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Article

Concerning a Novel Integral Operator and a Specific Category of Starlike Functions

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Abstract: In this study, a novel integral operator that extends the functionality of some existing integral operators is presented. Specifically, the integral operator acts as the inverse operator to the widely recognized Opoola differential operator. By making use of the integral operator, a certain subclass of analytic univalent functions defined in the unit disk is proposed and investigated. This new class encompasses some familiar subclasses, like the class of starlike and the class of convex functions, while some new ones are introduced. The investigation thereafter covers coefficient inequality, distortion, growth, covering, integral preserving, closure, subordinating factor sequence, and integral means properties. Furthermore, the radii problems associated with this class are successfully addressed. Additionally, a few remarks are provided, to show that the novel integral operator and the new class generalize some existing ones.

Keywords: analytic function; convex function; coefficient inequality; integral preserving property; radii problems; Opoola differential operator; closure property; distortion property; growth property; subordinating factor sequence property

MSC: 30C45; 30C50; 30C80; 11B65; 47B38

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1. Introduction

Geometric function theory (GFT), a branch of complex analysis, explains the characteristics of the geometric properties of the image domain of analytic functions. Therefore, a geometric function is an analytic function having certain geometric properties. Over the years, several subclasses of the normalized analytic functions have been defined and their properties explored in various horizons of research. Several geometric properties of analytic functions are featured in many standard texts, such as [1–8].

Integration is a methodology that is applied widely in the solutions of many physical problems. Integral operators have a significant impact in multiple areas of pure and applied mathematics. These areas include population biology [9,10], wave propagation theory [11], engineering fields [12], and statistics [13]. In fact, the areas of application of integral operators span many ramifications of human endeavor. The extensive use of integral operators can be attributed, *at least partially*, to the fact that an ordinary differential equation can be expressed in an equivalent form known as an integral equation. This equivalence serves as the foundation for the classical proof of the Picard–Lindelöf theorem, which ensures the uniqueness and existence of solutions to ordinary differential equations.

In the literature, various kinds of operators are in existence in GFT. These include differential operators, integral operators, convolution operators, and those that are a combination of two (or three) of the aforementioned operators. The introduction of operators in GFT opened the floodgates to new directions of research. The study of integral operators in GFT, however, dates back to the work of Alexander [14], where it was used to define some subclasses of normalized analytic functions. Thereafter, many authors have considered it in the definitions of many subclasses of normalized analytic functions. For instance, see [14–26] and the comprehensive report on operators by Shareef et al. [27], for some specifics.

The investigation carried out in this paper is into the class of analytic functions whose form is of the Maclaurin–Taylor's series representation,

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k \quad (j \in \mathbb{N},\ z \in \pounds := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}), \tag{1}$$

for $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\} = \mathbb{N} \cup \{0\}$. Let $\mathcal{A}(j)$ represent the class of functions of the form (1), normalized such that $f(0) = 0 = f'(0) - 1$ and, for simplicity, let $\mathcal{A}(1) = \mathcal{A}$.

Indeed, Study [4] introduced the subclass of analytic and univalent functions called the class of convex functions. A function f is called convex if it satisfies the condition

$$\Re\mathfrak{e}((zf''/f') + 1) > 0 \quad (z \in \pounds). \tag{2}$$

The class of convex functions is usually represented by the symbol $\mathcal{CV}$. In 1936, Robertson [28] generalized the class of convex functions by introducing the condition

$$\Re\mathfrak{e}((zf''/f') + 1) > \beta \in [0, 1) \quad (z \in \pounds). \tag{3}$$

A function satisfying condition (3) is called a convex function of order β.

In 1915, Alexander [4,14] introduced another subclass of analytic and univalent functions called the class of starlike functions. A function f is called starlike if it satisfies the condition

$$\Re\mathfrak{e}(zf'/f) > 0 \quad (z \in \pounds). \tag{4}$$

The class of starlike functions is usually represented by the symbol $\mathcal{ST}$. Again, Robertson [28] generalized the class of starlike functions by introducing the condition

$$\Re\mathfrak{e}(zf'/f) > \beta \in [0, 1) \quad (z \in \pounds). \tag{5}$$

A function satisfying condition (5) is called a starlike function of order β.

The first unification of geometric expressions in (2) and (4) can be ascribed to Mocanu [29], where the author studied the class

$$\mathcal{M}(\delta) := \left\{ f : f \in \mathcal{A},\ \Re\mathfrak{e}\left[(1-\delta)\left(\frac{zf'(z)}{f(z)}\right) + \delta\left(\frac{zf''(z)}{f'(z)} + 1\right)\right] > 0,\ \delta \in \mathbb{R}\ z \in \pounds \right\}.$$

Observe that $\mathcal{M}(0) = \mathcal{ST}$ and $\mathcal{M}(1) = \mathcal{CV}$. It was reported in Mocanu's work that if $f \in \mathcal{S}$ then $\frac{f'(z)f(z)}{z} \neq 0$ in £, and that if $\delta \geq 1$ then $f \in \mathcal{M}(\delta)$ is convex univalent in £. Subsequently, in 2014, Shanmugam et al. [30] studied the class

$$\mathcal{M}(\delta) := \left\{ f : f \in \mathcal{A},\ \Re\mathfrak{e}\left[(1-\delta)\left(\frac{zf'(z)}{f(z)}\right) + \delta\left(\frac{zf''(z)}{f'(z)} + 1\right)\right] > \beta \in [0, 1),\ \delta \in \mathbb{R}\ z \in \pounds \right\}$$

and reported some upper bounds of some Hankel determinants for the class. Some other works on the unification of some kinds of geometric expressions can be found in [31–39].

In 1991, Altintaş [40] considered the subclass $\mathcal{RT}(\delta,\beta)$ of analytic functions with negative coefficients that satisfy the condition

$$\mathfrak{Re}\left\{\frac{zf'(z)+\delta z^2 f''(z)}{\delta z f'(z)+(1-\delta)f(z)}\right\} > 0, \quad (\delta \in [0,1],\ z \in \pounds).$$

We observe that if f is an analytic function with negative coefficients, then $\mathcal{RT}(0,0) = \mathcal{ST}$, the class of starlike functions; $\mathcal{RT}(0,\beta) = \mathcal{ST}(\beta)$, the class of starlike functions or order β; $\mathcal{RT}(1,0) = \mathcal{CV}$, the class of convex functions; and $\mathcal{RT}(1,\beta) = \mathcal{CV}(\beta)$, the class of starlike functions of order β. Some of the properties observed by Altintaş [40] are coefficient inequality, distortion and growth theorems, closure results, and radius problems.

