Kernel Words and Gap Sequences of the Tribonacci Word on an Infinite Alphabet

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Abstract: In this paper, we propose a nuanced variation in the kernel words of the tribonacci sequence. Our primary objective is to investigate the intrinsic properties of the kernel words and associated gap sequences when the tribonacci sequence is expanded over an infinite alphabet.

Keywords: infinite tribonacci sequence; tribonacci number; kernel word; kernel number

MSC: 11B85; 68Q45

1. Introduction

Let \( t = \{ t_n \}_{n \geq 0} \) be the tribonacci sequence that is the fixed point of the Rauzy substitution \( \sigma \) defined by \( 0 \mapsto 01, \ 1 \mapsto 02, \ 2 \mapsto 0 \). The tribonacci sequence, as a natural progression from the well-known Fibonacci sequence, has garnered considerable scholarly attention as evident from works [1–5]. Rauzy’s contributions [2] to the domain encompass the dynamical and geometrical nuances of the tribonacci sequence. Further enriching this area, Tan and Wen [6] provided insights into the singular factorization and the Lyndon factorization, while concurrently exploring factor powers and calculating the free index. Pioneering works by Huang and Wen [7,8] delved into understanding the kernel words, gap sequences, and the numbers of repeated palindromes in the tribonacci sequence.

It is recognized that the tribonacci sequence augments the classical Fibonacci sequence from its binary form to a three-letter configuration. Beyond this, extensions involving \( k \) letters gave birth to the \( k \)-bonacci sequence, a topic that has witnessed rigorous explorations [9–11]. Notably, recent contributions by Ghareghani, Mohammad-Noori, and Sharifani [12,13] presented a broadened scope by generalizing the \( k \)-bonacci sequence to an infinite alphabet. Their studies shed light on intriguing aspects such as palindrome complexity, square factors, and critical factors.

In this paper, we are interested in the kernel words when \( k = 3 \). More precisely, we will consider this problem for a certain infinite tribonacci word defined on the infinite alphabet \( \{0, 1, 2, 3, \cdots \} \). This sequence can be comprehended as the fixed point of the morphism \( \phi \) of 0, characterized by:

\[
\begin{align*}
\phi(3i) &= (3i)(3i+1), \\
\phi(3i+1) &= (3i)(3i+2), \\
\phi(3i+2) &= (3i+3),
\end{align*}
\]

for all \( i \geq 0 \), i.e., \( T = \phi^{\infty}(0) = 0102013010234\cdots \). We will show more details in Section 1.2.

The concept of kernel words within the tribonacci sequence, introduced by Huang and Wen [8], emerged as a potent tool for scrutinizing its gap sequence. Building on similar lines, Ammar and Sellami [14] probed the kernel words inherent to the \( k \)-bonacci sequence. In subsequent sections, we shall delineate our findings on the kernel words and gap sequences particular to the infinite tribonacci sequence \( T \).

First of all, we start by recalling the basic definitions and notations.

1.1. Basic Notations

Let \( \mathcal{A} = \{a_0, a_1, \cdots, a_n, \cdots \} \) be an (infinite) alphabet. \( \mathcal{A}^k \) denotes the set of all words of length \( k \) on \( \mathcal{A} \), and \( \mathcal{A}^* = \cup_{k \geq 0} \mathcal{A}^k \) is the set of all words of any length on \( \mathcal{A} \), where
\( \mathcal{A}^0 = \{c\} \). We denote by \(|w|\) the length of a finite word \( w \in \mathcal{A}^* \). For any \( V, W \in \mathcal{A}^* \), we write \( V \prec W \) when the finite word \( V \) is a factor of the word \( W \); that is, when there exist words \( U, U' \in \mathcal{A}^* \), such that \( W = UVU' \). We say that \( V \) is a prefix (resp. suffix) of a word \( W \), and we write \( V < W \) (resp. \( V > W \)) if there exists a word \( U' \in \mathcal{A}^* \), such that \( W = VU' \) (resp. \( W = U'V \)).

Let \( u = u_0u_1u_2 \cdots u_n \) be a finite word (or \( u = u_0u_1u_2 \cdots u_n \cdots \) be a sequence). For any \( i \leq j \leq n \), we define \( u[i, j] := u_iu_{i+1} \cdots u_{j-1}u_j \), which means \( u[i, j] \) is the factor of \( u \) of length \( j - i + 1 \), starting from the \( i \)-th letter and ending at the \( j \)-th letter.

