \( G_j \) whenever \( v_i \) and \( v_j \) are adjacent in \( G \).

The operation denoted as \( G[G_1, G_2, \ldots, G_m] \) is also known as a generalized join graph operation [23] and \( G \)-join operation. If \( G = K_2 \), then the \( K_2 \)-join is the usual join operation.

Let us consider that the set of integers modulo \( n \), denoted as \( \mathbb{Z}_n \), is of order \( n - \phi(n) - 1 \), where \( \phi(n) \) is the Euler totient function. Let \( u_1, u_2, \ldots, u_k \) be the proper divisors of \( n \). For each integer \( r \) satisfying \( 1 \leq r \leq k \), let us consider the collections of elements given by

\[
A_{ur} = \{ x \in \mathbb{Z}_n : (x, n) = u_r \},
\]
where \((x, n)\) denotes the largest common divisor between the values \(x\) and \(n\). Also, we see that \(A_{ur} \cap A_{us} = \emptyset\) whenever \(r \neq x\). This fact implies that the sets \(A_{ur}, A_{u2}, \ldots, A_{uk}\) are pairwise disjoint and partition the vertex set of \(W(G)\) as

\[
V(W(G)) = A_{u1} \cup A_{u2} \cup \cdots \cup A_{uk}.
\]

The subsequent lemma provides insight into the size of \(A_{ur}\).

**Lemma 1** ([6]). \(|A_{ur}| = \phi\left(\frac{n}{m}\right)\), for \(1 \leq r \leq k\).

**Lemma 2** ([21]). Consider the set of proper divisors of \(n\) denoted as \(\{u_1, u_2, \ldots, u_k\}\). And, let \(n\) be expressed as \(n = p_1 p_2 \ldots p_m q_1^{k_1} q_2^{k_2} \ldots q_i^{k_i}\), where \(k_i \geq 2, m \geq 1\) and \(i \geq 0\). If \(u_r \in \{p_1, p_2, \ldots, p_m\}\), then the induced sub-graph of \(W(G)\) by \(A_r\) is \(K_{\phi\left(\frac{n}{m}\right)}\).

**Corollary 1** ([21]). Consider a proper divisor \(u_r\) of the positive integer \(n\). The following statements are true:

(i) The induced sub-graph \(W(G)\) of \(W(G)\), formed by the vertices in the set \(A_{ur}\), takes one of two forms: either \(K_{\phi\left(\frac{n}{m}\right)}\) or \(K_{\phi\left(\frac{n}{m}\right)}\), where \(r \in \{1, 2, \ldots, k\}\).

(ii) When \(r \neq x\), for \(r, x \in \{1, 2, \ldots, k\}\), a vertex within \(A_r\) is connected to either all or none of the vertices in \(A_{ux}\) in the graph \(W(G)\).

The Corollary 1 mentioned above demonstrates that the sub-graphs \(W(G)\) formed within the structure of \(W(G)\) can be categorized as either complete graphs or empty graphs. The subsequent lemma asserts that \(W(G)\) can be characterized as a composite structure involving complete graphs and their corresponding complementary graphs. We introduce the graph \(G^n\), which is constructed as a complete graph using the set of all proper divisors of \(n\) denoted as \(\{u_1, u_2, \ldots, u_k\}\).

**Lemma 3** ([21]). Consider the induced sub-graph \(W(G)\) of \(W(G)\) formed by the vertices in the set \(A_{ur}\), where \(1 \leq r \leq k\). Then,

\[
W(G) = \delta^n[\{W(G)\}, W(G_{u1}), W(G_{u2}), \ldots, W(G_{uk})\}].
\]

**Theorem 1** ([24]). Consider a graph \(G\) with a vertex set \(v(G) = \{1, 2, \ldots, t\}\). Let \(G_i\) represent \(r_i\)-regular graphs with an order of \(n_i\) (\(i = 1, 2, \ldots, t\)). If \(G = \{G_1, G_2, \ldots, G_t\}\), then the normalized Laplacian spectra of \(G\) is given by

\[
\sigma_N(G) = \left( \bigcup_{i=1}^{t} \left( \frac{N_i}{N_i + r_i} \sigma_N(G_i) \right) \right) \bigcup \sigma(Y_N(G)),
\]

where

\[
N_i = \begin{cases} \sum_{j \in N_G(i)} n_j, & N_G(i) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}
\]

and

\[
Y_N(G) = (y_{ij})_{n \times n} = \begin{cases} \frac{N_i}{N_i + r_i}, & i = j, \\ \sqrt{\frac{n_j n_i}{(N_i + r_i)(N_j + r_j)}}, & ij \in e(G), \\ 0, & \text{otherwise.} \end{cases}
\]

A graph represented as \(G\) is classified as a normalized Laplacian integral graph if all its normalized Laplacian eigenvalues are integers. The subsequent statement establishes a
condition for a \( G \)-join graph to possess this integral property, and the proof of this condition can be easily deduced from Theorem 1.

**Proposition 1.** The \( G \)-join graph, denoted as \( G[G_1, G_2, \ldots, G_t] \) is normalized Laplacian integral if and only if \( \frac{N_i}{X_i + r_i}, \frac{r_i}{X_i + r_i} \in \mathbb{Z} \) and matrix \( Y_N(G) \) is integral.

From Theorem 1, we can deduce that when \( G_i \) is isomorphic to \( K_i \), two conditions hold: \( \frac{N_i}{X_i + r_i} = 1 \) and \( \frac{r_i}{X_i + r_i} = 0 \). In this scenario, the graph \( G = G[G_1, G_2, \ldots, G_t] \) possesses a normalized Laplacian integral property if and only if the matrix \( Y_N(G) \) is integral.

3. Methodology

Graph theory research is highly active, representing an applied science that shares a concrete connection with the realms of pure and discrete mathematics. In our work, our methodology and strategies are conventional, revolving around utilizing existing literature in the field to formulate and investigate new problems.