The motivation to study this class was spurred by the background details discussed in the previous content. As such, the introduction of the novel integral operator into the definition of a new class studied in this paper justifies the extension of the class of Altintaş [40].

The Hadamard product (or convolution) of two analytic functions,

$$f \quad \text{in} \quad (1) \quad \text{and} \quad g(z) = z + \sum_{k=j+1}^{\infty} c_k z^k \quad (z \in \pounds), \tag{6}$$

is defined by the relation

$$(f \star g)(z) = z + \sum_{k=j+1}^{\infty} (a_k \times c_k) z^k = (g \star f)(z) \quad (z \in \pounds),$$

where '$\star$' symbolizes the Hadamard product. Also, let the notation '$\prec$' represent 'subordination'; then,

$$f \prec g \quad \text{if} \quad f = g \circ \varpi := g(\varpi(z)) \quad (z \in \pounds)$$

for analytic function

$$\varpi(z) = \varsigma_1 z + \varsigma_2 z^2 + \varsigma_3 z^3 + \cdots \quad (\varpi(0) = 0 \text{ and } |\varpi(z)| < 1). \tag{7}$$

Should g be univalent in £, then

$$f \prec g \iff f(0) = g(0) \text{ and } f(\pounds) \subset g(\pounds).$$

This paper is divided into five sections. The Section 1 contains the introduction and some of the preliminary details, the Section 2 houses the information and definitions of some operators, while the Section 3 covers the definition of the new class of functions, with some remarks. Furthermore, the Section 4 is on the main results, and we reach our conclusion in Section 5.

2. Some Key Definitions

The following definitions are fundamental to the study.

Definition 1 ([22])**.** *If we consider a function f that belongs to the set $\mathcal{A}$, then the Opoola differential operator $\mathcal{D}^{n,t}_{b,u}$ is a mapping that operates on $\mathcal{A}$ and transforms f into another function within the set $\mathcal{A}$. That is, $\mathcal{D}^{n,t}_{b,u}$ is defined by*

$$\begin{aligned}
&\mathcal{D}^{0,t}_{b,u} f(z) = f(z)\\
&\mathcal{D}^{1,t}_{b,u} f(z) = [1+(b-u-1)t]f(z) - zt(b-u) + ztf'(z) = \Delta_t(f(z))\\
&\mathcal{D}^{2,t}_{b,u} f(z) = \Delta_t(\mathcal{D}^{1,t}_{b,u} f(z))\\
&\mathcal{D}^{3,t}_{b,u} f(z) = \Delta_t(\mathcal{D}^{2,t}_{b,u} f(z)),
\end{aligned}$$

which, in general, means

$$\mathcal{D}_{b,u}^{n,t}f(z) = \Delta_t(\mathcal{D}_{b,u}^{n-1,t}f(z))$$

or, in an equivalent form,

$$\mathcal{D}_{b,u}^{n,t}f(z) = z + \sum_{k=2}^{\infty}[1+(k+b-u-1)t]^n a_k z^k, \tag{8}$$

where $z \in \pounds$; $b, t \geqslant 0$, $u \in [0, b]$, *and* $n \in \mathbb{N}_0$.

The Novel Integral Operator

As a right inverse operator to the Opoola differential operator, we therefore introduce a novel integral operator, defined as follows:

Definition 2. *Let* $f \in \mathcal{A}$, *then the integral operator* $\mathcal{I}_{b,u}^{n,t} : \mathcal{A} \longrightarrow \mathcal{A}$ *is defined by*

$$\begin{aligned}
\mathcal{I}_{b,u}^{0,t}f(z) &= f(z) \\
\mathcal{I}_{b,u}^{1,t}f(z) &= \frac{1}{tz^{(\frac{1}{t}+b-u-1)}}\int_0^z \zeta^{(\frac{1}{t}+b-u-2)}[(tb-tu)\zeta + f(\zeta)]d\zeta = I_t f(z) \quad (t>0) \\
\mathcal{I}_{b,u}^{2,t}f(z) &= I_t(\mathcal{I}_{b,u}^{1,t}f(z)) \\
\mathcal{I}_{b,u}^{3,t}f(z) &= I_t(\mathcal{I}_{b,u}^{2,t}f(z)),
\end{aligned}$$

which, in general, means

$$\mathcal{I}_{b,u}^{n,t}f(z) = I_t(\mathcal{I}_{b,u}^{n-1,t}f(z))$$

or, in an equivalent form,

$$\mathcal{I}_{b,u}^{n,t}f(z) = z + \sum_{k=2}^{\infty}\frac{1}{[1+(k+b-u-1)t]^n}a_k z^k. \tag{9}$$

Clearly, if $f \in \mathcal{A}(j)$, *then*

$$\mathcal{I}_{b,u}^{n,t}f(z) = z + \sum_{k=j+1}^{\infty}\frac{1}{[1+(k+b-u-1)t]^n}a_k z^k, \tag{10}$$

and, for brevity, we let

$$\mathcal{I}_{b,u}^{n,t}f(z) = z + \sum_{k=j+1}^{\infty}\phi^n(k,b,t,u)a_k z^k, \tag{11}$$

where

$$\phi^n(k,b,t,u) = \frac{1}{[1+(k+b-u-1)t]^n}$$

$$(z \in \pounds;\ b,t \geqslant 0,\ u \in [0,b],\ j \in \mathbb{N} \text{ and } n \in \mathbb{N}_0).$$

