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Spectral, Scattering and Dynamics: Gelfand–Levitan–Marchenko–Krein Equations

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Abstract: In this paper, we consider the Gelfand–Levitan–Marchenko–Krein approach. It is used for solving a variety of inverse problems, like inverse scattering or inverse problems for wave-type equations in both spectral and dynamic formulations. The approach is based on a reduction of the problem to the set of integral equations. While it is used in a wide range of applications, one of the most famous parts of the approach is given via the inverse scattering method, which utilizes solving the inverse problem for integrating the nonlinear Schrodinger equation. In this work, we present a short historical review that reflects the development of the approach, provide the variations of the method for 1D and 2D problems and consider some aspects of numerical solutions of the corresponding integral equations.

Keywords: Gelfand–Levitan–Krein–Marchenko equation; inverse coefficient problem; inverse scattering problem

MSC: 35R30; 65M32; 65R32; 65N21

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1. Introduction

The origin of Gelfand–Levitan–Krein–Marchenko (GLKM) approach is related to the inverse problem of recovering the differential operator via the spectral data. The first work in this field was conducted in 1929 by V.M. Ambartsumyan [1] and this was later followed by results of G. Borg [2] and N. Levinson [3,4]. The foundation of the GLKM method is connected with the paper [5], where I.M. Gelfand and B.M. Levitan presented a method to reconstruct a Sturm–Liouville operator by the spectral function and provided conditions sufficient for a given monotonic function to be the spectral function of the operator. A few years later, in 1954, a paper [6] by M.G. Krein was published, proposing algorithms for solving the inverse problem for a wave equation. These two works can be used to denote two directions of the method's development in the second half of the XX century. On the one hand, the continued study of inverse scattering and its applications was used by C.S. Gardner et al. in [7], where the authors used the theory of the inverse scattering problem to integrate the Korteweg de Vries equation and, later, the non-linear Schrodinger equation. This (and the fact the Schrodinger equation is widely used in photonics) determined the importance of developing the methods and algorithms of inverse scattering. On the other hand, results of M.G. Krein led to the dynamic version of the approach, which was applied to a variety of inverse problems for hyperbolic equations, and has strong connections with geophysical problems. The feature of the approach that is important for geophysical applications is its direct nature, with respect to methods, which reduces the inverse problem to optimization.

Since both "branches" of the approach have different variations and applications, the motivation behind this paper is to provide a structured and generalized review of the approach that considers the history of the method and its main variations (and multidimensional analogs as well) from both the theoretical and numerical points of view.

The paper is organized as follows. In the current section we, after a brief introduction, provide a review of the large amount of works related to the GLKM approach, divided into the two groups—papers related to the spectral and scattering inverse problems, and papers related to the inverse problems for hyperbolic equations.

In Section 2, we provide a description of the result that was previously obtained in a one-dimensional case. We provide the formulation of the I.M. Gelfand–B.M. Levitan equation in both spectral and dynamic formulations in Section 2.1. In Section 2.2, we consider the inverse scattering problem, which is reduced to the V.A. Marchenko equation and also describe the way to use this equation to obtain the solution of the Cauchy problem for the KdV equation (the inverse scattering method). We provide more details in Section 2.2, because the general scheme behind the results of Section 2.1 is covered by their two-dimensional analogs. Section 2.3 considers the M.G. Krein equation that arises in the 1D acoustic inverse problem. We also use this opportunity to consider the boundary-control method (which is often referred to as the BC method) for that inverse problem. We formulate the essence of the method and show that both the Krein method and the BC method have the same discrete form in one-dimensional cases. Then, in Section 2.5 we formulate how to use the already mentioned one-dimensional results in order to solve the inverse problem for the 1D seismic inverse problem.

Section 3 is devoted to the multi-dimensional analog of the Gelfand–Levitan equation (Section 3.1) and the Krein equation (Section 3.2), correspondingly. In Section 4, we present a review of the numerical methods that are applicable to solve the GLKM equation in both 1D and 2D. We also provide some numerical results to illustrate the two-dimensional variation of the GLKM method in Section 5. And finally, in Section 6 we provide some new results regarding using the approach to recover the speed of sound and the density from the acoustic equation in cases in which both functions depend on both space variables.

1.1. Inverse Spectral Problems—Inverse Scattering Problems and Method

Let us consider the short history of the development of the theory of inverse spectral and inverse scattering problems. In this section, we will also mention the inverse scattering method because it is very difficult to separate the scientific results obtained.

1929—V.M. Ambartsumyan [1] established that "a homogeneous string is uniquely determined by the set of eigenfrequencies". Specifically, if eigenvalues λ_n of the boundary value problem

$$-y'' + q(x)y = \lambda y, \quad x \in [0, \pi];$$
$$y'(0) = y'(\pi) = 0$$

are $\lambda_n = n^2$ and $q(x)$ is a real continuous function, then $q(x) \equiv 0$.

1943—W. Heisenberg [8,9] considered the inverse scattering problem and proved that in order to solve the inverse scattering problem, it is sufficient to know the asymptotic behavior of the wave function.

1946—G. Borg [2] investigated the inverse problem for a Sturm–Liouville operator. He shown that the result, provided by V.M. Ambartsumyan was quite rare, in the sense of recovering the potential, using only one spectrum. More specifically, he proved the following:

Theorem 1. *Let $\lambda_0 < \lambda_1 < \lambda_2 < \ldots$ be the eigenvalues of the operator*

$$-y'' + q(x)y = \lambda y, \quad 0 \le x \le \pi,$$
$$y'(0) - hy(0) = 0, \quad y'(\pi) + Hy(\pi) = 0, \quad h, H \in \mathbb{R},$$

Let $\mu_0 < \mu_1 < \mu_2 < \ldots$ be the eigenvalues of the operator

$$-y'' + q(x)y = \mu y, \quad 0 \le x \le \pi,$$
$$y'(0) - hy(0) = 0, \quad y'(\pi) + H_1 y(\pi) = 0, \quad H_1 \ne H \in \mathbb{R},$$

Then, the sequences $\{\lambda_m\}_{m=0}^{\infty}$ and $\{\mu_n\}_{n=0}^{\infty}$ uniquely determine the function $q(x)$ and the numbers h, H and H_1.

Over the next several years, the subject was actively studied and then advanced in 1949 by a series of interconnected works of N. Levinson and V. Bargmann. N. Levinson [3] provided simpler proofs for some of the results obtained by G. Borg [2]. Then, he [4] tackled an inverse problem of the quantum theory of scattering. He proved that in the absence of negative eigenvalues a scattering phase given for all positive energies and any fixed angular momentum uniquely determines the potential. Meanwhile V. Bargmann [10,11] proved that in the general case the spherically symmetric potential (at any fixed angular momentum) is not uniquely determined by the scattering phase.

1949—A.N. Tikhonov [12] continued (in a way) the works of G. Borg and N. Levinson in studying the properties of data that can guarantee the uniqueness of the solution. He proved the theorem of uniqueness of solving the inverse Sturm–Liouville problem on a semi-axis via a given Weyl function. These works were also the first where the inverse problem on a semi-axis was considered.

1950—V.A. Marchenko [13] investigated several questions for differential operators. This work can be considered the first in a series of fundamental results that formed the essence of the GLKM approach. He proved that the spectral function of a Sturm–Liouville operator (given in a half-line or finite interval) uniquely determines the operator. One of the main features of his work was to use transformation operators to investigate inverse problems.

1951—I.M. Gelfand and B.M. Levitan [5] developed a method to reconstruct a Sturm–Liouville operator via the spectral data. This work was the first that introduced the integral equations, named after authors. They also formulated conditions sufficient for a given monotonic function to be the spectral function of the operator (in a half-line or finite interval). It follows that for two sequences of real numbers $\{\lambda_n\}_0^{\infty}$, $\{\alpha_n\}_0^{\infty}$, $\alpha_n > 0$, to be the spectrum and normalization numbers of the Sturm–Liouville operator, it is sufficient that for some constants a_0, a_1, b_0, b_1

$$\sqrt{\lambda_n} = n + \frac{a_0}{n} + \frac{a_1}{n^3} + O\left(\frac{1}{n^4}\right), \qquad \alpha_n = \frac{\pi}{2} + \frac{b_0}{n^2} + \frac{b_1}{n^4} + O\left(\frac{1}{n^4}\right).$$

The results of I.M. Gelfand and B.M. Levitan were given in an intuitive account by N. Levinson in [14].

1952—V.A. Marchenko extended his results [15], obtained earlier, and provided a more systematic approach for usage of transmutation operators for studying the Sturm–Liouville inverse problems. The results of V.A. Marchenko generalize the results of G. Borg and N. Levinson and also explain the results of V. Bagrmann.

We should also mention a series of papers published in the same years by M.G. Krein [16–20] that are closely connected with works of Gelfand, Levitan and Marchenko. He developed an efficient method to construct the Sturm–Liouville operator via the spectral function and two spectra. Later, he used these results to solve the inverse problem for the string equations that started the usage of the approach for hyperbolic equations.
Over the next ten years, the method that originated in the mentioned works was extensively studied in a number of works [21–28].

The continued study of inverse scattering eventually led to a method that uses the solving of inverse problem for integrating several nonlinear equations. As a first step of the method's development, one can consider the work of E. Fermi et al. [29] in 1954, where the authors detected computationally an abnormally slow stochastization of a dynamic system in the form of a chain of nonlinear oscillators.

1962—R. Newton, using the results of I.M. Gelfand and B.M. Levitan, studied the construction of scattering potentials from the phase shifts at fixed energy [30]. A few years later, in 1965, M.D. Kruskal and N.J. Zabussky [31], using the work of E. Fermi et al., discovered using numerical simulation that collisions of solitons in the Korteweg–de Vries (KdV) equation are elastic. As a result, an endless series of conservation laws was discovered soon after.

1967—C.S. Gardner et al. [7] proposed the method of inverse scattering and integrated the KdV equation:

$$u_t - 6uu_x + u_{xxx} = 0, \qquad x \in \mathbb{R}, t > 0$$

with the initial condition

$$u(x, 0) = f(x)$$

by going from the potential of the one-dimensional Schrodinger equation

$$-\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + q(x)\psi = k^2\psi$$

to the reflection coefficient $r(k)$ of this potential. One of the crucial steps of the method is to solve the inverse problem and restore the potential of the Schrodinger equation, which was provided by results of Gelfand, Levitan and Marchenko.

Over the next ten years, such fundamental result led to a large number of papers where the inverse scattering method was studied and generalized.

1968–1969—P.D. Lax [32] discovered an algebraic mechanism forming a basis of the method of inverse scattering.

1970—B.B. Kadomtsev and V.I. Petviashvili obtained the generalization of the KdV equation when studying the stability of solitary waves in weakly dispersive media [33]. The equation was named after them.

1971—C.S. Gardner [34] introduced a theory of the KdV equation as a Hamiltonian system.

1971—V.E. Zakharov and A.B. Shabat [35] applied the method of inverse scattering to the nonlinear Schrodinger equation.

1971—V.E. Zakharov and L.D. Faddeev [36], independently of C.S. Gardner, constructed a theory of the KdV equation as a Hamiltonian system.