The utilization of linear algebra and matrix theory holds great importance in this context. Spectral graph theory, in particular, stands out as a formidable tool for visualizing the central objective of our research is to conduct a comprehensive analysis of the utilization of linear algebra and matrix theory holds great importance in this context. Spectral graph theory, in particular, stands out as a formidable tool for visualizing and gaining a deeper understanding of the principles underlying graph theory.

The centrality of our research is to conduct a comprehensive analysis of the normalized Laplacian spectra of the weakly zero-divisor graph for a finite commutative ring \( \mathbb{Z}_n \) across various values of \( n \). To achieve this, we use the concept of the normalized Laplacian spectra on the \( H \)-join operation of graphs, which was introduced by Wu et al. [24].

Pirzada et al. [10] investigated the normalized Laplacian spectra of the zero-divisor graph \( \Gamma(\mathbb{Z}_n) \) is a normalized Laplacian integral. We continue the study of spectral analysis of a weakly zero-divisor graph.

4. Normalized Laplacian Spectra of \( WT(\mathbb{Z}_n) \)

In this section, we compute the normalized Laplacian spectra of \( WT(\mathbb{Z}_n) \) for arbitrary \( n \). Let \( u_1, u_2, \ldots, u_k \) be the proper divisors of \( n \). For \( 1 \leq r \leq k \), we assign the weight \( |A_{u_i}| = \phi\left(\frac{n}{u_i}\right) \) to the vertex \( u_i \) of the graph \( \delta^*_n \). \( L(\delta^*_n) \) denotes the \( k \)th order weighted normalized Laplacian matrix of \( \delta^*_n \) defined in Theorem 1, which is given by

\[
L(\delta^*_n) = \begin{bmatrix}
    t_{1,1} & t_{1,2} & \cdots & t_{1,k} \\
    t_{2,1} & t_{2,2} & \cdots & t_{2,k} \\
    \vdots & \vdots & \ddots & \vdots \\
    t_{k,1} & t_{k,2} & \cdots & t_{k,k}
\end{bmatrix},
\tag{2}
\]

where

\[
t_{i,j} = \begin{cases} 
    \frac{M_{u_j}}{r_{u_i} + M_{u_i}} & i = j \\
    -\frac{\phi\left(\frac{n}{u_i}\right) \phi\left(\frac{n}{u_j}\right)}{(r_{u_i} + M_{u_i})(r_{u_j} + M_{u_j})} & u_i \sim u_j \in \delta^*_n \\
    0 & \text{otherwise.}
\end{cases}
\]

For \( 1 \leq i \neq j \leq k \) and \( M_{u_j} = \sum_{u_i \in N_G(u_j)} \phi\left(\frac{n}{u_i}\right) \).

The matrix denoted as \( L(\delta^*_n) \) is referred to as the weighted normalized Laplacian matrix associated with \( \delta^*_n \). When we observe the similarity between the matrices \( Y_N(G) \) and \( L(\delta^*_n) \), we can make an important observation.

**Remark 1.** \( Y_N(G) = L(\delta^*_n) \).
The primary outcome of this research article is the presentation and demonstration of the normalized Laplacian spectra for the weakly zero-divisor graph of $WT(Z_n)$.

**Theorem 2.** Let the proper divisors of $n$ be $u_1, u_2, \ldots, u_k$. Then, the normalized Laplacian spectra of $WT(Z_n)$ is given by

$$\sigma_N(WT(Z_n)) = \left( \bigcup_{i=1}^{k} \left( \frac{M_{u_i}}{r_{u_i} + r_{u_i} + M_{u_i}} \left( \sigma_N(WT(A_{u_i})) \setminus \{0\} \right) \right) \right) \cup \sigma_N(L(\delta_n^*)),$$

where $WT(A_{u_i})$ are $r_i$-regular graph and $\left( \frac{M_{u_i}}{r_{u_i} + r_{u_i} + M_{u_i}} \left( \sigma_N(WT(A_{u_i})) \setminus \{0\} \right) \right)$ represents that $\frac{M_{u_i}}{r_{u_i} + r_{u_i} + M_{u_i}}$ is added to each element of multiset $\frac{r_{u_i}}{r_{u_i} + M_{u_i}} \left( \sigma_N(WT(A_{u_i})) \setminus \{0\} \right)$.

**Proof.** Based on Lemma 3, we can observe that $WT(Z_n) = \delta_n^*[WT(A_{u_1}), WT(A_{u_2}), \ldots, WT(A_{u_k})]$. This implies that by utilizing the relationship $Y_Q(G) = L(\delta_n^*)$ and utilizing the implications of Theorem 1, the outcome is established. \(\square\)

Recall that the complete graph $K_m$ and its complement $\overline{K}_m$ on $m$ vertices with multiplicity have known normalized Laplacian spectra. Indeed,

$$\sigma_Q(K_m) = \left\{ \begin{array}{cc} 0 & 0 \ m-1 \ \ m-1 \end{array} \right\} \quad \text{and} \quad \sigma_Q(\overline{K}_m) = \left\{ \begin{array}{cc} 0 & 0 \end{array} \right\}.$$

By Corollary 1, $WT(A_{u_i})$ is isomorphic to either $K_{\phi(\frac{n}{u_i})}$ or $\overline{K}_{\phi(\frac{n}{u_i})}$. Consequently, as stated in Theorem 2, there are a total of $n - \phi(n) - 1$ normalized Laplacian spectra associated with $WT(Z_n)$. Among these, $n - \phi(n) - 1 - t$ has already been determined. The rest of the $t$ normalized Laplacian spectra of $WT(Z_n)$ are obtained from the roots of the characteristic polynomial of the matrix $L(\delta_n^*)$, as illustrated in Equation (2).

By applying Theorem 2, we can analyze the provided diagram below to calculate the normalized Laplacian spectra.