Remark 1. *The following properties hold for the operators in* (8) *and* (9)*:*

1. $\mathcal{D}_{b,u}^{0,t}f(z) = \mathcal{D}_{b,u}^{n,0}f(z) = \mathcal{D}_{b,u}^{0,0}f(z) = f(z) \in \mathcal{A}$ *in* (1).
2. $\mathcal{D}_{b,b}^{n,1}f(z) = \mathcal{D}_{u,u}^{n,1}f(z) = \mathcal{D}^n f(z)$ *is the Sălăgean differential operator in* [41].
3. $\mathcal{D}_{b,b}^{n,t}f(z) = \mathcal{D}_{u,u}^{n,t}f(z) = \mathcal{D}^{n,t}f(z)$ *is the Al-Oboudi differential operator in* [15].
4. $\mathcal{I}_{b,u}^{0,t}f(z) = \mathcal{I}_{b,u}^{n,0}f(z) = \mathcal{I}_{b,u}^{0,0}f(z) = f(z) \in \mathcal{A}$ *in* (1).
5. $\mathcal{I}_{b,b}^{n,1}f(z) = \mathcal{I}_{u,u}^{n,1}f(z) = \mathcal{I}^n f(z)$ *is the Sălăgean integral operator in* [41].
6. $\mathcal{I}_{b,b}^{n,t}f(z) = \mathcal{I}_{u,u}^{n,t}f(z) = \mathcal{I}^{n,t}f(z)$ *is the Al-Oboudi–Al-Qahtani integral operator in* [42–44].

7. $\mathcal{D}_{b,u}^{n,t}(\mathcal{I}_{b,u}^{n,t}f(z)) = \mathcal{I}_{b,u}^{n,t}(\mathcal{D}_{b,u}^{n,t}f(z)) = f(z)$.

3. A New Class of Analytic Functions

A function $f \in \mathcal{A}(j)$ is said to belong to the class $\mathcal{G}_j^n(b,t,u,\beta,\delta)$ if the geometric condition

$$\mathfrak{Re}\left\{\frac{z(\mathcal{I}_{b,u}^{n,t}f(z))' + \delta z^2(\mathcal{I}_{b,u}^{n,t}f(z))''}{\delta z(\mathcal{I}_{b,u}^{n,t}f(z))' + (1-\delta)(\mathcal{I}_{b,u}^{n,t}f(z))}\right\} > \beta \tag{12}$$

is satisfied for

$$z \in \pounds;\ b,t \geqslant 0,\ u \in [0,b],\ n \in \mathbb{N}_0,\ \delta \in [0,1],\ \beta \in [0,1) \text{ and } \mathcal{I}_{b,u}^{n,t}f(z) \tag{13}$$

is as defined in (10). From here onward, let all parameters be as defined in (13) unless otherwise stated.

Remark 2. *We note the following subclasses of* $\mathcal{G}_j^n(b,t,u,\beta,\delta)$*:*

1. $\mathcal{G}_1^0(b,t,u,0,0) = \mathcal{ST}$*, the class of starlike functions introduced in [3,4,8,14].*
2. $\mathcal{G}_1^0(b,t,u,\beta,0) = \mathcal{ST}(\beta)$*, the class of starlike functions of order* β *introduced in [28].*
3. $\mathcal{G}_1^n(b,1,b,0,0) = \mathcal{ST}(n)$*, the class of starlike functions defined by the Sălăgean integral operator in [41].*
4. $\mathcal{G}_1^n(b,1,b,\beta,0) = \mathcal{ST}(n,\beta)$*, the class of starlike functions of order* β *defined by the Sălăgean integral operator in [41].*
5. $\mathcal{G}_1^n(b,t,b,0,0) = \mathcal{ST}(n,t)$*, the class of starlike functions defined by the Al-Oboudi–Al-Qahtani integral operator in [42–44].*
6. $\mathcal{G}_1^n(b,t,b,\beta,0) = \mathcal{ST}(n,t,\beta)$*, the class of starlike functions of order* β *defined by the Al-Oboudi–Al-Qahtani integral operator in [15].*
7. $\mathcal{G}_1^0(b,t,u,0,1) = \mathcal{CV}$*, the class of convex functions reported in [3,4,8].*
8. $\mathcal{G}_1^0(b,t,u,\beta,1) = \mathcal{CV}(\beta)$*, the class of convex functions of order* β *introduced in [28].*
9. $\mathcal{G}_1^n(b,1,b,0,1) = \mathcal{CV}(n)$*, the class of convex functions defined by the Sălăgean integral operator in [41].*
10. $\mathcal{G}_1^n(b,1,b,\beta,1) = \mathcal{CV}(n,\beta)$*, the class of convex functions of order* β *defined by the Sălăgean integral operator in [41].*
11. $\mathcal{G}_1^n(b,t,b,0,1) = \mathcal{CV}(n,t)$*, the class of convex functions defined by the Al-Oboudi–Al-Qahtani integral operator in [42–44].*
12. $\mathcal{G}_1^n(b,t,b,\beta,1) = \mathcal{CV}(n,t,\beta)$*, the class of convex functions of order* β *defined by the Al-Oboudi–Al-Qahtani integral operator in [15].*
13. $\mathcal{G}_1^n(b,t,u,0,0) = \mathcal{ST}_1^n(b,t,u)$*, the class of starlike functions defined by the integral operator in* (9).
14. $\mathcal{G}_1^n(b,t,u,\beta,0) = \mathcal{ST}_1^n(b,t,u,\beta)$*, the class of starlike functions of order* β *defined by the integral operator in* (9).
15. $\mathcal{G}_1^n(b,t,u,0,1) = \mathcal{CV}_1^n(b,t,u)$*, the class of convex functions defined by the integral operator in* (9).
16. $\mathcal{G}_1^n(b,t,u,\beta,1) = \mathcal{CV}_1^n(b,t,u,\beta)$*, the class of convex functions of order* β *defined by the integral operator in* (9).
17. $\mathcal{G}_1^0(b,t,u,\beta,\delta) = \mathcal{G}_1^n(b,0,u,\beta,\delta) = \mathcal{G}(\beta,\delta)$*, the class studied by Altintaş [40] for functions* $f \in \mathcal{A}$ *whose coefficients are negative.*

The aforementioned subclasses and those in [20,21,23,25,31,45,46] are some of the motivations for the choice of the new class $\mathcal{G}_j^n(b,t,u,\beta,\delta)$*.*

In this paper, several conditions and properties of the class $\mathcal{G}_j^n(b,t,u,\beta,\delta)$—such as sufficient conditions for membership, distortion and growth theorems, closure results, radius problems, integral-preserving property, and conditions for subordinating factor sequence—are discussed.

4. Statement of the Results

The main results are presented as follows.