We denote by \( W^{-1} \) the inverse of \( W \); that is, \( W^{-1} = w_p^{-1} \cdots w_2^{-1}w_1^{-1} \) where \( W = w_1w_2 \cdots w_p \). If \( V \) is a suffix of \( W \), we can write \( WV^{-1} = U \), with \( W = UV \). This makes sense in \( \mathcal{A}^* \), since the reduced word associated with \( WV^{-1} \) belongs to \( \mathcal{A}^* \). Let \( (u_0u_1 \cdots u_{n-1}u_n)s = u_0u_1 \cdots u_{n-1} \), which deletes the last digit, and \( \phi(u_0u_1 \cdots u_{n-1}u_n) = u_1u_2 \cdots u_n \), which deletes the first digit.

Next, we give the extension of the tribonacci sequence to an infinite alphabet, which could be found in the much more general case in [12, 13].

### 1.2. The Tribonacci Sequence on an Infinite Alphabet

The tribonacci sequence \( t = \{t_n\}_{n \geq 0} \) is generated by the tribonacci morphism \( \sigma: 0 \mapsto 01, 1 \mapsto 02, 2 \mapsto 0 \), i.e., \( t = \sigma^\infty(0) \).

Consider the morphism \( \phi \) (over \( \mathbb{N} \)) which is given by

\[
\phi: (3i) \mapsto (3i)(3i + 1), (3i + 1) \mapsto (3i)(3i + 2) \quad \text{and} \quad (3i + 2) \mapsto (3i + 3),
\]

for all \( i \geq 0 \). We know that \( \phi(0) = 01, \phi^2(0) = \phi(\phi(0)) = \phi(01) = \phi(0)\phi(1) = 0102, \phi^3(0) = \phi(\phi^2(0)) = \phi(0102) = \phi(0)\phi(1)\phi(2) = 0102013 \), \( \cdots \). Then, it is obvious that \( \phi^n(0) < \phi^{n+1}(0) \). So, there exists a fixed point of \( \phi \) by iterating \( \phi \) with letter 0. The infinite tribonacci sequence \( T = \{T_i\}_{i \geq 0} \) is the fixed point of \( \phi \) starting by 0. The first several terms of \( T \) and \( t \) are

\[
T = 0102013010234010201343450102130102343435346 \cdots
\]

\[
t = 0102010010201012010010201001201001201020102010 \cdots
\]

It is easy to see that \( T \equiv t \) (mod 3), since \( \phi \) is reduced to \( \sigma \) while in modulo 3. In this sense, these two sequences are similar. Hence, the infinite tribonacci sequence \( T \) may inherit some combinatorial properties of the tribonacci sequence \( t \), which were studied sufficiently in [12, 13]. Let \( L_n := |\phi^n(0)| = |\sigma^n(0)| \), where \( L_0 = 1, L_1 = 2, \) and \( L_2 = 4 \). It is easy to see the sequence \( \{L_n\}_{n \geq 0} \) is formed by the tribonacci numbers [15], except for the initial numbers. So, for all \( n \geq 3 \), we have

\[
L_n = L_{n-1} + L_{n-2} + L_{n-3}.
\]

For convenience, we set \( L_{-1} = 1 \).

**Remark 1.** (1) It is easy to check that \( n \triangleright \phi^n(0) \) for any \( n \geq 0 \).
(2) For any \( n \geq 0 \) and \( i \geq 0 \), we have \( |\phi^n(3i)| = L_n \).

For application in Section 4, we need to introduce the following notation. We denote \( \ell_n := |\phi^n(1)| \), with \( \ell_0 = 1, \ell_1 = 2, \ell_2 = 3 \). It is easy to see that \( |\phi^n(3i+1)| = \ell_n \), for any \( n \geq 0 \) and \( i \geq 0 \). Next, we shall give the relationship between \( L_n \) and \( \ell_n \).