**Example 1.** The normalized Laplacian spectra of the weakly zero-divisor graph $WT(Z_{12})$ (Figure 1).

![Figure 1. The Graph $WT(Z_{12})$.](image)

Let $n = 12 = 2^2 \times 3$. First, we observe that $\delta_{12}^*$ is the complete graph on 4 vertices, i.e., $\{2, 3, 4, 6\}$. Then, by using Lemma 3, we have

$$WT(Z_{12}) = \delta_{12}^*[WT(A_2), WT(A_3), WT(A_4), WT(A_6)].$$
By using Lemma 2, we can also observe that $\text{WT}(\mathbb{Z}_{12}) = \delta_{12}^* \{K_2, K_2, K_2, K_1\}$. The cardinality $|V|$ of the vertex set $V$ of $\text{WT}(\mathbb{Z}_{12})$ is given by $\phi(2) + \phi(3) + \phi(4) + \phi(6) = 7$. It follows that $M_2 = \phi(2) + \phi(3) + \phi(4) = |V| - \phi(6) = 5$, $M_3 = |V| - \phi(4) = 5$, $M_4 = |V| - \phi(3) = 5$, $M_6 = |V| - \phi(2) = 6$. Also, we see that $r_3 = r_6 = 0$ and $r_2 = r_4 = 1$. Therefore, by Theorem 2, the normalized Laplacian spectra of $\text{WT}(\mathbb{Z}_{12})$ is

$$
\sigma_N(\text{WT}(\mathbb{Z}_{12})) = \left( \frac{M_2}{r_2 + M_2} + \frac{r_2}{r_2 + M_2} \left( \sigma_N(\text{WT}(A_2)) \setminus \{0\} \right) \right) \cup \left( \frac{M_3}{r_3 + M_3} + \frac{r_3}{r_3 + M_3} \left( \sigma_N(\text{WT}(A_3)) \setminus \{0\} \right) \right) \cup \left( \frac{M_4}{r_4 + M_4} + \frac{r_4}{r_4 + M_4} \left( \sigma_N(\text{WT}(A_4)) \setminus \{0\} \right) \right) \cup \left( \frac{M_6}{r_6 + M_6} + \frac{r_6}{r_6 + M_6} \left( \sigma_N(\text{WT}(A_6)) \setminus \{0\} \right) \right) \cup \sigma_N(\mathcal{L}(\delta_{12}^*)) \cup \sigma_N(\mathcal{L}(\delta_{12}^*)) \cup \sigma_N(\mathcal{L}(\delta_{12}^*)) \cup \sigma_N(\mathcal{L}(\delta_{12}^*)).
$$

The remaining normalized Laplacian eigenvalues are the eigenvalues of the matrix

$$
\mathcal{L}(\delta_{12}^*) = \begin{bmatrix}
\frac{5}{6} & -\sqrt{\frac{5}{15}} & -\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\
-\sqrt{\frac{5}{15}} & 1 & -\sqrt{\frac{5}{15}} & -\frac{1}{3\sqrt{2}} \\
-\frac{1}{3\sqrt{2}} & -\sqrt{\frac{5}{15}} & \frac{5}{6} & -\frac{1}{3\sqrt{2}} \\
-\frac{1}{3\sqrt{2}} & \frac{5}{6} & -\frac{1}{3\sqrt{2}} & 1
\end{bmatrix}.
$$

The approximated eigenvalues of the above matrix are

$$
\{0, 1.333, 1.667, 1.667\}.
$$

Now, we investigate the normalized Laplacian spectra of $\Gamma(\mathbb{Z}_n)$ for different values of $n$: when $n = p_1p_2$, $p_1^2p_2$, $p_1p_2p_3$ and $p_1^m p_2$, where $p_1$, $p_2$ and $p_3$ are distinct prime numbers with $p_1 < p_2 < p_3$ and $m \geq 2$ is a positive integer.

**Proposition 2.** The normalized Laplacian spectra of $\text{WT}(\mathbb{Z}_n)$ for $n = p_1p_2$ is given by

$$
\left\{ \begin{array}{ccc}
0 & 1 & 2 \\
1 & p_1 + p_2 - 4 & 1
\end{array} \right\}.
$$

**Proof.** Let $n = p_1p_2$, where $p < q$ and $p,q$ are distinct primes. First, we observe that $\delta_{p_1p_2}^*$ is the complete graph on two vertices so that $\delta_{p_1p_2}^*$ is $K_2$. Since $r_{p_1} = r_{p_2} = 0$ and $(M_{p_1}, M_{p_2}) = (p_1 - 1, p_2 - 1)$, by Theorem 2, the normalized Laplacian spectra of $\text{WT}(\mathbb{Z}_{p_1p_2})$ consist of the eigenvalue 1 with multiplicity $p_1 + p_2 - 4$ and the remaining two eigenvalues are given by the matrix

$$
\begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}.
$$

□
Proposition 3. The normalized Laplacian spectra of $\Gamma(Z_n)$, for $n = p_1^2p_2$ is given by

$$\left\{ \frac{|V|}{|V|-1} \phi(p_1p_2) + \phi(p_1) + \phi(p_2) - 3 \phi(p_1^2) - 1, 1, 1 \right\},$$

where $x_1, x_2, x_3$ and $x_4$ are the non-zero zeros of the characteristic polynomial of the below matrix (3) and $V$ is the vertex set of $\Gamma(Z_n)$.