1973—A.B. Shabat [37] constructed a class of quasi-linear equations that can be reduced to linear equations.

1974—S.P. Novikov [38], P.D. Lax [39] and V.A. Marchenko [40] studied a periodic Cauchy problem for the KdV equation.

The new result for the periodic problem obtained from the method introduced by C.S. Gardner et al. [7] was the Faddeev–Zakharov theorem: the eigenvalues of the operator L are commutating integrals of the KdV equation as a Hamiltonian system.

1974—V.E. Zakharov and A.B. Shabat [41] proposed a general scheme of the inverse scattering method to integrate the nonlinear differential equations.

1974—L.D. Faddeev [42] published a paper that contains the first multi-dimensional analogs of the Gelfand–Levitan method. Another example of solving scattering problems for more than one dimension can be found in [43].

1974—V.E. Zakharov and S.V. Manakov [44] showed that the nonlinear Shrodinger equation considered as a Hamiltonian system is fully integrable. This can be done using a scattering matrix of the one-dimensional Dirac operator.

1976—S.V. Manakov [45] generalized the Lax pair for two-dimensional time-dependent equations.

1976—V.E. Zakharov and S.V. Manakov [46] showed that each one-dimensional differential operator whose coefficient depends on an arbitrary set of parameters can be associated with a series of multidimensional nonlinear partial differential equations integrable via the inverse scattering method.

1976—P.D. Lax [47] considered the almost periodic behavior (in time) of the periodic solutions to the KdV equation. He presented a new proof based on Lennart's recursion.

1979—V.E. Zakharov and A.B. Shabat [48] extended the method developed in 1976 to spectral problems as rational functions of the spectral parameter. They obtained a description of new classes of equations integrable via the method of inverse scattering and an algorithm to construct their exact solutions.

In the early 1980s, new nonlinear equations integrable via the method of inverse scattering (in particular, the nonlinear Shrodinger equation, the sin-Gordon equation, etc.) were found.

1980—V.E. Zakharov et al. [49] provided a systematic description of the method of inverse scattering.

1982—L.P. Nizhnik and M.D. Pochinayko [50] investigated the nonlinear two-dimensional (in space) Shrodinger equation and used the inverse scattering method for its integration.

1984—A.P. Veselov and S.P. Novikov [51] considered a two-dimensional generalization of the KdV equation (the Veselov–Novikov equation) with the help of the two-dimensional potential Shrodinger operator.

1984—R.G. Novikov and G.M. Henkin [52] applied (and adapted) the inverse scattering method to obtain weakly localized solutions to a KdV equation in which the transmission coefficient of the scattering matrix may be zero for a finite set of pulses.

1984—A.P. Veselov and S.P. Novikov obtained a two-dimensional integrable extension of the KdV equation [51].

1985–1986—P. Grinevich et al. [53–56] gave the first results on the inverse scattering method for the Veselov–Novikov equation with decaying at infinity potential at fixed energy E.

The method of inverse scattering was investigated by Boiti et al. [57], Tsai [58], Nachman [59], Bogdanov et al. [60], Lassas et al. [61], Lassas et al. [62,63], Music [64] and Perry [65].

1986—R.G. Novikov [53] used it for physical inverse scattering, i.e., for inverse scattering in a physical sense, for the first time.

1992—J.-P. Francoise and R.G. Novikov [66] investigated the hierarchy of the Calogero–Moser system for the Kadomtsev–Petviashvili and Veselov–Novikov equations.

1999—R.G. Novikov [67] proposed some inverse scattering algorithm, which requires the solution of the linear integral equation with a specific kernel. That equation can be considered as a two-dimensional analog of the the Gelfand–Levitan–Marchenko–Krein equations.

2011—A.V. Kazeykina and R.G. Novikov [68] studied the asymptotics of solutions to a Cauchy problem for the Veselov–Novikov equation with positive energy.

2013— M. Music et al. [69] studiet nonlinear scattering transform for the two-dimensional Schrodinger equation at zero energy with a radial potential.

2020 and 2021—N. Bondarenko obtained the spectral data characterization for the matrix Sturm–Liouville operator with the general self-adjoint boundary conditions [70]. This result implies the characterization of the spectral data for the Sturm–Liouville operators on geometrical graphs of arbitrary structure with rationally dependent edge lengths. It is worth mentioning that the spectral data characterization for the Sturm–Liouville operator on a star-shaped graph was previously obtained in [71].

2021—S.A. Avdonin et al. [72] investigated the inverse problem of recovering the matrix potential from the dynamical Neumann-to-Dirichlet operator for a dynamical system with boundary control for the vector Schrodinger equation on the interval with a non-self-adjoint matrix potential.

2023—X.-C. Xu and N. Bondarenko [73] proved the local solvability and stability of the inverse Robin–Regge problem in the general case, taking eigenvalue multiplicities into account. The new approach was developed based on the reduction of this inverse problem to the recovery of the Sturm–Liouville potential from the Cauchy data.

1.2. Inverse Problems for Hyperbolic Equations

1954—M.G. Krein [6] was the first to use the GLK method for inverse problems for hyperbolic equations. He considered the so-called string problem and formulated theorems on the solvability of an inverse problem. The nonlinear inverse problem for the string equation was reduced to an integral equation (the Krein equation).

1968—B.S. Pariiskii [74] studied a one-dimensional inverse problem for the wave equation with a perturbation at some depth and derived the Krein equation.

1970–1971—B. Gopinath and M. Sondhi [75,76], independently of each other, proposed an integral equation (also in a time domain) for reconstructing human speech from acoustic measurements.

1971—A.S. Blagoveshchenskii [77] obtained another proof of Krein results. He showed that the dependence of the sought-for coefficient on the additional information is local. In contrast, earlier such problems were studied by using Fourier (or Laplace) transforms in time. Later, that differential equation coefficient was actually reconstructed from the properties of the eigenfunctions of the corresponding spectral problem.

1975—A.S. Alekseev and V.I. Dobrinskii [78] used a discrete analog of the Gelfand–Levitan method to study numerical algorithms for solving the one-dimensional inverse dynamic problem of seismology. Later, ideas from the I.M. Gelfand and B.M. Levitan approach were used to deal with monochromatic seismic problems [79].

1977—B.S. Pariiskii [80] published a detailed review of the numerical methods for solving Gelfand–Levitan equation.

1979—W. Symes [81] applied nonlinear integral equations for an inverse problem in time domain.

1980—R. Burridge [82] attempted to apply the Gelfand–Levitan–Marchenko equations for elasticity theory in a time domain and found a relation between them and the Gopinath–Sondhi equation.

1982—F. Santosa [83] developed an exact method for solving an inverse problem of plane wave propagation via the Gelfand–Levitan method, tested a numerical scheme for solving the integral equation, investigated the stability and analyzed the numerical errors and approximations.

1988—S.I. Kabanikhin [84] proposed a new algorithm for solving the Gelfand–Levitan equation using a sufficient condition for solvability of the inverse problem.

1991—V.G. Romanov and S.I. Kabanikhin [85] applied a dynamic version of the Gelfand–Levitan method to the one-dimensional inverse problem of geoelectrics for a quasi-stationary approximation of the system of Maxwell equations.

1998—A.S. Alekseev and V.S. Belonosov [86] used the spectral method to reconstruct the acoustic impedance in a one-dimensional wave equation.

1987—M.I. Belishev [87] developed the first multi-dimensional analogs of the Gelfand–Levitan–Krein equations for hyperbolic inverse problems.

1988—S.I. Kabanikhin [84,88] proposed another multi-dimensional analog of the Gelfand–Levitan–Krein equations.

1992—M.I. Belishev and A.S. Blagoveshchenskii [89] proposed a multidimensional analog of the Gelfand–Levitan equation based on the boundary-control method.

2004—S.I. Kabanikhin and M.A. Shishlenin [90] showed that the discrete analogs of the Krein and boundary-control methods are the same for the one-dimensional coefficient inverse problem of acoustics.

2005—S.I. Kabanikhin et al. [91] published a book on numerical methods for solving two-dimensional analogs of the Gelfand–Levitan–Krein equation for coefficient inverse problems for the wave and acoustics equations.

2011—S.I. Kabanikhin and M.A. Shishlenin [92] developed a numerical method for solving the Krein equation for the Nth approximation of the two-dimensional inverse acoustic problem. The Krein equation for the Nth approximation was obtained in matrix form, for which a numerical method was constructed based on the singular value decomposition method.

2011—M.A. Shishlenin and N.S. Novikov [93] conducted a comparative analysis of two numerical methods for solving the Gelfand–Levitan equation and developed the Monte-Carlo method for solving the Gelfand–Levitan equation.

2016—V. Druskin et al. [94] proposed the method of finding the velocity propagation speed in the acoustic wave equation based on the discrete form of the GLKM method.

2018—L. Borcea et al. [95] proposed new linear-algebraic algorithm that uses a reduced-order model to compare data with data corresponding to the Born model with single scattering.

2021—V. Druskin et al. [96] combined data-driven reduced-order models with the Lippmann–Schwinger integral equation to obtain a direct nonlinear inversion method. Numerically, it has been shown that the proposed inversion is much better than the Born inversion.

2021—V.G. Romanov [97] justified the scheme related to the construction of the infinite system of integral equations in the case when the potential is analytic in x.

The Gelfand–Levitan–Krein method was applied for solving acoustic [98–100], elasticity [101] and seismic [102,103] coefficient inverse problems. We also should mention series of works of A.V. Baev [104–107], where he recently obtained some new results, related to solving inverse problems for hyperbolic equations by using variations of GLK approach.

An advantage of the Gelfand–Levitan–Krein approach for solving coefficient inverse problems for hyperbolic equations is that the direct problems need not be solved many times. We note the boundary control (BC) method created by M.I. Belishev [108–111] and a globally convergent method of [112–118] by M.V. Klibanov. In the next section, we will briefly consider the BC method and its connection to the Krein approach in the one-dimensional case.