**Proposition 1.** For any \( n \geq 0 \), we have

\[
L_{n+1} = L_n + \ell_n.
\]
\[ \ell_{n+2} = L_{n+1} + L_n. \]  

**Proof.** Since \( \phi^{n+1}(0) = \phi^n(0) \phi^1(1), \) \( L_{n+1} = L_n + \ell_n. \) Meanwhile, from \( \phi^{n+2}(1) = \phi^{n+1}(0) \phi^{n+1}(2) = \phi^{n+1}(0) \phi^n(3), \) we have \( \ell_{n+2} = L_{n+1} + L_n. \) \( \square \)

**Proposition 2.** For any \( n \geq 3, \) we have

\[ \ell_n = \ell_{n-1} + \ell_{n-2} + \ell_{n-3}, \]  

with \( \ell_0 = 1, \ell_1 = 2, \) and \( \ell_2 = 3. \)

**Proof.** From Equation (3) and Equation (2), we have

\[
\ell_n &= L_{n-1} + L_{n-2} \\
&= L_n - L_{n-3} \\
&= (L_n - L_{n-1}) + (L_{n-1} - L_{n-2}) + (L_{n-2} - L_{n-3}) \\
&= \ell_{n-1} + \ell_{n-2} + \ell_{n-3},
\]

Hence, Equation (4) holds. \( \square \)

**Remark 2.** It is easy to see that the sequences \( \{L_n\}_n \geq 0 \) and \( \{\ell_n\}_n \geq 0 \) share the same recurrence relation except for the initial values.

In this paper, we are going to focus on the Kernel words and the gap sequences of the infinite tribonacci sequence \( T \) and the related properties. So, we need the following notations.

### 1.3. Gaps and Gap Sequences

Let \( \omega \) be a factor of the infinite tribonacci sequence \( T, \) and it occurs in the sequence \( T \) infinitely many times, which we arrange by the sequence \( \{\omega_p\}_{p \geq 1}, \) where \( \omega_p \) denotes the \( p \)-th occurrence of \( \omega. \)

Let the length of \( \omega \) be \( n, \) and \( \omega_p = x_{i+1} \cdots x_{i+n}, \) \( \omega_{p+1} = x_{j+1} \cdots x_{j+n}. \) The gap between \( \omega_p \) and \( \omega_{p+1}, \) denoted by \( G_p(\omega), \) is defined by

\[
G_p(\omega) = \begin{cases} 
\epsilon, & \text{when } i + n = j, \omega_p \text{ and } \omega_{p+1} \text{ are adjacent,} \\
x_{i+n+1} \cdots x_{j+p}, & \text{when } i + n < j, \omega_p \text{ and } \omega_{p+1} \text{ are separated,} \\
(x_{j+1} \cdots x_{i+n})^{-1}, & \text{when } i + n > j, \omega_p \text{ and } \omega_{p+1} \text{ are overlapped.}
\end{cases}
\]

The sequence \( \{G_p(\omega)\}_{p \geq 1} \) is called the gap sequence of the factor \( \omega. \)

For instance, \( G_1(3) = 0102 \) (separated), \( G_2(3) = \epsilon \) (adjacent), and \( G_2(343) = 3^{-1} \) (overlapped). When \( \omega_p \) and \( \omega_{p+1} \) are overlapped, we take the inverse word of the overlapped part as the gap \( G_p(\omega). \)

This paper is organized as follows. In Section 2, we give the kernel words of the infinite tribonacci sequence \( T \) and study the related properties. In Section 3, we show the expression and explore the properties of the gap sequence with respect to the kernel words. In Section 4, we give an application of the kernel words. In the last section, we present the conclusions.

### 2. Kernel Words of \( T \)

In this section, we extend the definition of kernel words to the infinite tribonacci sequence \( T \) in a natural manner and delve into their combinatorial characteristics.

For introducing the kernel words, we need the kernel number as follows.
Definition 1 ((Kernel number) Huang and Wen [8]). Let \( \{k_m\}_{m \geq 1} \) be the sequence of positive integers with \( k_1 = k_2 = k_3 = 1 \), and for \( m \geq 4 \),
\[
k_m = k_{m-1} + k_{m-2} + k_{m-3} - 1. \tag{5}
\]

The number \( k_m \) is called the \( m \)-th kernel number.

We notice that the numbers \( k_m \) are the same with the sequence A192804 in OEIS [16].

Remark 3. For any \( m \geq 4 \), we have \( L_{m-3} > k_m - 1 \).

Huang and Wen [8] gave the kernel words of the classical tribonacci sequence on the letters 0, 1, 2. We extend the kernel words of the infinite tribonacci sequence to an infinite alphabet.

Definition 2 (Kernel word). Let \( \{K_m\}_{m \geq 1} \) be the sequence of factors with \( K_1 = 0 \), \( K_2 = 1 \), \( K_3 = 2 \), and for \( m \geq 4 \),
\[
K_m = (m - 1)\phi^{m-3}(0)[1, k_m - 1]. \tag{6}
\]

We call \( K_m \) the kernel word with order \( m \).