Proof. Let $n = p_1^2p_2$, where $p_1 < p_2$ and $p_1, p_2$ are distinct primes. First, we can observe that $\delta_{p_1^2p_2}$ is the complete graph on four vertices $\{p_1, p_2, p_1^2\}$. Then, by using Lemma 3, we have

$$\Gamma(Z_n) = \delta_{p_1^2p_2}(\Gamma(A_{p_1}), \Gamma(A_{p_2}), \Gamma(A_{p_1^2}), \Gamma(A_{p_1p_2})).$$

By using Lemma 2, we can also observe that

$$\Gamma(Z_n) = \delta_{p_1^2p_2}(K_{p_1p_2}, K_{p_1}, K_{p_1}, K_{p_1}).$$

The cardinality $|V|$ of the vertex set $V$ of $\Gamma(Z_n)$ is given by $\phi(p_1p_2) + \phi(p_1^2) + \phi(p_2) + \phi(p_1)$. It follows that $M_{p_1} = \phi(p_1^2) + \phi(p_2) + \phi(p_1) = |V| - \phi(p_1p_2), M_{p_2} = |V| - \phi(p_1^2), M_{p_1^2} = |V| - \phi(p_2)$. Also, we have $r_{p_1} = \phi(p_1p_2) - 1, r_{p_1^2} = \phi(p_2) - 1, r_{p_1p_2} = \phi(p_1) - 1$ and $r_{p_2} = 0$. Therefore, by Theorem 2, the normalized Laplacian spectra of $\Gamma(Z_n)$ is

$$\sigma_N(\Gamma(Z_n)) = \left\{ \frac{|V|}{|V|-1} \phi(p_1p_2) + \phi(p_1) + \phi(p_2) - 3 \phi(p_1^2) - 1 \right\} \cup \sigma_N(\mathcal{L}(\delta_{p_1^2p_2})).$$

Thus, the remaining normalized Laplacian eigenvalues are the eigenvalues of the matrix

$$\sigma_N(\mathcal{L}(\delta_{p_1^2p_2})) = \begin{bmatrix} \frac{|V|}{|V|-1} & A & B & C \\ A & 1 & D & E \\ B & D & \frac{|V|}{|V|-1} & F \\ C & E & F & \frac{|V|}{|V|-1} \end{bmatrix}. \tag{3}$$

where $A = -\sqrt{\frac{\phi(p_1p_2)\phi(p_1^2)}{|V|-1(|V|-\phi(p_1^2))}}, B = -\sqrt{\frac{\phi(p_1p_2)\phi(p_2)}{|V|-1}}, C = -\sqrt{\frac{\phi(p_1p_2)\phi(p_1)}{|V|-1}},$

$D = -\sqrt{\frac{\phi(p_1^2)\phi(p_2)}{|V|-1(|V|-\phi(p_1^2))}}, E = -\sqrt{\frac{\phi(p_1p_2)\phi(p_1)}{|V|-1(|V|-\phi(p_1))}}$, and $F = -\sqrt{\frac{\phi(p_1p_2)\phi(p_2)}{|V|-1}}$. \hfill \Box

Proposition 4. The normalized Laplacian spectra of $\Gamma(Z_n)$, for $n = p_1p_2p_3$, with $p_1 < p_2 < p_3$ and $p_1, p_2, p_3$ are distinct primes is given by $\sigma_Q(\Gamma(Z_{p_1p_2p_3})) = \left\{ \frac{|V|}{|V|-1} \phi(p_1p_2) + \phi(p_1p_3) + \phi(p_2p_3) - 3, \phi(p_1) + \phi(p_2) + \phi(p_3) - 3 \right\}$, where $V$ is the vertex set of $\Gamma(Z_n)$. The roots of the characteristic polynomial of the below matrix (4) are the other normalized Laplacian eigenvalues of $\Gamma(Z_{p_1p_2p_3})$. 

\begin{equation*}
\left\{ \frac{|V|}{|V|-1} \phi(p_1p_2) + \phi(p_1p_3) + \phi(p_2p_3) - 3, \phi(p_1) + \phi(p_2) + \phi(p_3) - 3 \right\},
\end{equation*}
Proof. Let \( n = p_1 p_2 p_3 \). First, we observe that \( \delta_{p_1 p_2 p_3}^* \) is the complete graph on six vertices, i.e., \( p_1, p_2, p_3, p_1 p_2, p_1 p_3 \) and \( p_2 p_3 \). Then, by using Lemma 3, we have

\[
\omega(G(Z_{p_1 p_2 p_3})) = \delta_{p_1 p_2 p_3}^* [\omega(G(A_{p_1})), \omega(G(A_{p_2})), \omega(G(A_{p_3})), \omega(G(A_{p_1 p_2})), \omega(G(A_{p_1 p_3})), \omega(G(A_{p_2 p_3}))].
\]

By using Lemma 2, we can observe that

\[
\omega(G(Z_{p_1 p_2 p_3})) = \delta_{p_1 p_2 p_3}^* [\mathcal{R}_{\phi(p_3)}, \mathcal{R}_{\phi(p_1 p_3)}, \mathcal{R}_{\phi(p_1 p_2)}, K_{\phi(p_3)}, K_{\phi(p_2)}, K_{\phi(p_1)}].
\]

The cardinality \(|V|\) of the vertex set \( V \) of \( \omega(G(Z_{p_1 p_2 p_3})) \) is given by \( \phi(p_1) + \phi(p_2) + \phi(p_3) + \phi(p_1 p_2) + \phi(p_1 p_3) + \phi(p_2 p_3) \). It follows that \( M_{p_1} = \phi(p_1) + \phi(p_2) + \phi(p_3) + \phi(p_1 p_2) + \phi(p_1 p_3) = |V| - \phi(p_2 p_3), M_{p_2} = |V| - \phi(p_1 p_3), M_{p_3} = |V| - \phi(p_1 p_2), M_{p_1 p_2} = |V| - \phi(p_3), M_{p_1 p_3} = |V| - \phi(p_2) \) and \( M_{p_2 p_3} = |V| - \phi(p_1) \). Also, we see that \( r_{p_1} = r_{p_2} = r_{p_3} = 0 \) and \( r_{p_1 p_2} = \phi(p_3) - 1, r_{p_1 p_3} = \phi(p_2) - 1 \) and \( r_{p_2 p_3} = \phi(p_1) - 1 \). Therefore, by Theorem 2, the normalized Laplacian spectrum of \( G(Z_{p_1 p_2 p_3}) \) is

