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Article

Bifurcating Limit Cycles with a Perturbation of Systems Composed of Piecewise Smooth Differential Equations Consisting of Four Regions

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Abstract: Systems composed of piecewise smooth differential (PSD) mappings have quantitatively been searched for answers to a substantial issue of limit cycle (LC) bifurcations. In this paper, LC numbers (LCNs) of a PSD system (PSDS) consisting of four regions are dealt with. A Melnikov mapping whose order is one is implicitly obtained by finding its originators when the system is perturbed under any nth degree of real polynomials. Then, the approach employing the Picard–Fuchs mapping is utilized to attain a higher boundary of bifurcation LCNs of systems composed of PSD functions with a global center. The method we used could be implemented to examine the problems related to the LC of other PSDS.

Keywords: Melnikov function; limit cycles (LCs); Picard–Fuchs (PF) equation; piecewise smooth differential system (PSDS)

MSC: 34C07; 34C28

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1. Background and Literature

Planar Differential System (PDS) theory quantitatively searches for limit cycle numbers (LCNs) and their distributions. There exists a close association with the renowned 16th problem proposed by Hilbert [1]: given any polynomial differential system (PDS), what are the highest LCNs and their distributions? To tackle the issue, several approaches have been proposed, such as the approach based on the Melnikov function [2–7], the averaging method [8–11], the Picard–Fuchs (PF) equation method [12,13], the Chebyshev criterion [14–18], and the method to compute the Lyapunov constants [19–22].

The PF equation arises in the context of algebraic geometry and complex analysis, particularly when studying families of algebraic varieties or complex manifolds. In bifurcating LC analysis, the PF equation can be used to describe the behavior of LCs as parameters vary in a dynamical system. It helps in understanding how the properties of LCs change as system parameters are perturbed. In the study of LCs, the PF equation typically involves a complex parameter, and it describes how certain periods or integrals associated with the LCs vary with respect to this parameter. The solutions to the PF equation provide information about the monodromy of the LCs, which can be essential for understanding their stability and bifurcation behavior. In summary, the PF equation plays a role in the analysis of LCs by describing how certain complex integrals or periods associated with the cycles change as system parameters vary, shedding light on their bifurcation behavior.

To discuss further, Tian et al. studied LC bifurcations in systems composed of piecewise smooth second-order integrable mappings with invariant curves by utilizing the Melnikov mapping (MM) whose order is 1, deriving a lesser boundary for LCNs bifurcated based on the annulus period [2]. Gasull et al. considered mappings similar to the perturbed pendulum upon a cylinder under the perturbation of trigonometric polynomials and provided the higher boundaries pertinent to how many their related Melnikov mapping whose order is 1 generates zeros by covering areas related to oscillation and rotation [3]. Xiong and Han dealt with the issue of bifurcated LCs in the same nth-degree systems by utilizing a perturbation system composed of piecewise third-order polynomial mappings that has a generic heteroclinic hoop having a saddle point characterized by nilpotency and cusp and employed the expansion together with its coefficients of the first-order Melnikov mapping to attain at least $3n-1$ LC, [6].

Llibre et al. developed a theory called averaging related to orders of both first and second to investigate the periodical results of systems composed of discontinuous piecewise differential functions consisting of arbitrary numbers of systems and dimensions whose differentiability conditions are at minimum [11]. In [12], Horozov and Iliev derived features of the systems called the PF that are fulfilled by the four fundamental integral expressions to explicit a higher boundary regarding how many zeros the integral called Abelian of near-Hamiltonians contains. Yang offered the PF approach to the research of LC bifurcations for differential equations characterized by non-smoothness having two exchanging lines [23]. Grau et al. presented a benchmark that gives a simple sufficient requirement to obtain a class of Abelian integrals to attain Chebyshev's attribution [15]. The equivalence of the first two approaches was proven by [22].