Remark 4. (1) Notice that \( \phi^{m-3}(0)[1, k_m - 1] \prec \phi^{m-2}(0) \) with length \( k_m - 1 \), \( (m - 1) \) is the last digit of \( \phi^{m-1}(0) \), and
\[
\phi^m(0) = \phi^{m-1}(01) = \phi^{m-1}(0)\phi^{m-2}(02) = \phi^{m-1}(0)\phi^{m-2}(0)\phi^{m-3}(3).
\]

So, \( K_m = (m - 1)\phi^{m-3}(0)[1, k_m - 1] \prec \phi^{m-1}(0)\phi^{m-2}(0)[1, k_m - 1] \prec \phi^m(0) \prec T \). That is, for any \( m \in \mathbb{N} \), the word \( K_m \) is a factor of \( T \).

(2) It is obvious that \( |K_m| = k_m \).

(3) \( k_{m+4} > L_m > k_{m+2} \) for \( m \geq 1 \).

Proposition 3. For \( n \geq 0 \), \( L_{n+1} = k_{n+2} + k_{n+3} \).

Proof. When \( n = 0 \), \( L_0 = 1 \), \( k_2 = k_3 = 1 \), so \( L_0 + 1 = k_2 + k_3 \). Assuming that the result is true for all \( m \leq n \), then for \( m = n + 1 \), by Equation (5), we have
\[
L_{n+1} + 1 = L_n + L_{n-1} + L_{n-2} + 1
= (k_{n+2} + k_{n+3} - 1) + (k_{n+1} + k_{n+2} - 1) + (k_n + k_{n+1} - 1) + 1
= (k_{n+3} + k_{n+2} + k_{n+1} - 1) + (k_{n+2} + k_{n+1} + k_n - 1)
= k_{n+4} + k_{n+3}.
\]

Hence, the result holds. \( \square \)

Remark 5. For \( n \geq 1 \),
\[
L_n = k_{n+4} - k_{n+1}. \tag{7}
\]

Since \( k_{n+4} - k_{n+1} = k_{n+3} + k_{n+2} + k_{n+1} - 1 - k_{n+1} = k_{n+3} + k_{n+2} - 1 = L_n \).

Proposition 4. For any \( n \geq 1 \),
\[
k_{n+4} = 2k_{n+3} - k_n. \tag{8}
\]

\[
k_{n+4} = 2k_{n+3} - k_n. \tag{9}
\]
Theorem 1. By the definition of $k$, we have:

$$k_{n+4} - k_{n+3} = k_{n+2} + k_{n+1} - 1 = L_{n-1}.$$  \hspace{1cm} (10)

So, by Equation (7),

$$k_{n+4} - k_{n+3} = L_{n-1} = k_{n+3} - k_n$$

we have $k_{n+4} = 2k_{n+3} - k_n$. \hfill $\square$

Proposition 5. For any $n \geq m \geq 1$, with $n \equiv m \pmod{3}$, we have

$$K_m \triangleright K_n.$$ 

Proof. We use induction on $n$ to prove it. We know $K_1 = 0$, $K_4 = 30$, then the result is true for $n = 4$. We assume that the result is true for all positive integers less than $n$. Then, for $n = 3i + 1$, we have

$$K_{3i+1} = (3i)\phi^{3i-2}(0)[1,k_{3i+1} - 1] = (3i)\phi^{3i-3}(0)[1,k_{3i+1} - 1]$$

$$= (3i)\phi^{3i-3}(1)[1,k_{3i+1} - 1 - L_{3i-3})$$

$$= (3i)\phi^{3i-3}(0_*)(3i - 3)\phi^{3i-3}(1)[1,k_{3i+1} - L_{3i-3} - 1],$$

and by induction, $K_{3(i-1)+1} = (3i - 3)\phi^{3i-5}(0)[1,k_{3i-2} - 1].$

By Equation (7), for $n \geq 1$, $L_n = k_{n+4} - k_{n+1}$. So,

$$\mid \phi^{3i-5}(0)[1,k_{3i-2} - 1] \mid = \mid \phi^{3i-3}(1)[1,k_{3i+1} - L_{3i-3} - 1] \mid.$$