2. One-Dimensional Problems

2.1. I.M. Gelfand–B.M. Levitan Equation

First, we consider the direct Sturm–Liouville problem:

$$l_q y(x) := -y'' + q(x)y, \tag{1}$$

defined on a set of functions $y \in W_2^2(0,\pi)$ and satisfying the relations

$$\begin{gathered} l_q y(x) = \lambda y, \quad x \in (0,\pi), \\ y'(0) - hy(0) = 0, \quad y'(\pi) + Hy(\pi) = 0. \end{gathered} \tag{2}$$

Here, $h, H \in \mathbb{R}$, $q(x) \in L_2(0,\pi)$. Let λ_n be an eigenvalue and $\varphi(x,\lambda_n)$, an eigenfunction of the operator l_q and $\varphi(0,\lambda) = 1$, $\varphi'(0,\lambda) = h$. Let

$$\alpha_n = \int_0^\pi \varphi^2(x,\lambda_n)dx, \tag{3}$$

The set $\{\lambda_n, \alpha_n\}_{n\geq 0}$ is called the spectral data of the operator l_q, with the following asymptotic properties [119]:

$$\sqrt{\lambda_n} = n + \frac{\omega}{\pi n} + \frac{\beta_n}{n}, \quad \alpha_n = \frac{\pi}{2} + \frac{\beta_{1n}}{n}, \quad \{\beta_n\}, \{\beta_{1n}\} \in l_2,$$

$$\alpha_n > 0, \qquad \lambda_n \neq \lambda_m, \quad (n \neq m).$$

The inverse Sturm–Liouville problem consists in reconstructing the potential $q(x)$ and coefficients h, H from the spectral data. Using a function $F(x,t)$:

$$F(x,t) = \sum_{n=0}^{\infty} \left(\frac{\cos\sqrt{\lambda_n}x \cos\sqrt{\lambda_n}t}{\alpha_n} - \frac{\cos nx \cos nt}{\alpha_n^0} \right), \tag{4}$$

where

$$\alpha_n^0 = \begin{cases} \frac{\pi}{2}, & n > 0, \\ \pi, & n = 0, \end{cases}$$

one can reduce the inverse Sturm–Liouville problem to the Gelfand–Levitan equation:

$$G(x,t) + F(x,t) + \int_0^x G(x,s)F(s,t)ds = 0, \quad 0 < t < x. \tag{5}$$

The solution to the Gelfand–Levitan equation makes it possible to determine the solution to the inverse problem via the formula

$$q(x) = 2\frac{d}{dx}G(x,x), \quad h = G(+0,+0), \quad H = \omega - h - \frac{1}{2}\int_0^\pi q(t)\mathrm{d}t. \tag{6}$$

One can also consider the following coefficient dynamic inverse problem: find an even $q(x)$ that satisfies the following:

$$u_{tt} = u_{xx} - q(x)u, \qquad x \in \mathbb{R}, \quad t > 0; \tag{7}$$

$$u|_{t=0} = 0, \quad u_t|_{t=0} = \delta(x);$$

$$u|_{x=0} = f(t). \tag{8}$$

The inverse problem can be reduced to the Gelfand–Levitan equation [120]:

$$\tilde{w}(x,t) + \int_0^x [f'(t-\tau) + f'(t+\tau)]\tilde{w}(x,\tau)d\tau = -\frac{1}{2}[f'(t-x) + f'(t+x)], \quad t \in [0,x). \tag{9}$$

Here, $f(t)$ is an odd continuation of the inverse problem's data to the negative t and the derivative of $f(t)$ is taken at the points of continuity only. The solution to the inverse problems (7) and (8) can be found via the formula

$$q(x) = 4\frac{\mathrm{d}}{\mathrm{d}x}\tilde{w}(x,x-0), \quad x > 0. \tag{10}$$

We should also mention the paper of R.G. Novikov [121], where he proposed an explicit formula to solve the inverse scattering problem for the Sturm–Liouville operator (in dimension 1) up to smooth functions.

2.2. *V.A. Marchenko Equation—The Inverse Scattering Method*

Let us consider the direct scattering problem: given $q(x)$ such that $q(x) \to 0$ for $x \to \pm\infty$, find eigenfunctions and eigenvalues of the problem ([122]):

$$-y'' + q(x)y = k^2 y. \tag{11}$$

Equation (11) for $q(x) < 0$ and $k^2 > 0$ has a continuous spectrum of eigenvalues. If $k^2 < 0$ then the spectrum is discrete. Let us suppose that

$$\int_{-\infty}^{\infty} (1 + |x|)|q(x)|\mathrm{d}x < \infty.$$

Equation (11) has a fundamental system of the solutions:

$$\psi_1(x,k) \cong \mathrm{e}^{-ikx} + o(1), \qquad \psi_2(x,k) \cong \mathrm{e}^{ikx} + o(1), \qquad x \to +\infty, \tag{12}$$

$$\varphi_1(x,k) \cong \mathrm{e}^{-ikx} + o(1), \qquad \varphi_2(x,k) \cong \mathrm{e}^{ikx} + o(1), \qquad x \to -\infty. \tag{13}$$

Note that

$$\psi_1(x,k) = \psi_2^*(x,k) = \psi_2(x,-k), \tag{14}$$

$$\varphi_1(x,k) = \varphi_2^*(x,k) = \varphi_2(x,-k). \tag{15}$$

Here, * is the complex conjugation. Equation (11) has two linear independent solutions; therefore, each solution can be represented as a linear combination

$$\varphi(x,k) = a(k)\psi(x,k) + b(k)\psi^*(x,k), \tag{16}$$

$$\varphi^*(x,k) = c(k)\psi(x,k) + d(k)\psi^*(x,k). \tag{17}$$

Therefore, we have that

$$d(k) = a^*(k), \qquad c(k) = b^*(k).$$

It follows from (16) that

$$\frac{\varphi(x,k)}{a(k)} = \psi(x,k) + \frac{b(k)}{a(k)}\psi^*(x,k), \tag{18}$$

and using asymptotic for $x \to +\infty$ we obtain that

$$\frac{\varphi(x,k)}{a(k)} = \mathrm{e}^{-\mathrm{i}kx} + r(k)\mathrm{e}^{\mathrm{i}kx} + o(1). \tag{19}$$

Here, $r(k) = b(k)/a(k)$ is the reflection coefficient. For $x \to -\infty$, we have

$$\varphi(x,k) \cong \mathrm{e}^{-\mathrm{i}kx}. \tag{20}$$

Then, the last wave has the form

$$\frac{\varphi(x,k)}{a(k)} \cong t(k)\mathrm{e}^{-\mathrm{i}kx}.$$

Here, $t(k) = 1/a(k)$ is the completion rate. The discrete spectrum of the Schrodinger operator is $k_n^2 = -\chi_n^2$ ($\chi_n > 0$). If we consider the asymptotic for $x \to -\infty$ in the form

$$\varphi^{(n)}(x) = \mathrm{e}^{\chi_n x} + o(\mathrm{e}^{\chi_n x}), \tag{21}$$

then for $x \to +\infty$ we have the wave function in the form

$$\varphi^{(n)}(x) = b_n \mathrm{e}^{-\chi_n x} + o(\mathrm{e}^{-\chi_n x})$$

Eigenfunctions corresponding to the discrete spectrum and the eigenvalues are real-valued. Let us enumerate eigenvalues

$$\chi_1^2 > \chi_2^2 > \ldots > \chi_n^2 > 0$$

and suppose that the $\varphi^{(1)}(x)$ wave function corresponding to χ_1^2 has no zeros for $x \in (-\infty, \infty)$. Then, $\varphi^{(n)}(x)$ has $(n-1)$ zeros using oscillatory theorem [123] and we have that $b_n = (-1)^{n-1}|b_n|$.

The set

$$S = \{r(k),\ \chi_n,\ |b_n|,\ n = \overline{1,N}\}$$

is called scattering data. The direct scattering problem is to find scattering data via the given $q(x)$.

Let us consider the following functions

$$\chi_-(x,k) = \varphi(x,k)\mathrm{e}^{\mathrm{i}kx}, \tag{22}$$

$$\chi_+(x,k) = \psi(x,k)\mathrm{e}^{\mathrm{i}kx}, \tag{23}$$

and assume the following boundary conditions hold

$$\lim_{x\to-\infty} \chi_-(x,k) = 1, \tag{24}$$
$$\lim_{x\to+\infty} \chi_+(x,k) = 1. \tag{25}$$

Using the connection between Equation (11) and the Volterra integral equations, one can obtain that the functions φ, ψ are the solutions of the following integral equations [15,42]:

$$\begin{aligned}
\varphi(x,k) &= \mathrm{e}^{-\mathrm{i}kx} + \int\limits_{-\infty}^{x} \frac{\sin k(x-\xi)}{k} q(\xi)\varphi(\xi,k)\mathrm{d}\xi, \\
\psi(x,k) &= \mathrm{e}^{-\mathrm{i}kx} - \int\limits_{x}^{+\infty} \frac{\sin k(x-\xi)}{k} q(\xi)\psi(\xi,k)\mathrm{d}\xi.
\end{aligned} \tag{26}$$

Functions $\chi_-(x,k)$ and $\chi_+(x,k)$ solve the following equation

$$-\chi_{\pm xx}(x,k) + 2\mathrm{i}k\chi_{\pm x}(x,k) + q(x)\chi_\pm(x,k) = 0, \tag{27}$$

and instead of (26) we obtain

$$\chi_-(x,k) = 1 + \int\limits_{-\infty}^{x} \frac{\mathrm{e}^{2\mathrm{i}k(x-\xi)}-1}{2\mathrm{i}k} q(\xi)\chi_-(\xi,k)\mathrm{d}\xi.$$

$$\chi_+(x,k) = 1 - \int\limits_{x}^{\infty} \frac{\mathrm{e}^{2\mathrm{i}k(x-\xi)}-1}{2\mathrm{i}k} q(\xi)\chi_+(\xi,k)\mathrm{d}\xi.$$

For $k \to \infty$, we have

$$\chi_+(x,k) = 1 + \int\limits_{x}^{\infty} \frac{q(\xi)}{2\mathrm{i}k}\mathrm{d}\xi + o\Big(\frac{1}{k^2}\Big). \tag{28}$$

It follows from (16) that

$$\frac{\varphi(x,k)\mathrm{e}^{\mathrm{i}ky}}{a(k)} = \psi(x,k)\mathrm{e}^{\mathrm{i}ky} + r(k)\psi^*(x,k)\mathrm{e}^{\mathrm{i}ky}. \tag{29}$$

Let us integrate (29) with respect to k:

$$\int\limits_{-\infty}^{\infty} \frac{\varphi(x,k)\mathrm{e}^{\mathrm{i}ky}}{a(k)}\mathrm{d}k = \int\limits_{-\infty}^{\infty} \psi(x,k)\mathrm{e}^{\mathrm{i}ky}\mathrm{d}k + \int\limits_{-\infty}^{\infty} r(k)\psi^*(x,k)\mathrm{e}^{\mathrm{i}ky}\mathrm{d}k. \tag{30}$$

In the left-hand side of (30)

$$\int\limits_{-\infty}^{\infty} \frac{\varphi(x,k)\mathrm{e}^{\mathrm{i}ky}}{a(k)}\mathrm{d}k = 2\pi\mathrm{i}\sum_{n=1}^{N} \frac{\varphi(x,\mathrm{i}\chi_n)}{a_n(\mathrm{i}\chi_n)}\mathrm{e}^{-\chi_n y}. \tag{31}$$