Assume that the near-integrable differential system has the subsequent representation

$$\begin{cases} \dot{\nu} = p(\nu,\beta) + \varepsilon f(\nu,\beta), \\ \dot{\beta} = q(\nu,\beta) + \varepsilon g(\nu,\beta), \end{cases} \tag{1}$$

where in $0 < |\varepsilon| \ll 1$, $p, q, f, g \in C^\infty$. For $\varepsilon = 0$, an initial integral and its integration factor were characterized by $H(\nu,\beta)$ and $\mu(\nu,\beta)$ in Equation (1), respectively. Assume that Equation $(1)|_{\varepsilon=0}$ covers L_ξ that includes periodical orbits represented by a class that encloses ($\nu = 0$ and $\beta = 0$) is covered. We also have $H(\nu,\beta) = \xi$ delineating L_ξ and

$$\tilde{M}(\xi) = \oint_{L_\xi} \mu(\nu,\beta)[g(\nu,\beta)d\nu - f(\nu,\beta)d\beta], \tag{2}$$

was expressed as a Melnikov mapping whose order is 1 for (1), which has a substantial influence regarding the study of the bifurcations of LCs. For instance, if (2) is assumed to have a secluded 0 denoted by ξ_0. Equation (1) contains a LC close to $L(\xi_0)$.

In the simulations of real-world phenomena having discontinuity such as biology [24], oscillations characterized by non-linearity [25], and mechanics coping with impacts and frictions [26,27], investigating LCNs and their relevant locations has been concentrated on lately. The issue could be evaluated as the elongation of the tiny Hilbert's sixteenth question of the cases with discontinuity. An approach based on the Melnikov function was suggested to deal with systems composed of planar piecewise smooth Hamiltonian mappings consisting of two areas [28]. Deriving a mathematical representation for the Melnikov mapping whose order is 1 has a key function in the examination of LC bifurcations. This topic has been renewed very recently in some works such as [29,30]. Employing the theory called averaging based on the first order, the authors of [31] investigated LC bifurcations by using the steady isochronous epicenter whose orbits are periodical regarding

$$\dot{\nu} = -\beta + \nu^2\beta, \dot{\beta} = \nu + \nu\beta^2,$$

once perturbations of the LC bifurcations are applied to the systems composed of the entire second and third-order discontinuous polynomial differential functions having four regions.

The generic representation of a system composed of the piecewise smooth almost-integrable differentiable functions in the $\nu-\beta$ extents having two regions split by a β-axis was given by

$$(\dot{\nu},\dot{\beta}) = \begin{cases} (p^{+}(\nu,\beta)+\varepsilon f^{+}(\nu,\beta), q^{+}(\nu,\beta)+\varepsilon g^{+}(\nu,\beta)), \nu \geq 0, \\ (p^{-}(\nu,\beta)+\varepsilon f^{-}(\nu,\beta), q^{-}(\nu,\beta)+\varepsilon g^{-}(\nu,\beta)), \nu < 0, \end{cases} \tag{3}$$

where $p^{\pm}(\nu,\beta), q^{\pm}(\nu,\beta), f^{\pm}(\nu,\beta), g^{\pm}(\nu,\beta) \in C^{\infty}$. For $\varepsilon = 0$, an initial integration $H^{+}(\nu,\beta)$ (resp. $H^{-}(\nu,\beta)$) for $\nu \geq 0$, (resp. $\nu < 0$) and the integration quantity $\mu^{+}(\nu,\beta)$ (resp. $\mu^{-}(\nu,\beta)$) for $\nu \geq 0$ (resp. $\nu < 0$) are contained by $(3)|_{\varepsilon=0}$ that is assumed to have a collection of periodical orbits

$$L_{\xi} = L_{\xi}{}^{+} \bigcup L_{\xi}{}^{-},$$

enclosing ($\nu = 0$, $\beta = 0$). Figure 1 depicts that $L_{\xi}{}^{+}$ (resp. $L_{\xi}{}^{-}$) was delineated by $H^{+}(\nu,\beta) = \xi$ (resp. $H^{-}(\nu,\beta) = \bar{\xi}$). The Melnikov mapping whose order is 1 for (3) is expressed by

$$\bar{M}(\xi) = \frac{H^{+}_{\beta}(E)H^{-}_{\beta}(F)}{H^{-}_{\beta}(E)H^{+}_{\beta}(F)} \oint_{L^{+}_{\xi}} u^{+}\left[g^{+}d\nu - f^{+}d\beta\right] + \frac{H^{+}_{\beta}(E)}{H^{-}_{\beta}(E)} \oint_{L^{-}_{\xi}} u^{-}\left[g^{-}d\nu - f^{+}d\beta\right]. \tag{4}$$

The Relation (3) contains a LC close to L_{ξ_0} iff $\bar{M}(\xi)$ in (4) including the secluded 0 in ξ close to ξ_0, see [31,32].