\[
\sigma_N(\omega(G(Z_{p_1 p_2 p_3}))) = \left( \frac{M_{p_1}}{r_{p_1} + M_{p_1}} + \frac{r_{p_1}}{r_{p_1} + M_{p_1}}, \frac{M_{p_2}}{r_{p_2} + M_{p_2}} + \frac{r_{p_2}}{r_{p_2} + M_{p_2}}, \frac{M_{p_3}}{r_{p_3} + M_{p_3}} + \frac{r_{p_3}}{r_{p_3} + M_{p_3}}, \frac{M_{p_1 p_2}}{r_{p_1 p_2} + M_{p_1 p_2}} + \frac{r_{p_1 p_2}}{r_{p_1 p_2} + M_{p_1 p_2}}, \frac{M_{p_1 p_3}}{r_{p_1 p_3} + M_{p_1 p_3}} + \frac{r_{p_1 p_3}}{r_{p_1 p_3} + M_{p_1 p_3}}, \frac{M_{p_2 p_3}}{r_{p_2 p_3} + M_{p_2 p_3}} + \frac{r_{p_2 p_3}}{r_{p_2 p_3} + M_{p_2 p_3}} \right) \cup \sigma_N(\mathcal{L}(\delta_{p_1 p_2 p_3}^*))
\]

\[
\cup \sigma_N(\mathcal{L}(\delta_{p_1 p_2 p_3}^*)) = \left\{ \phi(p_1) + \phi(p_1 p_2) + \phi(p_1 p_3) - 3, \phi(p_1) + \phi(p_2) + \phi(p_3) - 3 \right\}
\]

Thus, the remaining normalized Laplacian eigenvalues are the eigenvalues of the matrix

\[
\sigma_N(\mathcal{L}(\delta_{p_1 p_2 p_3}^*)) = \left[ \begin{array}{cccccc}
\phi(p_1) & \phi(p_1 p_2) & \phi(p_1 p_3) & \phi(p_2) & \phi(p_2 p_3) & \phi(p_3) \\
\phi(p_1 p_2) & \phi(p_1 p_3) & \phi(p_2 p_3) & \phi(p_1) & \phi(p_2) & \phi(p_3) \\
\phi(p_1 p_3) & \phi(p_2 p_3) & \phi(p_1 p_2) & \phi(p_1) & \phi(p_2) & \phi(p_3) \\
\phi(p_2) & \phi(p_2 p_3) & \phi(p_1 p_2) & \phi(p_1) & \phi(p_2) & \phi(p_3) \\
\phi(p_2 p_3) & \phi(p_1 p_3) & \phi(p_2) & \phi(p_1) & \phi(p_2) & \phi(p_3) \\
\phi(p_3) & \phi(p_1) & \phi(p_2) & \phi(p_1) & \phi(p_2) & \phi(p_3)
\end{array} \right].
\]

\[\square\]

Theorem 3. The normalized Laplacian spectra of \( \omega(G(Z_n)) \), for \( n = p_1^m p_2 \ (m \geq 2, \text{ and } p_1, p_2 \text{ are distinct primes}) \) is given by

\[
\sigma_N(\omega(G(Z_n))) = \left\{ \frac{|V|}{|V|-1}, \left( \sum_{i=1}^{m} \phi(p_1^{m-1} p_2) + \sum_{i=1}^{m-1} \phi(p_1^{m-i}) \right) - (2m - 1) \phi(p_1^{m}) - 1 \right\}.
\]
where $V$ is the vertex set of $WT(Z_n)$. The roots of the characteristic polynomial of the below matrix (4) are the other normalized Laplacian eigenvalues of $WT(Z_n)$.

**Proof.** Let $n = p_1^m p_2^m (m \geq 2)$. The vertex set of the graph $\delta_{p_1^m p_2^m}$ comprises the elements from the set $\{p_1, p_1^2, \ldots, p_1^m, p_2, p_1 p_2, p_1^2 p_2, \ldots, p_1^{m-1} p_2\}$. Then, by using Lemma 3, we have

$$WT(Z_{p_1^m p_2^m}) = \delta_{p_1^m p_2^m}^* [WT(A_{p_1^m}), WT(A_{p_1^m}), \ldots, WT(A_{p_1^m}), WT(A_{p_1^m}), WT(A_{p_1^m}), \ldots, WT(A_{p_1^{m-1} p_2})].$$

By using Lemma 2, also we can observe that

$$WT(Z_{p_1^m p_2^m}) = \delta_{p_1^m p_2^m}^* [K_{\phi(p_1^{m-1} p_2)}, K_{\phi(p_1^{m-2} p_2)}, \ldots, K_{\phi(p_2)}, K_{\phi(p_1^{m-1})}, \ldots, K_{\phi(p_1)}].$$

The cardinality $|V|$ of the vertex set $V$ of $WT(Z_{p_1^m p_2^m})$ is given by $\phi(p_1^{m-1} p_2^m) + \phi(p_1^{m-2} p_2^m) + \cdots + \phi(p_2^m) + \phi(p_1^m) = \phi(p_1^{m-1} p_2^m) + \phi(p_1^{m-2} p_2^m) + \cdots + \phi(p_1 p_2^m)$. It follows that $M_{p_1} = \phi(p_1^{m-1}) + \phi(p_1^{m-2}) + \phi(p_1^{m-3}) + \cdots + \phi(p_1) + \phi(p_2)$. Therefore, by Theorem 2, the normalized Laplacian spectra of $WT(Z_{p_1^m p_2^m})$ is given by