Moreover, $\phi^{3i-3}(1) = \phi^{3i-4}(02) = \phi^{3i-4}(0)\phi^{3i-4}(2) = \phi^{3i-5}(0)\phi^{3i-5}(1)\phi^{3i-4}(2).$ By Proposition 3, we have $k_{3i-2} - 1 \leq L_{3i-5}$. So,

$$\phi^{3i-3}(1)[1,k_{3i+1} - L_{3i-3} - 1] = \phi^{3i-5}(0)[1,k_{3i-2} - 1].$$

Hence, $K_{3i-2} \triangleright K_{3i+1}$. By the induction hypothesis, we know the result is true for $n = 3i + 1$. It is similar with the cases $n = 3i$ and $n = 3i + 2$. \hfill $\square$

Remark 6. For any $n \geq 0$, we have $0 \triangleright K_{3n+1}, 1 \triangleright K_{3n+2}$, and $2 \triangleright K_{3n+3}$. Since $K_0 = 0$, $K_1 = 1$, and $K_2 = 2$. It is easy to see the result is true from the above Proposition 5.

Theorem 1. For $m \geq 4$, $K_m = (m - 1)\phi^{m-4}(0), K_{m-3}$.

Proof. By the definition of $K_m$, we have

$$K_m = (m - 1)\phi^{m-3}(0)[1,k_m - 1]$$

$$= (m - 1)\left(\phi^{m-4}(0)\phi^{m-4}(1)\right)[1,k_m - 1]$$

$$= (m - 1)\phi^{m-4}(0)\phi^{m-4}(1)[1,k_m - L_{m-4} - 1] \quad \text{(since } k_m > L_{m-4})$$

$$= (m - 1)\phi^{m-4}(0)[m - 4]\left(\phi^{m-5}(0)\phi^{m-5}(2)\right)[1,k_m - L_{m-4} - 1]$$

$$= (m - 1)\phi^{m-4}(0)[m - 4]\left(\phi^{m-5}(0)\phi^{m-5}(2)\right)[1,k_{m-3} - 1]$$

$$= (m - 1)\phi^{m-4}(0)[m - 4]\phi^{m-5}(0)[1,k_{m-3} - 1] \quad \text{(since } L_{m-5} > k_{m-3})$$

$$= (m - 1)\phi^{m-4}(0)[m - 4]\phi^{m-6}(0)[1,k_{m-3} - 1]$$
where

\[ (m - 1)^{\phi^{m-4}(0)} \left( (m - 4)\phi^{m-6}(0) [1, k_{m-3} - 1] \right) \]

(since \( L_{m-6} > k_{m-3} \))

\[ = (m - 1)\phi^{m-4}(0) \phi^{m-6}(0) K_{m-3}. \]

\[ \square \]

3. Gap Sequence with Respect to the Kernel Word \( K_n \)

Let \( K_n \) be a kernel word which is a factor of \( T \) by Remark 4. We can decompose \( T \) in the following way:

\[
T = 0 \ 1 \ 0 \ 2 \ 01 \ 30 \ 1023 \ 401 \ 02013343 \cdots
\]

\[ = K_1 G_1 K_2 G_2 K_3 G_3 \ K_4 \ G_4 \ K_5 \ G_5 \cdots \ K_n \ G_n \cdots, \]

where \( G_n \) is the gap sequence between \( K_n \) and \( K_{n+1} \). The aim of this section is to give the expression and properties of the gap sequence \( \{G_n\}_{n \geq 1} \).

It is easy to see \( G_1 = e, G_2 = 0, G_3 = 01 \).

Lemma 1. For any \( n \geq 1 \),

\[ \phi^n(0) = K_1 G_1 K_2 G_2 \cdots K_n G_n(n), \] (11)

and

\[ \phi^n(1) =^* K_{n+1} G_{n+1}(n + 1). \] (12)

Proof. It is easy to check that the results are true for \( n = 1 \). Assuming that the results are true for all \( m \leq n \), then for \( m = n + 1 \), we have

\[ \phi^{n+1}(0) = \phi^n(0) = \phi^n(0) \phi^n(0) \]

\[ = K_1 G_1 K_2 G_2 \cdots K_n G_n(n) K_{n+1} G_{n+1}(n + 1) \]

\[ = K_1 G_1 K_2 G_2 \cdots K_n G_n(n)^* K_{n+1} G_{n+1}(n + 1) \]

\[ = K_1 G_1 K_2 G_2 \cdots K_n G_n K_{n+1} G_{n+1}(n + 1). \]