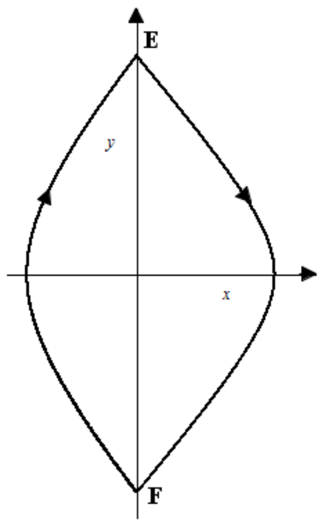

Figure 1. A state sketch of Equation $(3)|_{\varepsilon=0}$. For the sake of simplicity in its understanding we considered $\nu = x$ and $\beta = y$.

Considering the subsequent system in the plane having four regions

$$\begin{cases} \dot{\nu} = p^{1}(\nu,\beta)+\varepsilon f^{1}(\nu,\beta), \\ \dot{\beta} = q^{1}(\nu,\beta)+\varepsilon g^{1}(\nu,\beta), \end{cases} \quad \nu > 0, \beta > 0, \tag{5}$$

$$\begin{cases} \dot{\nu} = p^{2}(\nu,\beta)+\varepsilon f^{2}(\nu,\beta), \\ \dot{\beta} = q^{2}(\nu,\beta)+\varepsilon g^{2}(\nu,\beta), \end{cases} \quad \nu > 0, \beta < 0, \tag{6}$$

$$\begin{cases} \dot{\nu} = p^{3}(\nu,\beta)+\varepsilon f^{3}(\nu,\beta), \\ \dot{\beta} = q^{3}(\nu,\beta)+\varepsilon g^{3}(\nu,\beta), \end{cases} \quad \nu < 0, \beta < 0, \tag{7}$$

$$\begin{cases} \dot{\nu} = p^{4}(\nu,\beta)+\varepsilon f^{4}(\nu,\beta), \\ \dot{\beta} = q^{4}(\nu,\beta)+\varepsilon g^{4}(\nu,\beta), \end{cases} \quad \nu < 0, \beta > 0, \tag{8}$$

or

$$\begin{cases} \dot{\nu} = P(\nu,\beta) + \varepsilon f(\nu,\beta), \\ \dot{\beta} = Q(\nu,\beta) + \varepsilon g(\nu,\beta), \end{cases} \tag{9}$$

where $0 < |\varepsilon| \ll 1$, $p^\iota(\nu,\beta), q^\iota(\nu,\beta), f^\iota(\nu,\beta), g^\iota(\nu,\beta) \in C^\infty$, $\iota = 1,2,3,4$. We also have

$$P(\nu,\beta) = \begin{cases} p^1(\nu,\beta), \nu > 0, \beta > 0, \\ p^2(\nu,\beta), \nu > 0, \beta < 0, \\ p^3(\nu,\beta), \nu < 0, \beta < 0, \\ p^4(\nu,\beta), \nu < 0, \beta > 0, \end{cases}$$

$$Q(\nu,\beta) = \begin{cases} q^1(\nu,\beta), \nu > 0, \beta > 0, \\ q^2(\nu,\beta), \nu > 0, \beta < 0, \\ q^3(\nu,\beta), \nu < 0, \beta < 0, \\ q^4(\nu,\beta), \nu < 0, \beta > 0, \end{cases}$$

$$f(\nu,\beta) = \begin{cases} f^1(\nu,\beta), \nu > 0, \beta > 0, \\ f^2(\nu,\beta), \nu > 0, \beta < 0, \\ f^3(\nu,\beta), \nu < 0, \beta < 0, \\ f^4(\nu,\beta), \nu < 0, \beta > 0, \end{cases}$$

$$g(\nu,\beta) = \begin{cases} g^1(\nu,\beta), \nu > 0, \beta > 0, \\ g^2(\nu,\beta), \nu > 0, \beta < 0, \\ g^3(\nu,\beta), \nu < 0, \beta < 0, \\ g^4(\nu,\beta), \nu < 0, \beta > 0. \end{cases}$$