$$\sigma_Q(WT(Z_{p_1^m p_2^m})) = \left(\sum_{i=1}^{m} \frac{M_{p_1}}{r_{p_1} + M_{p_1}} \left(\sigma_N(WT(A_{p_1})) \setminus \{0\}\right) \cup \left(\sum_{i=1}^{m} \frac{M_{p_1^m}}{r_{p_1^m} + M_{p_1^m}} \left(\sigma_N(WT(A_{p_1^m})) \setminus \{0\}\right) \cup \cdots \cup \left(\sum_{i=1}^{m} \frac{M_{p_1^m}}{r_{p_1^m} + M_{p_1^m}} \left(\sigma_N(WT(A_{p_1^m})) \setminus \{0\}\right) \cup \left(\sum_{i=1}^{m} \frac{M_{p_1^m}}{r_{p_1^m} + M_{p_1^m}} \left(\sigma_N(WT(A_{p_1^m})) \setminus \{0\}\right) \cup \cdots \cup \left(\sum_{i=1}^{m} \frac{M_{p_1^m}}{r_{p_1^m} + M_{p_1^m}} \left(\sigma_N(WT(A_{p_1^m})) \setminus \{0\}\right) \cup \sigma_N(L(\delta_{p_1^m p_2^m}^*)) \right) \right)$$

$$= \left\{ \left(\sum_{i=1}^{m} \phi(p_1^{m-1} p_2^m) + \sum_{i=1}^{m-1} \phi(p_1^{m-i}) \right) (2m - 1) \phi(p_1^m) - 1 \right\} \cup \sigma_Q(L(\delta_{p_1^m p_2^m}^*)).$$
The rest of the 2m normalized Laplacian eigenvalues are the eigenvalues of the matrix
\[ \sigma_N(\mathcal{L}(\delta^m_{p_i^1 p_2})) = \]
\[
\begin{bmatrix}
\sqrt{\phi (p_1)} & \sqrt{\phi (m)} & \ldots & \sqrt{\phi (p_2)} \\
\sqrt{\phi (m)} & \sqrt{\phi (p_2)} & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
\sqrt{\phi (p_2)} & 1 & \ldots & \sqrt{\phi (m)} \\
\end{bmatrix}
\]

\[ \text{also,} \]
\[ \text{Theorem 4. Let} \ n = p_1 p_2 \cdots p_m q_1^{k_1} q_2^{k_2} \cdots q_r^{k_r} (k_i \geq 2, m \geq 1, r \geq 0), \text{where} \ p_i \text{’s and} \ q_i \text{’s are distinct primes. Suppose} \ \{u_1, u_2, \ldots, u_{\tau(n) - 2}\} \text{is the set of all proper divisors of} \ n. \text{Then, the normalized Laplacian spectra of} \ \Gamma(Z_n) \text{is given by}
\]
\[ \left\{ \frac{|V|}{\sum_{i=1}^{m} \phi \left( \frac{n}{u_i} \right) - m \sum_{i \neq p_i} \phi \left( \frac{n}{u_i} \right) - (\tau(n) - 2 - m) } \right\}, \]

where \ V \text{ is the vertex set of} \ \Gamma(Z_n). \text{The roots of the characteristic polynomial of the below matrix} (4) \text{are the other normalized Laplacian eigenvalues of} \ \Gamma(Z_n).

\[ \text{Proof.}\ \text{Suppose that} \ n = p_1 p_2 \cdots p_m q_1^{k_1} q_2^{k_2} \cdots q_r^{k_r} (k_i \geq 2, m \geq 1, r \geq 0), \text{where} \ p_i \text{’s and} \ q_i \text{’s are distinct primes. Let} \ \mathcal{D} = \{p_1, p_2, \ldots, p_m\}, \text{then by using Theorem 2, the normalized Laplacian spectra of} \ \Gamma(Z_n) \text{is}
\]
\[ \sigma_N(\Gamma(Z_n)) = \bigcup_{u_i \in \mathcal{D}} \left( \frac{M_{u_i}}{r_{u_i} + r_{u_i}} + \frac{r_{u_i}}{r_{u_i} + M_{u_i}} \left( \sigma_N(\Gamma(A_{u_i})) \setminus \{0\} \right) \right) \bigcup \left( \frac{M_{u_i}}{r_{u_i} + r_{u_i}} + \frac{r_{u_i}}{r_{u_i} + M_{u_i}} \left( \sigma_N(\Gamma(A_{u_i})) \setminus \{0\} \right) \right) \bigcup \sigma_N(\mathcal{L}(\delta^m_{u_i})). \]

Using Lemma 1, Lemma 2 and Corollary 1, we can derive the following results: for each \ u_i \in \mathcal{D}, \text{we have} \ \Gamma(A_{u_i}) = K_{\phi \left( \frac{n}{u_i} \right)} \text{and for} \ u_i \notin \mathcal{D}, \text{we have} \ \Gamma(A_{u_i}) = K_{\phi \left( \frac{n}{u_i} \right)} \text{. It can be noted that the size, denoted as} |V|, \text{of the vertex set} V \text{in the graph} \ \Gamma(Z_n) \text{is equal to the sum of} \ \phi \left( \frac{n}{u_i} \right) \text{for} i \text{ranging from 1 to} \ \tau(n) - 2, \text{i.e.,} \ |V| = \left( \sum_{i=1}^{\tau(n) - 2} \phi \left( \frac{n}{u_i} \right) \right). \text{Also, note that for} 1 \leq i \leq \tau(n) - 2, \text{we have}
\]
\[ M_{u_i} = \sum_{j=1, j \neq i}^{\tau(n) - 2} \phi \left( \frac{n}{u_i} \right) - |V| \phi \left( \frac{n}{u_i} \right). \]