We have
$$\varphi(x, \mathrm{i}\chi_n) = b_n\psi^*(x, \mathrm{i}\chi_n) = b_n\psi(x, -\mathrm{i}\chi_n),$$
then (31) can be rewritten in the form
$$\int\limits_{-\infty}^{\infty} \frac{\varphi(x,k)\mathrm{e}^{iky}}{a(k)}\mathrm{d}k = 2\pi\mathrm{i}\sum_{n=1}^{N} \frac{b_n\psi(x,-\mathrm{i}\chi_n)}{a_n(\mathrm{i}\chi_n)}\mathrm{e}^{-\chi_n y}. \tag{32}$$
Let us introduce a new function $K(x, y)$, such that
$$\psi(x,k) = \mathrm{e}^{-\mathrm{i}kx} + \int\limits_{x}^{\infty} K(x,y)\mathrm{e}^{-\mathrm{i}ky}\mathrm{d}y, \tag{33}$$
then (32) is rewritten as follows
$$\int\limits_{-\infty}^{\infty} \frac{\varphi(x,k)\mathrm{e}^{iky}}{a(k)}\mathrm{d}k = 2\pi\mathrm{i}\sum_{n=1}^{N} \frac{b_n\mathrm{e}^{-\chi_n y}}{a_n(\mathrm{i}\chi_n)}\left[\mathrm{e}^{-\chi_n x} + \int\limits_{x}^{\infty} K(x,z)\mathrm{e}^{-\chi_n z}\mathrm{d}z\right]. \tag{34}$$
It follows from (30) that
$$\int\limits_{-\infty}^{\infty} \frac{\varphi(x,k)\mathrm{e}^{iky}}{a(k)}\mathrm{d}k = 2\pi\int\limits_{x}^{\infty} K(x,z)\delta(y-z)\mathrm{d}z+$$
$$+\int\limits_{-\infty}^{\infty} r(k)\mathrm{e}^{\mathrm{i}k(y+x)}\mathrm{d}k + \int\limits_{x}^{\infty} K(x,z)\left[\int\limits_{-\infty}^{\infty} r(k)\mathrm{e}^{\mathrm{i}k(z+y)}\mathrm{d}k\right]\mathrm{d}z. \tag{35}$$
Therefore, we obtain
$$2\pi\mathrm{i}\sum_{n=1}^{N} \frac{b_n}{a_k(\mathrm{i}\chi_n)}\mathrm{e}^{-\chi_n(x+y)} + 2\pi\mathrm{i}\int\limits_{x}^{\infty} K(x,z)\sum_{n=1}^{\infty} \frac{b_n}{a_n(\mathrm{i}\chi_n)}\mathrm{e}^{-\chi_n(x+y)}\mathrm{d}z=$$
$$=2\pi K(x,y) + \int\limits_{-\infty}^{\infty} r(k)\mathrm{e}^{\mathrm{i}k(y+x)}\mathrm{d}k + \int\limits_{-\infty}^{\infty} K(x,z)\left[\int\limits_{-\infty}^{\infty} r(k)\mathrm{e}^{\mathrm{i}k(z+y)}\mathrm{d}k\right]\mathrm{d}z. \tag{36}$$
Define the function $F(x)$ which consists of scattering data
$$F(x) = \sum_{n=1}^{N} \frac{b_n\mathrm{e}^{-\chi_n x}}{\mathrm{i}a_n(\mathrm{i}\chi_n)} + \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} r(k)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k, \tag{37}$$
then Equation (36) can be rewritten in the form of the V.A. Marchenko integral equation
$$K(x,y) + F(x+y) + \int\limits_{x}^{\infty} K(x,s)F(s+y)\mathrm{d}s = 0. \tag{38}$$
Using (33), we obtain that
$$\chi_+(x,k) = 1 + \int\limits_{x}^{\infty} K(x,y)\mathrm{e}^{\mathrm{i}k(x-y)}\mathrm{d}y = 1 - \frac{1}{\mathrm{i}k}K(x,y)\mathrm{e}^{\mathrm{i}k(x-y)}\Big|_{y=x}^{y=\infty} + o\left(\frac{1}{k}\right). \tag{39}$$
We have
$$\frac{1}{\mathrm{i}k}K(x,y)\mathrm{e}^{\mathrm{i}k(x-y)} \to 0, \quad y \to \infty$$

then

$$\chi_+(x,k) = 1 + \frac{1}{\mathrm{i}k}K(x,x). \tag{40}$$

It follows from (28) that

$$K(x,x) = \frac{1}{2}\int\limits_x^\infty q(\xi)\mathrm{d}\xi. \tag{41}$$

Then, the function $q(x)$ in the equation is reconstructed via the formula

$$q(x) = -2\frac{\mathrm{d}}{\mathrm{d}x}K(x,x).$$

Now, before moving further to the inverse scattering method, we would like to illustrate the connection between the considered inverse problems for 1D via Figure 1.

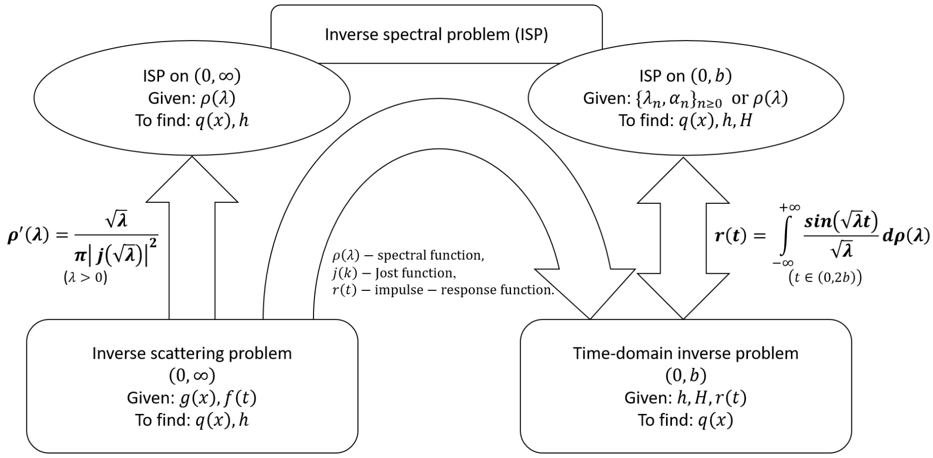

Figure 1. The connection between the inverse spectral problem, the inverse scattering problem and the inverse problem in time domain.

Let us consider the inverse scattering method, which appears while studying some nonlinear equations of mathematical physics. The method, proposed by C.S. Gardner, J.M. Green, M.D. Kruskal and R.M. Miura in 1967 [7] represent the nonlinear equation under study as a compatibility condition for a system of linear equations. An initial version of the method based on the theory of scattering for differential operators (hence, the name of the method) was applied to the Korteweg–de Vries equation

$$u_t - 6uu_x + u_{xxx} = 0, \tag{42}$$

and can be represented in the form of the solution of linear equations

$$\begin{gathered} \psi_{xx} + (\lambda - u)\psi = 0, \\ \psi_t + \psi_{xxx} - 3(\lambda + u)\psi_x = C(t)\psi. \end{gathered} \tag{43}$$

It is a compatibility condition of the overspecified linear system of equations

$$(L-\lambda)\psi = 0, \tag{44}$$

$$\psi_t + A\psi = 0,$$

where

$$L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u(x,t), \quad A = \frac{\mathrm{d}^3}{\mathrm{d}x^3} - 3\left[u\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}u\right],$$

and is equivalent to the operator relation (Lax representation)

$$\frac{\partial L}{\partial t} = [L, A]. \tag{45}$$

Let us consider the initial condition for the Korteweg–de Vries Equation (42)

$$u(x, 0) = f(x). \tag{46}$$

We suppose that

$$\int_{-\infty}^{\infty} (1 + |x|)|f(x)|\mathrm{d}x < \infty.$$

Let it be that for the known $f(x)$ we find from the scattering data

$$S_n(0) = \{\lambda_n(0); r(k, 0); b_n(0); n = \overline{1, N}\}. \tag{47}$$

The wave function in Equation (43) depends on the time variable t:

$$\varphi(x, k, t) = a(k, t)\psi(x, k, t) + b(k, t)\psi^*(x, k, t), \tag{48}$$

and we have the following asymptotics when $x \to +\infty$

$$\varphi(x, k, t) = a(k, t)\mathrm{e}^{-\mathrm{i}kx} + b(k, t)\mathrm{e}^{\mathrm{i}kx} + o(1). \tag{49}$$

Substituting (49) into (43) we obtain

$$\begin{aligned} \dot{a} + 4\mathrm{i}k^3 a - ca &= 0, \\ \dot{b} - 4\mathrm{i}k^3 b - cb &= 0. \end{aligned} \tag{50}$$

Therefore, solving (50) we obtain

$$r(k, t) = \frac{b(k, t)}{a(k, t)} = r(k, 0)\mathrm{e}^{8\mathrm{i}k^3 t}. \tag{51}$$

$$b_n(t) = b_n(0)\mathrm{e}^{8\chi_n^3 t}, \qquad n = \overline{1, N}. \tag{52}$$

Therefore, if by the given initial data $f(x)$ we find $S(t = 0)$, then $S(t)$ has the following form

$$S(t) = \left\{ r(k, 0)\mathrm{e}^{8\mathrm{i}k^3 t};\ \chi_n(0);\ b_n(0)\mathrm{e}^{8\chi_n^3 t}, \qquad n = \overline{1, N} \right\}. \tag{53}$$

Let us denote

$$F(x, t) = \sum_{n=1}^{N} \frac{b_n(0)\mathrm{e}^{-\chi_n x + 8\chi_n^3 t}}{\mathrm{i}a_n(\mathrm{i}\chi_n)} + \frac{1}{2\pi}\int_{-\infty}^{\infty} r(k, 0)\mathrm{e}^{\mathrm{i}kx + 8\mathrm{i}k^3 t}\mathrm{d}k. \tag{54}$$

Then, we solve the Marchenko integral equation for the function $K(x, y, t)$ to solve the inverse problem of scattering:

$$K(x, y, t) + F(x + y, t) + \int_{x}^{\infty} K(x, s, t)F(s + y, t)\mathrm{d}s = 0. \tag{55}$$

Then, we find the solution of the KdV equation via the formula

$$u(x, t) = -2\frac{\partial}{\partial x}K(x, x, t). \tag{56}$$

Therefore, in the first step we find the scattering data $S(t=0)$ for the known $f(x)$. In the second step, we find the scattering data $S(t)$. Then, the function $F(x,t)$ is found via (54). In the fourth step, we solve the Marchenko Equation (55). In the last step, we find the solution of the KdV equation via Equation (56). The scheme of the method is illustrated in Figure 2:

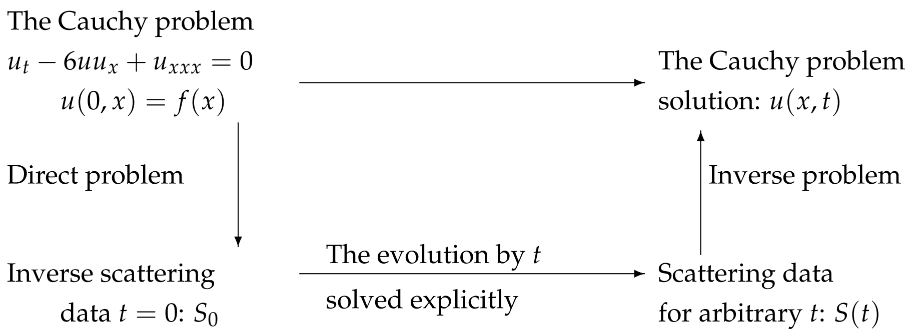

Figure 2. Scheme of solving the KdV equation by solving the inverse scattering problem.