4.1. Coefficient Inequality

Theorem 1. *Let $f \in \mathcal{A}(j)$ be of the form (1). Then, f belongs to the class $\mathcal{G}_j^n(b,t,u,\beta,\delta)$ if*

$$\sum_{k=j+1}^{\infty} \psi^n(k,b,t,u,\beta,\delta)|a_k| \leqslant 1-\beta, \tag{14}$$

where

$$\psi^n(k,b,t,u,\beta,\delta) = \frac{[\delta(k-1)^2+(k-1)]-[\delta(k-1)+1](1-\beta)}{[1+(k+b-u-1)t]^n}. \tag{15}$$

See (13) for declarations.

Proof. Assume that inequality (14) is true; then, for $|z|=1$, we have from (12) and (10) that

$$\left|\frac{z(\mathcal{I}_{b,u}^{n,t}f(z))' + \delta z^2(\mathcal{I}_{b,u}^{n,t}f(z))''}{\delta z(\mathcal{I}_{b,u}^{n,t}f(z))' + (1-\delta)(\mathcal{I}_{b,u}^{n,t}f(z))} - 1\right| = \left|\frac{\sum\limits_{k=j+1}^{\infty}\frac{\delta(k-1)^2+(k-1)}{[1+(k+b-u-1)t]^n}a_k z^{k-1}}{1+\sum\limits_{k=j+1}^{\infty}\frac{\delta(k-1)+1}{[1+(k+b-u-1)t]^n}a_k z^{k-1}}\right|$$

$$\leqslant \frac{\sum\limits_{k=j+1}^{\infty}\frac{\delta(k-1)^2+(k-1)}{[1+(k+b-u-1)t]^n}|a_k|}{1+\sum\limits_{k=j+1}^{\infty}\frac{\delta(k-1)+1}{[1+(k+b-u-1)t]^n}|a_k|} \leqslant 1-\beta,$$

so that with further simplification and by maximum modulus principle, we can conclude that $f \in \mathcal{G}_j^n(b,t,u,\beta,\delta)$. □

Corollary 1. *The inequality is sharp for the extremal function*

$$f_k(z) = z + \frac{1-\beta}{\psi^n(k,b,t,u,\beta,\delta)}z^k \quad (k \geqslant j+1,\ j\in\mathbb{N},\ z\in\pounds), \tag{16}$$

which implies that for $f \in \mathcal{G}_j^n(b,t,u,\beta,\delta)$,

$$|a_k| \leqslant \frac{1-\beta}{\psi^n(k,b,t,u,\beta,\delta)} \quad (k \geqslant j+1,\ j\in\mathbb{N}).$$

4.2. Growth Theorem

Theorem 2. *Let $f \in \mathcal{G}_j^n(b,t,u,\beta,\delta)$. Then,*

$$|z| - \frac{|z|^{j+1}\phi^n(1,b,t,u)(1-\beta)}{\psi^n(j+1,b,t,u,\beta,\delta)} \leqslant |\mathcal{I}_{b,u}^{n,t}f(z)| \leqslant |z| + \frac{|z|^{j+1}\phi^n(1,b,t,u)(1-\beta)}{\psi^n(j+1,b,t,u,\beta,\delta)}. \tag{17}$$

The inequality is sharp for the extremal function

$$f_{j+1}(z) = z + \frac{1-\beta}{\psi^n(j+1,b,t,u,\beta,\delta)}z^{j+1} \quad (j\in\mathbb{N}). \tag{18}$$

Proof. Following from (14),

$$\psi^n(j+1,b,t,u,\beta,\delta)\sum_{k=j+1}^{\infty}|a_k| \leqslant \sum_{k=j+1}^{\infty}\psi^n(k,b,t,u,\beta,\delta)|a_k| \leqslant (1-\beta),$$

so that

$$\sum_{k=j+1}^{\infty} |a_k| \leqslant \frac{1-\beta}{\psi^n(j+1,b,t,u,\beta,\delta)}. \tag{19}$$

As $|z|^k < |z| < 1$, then from (11),

$$|\mathcal{I}_{b,u}^{n,t} f(z)| = \left| z + \sum_{k=j+1}^{\infty} \phi^n(k,b,t,u) a_k z^k \right| \leqslant |z| + |z|^{j+1} \phi^n(j+1,b,t,u) \sum_{k=j+1}^{\infty} |a_k|, \tag{20}$$

and putting (19) into (20) gives

$$|\mathcal{I}_{b,u}^{n,t} f(z)| \leqslant |z| + \frac{|z|^{j+1} \phi^n(j+1,b,t,u)(1-\beta)}{\psi^n(j+1,b,t,u,\beta,\delta)}.$$

Likewise,

$$|\mathcal{I}_{b,u}^{n,t} f(z)| \leqslant |z| - \frac{|z|^{j+1} \phi^n(j+1,b,t,u)(1-\beta)}{\psi^n(j+1,b,t,u,\beta,\delta)},$$

and the proof is complete. □

4.3. Distortion Theorem

Theorem 3. *Let $f \in \mathcal{G}_j^n(b,t,u,\beta,\delta)$. Then,*

$$1 - \frac{|z|^j (j+1) \phi^n(j+1,b,t,u)(1-\beta)}{\psi^n(j+1,b,t,u,\beta,\delta)} \leqslant |(\mathcal{I}_{b,u}^{n,t} f(z))'| \leqslant 1 + \frac{|z|^j (j+1) \phi^n(j+1,b,t,u)(1-\beta)}{\psi^n(j+1,b,t,u,\beta,\delta)}.$$

The inequality is sharp for the extremal function in (18).