Also, \ r_{u_i} = 0 \text{for} u_i \in \mathcal{D} \text{and} r_{u_i} = \phi \left( \frac{n}{u_i} \right) - 1 \text{for} u_i \notin \mathcal{D}. \text{Thus, we obtain}
\[
\sigma_Q(\text{WT}(\mathbb{Z}_n)) = \bigcup_{u_i \in \mathcal{D}} \left( 1 + \left( \sigma_N(K_{\phi(\frac{n}{m})}) \setminus \{0\} \right) \right) \bigcup_{u_i \in \mathcal{D}} \left( \frac{|V| - \phi\left( \frac{n}{m} \right)}{|V| - 1} + \left( \sigma_Q(K_{\phi(\frac{n}{m})}) \setminus \{0\} \right) \right) \\
\mathcal{N}(\delta_n) \\
= \bigcup_{u_i \in \mathcal{D}} \left( 1 + \left( \phi\left( \frac{n}{m} \right) \right) \setminus \{0\} \right) \bigcup_{u_i \in \mathcal{D}} \left( \frac{|V| - \phi\left( \frac{n}{m} \right)}{|V| - 1} + \left( \phi\left( \frac{n}{m} \right) \setminus \{0\} \right) \right) \bigcup \mathcal{N}(\mathcal{L}(\delta_n)) \\
= \bigcup_{u_i \in \mathcal{D}} \left( \phi\left( \frac{n}{m} \right) - 1 \right) \bigcup \mathcal{N}(\mathcal{L}(\delta_n)) \\
= \left( \sum_{i=1}^{m} \phi\left( \frac{n}{m} \right) - m \sum_{u_i \neq p} \phi\left( \frac{n}{m} \right) - (n - 2 - m) \right) \bigcup \mathcal{N}(\mathcal{L}(\delta_n)).
\]

Thus, the remaining normalized Laplacian eigenvalues are the eigenvalues of the matrix \(\mathcal{N}(\mathcal{L}(\delta_n)) =\)

\[
\begin{bmatrix}
1 & -\sqrt{\frac{\phi(n)}{|V|-\phi(n)}} & \cdots & -\sqrt{\frac{\phi(n)}{|V|-\phi(n)}} & \cdots & -\sqrt{\frac{\phi(n)}{|V|-\phi(n)}} \\
-\sqrt{\frac{\phi(n)}{|V|-\phi(n)}} & 1 & \cdots & -\sqrt{\frac{\phi(n)}{|V|-\phi(n)}} & \cdots & -\sqrt{\frac{\phi(n)}{|V|-\phi(n)}} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
-\sqrt{\frac{\phi(n)}{|V|-\phi(n)}} & -\sqrt{\frac{\phi(n)}{|V|-\phi(n)}} & 1 & \cdots & -\sqrt{\frac{\phi(n)}{|V|-\phi(n)}} & \cdots \\
-\sqrt{\frac{\phi(n)}{|V|-\phi(n)}} & -\sqrt{\frac{\phi(n)}{|V|-\phi(n)}} & -\sqrt{\frac{\phi(n)}{|V|-\phi(n)}} & 1 & \cdots & -\sqrt{\frac{\phi(n)}{|V|-\phi(n)}} \\
-\sqrt{\frac{\phi(n)}{|V|-\phi(n)}} & -\sqrt{\frac{\phi(n)}{|V|-\phi(n)}} & -\sqrt{\frac{\phi(n)}{|V|-\phi(n)}} & -\sqrt{\frac{\phi(n)}{|V|-\phi(n)}} & 1 & \cdots \\
\end{bmatrix}
\]

**Example 2.** The normalized Laplacian spectra of weakly zero-divisor graph \(\text{WT}(\mathbb{Z}_{30})\) (Figure 2a,b).

Let \(n = 30 = 2 \times 3 \times 5\). First, we observe that \(\delta_{30}\) is the complete graph on six vertices, i.e., \(\{2, 3, 5, 6, 10, 15\}\). Then, by using Lemma 3, we have

\[\text{WT}(\mathbb{Z}_{30}) = \delta_{30}[\text{WT}(A_2), \text{WT}(A_3), \text{WT}(A_5), \text{WT}(A_6), \text{WT}(A_{10}), \text{WT}(A_{15})].\]

By using Lemma 2, we can also observe that

\[\text{WT}(\mathbb{Z}_{30}) = \delta_{30}[\mathcal{K}_8, \mathcal{K}_4, \mathcal{K}_2, \mathcal{K}_4, \mathcal{K}_2, \mathcal{K}_1].\]

The cardinality \(|V|\) of the vertex set \(V\) of \(\text{WT}(\mathbb{Z}_{30})\) is given by \(\phi(2) + \phi(3) + \phi(5) + \phi(6) + \phi(10) + \phi(15) = 13\). It follows that \(M_2 = \phi(2) + \phi(3) + \phi(5) + \phi(6) + \phi(10) = |V| - \phi(15) = 5\), \(M_3 = |V| - \phi(10) = 9\), \(M_5 = |V| - \phi(6) = 11\), \(M_6 = |V| - \phi(5) = 9\), \(M_{10} = |V| - \phi(3) = 11\) and \(M_{15} = |V| - \phi(2) = 12\). Also, we see that \(r_2 = r_3 = r_5 = 0, r_{15} = 0\) and \(r_6 = 3, r_{10} = 1\). Therefore, by Theorem 2, the normalized Laplacian spectra of \(\text{WT}(\mathbb{Z}_{30})\)
\[ \sigma_N(\text{WT}(\mathbb{Z}_{30})) = \left( \frac{M_2}{r_2 + M_2} + \frac{r_2}{r_2 + M_2} \left( \sigma_N(\text{WT}(A_2)) \setminus \{0\} \right) \right) \]

\[ \bigcup \left( \frac{M_3}{r_3 + M_3} + \frac{r_3}{r_3 + M_3} \left( \sigma_N(\text{WT}(A_3)) \setminus \{0\} \right) \right) \]

\[ \bigcup \left( \frac{M_5}{r_5 + M_5} + \frac{r_5}{r_5 + M_5} \left( \sigma_N(\text{WT}(A_5)) \setminus \{0\} \right) \right) \]

\[ \bigcup \left( \frac{M_6}{r_6 + M_6} + \frac{r_6}{r_6 + M_6} \left( \sigma_N(\text{WT}(A_6)) \setminus \{0\} \right) \right) \]

\[ \bigcup \left( \frac{M_{10}}{r_{10} + M_{10}} + \frac{r_{10}}{r_{10} + M_{10}} \left( \sigma_N(\text{WT}(A_{10})) \setminus \{0\} \right) \right) \]

\[ \bigcup \left( \frac{M_{15}}{r_{15} + M_{15}} + \frac{r_{15}}{r_{15} + M_{15}} \left( \sigma_N(\text{WT}(A_{15})) \setminus \{0\} \right) \right) \bigcup \sigma_N(\mathbb{L}(\delta_{30})) \]

\[ = \left\{ \frac{1}{11}, \frac{12}{4} \right\} \bigcup \sigma_N(\mathbb{L}(\delta_{30})). \]