So the result is true for Equation (11). Hence,

\[ \phi^{n+2}(0) = K_1 G_1 K_2 G_2 \cdots K_{n+1} G_{n+1} K_{n+2} G_{n+2}(n + 2) \]

\[ = K_1 G_1 K_2 G_2 \cdots K_{n+1} G_{n+1}(n + 1)^* K_{n+2} G_{n+2}(n + 2). \] (13)

Meanwhile, we know

\[ \phi^{n+2}(0) = \phi^{n+1}(0) = \phi^{n+1}(0) \phi^{n+1}(1) \]

\[ = K_1 G_1 K_2 G_2 \cdots K_{n+1} G_{n+1}(n + 1) \phi^{n+1}(1). \] (14)

Combining Equations (13) and (14), we have

\[ \phi^{n+1}(1) =^* K_{n+2} G_{n+2}(n + 2), \]

which implies the result is true for Equation (12). \( \square \)

Proposition 6. For any \( n \geq 4 \), \( K_n = (n - 1)K_1 G_1 K_2 G_2 \cdots K_{n-4} G_{n-4} K_{n-3} \).

Proof. By Theorem 1 and Lemma 1, we have

\[ K_n = (n - 1) \phi^{n-4}(0) \phi^{n-6}(0) K_{n-3} \]

\[ = (n - 1) K_1 G_1 K_2 G_2 \cdots K_{n-4} G_{n-4}(n - 4) K_{n-3} \]

\[ = (n - 1) K_1 G_1 K_2 G_2 \cdots K_{n-4} G_{n-4} K_{n-3}. \]
Theorem 2. For any \( n \geq 4 \), \( G_n = G_{n-3}K_{n-2}G_{n-2}(n-2)\phi^{n-3}(3)_+ \).

Proof. According to the iteration of \( \phi \), by Lemma 1 and Proposition 6, we have

\[
\phi^n(0) = \phi^{n-1}(01) = \phi^{n-1}(0)\phi^{n-2}(02) = \phi^{n-1}(0)\phi^{n-2}(0)\phi^{n-3}(3) = (K_1G_1 \cdots K_{n-1}G_{n-1}(n-1))(K_1G_1 \cdots K_{n-3}G_{n-3}K_{n-2}G_{n-2}(n-2))\phi^{n-3}(3) = K_1G_1 \cdots K_{n-1}G_{n-1}((n-1)K_1G_1 \cdots K_{n-3})G_{n-3}K_{n-2}G_{n-2}(n-2)\phi^{n-3}(3) = K_1G_1 \cdots K_{n-1}G_{n-1}K_nG_{n-3}K_{n-2}G_{n-2}(n-2)\phi^{n-3}(3).
\]

Moreover, by Lemma 1, we have \( \phi^n(0) = K_1G_1K_2G_2 \cdots K_nG_n(n) \). So,

\[
G_n = G_{n-3}K_{n-2}G_{n-2}(n-2)\phi^{n-3}(3)_+,
\]

since \( n > \phi^{n-3}(3) \).

Remark 7. For any \( n \geq 4 \), \( G_{n-3} < G_n \).

Next, we shall study the length of the gap sequence \( \{G_n\}_{n \geq 1} \). Let \( g_n := |G_n| \) for \( n \geq 1 \). It is easy to see that \( g_1 = 0, g_2 = 1, \) and \( g_3 = 2 \). For convenience, we set \( g_0 = 0 \). We notice that the numbers \( g_n \) are the same with the sequence A008937 in OEIS [17]. So, for any \( n \geq 4 \), we have

\[
g_n = \sum_{k=1}^{n-3} L_k. 
\]  

Corollary 1. For any \( n \geq 4 \), we have

\[
g_n = g_{n-3} + k_{n-2} + g_{n-2} + L_{n-3}. 
\]  

Remark 8. (1) For any \( n \geq 1 \), we have

\[
L_n = L_{n-1} + k_n + g_n,
\]  

since \( \phi^n(0) = \phi^{n-1}(01) = \phi^{n-1}(0)K_nG_n(n) \) from Equation (12).