Proof. As $|z|^k < |z| < 1$, then from (11),

$$\left|(\mathcal{I}_{b,u}^{n,t} f(z))'\right| = \left| 1 + \sum_{k=j+1}^{\infty} \phi^n(k,b,t,u) k a_k z^{k-1} \right| \leqslant 1 + |z|^j (j+1) \phi^n(j+1,b,t,u) \sum_{k=j+1}^{\infty} |a_k|, \tag{21}$$

and putting (19) into (21) gives

$$\left|(\mathcal{I}_{b,u}^{n,t} f(z))'\right| \leqslant 1 + \frac{|z|^j (j+1) \phi^n(j+1,b,t,u)(1-\beta)}{\psi^n(j+1,b,t,u,\beta,\delta)}.$$

Likewise,

$$\left|(\mathcal{I}_{b,u}^{n,t} f(z))'\right| \leqslant 1 - \frac{|z|^j (j+1) \phi^n(j+1,b,t,u)(1-\beta)}{\psi^n(j+1,b,t,u,\beta,\delta)},$$

and the proof is complete. □

4.4. Covering Theorem

Theorem 4. *Let $f \in \mathcal{G}_j^n(b,t,u,\beta,\delta)$. Then, $\mathcal{I}_{b,u}^{n,t} f(z)$ maps £ onto a domain that covers the disk*

$$|\mathcal{I}_{b,u}^{n,t} f(z)| < \frac{\psi^n(j+1,b,t,u,\beta,\delta) - \phi^n(j+1,b,b,u)(1-\beta)}{\psi^n(j+1,b,t,u,\beta,\delta)}.$$

The inequality is sharp for the extremal function in (18).

Proof. Letting $|z| \longrightarrow 1^-$ in (17) completes the proof. □

4.5. Radii Problems

This section establishes the radii of convexity, starlikeness, and close-to-convexity of order $\beta \in [0,1)$.

Theorem 5. *Let $f \in \mathcal{G}_j^n(b,t,u,\beta,\delta)$. Then, f is starlike of order β in the disk*

$$|z| < \inf_{k \geqslant j+1} \left\{ \frac{\psi^n(k,b,t,u,\beta,\delta)}{(k-\beta)} \right\}^{\frac{1}{k-1}} \quad (j \in \mathbb{N}).$$

The inequality is sharp for the extremal function in (16).

Proof. It is sufficient to show that

$$\left| \frac{zf'(z) - f(z)}{f(z)} \right| < 1 - \beta. \tag{22}$$

Using (1) in (22) gives

$$\sum_{k=j+1}^{\infty} \left(\frac{k-\beta}{1-\beta} \right) |a_k| |z|^{k-1} \leqslant 1. \tag{23}$$

Evidently, inequalities (14) and (23) are only valid if

$$\frac{k-\beta}{1-\beta} |z|^{k-1} \leqslant \frac{\psi^n(k,b,t,u,\beta,\delta)}{(1-\beta)},$$

and isolating $|z|^{k-1}$ completes the proof. □

Theorem 6. *Let $f \in \mathcal{G}_j^n(b,t,u,\beta,\delta)$. Then, f is convex of order β in the disk*

$$|z| < \inf_{k \geqslant j+1} \left\{ \frac{\psi^n(k,b,t,u,\beta,\delta)}{k(k-\beta)} \right\}^{\frac{1}{k-1}} \quad (j \in \mathbb{N}).$$

The inequality is sharp for the extremal function in (16).

Proof. It is sufficient to show that

$$\left| \frac{zf''(z)}{f'(z)} \right| < 1 - \beta. \tag{24}$$

Using (1) in (24) gives

$$\sum_{k=j+1}^{\infty} \left(\frac{k(k-\beta)}{1-\beta} \right) |a_k| |z|^{k-1} \leqslant 1.$$

Evidently, inequalities (14) and (26) are only valid if

$$\frac{k(k-\beta)}{1-\beta} |z|^{k-1} \leqslant \frac{\psi^n(k,b,t,u,\beta,\delta)}{(1-\beta)},$$

and isolating $|z|^{k-1}$ completes the proof. □

Theorem 7. *Let $f \in \mathcal{G}_j^n(b,t,u,\beta,\delta)$. Then, f is close-to-convex of order β in the disk*

$$|z| < \inf_{k \geqslant j+1} \left\{ \frac{\psi^n(k,b,t,u,\beta,\delta)}{k} \right\}^{\frac{1}{k-1}} \quad (j \in \mathbb{N}).$$

The inequality is sharp for the extremal function in (16).

Proof. It is sufficient to show that

$$\left|f' - 1\right| < 1 - \beta. \tag{25}$$

Using (1) in (25) gives

$$\sum_{k=j+1}^{\infty} \left(\frac{k}{1-\beta} \right) |a_k||z|^{k-1} \leqslant 1. \tag{26}$$

Evidently, inequalities (14) and (26) are only valid if

$$\frac{k}{1-\beta}|z|^{k-1} \leqslant \frac{\psi^n(k,b,t,u,\beta,\delta)}{1-\beta},$$

and isolating $|z|^{k-1}$ completes the proof. □

4.6. Subordinating Factor Sequence

Definition 3 ([47])**.** *The sequence $\{c_k\}_{k=1}^{\infty}$ of complex numbers is called a* subordinating factor sequence *if, whenever*

$$h(z) = \sum_{k=1}^{\infty} g_k z^k \quad (g_1 = 1,\ z \in \pounds)$$

is analytic and univalently convex in £,

$$\sum_{k=1}^{\infty} c_k g_k \prec h(z).$$

Lemma 1 ([47])**.** *From Definition 3, the sequence $\{c_k\}_{k=1}^{\infty}$ is called a subordinating factor sequence if and only if*

$$\Re\mathfrak{e}\left(1 + 2\sum_{k=1}^{\infty} c_k z^k\right) > 0 \quad (z \in \pounds).$$

Theorem 8. *Let $f \in \mathcal{G}_j^n(b,t,u,\beta,\delta)$ and let $h(z)$ be a convex function. Then,*

$$\frac{\psi^n(j+1,b,t,u,\beta,\delta)}{2\{(1-\beta)+\psi^n(j+1,b,t,u,\beta,\delta)\}}(f \star h)(z) \prec h(z) \tag{27}$$

and

$$\Re\mathfrak{e} f(z) > -\frac{(1-\beta)+\psi^n(j+1,b,t,u,\beta,\delta)}{\psi^n(j+1,b,t,u,\beta,\delta)}.$$

The constant factor

$$\frac{\psi^n(j+1,b,t,u,\beta,\delta)}{2\{(1-\beta)+\psi^n(j+1,b,t,u,\beta,\delta)\}} \tag{28}$$

in (27) cannot be replaced by a larger value.

The proof adopts the technique of Srivastava and Attiya [48].