To solve these problems efficiently, numerical calculations are required. An advantage of the inverse scattering method is that it allows advancing in time as far as needed without loss of accuracy.

2.3. Krein Equation

We consider the following inverse problem:

$$u_{tt} = u_{xx} - \frac{\sigma'(x)}{\sigma(x)} u_x, \qquad x > 0, \quad t > 0; \tag{57}$$

$$u\big|_{t<0} \equiv 0; \tag{58}$$

$$u_x\big|_{x=+0} = \delta(t); \tag{59}$$

$$u(+0,t) = f(t). \tag{60}$$

In [77], the inverse problem of acoustics is reduced to the Krein equation

$$-2f(+0)\Phi(x,t) - \int_{-x}^{x} f'(t-s)\Phi(x,s)\mathrm{d}s = \frac{1}{\sigma(+0)}, \qquad |t| < x, \tag{61}$$

in which the function $r(t)$ is oddly extended to the negative t.

The solution to the inverse problems (57)–(60) can be found via the formula

$$\sigma(x) = \frac{1}{4\sigma(+0)\Phi^2(x, x-0)}. \tag{62}$$

2.4. Boundary-Control Method in One-Dimensional Case

Let us consider the inverse problem with arbitrary source:

$$\begin{gathered}
\frac{\partial^2 u^g}{\partial t^2} = \frac{\partial^2 u^g}{\partial x^2} - \frac{\sigma'(x)}{\sigma(x)} \frac{\partial u^g}{\partial x}, \qquad x > 0, \quad t > 0; \\
u^g\big|_{t<0} \equiv 0; \\
\frac{\partial u^g}{\partial x}\Big|_{x=+0} = g(t); \\
u^g(0,t) = f^g(t).
\end{gathered}$$

$f^g(t)$ is data measured for $g(t)$. The inverse problem consists in finding $\sigma(x)$ by the given functions $g(t)$ and $f^g(t)$. Let us define

$$w^{gh}(s,t) = \int_0^L \frac{u^g(x,s)\,u^h(x,t)}{\sigma(x)}dx := \Big(u^g(\cdot,s), u^h(\cdot,t)\Big)_{H'} \tag{63}$$

which for $s = t = \ell$ can be expressed as:

$$w^{gh}(\ell,\ell) = \frac{1}{2\sigma(+0)}\int_0^\ell \int_\xi^{2\ell-\xi} \Big[h(\tau)f^g(\xi) - g(\xi)f^h(\tau)\Big]d\tau d\xi. \tag{64}$$

Note, we can find the response to the arbitrary source $g(t)$ by knowing response $f(t)$ (57)–(60):

$$u^g(0,t) := f^g(t) = \int_0^t f(t-s)\,g(s)ds. \tag{65}$$

Let us consider dense system of functions $\{\psi_k(t)\} \in L_2(0,\ell)$, $k = \overline{1,M}$ and

$$u^g(x,\ell) = \theta(\ell - x). \tag{66}$$

Here, $\theta(\cdot)$ is a Heaviside theta-function. Therefore, we find the approximate source

$$g(t) \approx g_M(t) = \sum_{k=1}^M \alpha_k \psi_k(t). \tag{67}$$

The coefficients $\{\alpha_k\}$, $k = \overline{1,M}$ solve the following system

$$\Gamma\alpha = b, \tag{68}$$

where

$$\Gamma_{jk} = \Big(u_j(\cdot,T), u_k(\cdot,T)\Big)_{H'}, \quad b_j = \Big(a(\cdot), u_j(\cdot,T)\Big)_{H'}, \quad j,k = \overline{1,M}. \tag{69}$$

Using (65), (69) and taking into account that $\{\psi_k(t)\}$ are given on $[0,\ell]$, one can show that coefficients of matrix Γ and components of vector b are defined from $\{\psi_k(t)\}$, $k = \overline{1,M}$:

$$\begin{aligned}\Gamma_{jk} = {}& -\frac{f(+0)}{\sigma(+0)}\int_0^\ell \int_0^{\ell-\tau}\psi_j(\xi)d\xi \int_0^{\ell-\tau}\psi_k(\eta)d\eta d\tau - \\ & -\frac{1}{2\sigma(+0)}\int_0^\ell \int_0^{\ell-\eta}\psi_k(\eta')d\eta' \int_0^\ell \left[r'(\tau+\eta) + r'(|\tau-\eta|)\right] \times \int_0^{\ell-\tau}\psi_j(\xi)d\xi d\tau d\eta;\end{aligned} \tag{70}$$

$$b_j = -\frac{1}{\sigma(+0)}\int_0^\ell (\ell - t)\psi_j(t)dt. \tag{71}$$

Here, $j,k = \overline{1,M}$. One can obtain:

$$\|a\|_H^2 \approx \|a_M\|_H^2 = \sum_{k=1}^M \alpha_k b_k. \tag{72}$$

Therefore

$$\int_0^\ell \frac{dx}{\sigma(x)} = \|a\|_H^2 \approx \|a_M\|_H^2 = \sum_{k=1}^M \alpha_k b_k. \tag{73}$$

It follows from (70) and (71) that all components of system (68) depend on parameter ℓ. Therefore, differentiating by ℓ we derive from (73):

$$\frac{d}{d\ell}\|a\|_H^2(\ell) = \frac{d}{d\ell}\int_0^\ell \frac{dx}{\sigma(x)} = \frac{1}{\sigma(\ell)} \approx \frac{d}{d\ell}\left[\sum_{k=1}^M \alpha_k b_k\right]. \tag{74}$$

Using (73) and (74), an approximate solution of the inverse problem is given by the formula:

$$\sigma(\ell) \approx \left\{ \frac{d}{d\ell} \left[\sum_{k=1}^{M} \alpha_k b_k \right] \right\}^{-1}. \tag{75}$$

Now, we will consider the connection between the BC method and the Krein equation for the 1D acoustic problem. In order to do that, we consider the discrete version of Equations (61) and (68).

Let $h = \ell/N$, $\psi_k^n = \psi_k(nh)$, $k = \overline{1,N}$, $n = \overline{0,N-1}$. We consider the following basis functions:

$$\psi_k^n = \frac{1}{h^2}\left[\delta_{N-k+1,n} - \delta_{N-k,n}\right], \quad n = \overline{0,N-1}, \qquad k = \overline{1,N},$$

where

$$\delta_{n,k} = \begin{cases} 1, & n = k \\ 0, & n \neq k \end{cases}.$$

We use the following formula for numerical integration:

$$\int_0^\ell f(t)dt = h \sum_{n=0}^{N-1} f^n + o(h).$$

Let us find the discrete analogs of components of vector b and matrix Γ.

Lemma 1. *The components of the vector can be represented in the form*

$$b_j = \frac{1}{\sigma_0} + o(h), \quad j = \overline{1,N}. \tag{76}$$

Lemma 2. *The following equality holds true*

$$\sum_{n=0}^{N-m-1} \psi_k^n = -\frac{1}{h^2}\delta_{m+1,k}. \tag{77}$$

Lemma 3. *The coefficients Γ_{jk} can be represented in the form*

$$\Gamma_{jk} = -\frac{f^0}{h\sigma_0}\delta_{k,j} - \frac{1}{2\sigma_0}\left[f'^{j+k-2} + f'^{|j-k|} \right] + o(h). \tag{78}$$

Using (78), we rewrite the initial Equation (68) in discrete form:

$$2\frac{f^0}{\sigma_0}\alpha_j + \frac{h}{\sigma_0}\sum_{k=1}^{N}\left[f'^{j+k-2} + f'^{|j-k|} \right]\alpha_k = -2\frac{h}{\sigma_0}, \qquad j = \overline{1,N}. \tag{79}$$

Denote $\alpha_k^N = \alpha_k(Nh)$. Then, we obtain that

$$\sum_{k=1}^{N} \alpha_k^N b_k^N = \frac{1}{\sigma_0}\sum_{k=1}^{N}\alpha_k^N. \tag{80}$$

Let us find the finite-difference derivative of (80):

$$\left(\sum_{k=1}^{N} \alpha_k^N b_k^N \right)_{\bar{\ell}} = \frac{1}{h\sigma_0}\left[\sum_{k=1}^{N}\alpha_k^N - \sum_{k=1}^{N-1}\alpha_k^{N-1} \right] = \frac{1}{\sigma_0}\sum_{k=1}^{N-1}\frac{\alpha_k^N - \alpha_k^{N-1}}{h} + \frac{\alpha_N^N}{h}. \tag{81}$$

Let us denote

$$\beta_k^N = \frac{1}{2h\sigma_0}\alpha_k^N.$$

Then, Equation (79) has the form

$$2f^0\beta_j^N + h\sum_{k=1}^{N}\left[f'^{j+k-2} + f'^{|j-k|}\right]\beta_k^N = -\frac{1}{\sigma_0}, \qquad j=\overline{1,N}. \tag{82}$$

Equation (82) is solved for fixed N. Then, the solution to the inverse problem can be found in discrete form

$$\sigma_N = \left[\Big(\sum_{k=1}^{N}\alpha_k^N b_k^N\Big)_{\bar{\ell}}\right]^{-1} = \frac{1}{2}\Big[\sum_{k=1}^{N-1}(\beta_k^N - \beta_k^{N-1}) + \beta_N^N\Big]^{-1}.$$

$$\sigma_N = \frac{1}{4\sigma_0\,[\Psi_N^N]^2}$$

Note that the system of Equation (82) coincides with the Krein Equation (61). Thus, we have shown that the basic relations of the boundary-control method and the Krein method coincide in the one-dimensional case in a discrete form. Both methods allow us to find a solution to the inverse problem at a specific point x_0 of depth without any special calculations of unknown coefficients in the interval $(0, x_0)$.

2.5. One-Dimensional Inverse Seismic Problem

A.S. Alekseev used the one-dimensional Gelfand–Levitan approach for solving the theory of elasticity's inverse problem [22]. The problem is to determine elastic properties of the medium from the following system:

$$\begin{gathered}
\rho\frac{\partial^2\mathbf{U}}{\partial t^2} = (\lambda+\mu)\mathrm{grad\ div}\mathbf{U} + \mu\Delta\mathbf{U} + \mathrm{grad}\lambda\mathrm{div}\mathbf{U} + \sum_{i=1}^{3}\mathrm{grad}\mu\Big(\frac{\partial\mathbf{U}}{\partial x_i} + \mathrm{grad}U_{x_i}\Big)\mathbf{e}_i;\\
\mathbf{U}(x,y,z,t)|_{t<0} \equiv 0;\\
\sigma_z|_{z=0} = \lambda_0\left[\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z}\right] + 2\mu_0\frac{\partial U_z}{\partial z} = g_1(x,y,t);\\
\tau_{xz}|_{z=0} = \mu_0\left(\frac{\partial U_x}{\partial z} + \frac{\partial U_z}{\partial x}\right) = g_2(x,y,t);\\
\tau_{yz}|_{z=0} = \mu_0\left(\frac{\partial U_y}{\partial z} + \frac{\partial U_z}{\partial y}\right) = g_3(x,y,t);\\
U_x(x,y,0,t) = f_1(x,y,t);\quad U_y(x,y,0,t) = f_2(x,y,t);\quad U_z(x,y,0,t) = f_3(x,y,t).
\end{gathered} \tag{83}$$

The first Equation in (83) describes the propagation of elastic waves through the medium. Functions $g_i, i=1,2,3$ set the seismic load, which causes the propagation of the seismic waves through the medium. The problem is to determine Lame parameters λ,μ,ρ by using the measurements $f_j, j=1,2,3$, given by the last ratios in (83) that correspond to measuring the components of the displacement vector $\mathbf{U}$ by using receivers located on the surface $z=0$.