(2) For any \( n \geq 2 \), we have

\[
g_{n+1} - g_n = L_{n-2}.
\]

Moreover, for any \( n \geq 1 \),

\[
g_{n+4} = 2g_{n+3} - g_n.
\]

Proposition 7. For any \( n \geq 1 \), we have

\[
k_n = g_{n+1} - g_n - g_{n-1}.
\]

Proof. By Equation (17) and Equation (15), we have

\[
k_n = L_n - L_{n-1} - g_n = L_{n-2} + L_{n-3} - g_n.
\]
\[
\phi_n = \left( \sum_{k=1}^{n-2} L_k - \sum_{k=1}^{n-4} L_k \right) - g_n
\]
\[
g_{n+1} - g_{n-1} - g_n.
\]

\[\square\]

4. Application

In this section, as an application of our main results, we mainly use the properties of the kernel words to decompose the infinite tribonacci sequence \( T \) and study the related factor properties.

Theorem 3.

\[
T = 0 \ 02 \ 02 \ 02 \ 02 \ 02 \ 02 \prod_{i=0}^{\infty} \left[ (\phi^i(0)\phi^{i-3}(0)\phi^{i-6}(0) \cdots \phi^0(0))^{-1} \phi^{i+3}(1), K_{i+5} \right],
\]
where \( i \equiv i' \pmod{3} \), \( i' \in \{0,1,2\} \).

To prove Theorem 3, we need the following lemmas.

Lemma 2. For any \( n \geq 3 \),

\[
|\phi^{n+3}(1)| - 1 - \left( |\phi^n(0)| + |\phi^{n-3}(0)| + |\phi^{n-6}(0)| + \cdots + |\phi^0(0)| \right) + k_{n+5} = L_{n+3}. \quad (18)
\]

Proof. It is easy to check that Equation (18) holds for \( n = 3 \). Assuming that Equation (18) holds for all \( m < n \), we shall check it for \( m = n \). If \( m = 3l \), then \( m' = 0 \). By the induction hypothesis, we know Equation (18) holds for \( n - 3 \), i.e.,

\[
\ell_n - 1 - (L_{n-3} + L_{n-6} + \cdots + L_0) + k_{n+2} = L_n.
\]

From Equation (3), we have \( \ell_n = L_{n-1} + L_{n-2} \). So,

\[
L_{n-1} + L_{n-2} - 1 - (L_{n-3} + L_{n-6} + \cdots + L_0) + k_{n+2} = L_n.
\]

Hence, by Proposition 3,

\[
L_{n-3} + L_{n-6} + \cdots + L_0 = L_{n-1} + L_{n-2} - 1 + k_{n+2} - L_n
\]
\[
= L_{n-1} + L_{n-2} - 1 + k_{n+2} - (L_{n-1} + L_{n-2} + L_{n-3})
\]
\[
= k_{n+2} - (L_{n-3} + 1)
\]
\[
= k_{n+2} - (k_n + k_{n-1})
\]
\[
= k_{n+1} + k_n + k_{n-1} - 1 - (k_n + k_{n-1})
\]
\[
= k_{n+1} - 1. \quad (19)
\]

By Equation (8), we have

\[
k_{n+5} = k_{n+4} + L_n. \quad (20)
\]

Then, by Equation (3), Equation (19), Equation (20) and Equation (7), we have

\[
|\phi^{n+3}(1)| - 1 - \left( |\phi^n(0)| + |\phi^{n-3}(0)| + |\phi^{n-6}(0)| + \cdots + |\phi^0(0)| \right) + k_{n+5}
\]
\[
= \ell_{n+3} - 1 - (L_n + L_{n-3} + \cdots + L_0) + k_{n+5}
\]
\[
= L_{n+2} + L_{n+1} - 1 - (L_n + L_{n-3} + \cdots + L_0) + k_{n+5}
\]
\[
= L_{n+2} + L_{n+1} - L_n - (k_{n+1} - 1) + k_{n+5}
\]
\[
= L_{n+2} + L_{n+1} - L_n - k_{n+1} + k_{n+5}
\]
Therefore, Equation (18) holds. 

By Lemma 3, we only need to show where \( n \equiv \phi \pmod{3} \). It is easy to check that Equation (21) holds for \( n \equiv \phi \pmod{3} \), \( n' \in \{0, 1, 2\} \).