The Relation $(5)|_{\varepsilon=0}$, $(6)|_{\varepsilon=0}$, $(6)|_{\varepsilon=0}$, and $(8)|_{\varepsilon=0}$ include first integrals and integration factors that correspond to $H^1(\nu,\beta)$ and $\mu^1(\nu,\beta)$, $H^2(\nu,\beta)$ and $\mu^2(\nu,\beta)$, $H^3(\nu,\beta)$ and $\mu^3(\nu,\beta)$, $H^4(\nu,\beta)$ and $\mu^4(\nu,\beta)$, respectively. Note that

$$\frac{\partial H^1(\nu,\beta)}{\partial\beta} = \mu^1(\nu,\beta)p^1(\nu,\beta),$$

$$\frac{\partial H^1(\nu,\beta)}{\partial\nu} = -\mu^1(\nu,\beta)q^1(\nu,\beta),$$

$$\frac{\partial H^2(\nu,\beta)}{\partial\beta} = \mu^2(\nu,\beta)p^2(\nu,\beta),$$

$$\frac{\partial H^2(\nu,\beta)}{\partial\nu} = -\mu^2(\nu,\beta)q^2(\nu,\beta),$$

$$\frac{\partial H^3(\nu,\beta)}{\partial\beta} = \mu^3(\nu,\beta)p^3(\nu,\beta),$$

$$\frac{\partial H^3(\nu,\beta)}{\partial\nu} = -\mu^3(\nu,\beta)q^3(\nu,\beta),$$

$$\frac{\partial H^4(\nu,\beta)}{\partial\beta} = \mu^4(\nu,\beta)p^4(\nu,\beta),$$

$$\frac{\partial H^2(\nu,\beta)}{\partial\nu} = -\mu^4(\nu,\beta)q^4(\nu,\beta).$$

To discuss further, multiplying (5) by $\mu^1(\nu,\beta)$, and $dt_1 = \mu^1(\nu,\beta)dt$. Then, subscript one will be omitted subsequently. One obtains

$$\begin{cases} \dot{\nu} = H^1_\beta(\nu,\beta) + \varepsilon\mu^1(\nu,\beta)f^1(\nu,\beta), \\ \dot{\beta} = -H^1_\beta(\nu,\beta) + \varepsilon\mu^1(\nu,\beta)g^1(\nu,\beta), \end{cases} \quad \nu > 0, \beta > 0, \tag{10}$$

$$\begin{cases} \dot{\nu} = H^2_\beta(\nu,\beta) + \varepsilon\mu^2(\nu,\beta)f^2(\nu,\beta), \\ \dot{\beta} = -H^2_\beta(\nu,\beta) + \varepsilon\mu^2(\nu,\beta)g^2(\nu,\beta), \end{cases} \quad \nu > 0, \beta < 0, \tag{11}$$

$$\begin{cases} \dot{\nu} = H^3_\beta(\nu,\beta) + \varepsilon\mu^3(\nu,\beta)f^3(\nu,\beta), \\ \dot{\beta} = -H^3_\beta(\nu,\beta) + \varepsilon\mu^3(\nu,\beta)g^3(\nu,\beta), \end{cases} \quad \nu < 0, \beta < 0, \tag{12}$$

and

$$\begin{cases} \dot{\nu} = H^4_\beta(\nu,\beta) + \varepsilon\mu^4(\nu,\beta)f^4(\nu,\beta), \\ \dot{\beta} = -H^4_\beta(\nu,\beta) + \varepsilon\mu^4(\nu,\beta)g^4(\nu,\beta), \end{cases} \quad \nu < 0, \beta > 0. \tag{13}$$