If the desirable elastic parameters λ,μ,ρ depend only on the depth z, then we can solve the inverse problem (83) by reducing it to several families of integral equations of Gelfand–Levitan type. As shown in [22], if we apply it to the medium surficial moment of force with the intensity $\delta(t)$, then the propagated wave is the SH wave. The inverse problem can be reduced then, using the Hankel transform and the symmetry of the problem, to the following family of problems:

$$\begin{gathered}\frac{1}{v_s^2}\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial z^2} + \frac{\partial \ln(\mu)}{\partial z}\frac{\partial U}{\partial z} - k^2 U; \\ U(z,t;k)|_{t<0} \equiv 0; \\ \frac{\partial U}{\partial z}|_{z=0} = \frac{1}{4\pi\mu_0}\delta(t); \\ U(z,t;k)|_{z=0} = f_k(t).\end{gathered} \tag{84}$$

Here, $v_s = \sqrt{\frac{\mu}{\rho}}$ is the shear wave's velocity, k is the integer parameter, μ_0 is the known value of the function $\mu(z)$ at the surface $z = 0$ and $f_k(t)$ are known functions. We can use travel-time coordinates to rewrite the first equation in (84) as follows:

$$\frac{\partial^2 V}{\partial t^2} = \frac{\partial^2 V}{\partial x^2} - q(x;k)V; \tag{85}$$

Here, $x = \int_0^z \frac{d\xi}{v_s(\xi)}$, $q(x;k) = k^2 v_s^2 - \frac{1}{2}\frac{\sigma''}{\sigma} + \frac{3}{4}(\frac{\sigma'}{\sigma})^2$ and $\sigma(x) = \frac{1}{\sqrt{\mu\rho}}$ is the acoustic impedance of the medium. A.S. Alekseev proposed solving the inverse problem for Equation (85) by reducing it to the inverse Sturm–Liouville problem and using the Gelfand–Levitan approach (9) in the spectral domain. This allows us to calculate shear wave velocity $v_s(x)$ and the density of the medium $\rho(x)$. After that, as shown in [22], we can reconstruct p-wave velocity $v_p(z) = \sqrt{\frac{\lambda+2\mu}{\rho}}(z)$ in the same manner by using the point force type source with intensity $\delta(t)$.

3. Two-Dimensional Analogs of the Approach

3.1. A Two-Dimensional Analog of Gelfand–Levitan Equation

Consider the following sequence of direct problems ($k = 0, \pm 1, \pm 2, \ldots$):

$$u_{tt}^{(k)} = u_{xx}^{(k)} + u_{yy}^{(k)} - q(x,y)u^{(k)}, \qquad x \in \mathrm{R}, \quad y \in \mathrm{R}, \quad t > 0; \tag{86}$$

$$u^{(k)}|_{t=0} = 0, \quad u_t^{(k)}|_{t=0} = \delta(x)\mathrm{e}^{iky}. \tag{87}$$

We suppose that $q(x,y)$ is a 2π-periodic function with respect to y. Consider an inverse problem: determine the even function $q(x,y)$ from the additional information

$$u^{(k)}|_{x=0} = f^{(k)}(y,t), \qquad k = 0, \pm 1, \pm 2, \ldots \tag{88}$$

The uniqueness of the solution to the inverse problems (86)–(88) can be proved using a technique proposed in [124,125], based on properties of the Dirichlet-to-Neumann map and the finite dependance of the solution on the boundary conditions. Now, we consider the following auxiliary sequence of direct problems ($m = 0, \pm 1, \pm 2, \ldots$) [91,126]:

$$w_{tt}^{(m)} = w_{xx}^{(m)} + w_{yy}^{(m)} - q(x,y)w^{(m)}, \qquad x > 0, \quad y \in \mathrm{R}, \quad t \in \mathrm{R}; \tag{89}$$

$$w^{(m)}|_{x=0} = \mathrm{e}^{imy}\delta(t), \qquad w_x^{(m)}|_{x=0} = 0. \tag{90}$$

Now, using the d'Alembert formula for problems (89) and (90), we can obtain [91,126]:

$$w^{(m)}(x,y,t) = \frac{1}{2}\mathrm{e}^{\mathrm{i}my}[\delta(t-x)+\delta(t+x)]+ \\ +\frac{1}{2}\int_0^x \int_{t-x+\xi}^{t+x-\xi} \big[-w_{yy}^{(m)} + q(x,y)w^{(m)}\big](\xi,y,\tau)\mathrm{d}\xi\mathrm{d}\tau. \quad (91)$$

The following condition takes place: $w^{(m)}(x,y,t) \equiv 0, 0 < |x| < t, y \in \mathbb{R}$.
We denote

$$\tilde{w}^{(m)}(x,y,t) = w^{(m)}(x,y,t) - \frac{1}{2}\mathrm{e}^{\mathrm{i}my}\big[\delta(t-x)+\delta(t+x)\big]. \quad (92)$$

Using (92) in (91), we can obtain:

$$\tilde{w}^{(m)}(x,y,t) = \frac{1}{4}\mathrm{e}^{\mathrm{i}my}\theta(x-|t|)\Big[xm^2 + Q(x,y,t)\Big]+ \\ +\frac{1}{2}\int_0^x \int_{t-x+\xi}^{t+x-\xi} \big[-\tilde{w}_{yy}^{(m)} + q(x,y)\tilde{w}^{(m)}\big](\xi,y,\tau)\mathrm{d}\xi\mathrm{d}\tau. \quad (93)$$

Here,

$$Q(x,y,t) = \int_0^{\frac{x+t}{2}} q(\xi,y)\mathrm{d}\xi + \int_0^{\frac{x-t}{2}} q(\xi,y)\mathrm{d}\xi. \quad (94)$$

Then,

$$\tilde{w}^{(m)}(x,y,x-0) = \frac{1}{4}\mathrm{e}^{\mathrm{i}my}\left[xm^2 + \int_0^x q(\xi,y)\mathrm{d}\xi\right]. \quad (95)$$

Inverse problems (86)–(88) can be reduced formally to a system of integral equations ($k = 0, \pm1, \pm2, \ldots$) of the first kind

$$\int_{-x}^{x} \sum_m f_m^{(k)}(t-s)\tilde{w}^{(m)}(x,y,s)\mathrm{d}s = -\frac{1}{2}\Big[f^{(k)}(y,t-x) + f^{(k)}(y,t+x)\Big], \quad (96)$$

or the second kind

$$\tilde{w}^{(k)}(x,y,t) + \int_{-x}^{x} \sum_m f_m^{(k)\prime}(t-s)\tilde{w}^{(m)}(x,y,s)\mathrm{d}s = -\frac{1}{2}\Big[f_t^{(k)}(y,t-x) + f_t^{(k)}(y,t+x)\Big]. \quad (97)$$

Here, $|t| < x$, $y \in \mathrm{R}$. The systems of Equations (96) and (97) are two-dimensional analogs of the Gelfand–Levitan equation. Note that, according to Equation (95), $q(x,y)$ can be calculated, for instance, via the formula

$$q(x,y) = 4\frac{\mathrm{d}}{\mathrm{d}x}\tilde{w}^{(0)}(x,y,x-0). \quad (98)$$

3.2. A Two-Dimensional Analog of Krein Equation

Consider the following sequence of the direct problems ($k = 0, \pm1, \pm2, \ldots$) [92]:

$$u_{tt}^{(k)} = u_{xx}^{(k)} + u_{yy}^{(k)} - \nabla \ln\rho(x,y)\,\nabla u^{(k)}, \qquad x > 0, \quad y \in \mathrm{R}, \quad t > 0; \quad (99)$$

$$u^{(k)}|_{t<0} \equiv 0, \qquad u_x^{(k)}(+0,y,t) = \mathrm{e}^{\mathrm{i}ky}\,\delta(t). \quad (100)$$

An inverse problem is to determine the function $\rho(x,y)$ from the additional information

$$u^{(k)}(+0,y,t) = f^{(k)}(y,t), \qquad k = 0,\pm1,\pm2,\ldots \quad (101)$$

We suppose that $\ln\rho(x,y)$ is a 2π-periodic function. The necessary condition of the inverse problem's solvability can be obtained [84,88,92]:

$$f^{(k)}(y,+0) = -\mathrm{e}^{\mathrm{i}ky}, \qquad y \in (-\pi,\pi), \quad k \in \mathbb{Z}. \tag{102}$$

We also consider the following sequence of the auxiliary problem:

$$w_{tt}^{(m)} = w_{xx}^{(m)} + w_{yy}^{(m)} - \nabla \ln\rho(x,y)\,\nabla w^{(m)}, \qquad x > 0, \quad y \in (-\pi,\pi), \quad t \in \mathbb{R}, \quad m \in \mathbb{Z}; \tag{103}$$

$$w^{(m)}|_{x=0} = \mathrm{e}^{\mathrm{i}my}\delta(t), \qquad w_x^{(m)}|_{x=0} = 0. \tag{104}$$

Then, we have the following form of the solution of problems (103) and (104):

$$w^{(m)}(x,y,t) = \frac{1}{2}\mathrm{e}^{\mathrm{i}my}\sqrt{\frac{\rho(x,y)}{\rho(0,y)}}[\delta(x+t)+\delta(x-t)] + \widetilde{w}^{(m)}(x,y,t), \tag{105}$$

Here, $\widetilde{w}^{(m)}(x,y,t)$ is the piecewise-smooth function. Solutions of sequences (99), (100) and (103), (104) are connected:

$$u^{(k)}(x,y,t) = \int_R \sum_m f_m^{(k)}(t-s)w^{(m)}(x,y,s)ds, \quad x > 0,\ y \in (-\pi,\pi),\ t \in \mathbb{R}; \tag{106}$$

Here,

$$f^{(k)}(y,t) = \sum_m f_m^{(k)}(t)\mathrm{e}^{\mathrm{i}my}. \tag{107}$$

Then, we extend functions $f^{(k)}$ and $u^{(k)}$ for $t < 0$ as an odd continuation:

$$f^{(k)}(y,t) = -f^{(k)}(y,-t), \quad t < 0; \tag{108}$$

$$u^{(k)}(x,y,t) = -u^{(k)}(x,y,-t), \quad t < 0; \tag{109}$$

Now, we apply the operator

$$\int_0^x \frac{(.)}{\rho(\xi,y)}d\xi \tag{110}$$

to equality (106). Let us denote

$$V^{(m)}(x,y,t) = \int_0^x \frac{w^{(m)}(\xi,y,t)}{\rho(\xi,y)}d\xi, \tag{111}$$