**Proof.** It is easy to check that Equation (21) holds for \( n = 3 \). Assume that Equation (21) holds for all \( m < n \). We shall check it for \( m = n \). If \( m = 3i \), then \( m' = 0 \). By the induction hypothesis, we have \( \phi^{n-3}(0)\phi^{n-6}(0)\cdots \phi^{i}(0) < \phi^{n+3}(1) \). Hence,

\[
\phi^n(0)\phi^{n-3}(0)\phi^{n-6}(0)\cdots \phi^{i}(0) < \phi^n(0)\phi^i(1) = \phi^{n+1}(1).
\]

We know \( \phi^{n+1}(0) < \phi^{n+3}(1) \), since

\[
\phi^{n+3}(1) = \phi^{n+2}(0)\phi^{n+2}(2) = \phi^{n+1}(0)\phi^{n+1}(1)\phi^{n+2}(2).
\]

Hence, Equation (21) holds. The case \( n = 3i + 1 \) and \( n = 3i + 2 \) are similar with the case \( n = 3i \) for any \( i \geq 0 \). Therefore, Equation (21) holds. 

**Lemma 4.** For any \( n \geq 3 \),

\[
\phi^{n+1}(0)[1, \phi_{n+4} - 1] = \phi^n(0)\phi^{n-3}(0)\phi^{n-6}(0)\cdots \phi^n(0),
\]

where \( n \equiv n' \pmod{3} \), \( n' \in \{0, 1, 2\} \).

**Proof.** By Lemma 3, we only need to show \( |\phi^n(0)| + |\phi^{n-3}(0)| + \cdots + |\phi^n(0)| = k_{n+4} - 1 \), i.e.,

\[
L_n + L_{n-3} + \cdots + L_{n'} = k_{n+4} - 1.
\]

It is direct from Equation (19). 

**Lemma 5.** For any \( n \geq 0 \),

\[
\phi^{n+4}(0)\phi^{n+1}(0)\cdots \phi^{(n+4)'}(0) = 0 \bigg/ 02 \phi^2(1)K_4 \prod_{i=0}^{n-1} \left[ \left( \phi^i(0)\phi^{i-3}(0)\phi^{i-6}(0)\cdots \phi^i(0) \right)^{-1} \phi^{i+3}(1)K_{i+5} \right],
\]

where \( i \equiv i' \pmod{3} \), \( i' \in \{0, 1, 2\} \).

**Proof.** When \( n = 0 \), it is easy to check that the result holds. Now, supposing the result holds for all \( m < n \), we shall prove it for \( n \). By Lemma 4, we have

\[
0 \bigg/ 02 \phi^2(1)K_4 \prod_{i=0}^{n-1} \left[ \left( \phi^i(0)\phi^{i-3}(0)\phi^{i-6}(0)\cdots \phi^i(0) \right)^{-1} \phi^{i+3}(1)K_{i+5} \right] = 0 \bigg/ 02 \phi^2(1)K_4 \prod_{i=0}^{n-1} \left[ \left( \phi^i(0)\phi^{i-3}(0)\phi^{i-6}(0)\cdots \phi^i(0) \right)^{-1} \phi^{i+3}(1)K_{i+5} \right]
\]

\[
\left( \phi^n(0)\phi^{n-3}(0)\phi^{n-6}(0)\cdots \phi^n(0) \right)^{-1} \phi^{n+3}(1)K_{n+5}
\]
\begin{align*}
\phi^n + 3(0) & \left( \phi^n(0) \cdots \phi^n(0) \right) \left( \phi^n(0) \phi^n+3(0) \cdots \phi^n(0) \right)^{-1} \phi^n+3(1) K_{n+5} \\
= \phi^n + 3(0) \phi^n+3(1) \phi^n+2(0) |1, k_{n+5} - 1] & \text{(by the definition of } K_{n+5}) \\
= \phi^n + 4(0) \phi^n+2(0) |1, k_{n+5} - 1] \\
= \phi^n + 4(0) \phi^n+1(0) \cdots \phi^{n+4} \gamma(0). \\
\end{align*}

Therefore, the result holds. \qed

**Proof of Theorem 3.** It is directly from the above lemmas. \qed

5. Conclusions

In this work, we presented a comprehensive study of the kernel words and their associated gap sequences within the infinite tribonacci sequence $T$. By delving into the combinatorial attributes of these kernel words and gap sequences, we were able to shed light on the local isomorphism and overlap characteristics of factors embedded in the sequence. Furthermore, we showcased how kernel words can be effectively applied to reconstruct the infinite tribonacci sequence $T$. This method of factorization facilitates a detailed analysis of the sequence, highlighting repetitive patterns and its intrinsic structural characteristics. Such insights not only deepen our understanding of the sequence’s internal architecture but also hint at potential intersections with other mathematical domains. These connections and further extrapolations will be the focal point of our subsequent investigations.

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