To make Equation $(9)|_{\varepsilon=0}$ contain a collection of orbits that are periodically close to $(0,0)$, an interval, $\sum = (\nu,\beta)$, and $D = (d(\xi),0)$, $C = (0,c(\xi))$, $B = (b(\xi),0)$, $A = (0,a(\xi))$, exist for all $\xi \in \Sigma$ is assumed,

$$H^4(D) = H^4(A), H^3(C) = H^3(D), H^2(B) = H^2(C), H^1(A) = H^1(B) = \xi,$$

when $c(\xi)a(\xi) < 0$ and $d(\xi)b(\xi) < 0$. Considering $(5)|_{\varepsilon=0}$ having L^1_ξ, an arc in the form of an orbit, starts with A and ends at B delineated by $H^1(\nu,\beta) = \xi, \xi \in \Sigma, \nu > 0, \beta > 0$; $(6)|_{\varepsilon=0}$ having L^2_ξ, an arc in the form of an orbit, starts with B and ends at C delineated by $H^2(\nu,\beta) = H^2(B), \nu > 0, \beta < 0$; $(7)|_{\varepsilon=0}$ having an orbital arc L^3_ξ starts with C and ends at D delineated by $H^3(\nu,\beta) = H^3(C), \nu < 0, \beta < 0$, and Equation $(8)|_{\varepsilon=0}$ includes L^4_ξ, an arc in the form of an orbit, begins with D and finishes at A delineated by $H^4(\nu,\beta) = H^4(D)$, $\nu < 0, \beta > 0$. Hence,

$$L_\xi = L^4_\xi \bigcup L^3_\xi \bigcup L^2_\xi \bigcup L^1_\xi,$$

is an orbit (periodic) of $(9)|_{\varepsilon=0}$ enclosing $(\nu = 0, \beta = 0)$ for $\xi \in \Sigma$.

Therefore, $\{L_\xi, \xi \in \Sigma\}$ denotes a collection of periodic orbits of $(9)|_{\varepsilon=0}$ satisfying

$$\lim_{\xi \to 0} L_\xi = O,$$

where O and each L_ξ denote the origin and piecewise smooth. Figure 2 depicts that L_ξ including an orientation in a clockwise manner is supposed. An interesting and important problem was to examine LCNs that was bifurcated based on $\{L_\xi, \xi \in \Sigma\}$.

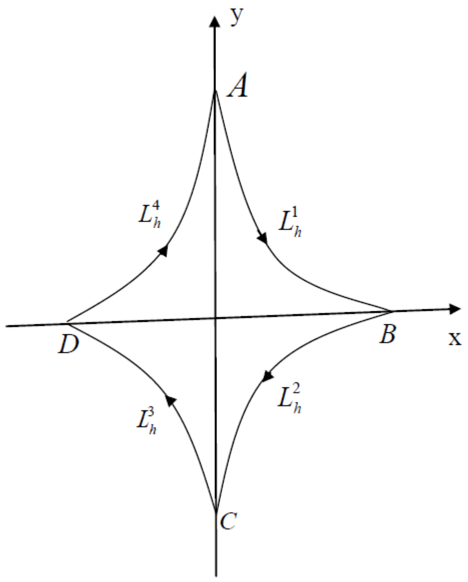

Figure 2. The shut trajectory of Equation $(9)|_{\varepsilon=0}$. For the sake of simplicity in its understanding we considered $\nu = x$ and $\beta = y$.

The remainder of this paper is unfolded as comes next. Section 2 provides a discussion and the main aim of this article in the form of a theorem. Toward this goal and to prove the theorem theoretically, Section 3 discusses that $M(\xi)$, a Melnikov mapping whose order is 1, and the PF mappings are expressed and fulfilled by the originators of $M(\xi)$. This work follows the discussions given recently in [33–35]. In terms of novelty, here by considering some piecewise smooth Liênard mappings with a small enough $|\varepsilon|$, a higher boundary for the LCNs bifurcated based on the annulus period is given for different values of n. The proof of the main result and the related discussions are furnished in Section 4. Finally, some concluding comments are brought forward in Section 5.

2. Primary Outcomes

The Melnikov mapping (MM) is a mathematical technique used in the study of bifurcating LCs in dynamical systems. It is named after mathematician V. K. Melnikov, who developed the method [36]. The MM is particularly useful for analyzing the stability of LCs in systems described by ordinary differential equations. When a dynamical system undergoes a bifurcation, such as a saddle-node or Hopf bifurcation, LCs can emerge or disappear. The MM is used to understand the conditions under which these LCs are born or annihilated.

The key idea behind the MM is to calculate the transverse intersections between the unstable and stable manifolds of the saddle equilibrium point (or the periodic orbit created by a Hopf bifurcation). These transverse intersections, called "Melnikov integrals", are computed along a certain direction in phase space. If these integrals are nonzero, they indicate that the unstable and stable manifolds intersect transversely, implying the potential existence of a LC in the vicinity. By analyzing the Melnikov integrals and their behavior as system parameters change, researchers can determine whether LCs are created or destroyed during a bifurcation. This information is crucial for understanding the bifurcation dynamics and the stability of LCs in dynamical systems.