Then, we can obtain:

$$\begin{aligned}\frac{\partial}{\partial t}\int_0^x \frac{u^{(k)}(\xi,y,t)}{\rho(\xi,y)}d\xi &= \frac{\partial}{\partial t}\int_R \sum_m V^{(m)}(x,y,s)f_m^{(k)}(t-s)ds= \\ &= -2V^{(k)}(x,y,t) + \int_{-x}^x \sum_m V^{(m)}(x,y,s)f_m^{(k)\prime}(t-s)ds. \end{aligned}\tag{112}$$

It was shown [84,92] that the left part of Equation (112) does not depend on x, t, and satisfies the following ratio:

$$\frac{\partial}{\partial t}\int_{-\pi}^{\pi}\int_0^x \frac{u^{(k)}(\xi,y,t)}{\rho(\xi,y)}d\xi dy = -\int_{-\pi}^{\pi}\frac{e^{iky}}{\rho(0,y)}\mathrm{d}y. \tag{113}$$

Let us denote

$$\Phi^{(m)}(x,t) = \int_{-\pi}^{\pi} V^{(m)}(x,y,t)\mathrm{d}y. \tag{114}$$

We can obtain from (112) and (113):

$$2\Phi^{(k)}(x,t) - \sum_m \int_{-x}^{x} (f_m^k)'(t-s)\Phi^{(m)}(x,s)ds = -\int_{-\pi}^{\pi} \frac{\mathrm{e}^{iky}}{\rho(0,y)}\mathrm{d}y, \quad k \in \mathbb{Z}. \tag{115}$$

For every fixed value of x, Equation (115) is a linear Fredholm integral equation of the second kind. The set of Equation (115) is the multi-dimensional analog of the M.G. Krein equation [84,91]. It was proved [91,126] that

$$V^{(m)}(x,y,x-0) = \frac{\mathrm{e}^{imy}}{2\sqrt{\rho(x,y)\rho(0,y)}}. \tag{116}$$

Therefore,

$$\Phi^{(m)}(x,x-0) = \int_{-\pi}^{\pi} \frac{\mathrm{e}^{imy}}{2\sqrt{\rho(x,y)\rho(0,y)}}\mathrm{d}y. \tag{117}$$

The solution to the inverse problem $\rho(x,y)$ can be found via the formula

$$\rho(x,y) = \frac{\pi^2}{\rho(0,y)}\Big[\sum_m \Phi^m(x,x-0)\mathrm{e}^{-imy}\Big]^{-2}. \tag{118}$$

To find the solution to the inverse problem $\rho(x,y)$ at a point $x_0 > 0$, we solve the system (115) setting $x = x_0$ and calculate $\rho(x_0,y)$ via Equation (118).

To numerically solve the two-dimensional analog of the Krein equation (see Figures 3–6), we use the Nth approximation [127,128] of the Krein equation [91]. That is, in the system (115) we set $\Phi^k(x,t) \equiv 0$ for all $N < |k|$ [92]. Discrete analogs of the Gelfand–Levitan equation are investigated in [91,129–132].

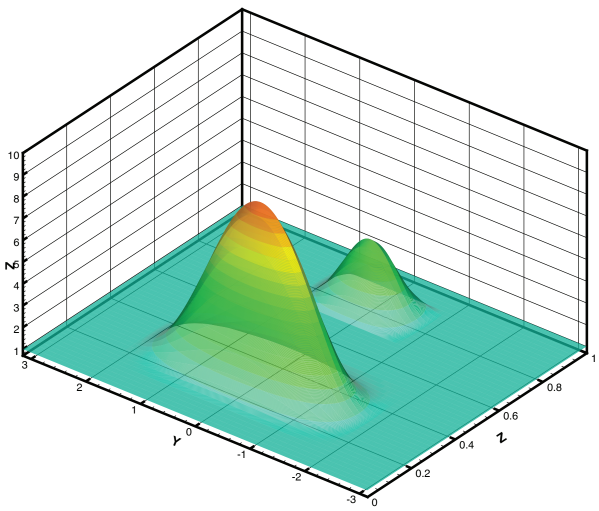

Figure 3. Exact solution of the inverse problem.

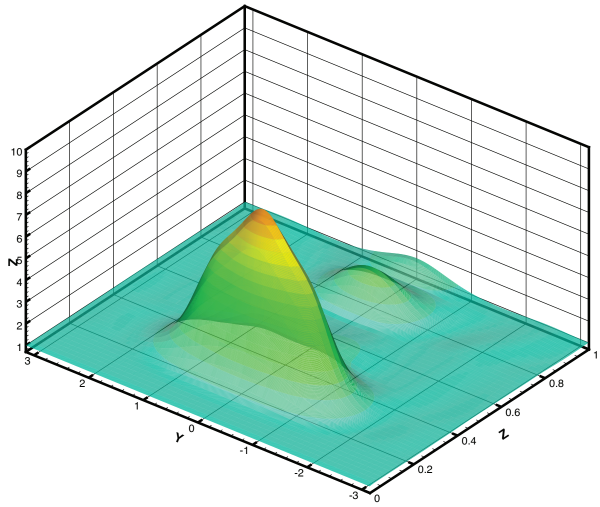

Figure 4. Approximate solution of the inverse problem, $N = 5$, $\varepsilon = 0$.

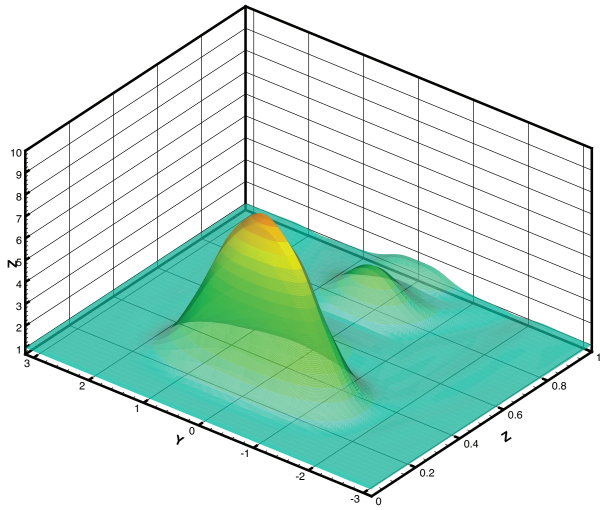

Figure 5. Approximate solution of the inverse problem, $N = 10$, $\varepsilon = 0$.

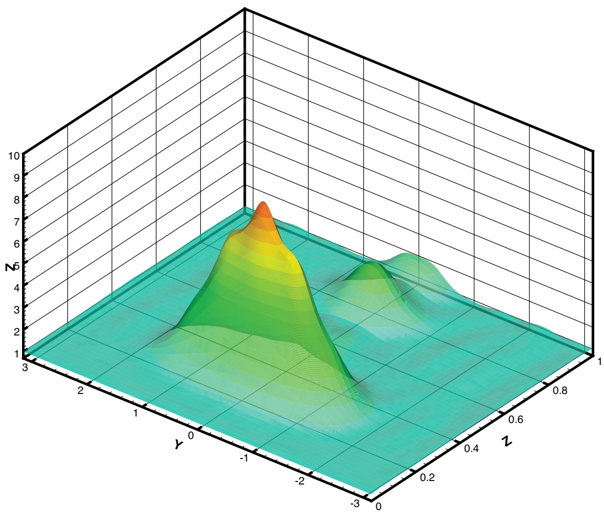

Figure 6. Approximate solution of the inverse problem, $N = 10$, $\varepsilon = 0.05$.

4. Numerical Methods for Solving Gelfand–Levitan and Krein Equations

In this section, we present a short review of numerical methods that can be used for solving arising equations. In order to find the solution of one of the considered inverse problems, we have to solve the family of corresponding integral equations for every value of the parameter x, which is correlated to the current depth. On the one hand, Gelfand–Levitan ((9) and (91)) and Krein ((61) and (115)) equations, despite their differences, have similar basic structure, which is provided by the form of the Fredholm linear integral equation with a convolution-type kernel and the fact that we need only one component of the solution to solve the inverse problem. On the other hand, this structure makes several numerical techniques suitable for solving these equations. The efficiency of one or another algorithm results from its ability to utilize the structure of the equation.

In [133,134], the authors proposed the Monte-Carlo method for solving the 2D analog of the Gelfand–Levitan Equation (91). The idea is to represent the solution of the equation by the sum of Neumann series. This sum is calculated as mean value of some random variable. Such a scheme allows one to estimate specific components of the solution, which is important due to the fact that we only need one component of the solution of the Gelfand–Levitan-type equation to restore the solution of the inverse problem. Another stochastic approach was proposed in [135], where Equation (91) was solved by a randomized version of the Kaczmarz algorithm. Randomization of the well-known Kaczmarz projection method allows one to achieve (under some assumptions) second-order complexity of the method. It is also possible to modify the algorithm to use more information about the structure of the discrete version of Equation (91), like block structure of the matrix. Moreover, unlike most Monte-Carlo methods, this algorithm does not require the convergence of Neumann series.

We should also mention the approaches based on the convolution-type kernels of the Gelfand–Levitan and Krein equations. Such kernels allow one to reduce the equations to linear systems with Toeplitz (or block-Toeplitz) matrices. In [136], Equation (115) was solved via the block version of the Levinson–Durbin recursion method. It was shown that such an approach allows one to use the connection between Equation (115) for different values of depth parameter x. The proposed recursion method allows one to calculate the solutions of the whole family of Krein equations during the solution of only one linear system. Both methods allow one to obtain the solution of the inverse problem in every point of the mesh for $O(N^2)$ operations, where N is the number of mesh points.