Proof. Using (1), let

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k \in \mathcal{G}_j^n(b,t,u,\beta,\delta) \quad \text{and} \quad h(z) = z + \sum_{k=j+1}^{\infty} g_k z^k \in \mathcal{CV},$$

then, from (27),

$$\begin{aligned}
&\frac{\psi^n(j+1,b,t,u,\beta,\delta)}{2\{(1-\beta)+\psi^n(j+1,b,t,u,\beta,\delta)\}}(f \star h)(z) \\
&\qquad = \frac{\psi^n(j+1,b,t,u,\beta,\delta)}{2\{(1-\beta)+\psi^n(j+1,b,t,u,\beta,\delta)\}}\left(z + \sum_{k=j+1}^{\infty} a_k g_k z^k\right) \\
&\qquad = \sum_{k=j}^{\infty} \frac{\psi^n(j+1,b,t,u,\beta,\delta)}{2\{(1-\beta)+\psi^n(j+1,b,t,u,\beta,\delta)\}} a_k g_k z^k.
\end{aligned}$$

By Definition 3, the subordination result in (27) holds if

$$\left\{\frac{\psi^n(j+1,b,t,u,\beta,\delta)}{2\{(1-\beta)+\psi^n(j+1,b,t,u,\beta,\delta)\}} a_k\right\}_{k=j}^{\infty}$$

is a *subordinating factor sequence* when $a_1 = 1$. Applying Lemma 1 gives an equivalence inequality,

$$\Re\left(1 + \sum_{k=j+1}^{\infty} \frac{\psi^n(j+1,b,t,u,\beta,\delta)}{(1-\beta)+\psi^n(j+1,b,t,u,\beta,\delta)} a_k z^k\right) > 0. \tag{29}$$

Observe that $\psi^n(j+1,b,t,u,\beta,\delta)$ is an increasing function of j $(j \geqslant 1)$; thus, in particular,

$$\psi^n(j+1,b,t,u,\beta,\delta) \leqslant \psi^n(k,b,t,u,\beta,\delta) \quad (j \geqslant 1,\ k \geqslant j+1);$$

hence, for $|z| = r < 1$, using triangle inequality, and condition (1),

$$\begin{aligned}
&\mathfrak{Re}\left(1 + \sum_{k=j+1}^{\infty} \frac{\psi^n(j+1,b,t,u,\beta,\delta)}{(1-\beta)+\psi^n(j+1,b,t,u,\beta,\delta)} a_k z^k\right) \\
&= \mathfrak{Re}\left(1 + \frac{\psi^n(j+1,b,t,u,\beta,\delta)}{(1-\beta)+\psi^n(j+1,b,t,u,\beta,\delta)} \sum_{k=j+1}^{\infty} a_k z^k\right) \\
&= \mathfrak{Re}\left(1 + \frac{\psi^n(j+1,b,t,u,\beta,\delta)}{(1-\beta)+\psi^n(j+1,b,t,u,\beta,\delta)} z + \frac{\sum_{k=j+1}^{\infty} \psi^n(j+1,b,t,u,\beta,\delta) a_k z^k}{(1-\beta)+\psi^n(j+1,b,t,u,\beta,\delta)}\right) \\
&\geqq 1 - \frac{\psi^n(j+1,b,t,u,\beta,\delta)}{(1-\beta)+\psi^n(j+1,b,t,u,\beta,\delta)} r - \frac{\sum_{k=j+1}^{\infty} \psi^n(j+1,b,t,u,\beta,\delta)|a_k|}{(1-\beta)+\psi^n(j+1,b,t,u,\beta,\delta)} r^k \\
&> 1 - \frac{\psi^n(j+1,b,t,u,\beta,\delta)}{(1-\beta)+\psi^n(j+1,b,t,u,\beta,\delta)} r - \frac{(1-\beta)}{(1-\beta)+\psi^n(j+1,b,t,u,\beta,\delta)} r \\
&= 1 - r > 0.
\end{aligned}$$

This evidently proves inequality (29) as well as the subordination result in (27). To prove the sharpness of constant (28), consider the function

$$f_{j+1}(z) = z + \frac{(1-\beta)}{\psi^n(j+1,b,t,u,\beta,\delta)} z^{j+1} \in \mathcal{G}_j^n(b,t,u,\beta,\delta)$$

(see (18)) and the convex function

$$h_0(z) = \frac{z}{1-z} = z + \sum_{k=j+1}^{\infty} z^k \in \mathcal{CV}$$

in (27), so that

$$\frac{\psi^n(j+1,b,t,u,\beta,\delta)}{2\{(1-\beta)+\psi^n(j+1,b,t,u,\beta,\delta)\}} f_{j+1}(z) \prec h_0(z) = \frac{z}{1-z}. \tag{30}$$

It can *easily* be verified that for function $f_{j+1}(z)$,

$$\min_{|z|\leqslant r}\left\{\Re\mathfrak{e}\left(\frac{\psi^n(j+1,b,t,u,\beta,\delta)}{2\{(1-\beta)+\psi^n(j+1,b,t,u,\beta,\delta)\}} f_{j+1}(z)\right)\right\} = -\frac{1}{2} \quad (z \in \pounds).$$

This shows that the constant $\frac{\psi^n(j+1,b,t,u,\beta,\delta)}{2\{(1-\beta)(1-\lambda)+\psi^n(j+1,b,t,u,\beta,\delta)\}}$ cannot be replaced by any larger value. □

4.7. Closure Properties

Some conditions are given in this section, to show that some certain functions belong to the new class $\mathcal{G}_j^n(b,t,u,\beta,\delta)$.