From [37,38], $M(\xi)$, which is a MM whose order is 1 for (9), is attained as follows.

Proposition 1. *Assumed that the parts represented by (I) and (II) hold, a MM whose order is 1 for Equation (9) were expressed by*

$$\begin{aligned} M(\xi) = {} & \frac{H_\beta^1(A)H_\nu^2(B)H_\beta^3(C)H_\nu^4(D)}{H_\beta^4(A)H_\nu^1(B)H_\beta^2(C)H_\nu^3(D)} \int_{\widetilde{AB}} \mu^1 \left[g^1 d\nu - f^1 d\beta\right] \\ & + \frac{H_\beta^1(A)H_\beta^3(C)H_\nu^4(D)}{H_\beta^4(A)H_\beta^2(C)H_\nu^3(D)} \int_{\widetilde{BC}} \mu^2 \left[g^2 d\nu - f^2 d\beta\right] \\ & + \frac{H_\beta^1(A)H_\nu^4(D)}{H_\beta^4(A)H_\nu^3(D)} \int_{\widetilde{CD}} \mu^3 \left[g^3 d\nu - f^3 d\beta\right] \\ & + \frac{H_\beta^1(A)}{H_\beta^4(A)} \int_{\widetilde{DA}} \mu^4 \left[g^4 d\nu - f^4 d\beta\right], \xi \in \Sigma. \end{aligned} \tag{14}$$

Additionally, when $M(\xi_0) = 0$ and $M'(\xi_0) \neq 0$ for some $\xi_0 \in \Sigma$. Equation (9) contains a unique LC close to L_{ξ_0} for all $|\varepsilon|$ small enough.

In this work, by employing the first-order MM (14), the LCNs for the subsequent systems that are characterized by piecewise smooth Liénard mappings are examined.

$$\begin{pmatrix}\dot{\nu}\\ \dot{\beta}\end{pmatrix} = \begin{cases} \begin{pmatrix}\beta\\ -\nu-\nu^3+\varepsilon g^1(\nu,\beta)\beta\end{pmatrix}, \nu>0,\beta>0,\\ \begin{pmatrix}\beta\\ -\nu-\nu^3+\varepsilon g^2(\nu,\beta)\beta\end{pmatrix}, \nu>0,\beta<0,\\ \begin{pmatrix}\beta\\ -\nu-\nu^3+\varepsilon g^3(\nu,\beta)\beta\end{pmatrix}, \nu<0,\beta<0,\\ \begin{pmatrix}\beta\\ -\nu-\nu^3+\varepsilon g^4(\nu,\beta)\beta\end{pmatrix}, \nu<0,\beta>0,\end{cases} \tag{15}$$

where $g^k(\nu) = \sum\limits_{\iota=0}^{n} a_\iota^k \nu^\iota$, $k = 1,2,3,4$. The Equation $(15)|_{\varepsilon=0}$ leads to

$$H(\nu,\beta) = \frac{1}{2}\beta^2 + \frac{1}{2}\nu^2 + \frac{1}{4}\nu^4 = \xi, \xi \in (0,+\infty). \tag{16}$$

Figure 3 depicts that (0, 0) was called the epicenter, implementing Equation (14) and the PF mapping, a higher boundary for the LCNs bifurcating based on the around of the annulus period for Equation $(15)|_{\varepsilon=0}$'s origin.

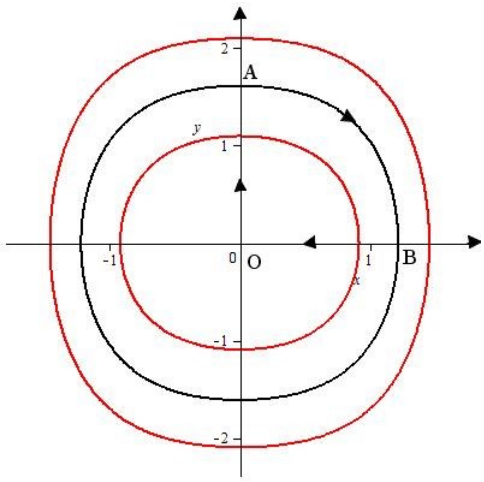

Figure 3. The state sketch of Equation $(15)|_{\varepsilon=0}$.