5. Numerical Calculations

In this section, we present an example of the numerical reconstruction of the inverse problem's solution, based on the GLK approach. For the numerical solution of inverse problems (99)–(101), we used a regularization technique, based on a projection of the problem on N-dimensional subspace, produced by the basis $\{e^{iky}\}_{k=0,\pm1,\ldots,\pm N}$ [91,92,128]. This approach reduces the two-dimensional problem to a finite system of one-dimensional inverse problems [91]. We suppose that the solution of problems (99) and (100) can be represented as a series:

$$u^{(k)}(x,y,t) = \sum_m u_m^{(k)}(x,t)e^{imy}; \tag{119}$$

We also suppose that the function $\rho(x,y)$ also has the same representation:

$$\ln \rho(x,y) = \sum_m a_m(x)e^{imy}; \tag{120}$$

In this case, problems (99)–(101) can be rewritten as follows

$$\frac{\partial^2 u_n^{(k)}}{\partial t^2} = \frac{\partial^2 u_n^{(k)}}{\partial x^2} - n^2 u_n^{(k)}(x,t) - \sum_{m\in\mathbb{Z}} \frac{\partial a_m}{\partial x}(x)\frac{\partial u_{n-m}^{(k)}}{\partial x}(x,t) +$$
$$+ \sum_{m\in\mathbb{Z}} m(m-n)a_m(x)u_{n-m}^{(k)}(x,t), \quad x\in\mathbb{R}, \quad t>0, \quad k,n\in\mathbb{Z};$$

$$u_n^{(k)}|_{t=0} = 0, \quad \frac{\partial u_n^{(k)}}{\partial t}|_{t=0} = \delta_{kn}\delta(x);$$
$$u_n^{(k)}|_{x=0} = f_n^{(k)}(t).$$

Here, δ_{kn} is the Kronecker symbol:

$$\delta_{kn} = \begin{cases} 1, & k=n \\ 0, & \text{else} \end{cases}.$$

Now, we suppose that all the Fourier coefficients with number greater than N vanish and consider then the following problem:

$$\frac{\partial^2 \overline{V}_N^{(k)}}{\partial t^2} = \frac{\partial^2 \overline{V}_N^{(k)}}{\partial x^2} - K\overline{V}_N^{(k)} + A(x)\overline{V}_N^{(k)} - B(x)\frac{\partial \overline{V}_N^{(k)}}{\partial x}, \qquad x>0, \quad t>0; \tag{121}$$

$$\overline{V}_N^{(k)}|_{t<0} \equiv 0, \quad \frac{\partial \overline{V}_N^{(k)}}{\partial x}|_{x=0} = I_N^{(k)}\delta(t); \tag{122}$$

$$\overline{V}_N^{(k)}|_{x=0} = \overline{F}_N^{(k)}. \tag{123}$$

Problems (121)–(123) are called an N-approximation of inverse problems (99) and (100). Here, A, K, B are square matrices of size $2N+1$ with elements:

$$K_{km} = m^2\delta_{km}; \tag{124}$$

$$A_{km}(x) = m(k-m)a_{k-m}(x), \qquad k,m = -N\ldots N; \tag{125}$$

$$B_{km}(x) = \frac{\partial a_{k-m}}{\partial x}, \qquad k,m = -N\ldots N. \tag{126}$$

Using the technique proposed in [137], one can obtain that, as $N \to \infty$, the N-approximation converges to the solution of systems (99) and (100). The N-approximation of the Krein Equation (115) can be also obtained:

$$\Phi^k(x,t) = \frac{1}{2}\int_{-x}^{x} \sum_{|m|<N} (f_m^k)'(t-s)\Phi^m(x,s)ds + G^k, \quad k = -N,\ldots,N; \tag{127}$$

Numerical calculations (see Figures 3–6) are used to find an approximate solution to the inverse problem. The two-dimensional inverse problem is approximated via a finite system of one-dimensional inverse problems [91,92,128]. The problem is solved in the domain $x \in (0,1)$, $y \in (-\pi,\pi)$, $t \in (0,2)$. Equation (127) is approximated via an SLAE, the size of which depends on both the number of grid points and the number of harmonics considered. The spatial dimension of the grid is 100×100. The number of Fourier harmonics can be associated with the number of sources and receivers that one has on the surface $x = 0$. While the dependence of the solution on the number of harmonics required was considered in several works related to the numerical solution of 2D problems (that we mentioned in previous section), in this paper we consider some values of that number only to illustrate the behaviour of the numerical reconstruction. The number of Fourier harmonics is $N = 5$ in Figure 4 and $N = 10$ in Figures 5 and 6.

In order to illustrate the impact of the noise on the reconstruction, we consider the noisy data in the following form:

$$f^{\varepsilon}(y,t) = f(y,t) + \varepsilon\alpha(y,t)(f_{\max} - f_{\min}). \tag{128}$$

Here, ε is a noise level in the data, $\alpha(y,t)$ is a uniformly distributed random number on the interval $[-1,1]$ for a fixed y and t and $f_{\max}$ and $f_{\min}$ are the maximum and minimum of the exact data, respectively. Below, we provide the results of the inverse problem's solution for 5% error in the data. We chose this number in an arbitrary way to illustrate the stability of the method. On the other hand, such a level of errors, introduced in the data without any pre-processing, could be considered as a way to simulate the real-case scenario.

One can see that the accuracy of the numerical reconstruction was acceptable. However, the second peak, which is located further from the daylight surface (and, thus, receivers), was reconstructed worse than the first one. On the one hand, this result fits well into the physics beside the inverse problem—the large obstacle deters part of the information about the object to get to the receivers, located on the daylight surface. On the other hand, from a mathematical point of view the convergence of the numerical solution to the exact one can be provided by increasing the numbers of Fourier coefficients and grid points, but only in the case of smooth parameters. The stability of the method is also acceptable and can be improved further by using some processing of the noised data (because the kernel of the equation depends on the first derivative of the data).

6. Reconstruction of the Velocity $c(x,y)$ and the Density $\rho(x,y)$

Let us consider the following inverse problem: find the velocity $c(x,y)$ and the density $\rho(x,y)$ from the sequence of relations ($k = 0, \pm 1, \pm 2, \ldots$):

$$c^{-2}(x,y)u^{(k)}_{tt} = \Delta u^{(k)} - \nabla \ln \rho(x,y) \nabla u^{(k)}, \qquad x \in \mathrm{R}, \quad y \in \mathrm{R}, \quad t > 0;$$

$$u^{(k)}|_{t=0} = 0, \qquad u^{(k)}_t|_{t=0} = \mathrm{e}^{\mathrm{i}ky}\,\delta(x).$$

$$u^{(k)}(0,y,t) = f^{(k)}(y,t), \qquad u^{(k)}_x(+0,y,t) = 0.$$

Let $\tau(x,y)$ be a solution of the Cauchy problem for the the eikonal equation

$$\begin{gathered} \tau_x^2 + \tau_y^2 = c^{-2}(x,y), \qquad x > 0, \quad y \in \mathrm{R}; \\ \tau|_{x=0} = 0, \qquad \tau_x|_{x=0} = c^{-1}(0,y), \qquad y \in \mathrm{R}. \end{gathered} \tag{129}$$

Let us introduce new variables $z = \tau(x,y)$, $y = y$ and new functions

$$v^{(k)}(z,y,t) = u^{(k)}(x,y,t), \qquad b(z,y) = c(x,y). \tag{130}$$

Since the velocity is supposed to be strictly positive, this change of variables is not degenerate at least in some interval $x \in (0,h)$.

Let us consider the sequence of the auxiliary problems ($m = 0, \pm 1, \pm 2, \ldots$) [91,126]:

$$\begin{gathered} w^{(m)}_{tt} = w^{(m)}_{zz} + b^2 w^{(m)}_{yy} + q w^{(m)}_{yz} + p w^{(m)}_z + r w^{(m)}_y, \qquad z > 0, \quad y \in \mathrm{R}, \quad t \in \mathrm{R}; \\ w^{(m)}(0,y,t) = \mathrm{e}^{\mathrm{i}my}\,\delta(t), \qquad w^{(m)}_z(0,y,t) = 0. \end{gathered} \tag{131}$$

Here,

$$q(z,y) = 2b^2\tau_y, \tag{132}$$

$$p(z,y) = b^2(z,y)(\tau_{xx} + \tau_{zz}) - (\ln\rho)_z - \frac{1}{2}q(z,y)(\ln\rho)_y, \tag{133}$$

$$r(z,y) = -b^2(z,y)(\ln\rho)_y - \frac{1}{2}q(z,y)(\ln\rho)_z. \tag{134}$$

We suppose that $c(0,y) = b(0,y)$ is known and for simplicity $b(0,y) \equiv 1$ for $y \in \mathrm{R}$.

In the neighborhood of the plane $t = z$, the solution of the direct problem (131) has the form [91,126]:

$$w^{(m)}(z,y,t) = S^{(m)}(z,y)\delta(z-t) + Q^{(m)}(z,y)\theta(z-t) + \tilde{w}^{(m)}(z,y,t). \tag{135}$$

Here, $\tilde{w}^{(m)}$ is continuous function and functions $S^{(m)}$ and $Q^{(m)}$ solve the following problems:

$$\begin{gathered} 2(S^{(m)})_z + q(z,y)(S^{(m)})_y + Sp(z,y) = 0, \qquad t > 0, \quad y \in \mathbb{R}; \\ S^{(m)}|_{t=0} = \frac{1}{2}\mathrm{e}^{\mathrm{i}my}. \end{gathered} \tag{136}$$

$$\begin{aligned} &2(Q^{(m)})_z + q(z,y)(Q^{(m)})_y + Qh(z,y) = \\ &= -\Big[(S^{(m)})_{zz} + b(z,y)(S^{(m)})_{yy} + q(S^{(m)})_{yz} + p(S^{(m)})_z + q(S^{(m)})_y\Big], t > 0, \quad y \in \mathrm{R}; \end{aligned} \tag{137}$$

$$Q^{(m)}|_{t=0} = 0. \tag{138}$$

The 2D analogy of the M.G. Krein equation follows from (135) ($m = 0, \pm 1, \pm 2, \ldots$):

$$\sum_m S^{(m)}(z,y) f_m^{(k)'}(t-z) + \tilde{w}^{(k)}(z,y,t) + \sum_m \int_{-z}^{z} f_m^{(k)'}(t-s)\tilde{w}^{(m)}(z,y,s)\mathrm{d}s = 0, \quad |t| < z. \tag{139}$$

So, to solve the inverse problem we can solve the system (136)–(139) using the projection method and find functions $c(x,y)$ and $\rho(x,y)$ using the following iterative algorithm.

First, we introduce the N-approximation of the system (136)–(139), e.g., let $\tilde{w}^{(m)}$, $S^{(m)}$ and $Q^{(m)}$ be equal to 0 for all $|m| > N$. Let us suppose that $c_n(x,y)$ is known. Then, we calculate $\tau_n(x,y)$ from (129), $b_n(z,y)$ from (130) and $q_n(z,y)$ and $p_n(z,y)$ from (134). Function $S_n^{(m)}(t,y)$ is calculated from (136). Then, solving the 2D analogy of M.G. Krein Equation (139), we find $\tilde{w}_n^{(m)}(z,y,t)$ for $|m| \leq N$. It follows from (135) that $Q_n^{(m)}(t,y) = \tilde{w}_n^{(m)}(t+0,y,t)$. Then, from Equations (136) and (137), we find function $b_{n+1}(z,y)$ and after that the new value $c_{n+1}(x,y) = b_{n+1}(z,y)$ is calculated.

7. Conclusions

In this paper, we reviewed existing works related to the Gelfand–Levitan–Marchenko–Krein approach that allows one to solve some inverse problems by reducing them to sets of integral equations. We discussed spectral and dynamic variations of the method, as well as the connection between the GLKM approach and the BC method and the inverse scattering method that utilizes the connection of some nonlinear equations (we considered the KdV equation in the manuscript) and the inverse scattering problem to formulate the algorithm to integrate the equations. Also, we mentioned different approaches to the numerical solution of the GLKM equations.

When considering the further development of the approach, one should mention that, while the one-dimensional version of the method is well-developed, there are still a lot of aspects that have to be considered in the multi-dimensional case, both in theory and numerics. Also, the fact that the method belongs to the class of the direct ones provides the natural usage of the approach as the first step of the data processing and parameter estimations that can be later improved via other techniques of solving the inverse problems.

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