Theorem 9. *Let*

$$\left.\begin{aligned} f(z) &= z + \sum_{k=j+1}^{\infty} a_k z^k \\ g(z) &= z + \sum_{k=j+1}^{\infty} c_k z^k \end{aligned}\right\} \in \mathcal{G}_j^n(b,t,u,\beta,\delta); \tag{31}$$

then, for $i \in [0,1]$, the function

$$\ell_i(z) = (1-i)f(z) + ig(z) = z + \sum_{k=j+1}^{\infty} \{(1-i)a_k + ic_k\} z^k \in \mathcal{G}_j^n(b,t,u,\beta,\delta). \tag{32}$$

Proof. As $f, g \in \mathcal{G}_j^n(b,t,u,\beta,\delta)$, then it follows from (14) and (32) that

$$\begin{aligned} &\sum_{k=j+1}^{\infty} \psi^n(k,b,t,u,\beta,\delta)\big|\{(1-i)a_k + ic_k\}\big| \\ &\qquad \leqslant \sum_{k=j+1}^{\infty} \psi^n(k,b,t,u,\beta,\delta)\{(1-i)|a_k| + i|c_k|\} \\ &\qquad = (1-i)\sum_{k=j+1}^{\infty} \psi^n(k,b,t,u,\beta,\delta)|a_k| + i\sum_{k=j+1}^{\infty} \psi^n(k,b,t,u,\beta,\delta)|c_k| \\ &\qquad \leqslant (1-i)\{(1-\beta)\} + i\{(1-\beta)\} = (1-\beta). \end{aligned}$$

This shows that $\ell_i \in \mathcal{G}_j^n(b,t,u,\beta,\delta)$. □

Theorem 10. *Let the functions*

$$f_i(z) = z + \sum_{k=j+1}^{\infty} a_{k,i} z^k \quad (i \in \{1,2,3,\ldots,l\}) \tag{33}$$

belong to the class $\mathcal{G}_j^n(b,t,u,\beta,\delta)$*; then, for* $\sum_{i=1}^{l}\eta_i = 1$*, the function*

$$p(z) = \sum_{i=1}^{l} \eta_i f_i(z) \in \mathcal{G}_j^n(b,t,u,\beta,\delta) \quad (z \in \pounds). \tag{34}$$

Proof. Firstly, observe that for functions f_i in (33),

$$\sum_{k=j+1}^{\infty} \frac{\psi^n(k,b,t,u,\beta,\delta)}{(1-\beta)} |a_{k,i}| \leqslant 1, \tag{35}$$

and that (34) can be expressed as

$$p(z) = \sum_{j=1}^{l} \eta_j \left(z + \sum_{k=j+1}^{\infty} a_{k,i} z^k \right) = z + \sum_{i=1}^{l} \sum_{k=j+1}^{\infty} \eta_i a_{k,i} z^k = z + \sum_{k=j+1}^{\infty} \left(\sum_{i=1}^{l} \eta_i a_{k,i} \right) z^k. \tag{36}$$

Putting (36) into (35) leads to

$$\sum_{k=j+1}^{\infty} \frac{\psi^n(k,b,t,u,\beta,\delta)}{(1-\beta)} \left| \sum_{i=1}^{l} \eta_i a_{k,i} \right| = \sum_{k=j+1}^{\infty} \frac{\psi^n(k,b,t,u,\beta,\delta)}{(1-\beta)} |a_{k,i}| \leqslant 1.$$

This shows that $p \in \mathcal{G}_j^n(b,t,u,\beta,\delta)$. □

Theorem 11. *Let the functions* $f_i(z)$ *be as defined in* (33) *and belong to the class* $\mathcal{G}_j^n(b,t,u,\beta,\delta)$*; then, the arithmetic mean* $m(z)$ *of the functions* $f_i(z)$ *defined by*

$$m(z) = \frac{1}{l} \sum_{i=1}^{l} f_i(z) \quad (z \in \pounds) \tag{37}$$

is in $\mathcal{G}_j^n(b,t,u,\beta,\delta)$.

Proof. Observe that (37) can be simplified as

$$m(z) = \frac{1}{l} \sum_{i=1}^{l} \left(z + \sum_{k=j+1}^{\infty} a_{k,i} z^k \right) = z + \sum_{k=j+1}^{\infty} \left(\frac{1}{l} \sum_{i=1}^{l} a_{k,i} \right) z^k. \tag{38}$$

As $f_i \in \mathcal{G}_j^n(b,t,u,\beta,\delta)$ for all $i \in \{1,2,3,\ldots,l\}$, then (38), in view of (14), gives

$$\sum_{k=j+1}^{\infty} \frac{\psi^n(k,b,t,u,\beta,\delta)}{(1-\beta)} \left| \frac{1}{l} \sum_{i=1}^{l} a_{k,i} \right| = \frac{1}{l} \sum_{i=1}^{l} \left\{ \sum_{k=j+1}^{\infty} \frac{\psi^n(k,b,t,u,\beta,\delta)}{(1-\beta)} |a_{k,i}| \right\} \leqslant \frac{1}{l} \sum_{j=1}^{l} (1) = 1.$$

This shows that $m \in \mathcal{G}_j^n(b,t,u,\beta,\delta)$. □

Theorem 12. *For* $f, g \in \mathcal{G}_j^n(b,t,u,\beta,\delta)$ *in* (31)*, the weighted mean* w_i *of functions* f *and* g *defined by*

$$w_i(z) = \frac{(1-i)f(z) + (1+i)g(z)}{2} \quad (i \in \mathbb{N},\ z \in \pounds) \tag{39}$$

is in $\mathcal{G}_j^n(b,t,u,\beta,\delta)$.

Proof. Using (31) in (39) leads to

$$w_i(z) = z + \sum_{k=j+1}^{\infty} \frac{(1-i)a_k + (1+i)c_k}{2} z^k. \tag{40}$$

To show that w_i is in $\mathcal{G}_j^n(b,t,u,\beta,\delta)$ is to show that

$$\sum_{k=j+1}^{\infty} \frac{\psi^n(k,b,t,u,\beta,\delta)}{(1-\beta)} \left| \frac{(1-i)a_k + (1+i)c_k}{2} \right| \leqslant 1.$$

This follows from (14), to give

$$\begin{aligned}
&\sum_{k=j+1}^{\infty} \frac{\psi^n(k,b,t,u,\beta,\delta)}{(1-\beta)} \left\{ \frac{(1-i)|a_k| + (1+i)|c_k|}{2} \right\} \\
&= \frac{(1-i)}{2} \sum_{k=j+1}^{\infty} \frac{\psi^n(k,b,t,u,\beta,\delta)}{(1-\beta)} |a_k| + \frac{(1+i)}{2} \sum_{k=j+1}^{\infty} \frac{\psi^n(k,b,t,u,\beta,\delta)}{(1-\beta)} |c_k| \\
&\leqslant \frac{(1-i)}{2} + \frac{(1+i)}{2} = 1.
\end{aligned}$$

This shows that $w_i \in \mathcal{G}_j^n(b,t,u,\beta,\delta)$. □

Theorem 13. *For $f, g \in \mathcal{G}_j^n(b,t,u,\beta,\delta)$ in (31), the function*

$$v = (f \star g) \in \mathcal{G}_j^n(b,t,u,\beta,\delta)$$

if $|c_k| \leqslant 1$.