The following theorem is the major contribution of this article.

Theorem 1. *Considering (15) with a small enough $|\varepsilon|$ and using (14), a higher boundary for the LCNs bifurcated based on the annulus period for (16) $|_{\varepsilon=0}$'s origin equals $8 + \left[\frac{n}{4}\right] + 5\left[\frac{n-1}{4}\right] + 2\left[\frac{n-2}{4}\right]$, when $n \geq 2$; and 2 if $n = 1$; and 0 when $n = 0$.*

3. M(ξ) Represented Algebraically through Mathematical Analysis

For $\xi \in (0,+\infty)$, we have

$$I_{\iota,\kappa}(\xi) = \int_{L_\xi^1} \nu^\iota \beta^\iota d\nu,\ \iota,\kappa \in \mathbb{N}.$$

Based on Proposition 1, a MM whose order is 1 for Equation (15) is obtained as follows

$$\begin{aligned} M(\xi) &= \int\limits_{L_\xi^1} \beta g^1(\nu)d\nu + \int\limits_{L_\xi^2} \beta g^2(\nu)d\nu + \int\limits_{L_\xi^3} \beta g^3(\nu)d\nu + \int\limits_{L_\xi^4} \beta g^4(\nu)d\nu \\ &= \sum_{\iota=0}^{n} \left[a_\iota^1 \int\limits_{L_\xi^1} \nu^\iota \beta d\nu + a_\iota^2 \int\limits_{L_\xi^2} \nu^\iota \beta d\nu + a_\iota^3 \int\limits_{L_\xi^3} \nu^\iota \beta d\nu + a_\iota^4 \int\limits_{L_\xi^4} \nu^\iota \beta d\nu \right], \end{aligned} \tag{17}$$

where

$$\begin{aligned} L_\xi^1 &= \{(\nu,\beta)|H(\nu,\beta) = \xi, \xi \in (0,+\infty), \nu > 0, \beta > 0\}, \\ L_\xi^2 &= \{(\nu,\beta)|H(\nu,\beta) = \xi, \xi \in (0,+\infty), \nu > 0, \beta < 0\}, \\ L_\xi^3 &= \{(\nu,\beta)|H(\nu,\beta) = \xi, \xi \in (0,+\infty), \nu < 0, \beta < 0\}, \\ L_\xi^4 &= \{(\nu,\beta)|H(\nu,\beta) = \xi, \xi \in (0,+\infty), \nu < 0, \beta > 0\}. \end{aligned}$$

One also can write now

$$I_{\iota,3}(\xi) = I_{\iota,4}(\xi) = (-1)^\iota I_{\iota,1}(\xi), I_{\iota,2}(\xi) = I_{\iota,1}(\xi).$$

By conducting direct computations. Thus, Equation (17) could be rewritten as

$$M(\xi) = \sum_{\iota=0}^{n} \left[a_\iota^1 + a_\iota^2 + (-1)^\iota a_\iota^3 + (-1)^\iota a_\iota^4 \right] I_{\iota,1}(\xi) := \sum_{\iota=0}^{n} a_\iota I_{\iota,1}(\xi). \tag{18}$$

$M(\xi)$ having zeroes, where $\xi \in (0,+\infty)$, needs to be predicted. To this end, the derivation of the algebraic representation of $M(\xi)$ in Equation (18) is conducted.

Lemma 1. *For $\xi \in (0,+\infty)$, we have*

$$\begin{cases} I_{2l,1} = \tilde{\sigma}(\xi)I_{0,1}(\xi) + \tilde{\varsigma}(\xi)I_{2,1}(\xi),\ l \geq 2,\ n = 2l, \\ \quad I_{2l+1,1} = \tilde{\gamma}(\xi)I_{1,1}(\xi),\ l \geq 1,\ n = \ 2l+1, \end{cases} \tag{19}$$

where $\tilde{\sigma}(\xi)$, $\tilde{\varsigma}(\xi)$ and $\tilde{\gamma}(\xi)$ represent ξ-th degree polynomials having

$$deg\tilde{\varsigma}(\xi) \leq \left[\frac{n-2}{4}\right],\ deg\tilde{\sigma}(\xi) \leq \left[\frac{n}{4}\right],\ deg\tilde{\gamma}(\xi) \leq \left[\frac{n-1}{4}\right].$$