The remaining normalized Laplacian eigenvalues are the eigenvalues of the matrix \( \mathbb{L}(\delta_{30}) \)

\[
\begin{bmatrix}
   1 & -\frac{4}{3} \sqrt{\frac{3}{5}} & -\frac{4}{3} \sqrt{\frac{2}{11}} & -\frac{4}{3} \sqrt{\frac{2}{11}} & -\frac{2}{3} \sqrt{\frac{2}{15}} & -\frac{1}{3} \sqrt{\frac{2}{15}} & -\frac{1}{3} \sqrt{\frac{2}{15}} \\
   -\frac{4}{3} \sqrt{\frac{3}{5}} & 1 & -\frac{2}{3} \sqrt{\frac{2}{11}} & -\frac{2}{3} \sqrt{\frac{2}{11}} & -\frac{1}{3} \sqrt{\frac{2}{15}} & -\frac{1}{3} \sqrt{\frac{2}{15}} & -\frac{1}{3} \sqrt{\frac{2}{15}} \\
   -\frac{4}{3} \sqrt{\frac{2}{11}} & -\frac{2}{3} \sqrt{\frac{2}{11}} & 1 & -\frac{1}{3} \sqrt{\frac{2}{33}} & -\frac{1}{3} \sqrt{\frac{2}{33}} & -\frac{1}{3} \sqrt{\frac{2}{33}} & -\frac{1}{3} \sqrt{\frac{2}{33}} \\
   -\frac{2}{3} \sqrt{\frac{2}{11}} & -\frac{1}{3} \sqrt{\frac{2}{11}} & -\frac{1}{3} \sqrt{\frac{2}{11}} & 1 & -\frac{1}{3} \sqrt{\frac{2}{33}} & -\frac{1}{3} \sqrt{\frac{2}{33}} & -\frac{1}{3} \sqrt{\frac{2}{33}} \\
   -\frac{2}{3} \sqrt{\frac{2}{15}} & -\frac{1}{3} \sqrt{\frac{2}{15}} & -\frac{1}{3} \sqrt{\frac{2}{15}} & -\frac{1}{3} \sqrt{\frac{2}{15}} & 1 & -\frac{1}{3} \sqrt{\frac{2}{33}} & -\frac{1}{3} \sqrt{\frac{2}{33}} \\
   -\frac{1}{3} \sqrt{\frac{2}{15}} & -\frac{1}{3} \sqrt{\frac{2}{15}} & -\frac{1}{3} \sqrt{\frac{2}{15}} & -\frac{1}{3} \sqrt{\frac{2}{15}} & -\frac{1}{3} \sqrt{\frac{2}{15}} & 1 & -\frac{1}{3} \sqrt{\frac{2}{33}} \\
   -\frac{1}{3} \sqrt{\frac{2}{15}} & -\frac{1}{3} \sqrt{\frac{2}{15}} & -\frac{1}{3} \sqrt{\frac{2}{15}} & -\frac{1}{3} \sqrt{\frac{2}{15}} & -\frac{1}{3} \sqrt{\frac{2}{15}} & -\frac{1}{3} \sqrt{\frac{2}{15}} & 1
\end{bmatrix}
\]

The approximated eigenvalues of the above matrix are

\[ \{-0.797, 1.014, 1.083, 1.135, 1.289, 1.943\}. \]

Figure 2. (a) The Graph \( \text{WT}(\mathbb{Z}_{30}) \); (b) The Graph \( \delta_{30}^* \).
The major innovation of this paper lies in its exploration of the normalized Laplacian spectra of the weakly zero-divisor graph $W Γ(R)$ for finite commutative rings $R$ with $1 ≠ 0$. While previous research has investigated zero-divisor graphs, our work introduces a novel perspective by focusing on the weaker condition of weakly zero-divisors and employing spectral graph theory techniques. Specifically, we introduce the use of the normalized Laplacian spectra in conjunction with the H-join operation graphs, as introduced by Wu et al. [24] to unveil previously uncharted spectral characteristics of these graphs. This innovative approach not only provides a deeper understanding of the underlying algebraic structures but also opens up new avenues for research at the intersection of algebra and graph theory. Furthermore, our investigation extends to the special case of $W Γ(Z_n)$ for various values of $n$, offering a comprehensive analysis that adds substantial value to the field.

5. Conclusions

This paper centers on the exploration of an algebraic graph theory problem. We provide a solid foundation by offering background information and a comprehensive survey of relevant literature. The structure and presentation of this paper adhere to conventional norms and practices. Our study has delved into the normalized Laplacian spectra of weakly zero-divisor graphs in the context of finite commutative rings with non-zero identities. We focused our investigation on the weakly zero-divisor graph denoted as $W Γ(R)$, where $R$ represents such a ring. We established that the weakly zero-divisor graph is a valuable construct for understanding the structural properties of finite commutative rings. By exploring the normalized Laplacian spectra of these graphs, we gained insights into their spectral characteristics, shedding light on their algebraic and combinatorial properties.

Notably, our research extended to various finite commutative rings represented by $Z_n$, where $n$ took on different values. Through this extensive exploration, we observed how the spectral properties of the weakly zero-divisor graph varied with changes in the underlying ring structure. The findings from this study not only contribute to the field of algebraic graph theory but also have practical implications in areas such as network analysis and design. Understanding the spectral behavior of weakly zero-divisor graphs can aid in optimizing communication networks and other systems where graph theory plays a role.

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