Proof. Using (31) for $f, g \in \mathcal{G}_j^n(b,t,u,\beta,\delta)$ implies that

$$v(z) = (f \star g)(z) = z + \sum_{k=j+1}^{\infty} a_k c_k z^k.$$

Observe that as $|c_k| \leqslant 1$ then $|a_k||c_k| \leqslant |a_k|\ \forall k \in \{2,3,4,\ldots\}$, so that, in view of (14),

$$\sum_{k=j+1}^{\infty} \frac{\psi^n(k,b,t,u,\beta,\delta)}{(1-\beta)} |a_k c_k| \leqslant \sum_{k=j+1}^{\infty} \frac{\psi^n(k,b,t,u,\beta,\delta)}{(1-\beta)} |a_k| \leqslant 1,$$

which completes the proof. □

4.8. Integral Preserving Theorem

In this section, some known integral functions are proved to belong to the new class, $\mathcal{G}_j^n(b,t,u,\beta,\delta)$.

Definition 4 ([19]). *Let $f \in \mathcal{A}(j)$. Then, the Bernardi integral operator* $\mathcal{B}_\tau : \mathcal{A}(j) \longrightarrow \mathcal{A}(j) \quad (j \in \mathbb{N},\ \tau > -1)$ *is defined by*

$$\mathcal{B}_\tau f(z) = \frac{1+\tau}{z^\tau} \int_0^z \eta^{\tau-1} f(\eta) d\eta = z + \sum_{k=j+1}^{\infty} \frac{1+\tau}{k+\tau} a_k z^k \quad (z \in \pounds). \tag{41}$$

Theorem 14. *Let $f \in \mathcal{G}_j^n(b,t,u,\beta,\delta)$. Then, $\mathcal{B}_\tau f(z)$ is in $\mathcal{G}_j^n(b,t,u,\beta,\delta)$.*

Proof. As $f \in \mathcal{G}_j^n(b,t,u,\beta,\delta)$, then from (41),

$$\frac{1+\tau}{k+\tau}|a_k| < |a_k|.$$

Observe that $\left|\frac{1+\tau}{k+\tau}\right| < 1\ \forall k$, so that the application of Theorem 13 completes the proof. □

4.9. Integral Means Inequality

Lemma 2 ([49])**.** *Let $f, h \in \mathcal{A}(j)$ with $f(z) \prec h(z)$. Then,*

$$\int_0^{2\pi} |f(z)|^l d\rho \leqslant \int_0^{2\pi} |h(z)|^l d\rho,$$

where $z = re^{i\rho}$, $\rho > 0$ and $0 < r < 1$.

Theorem 15. *Let $f \in \mathcal{G}_j^n(b,t,u,\beta,\delta)$ and*

$$h(z) = z + \frac{(1-\beta)}{\psi^n(k,b,t,u,\beta,\delta)} z^k \quad (k \geqslant j+1)$$

from (16). Then, for $\rho > 0$, $z = re^{i\rho}$, and $0 < r < 1$,

$$\int_0^{2\pi} |f(z)|^l d\rho \leqslant \int_0^{2\pi} |h(z)|^l d\rho. \tag{42}$$

Proof. As $f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k \in \mathcal{G}_j^n(b,t,u,\beta,\delta)$, then Lemma 2 implies that (42) can be expressed as

$$\int_0^{2\pi} \left|1 + \sum_{k=j+1}^{\infty} a_k z^{k-1}\right|^l d\rho \leqslant \int_0^{2\pi} \left|1 + \frac{(1-\beta)}{\psi^n(k,b,t,u,\beta,\delta)} z^{k-1}\right|^l d\rho;$$

therefore, it is sufficient (see Lemma 2) to show that

$$1 + \sum_{k=j+1}^{\infty} a_k z^{k-1} \prec 1 + \frac{(1-\beta)}{\psi^n(k,b,t,u,\beta,\delta)} z^{k-1}.$$

Ultimately,

$$1 + \sum_{k=j+1}^{\infty} a_k z^{k-1} = 1 + \frac{(1-\beta)}{\psi^n(k,b,t,u,\beta,\delta)} [\varpi(z)]^{k-1}$$

for $\varpi(z)$ in (7). Simple computation leads to

$$[\varpi(z)]^{k-1} = \sum_{k=j+1}^{\infty} \frac{\psi^n(k,b,t,u,\beta,\delta)}{(1-\beta)} a_k z^{k-1},$$

so that, in view of (14),

$$|\varpi(z)|^{k-1} = \left|\sum_{k=j+1}^{\infty} \frac{\psi^n(k,b,t,u,\beta,\delta)}{(1-\beta)} a_k z^{k-1}\right| \leqslant |z|^j \sum_{k=j+1}^{\infty} \frac{\psi^n(k,b,t,u,\beta,\delta)}{(1-\beta)} |a_k| \leqslant |z| < 1$$

and the proof is complete. □

5. Conclusions

We presented a novel integral operator that extends some well-known integral operators. The integral operator is the right inverse operator of the Opoola differential operator introduced in [22]. This new operator allowed us to define a specific subclass of analytic univalent functions, after which, we explored various geometric properties of the class. More importantly, this class encompasses some well-known classes, such as the class of starlike functions and the class of convex functions, and some new ones. The investigated properties included coefficient inequality, distortion, growth, covering, integral preserving, subordinating factor sequence, integral means, and some closure conditions. Furthermore, the radii problems associated with the new class were successfully solved. Ultimately, this investigation is a contribution to knowledge, through the introduction of a new integral operator, exploring its applications in GFT, and providing insights into the geometric properties of the new class of analytic functions. For some related examples, see [45,46,50–53].

The findings presented in this research paper have unveiled fresh concepts that could pave the way for further investigations. We have also provided opportunities for researchers to extend the scope of this operator and generate new results in both univalent and multivalent function theory.

We note, however, that the results in this paper are limited to the study of the subclass of analytic and univalent functions.

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