Proof. Differentiating Equation (16) regarding ν, one obtains

$$\beta\frac{\partial\beta}{\partial\nu} + \nu + \nu^3 = 0. \tag{20}$$

Multiplication of Equation (20) by $\nu^{\iota-3}\beta^\kappa d\nu$, integrating over L_ξ^1, one has

$$\int\limits_{L_\xi^1} \nu^{\iota-3}\beta^{\kappa+1}d\beta + \int\limits_{L_\xi^1} \nu^{\iota-2}\beta^\kappa d\nu + \int\limits_{L_\xi^1} \nu^\iota\beta^\kappa d\nu = 0.$$

Assume that $L_\xi = \{(\nu,\beta) : H(\nu,\beta) = \xi\}$ bisects positive ν and β-axes at the set of points denoted by A and B, respectively. Figure 3 depicts that Ω is supposed to be the interior of $L^1_\xi \bigcup \overrightarrow{BO} \bigcup \overrightarrow{OA}$. One gets for $\iota \geq 4$, the following

$$\begin{aligned}\int\limits_{L^1_\xi} \nu^{\iota-3}\beta^{\kappa+1}d\beta &= \oint\limits_{L^1_\xi \bigcup \overrightarrow{BO} \bigcup \overrightarrow{OA}} \nu^{\iota-3}\beta^{\kappa+1}d\beta - \int\limits_{\overrightarrow{BO}} \nu^{\iota-3}\beta^{\kappa+1}d\beta - \int\limits_{\overrightarrow{OA}} \nu^{\iota-3}\beta^{\kappa+1}d\beta \\ &= \oint\limits_{L^1_\xi \bigcup \overrightarrow{BO} \bigcup \overrightarrow{OA}} \nu^{\iota-3}\beta^{\kappa+1}d\beta = -(\iota-3)\iint\limits_{\Omega} \nu^{(\iota-4)}\beta^{\kappa+1}d\nu d\beta \\ &= -\frac{\iota-3}{\kappa+2}\oint\limits_{L^1_\xi \bigcup \overrightarrow{BO} \bigcup \overrightarrow{OA}} \nu^{\iota-4}\beta^{\kappa+2}d\nu = -\frac{\iota-3}{\kappa+2}\int\limits_{L^1_\xi} \nu^{\iota-4}\beta^{\kappa+2}d\nu.\end{aligned}$$

Hence,

$$I_{\iota,\kappa} = \frac{\iota-3}{\kappa+2} I_{\iota-4,\kappa+2} - I_{\iota-2,\kappa}. \tag{21}$$

Multiplication of Equation (16) by $\nu^\iota\beta^{\kappa-2}d\nu$ and integrating over L^1_ξ leads to

$$I_{\iota,\kappa} = 2\xi I_{\iota,\kappa-2} - I_{\iota+2,\kappa-2} - \frac{1}{2}I_{\iota+4,\kappa-2}. \tag{22}$$

Equations (21) and (22) are finally reduced to

$$I_{\iota,\kappa} = \frac{2}{\iota+2\kappa+1}[2(\iota-3)\xi I_{\iota-4,\kappa} - (\iota+\kappa-1)I_{\iota-2,\kappa}]. \tag{23}$$

The first equality in (19) is proven with no loss of generality. Given (23), one has

$$I_{6,1} = \frac{4}{3}\xi I_{2,1} - \frac{4}{3}I_{4,1}, I_{4,1} = \frac{4}{7}\xi I_{0,1} - \frac{8}{7}I_{2,1}. \tag{24}$$

The first equality in (19) is now proven via induction over l. So, (24) implied that it held for $l = 2, 3$. Suppose that equivalence held for $l \leq k-1$, $(k \geq 4)$. By using (23) one has for $l = k$ the following

$$I_{2k,1} = 2(3+2k)^{-1}[-2kI_{2k-2,1} + 2(2k-3)\xi I_{2k-4,1}]. \tag{25}$$

Using the induction hypothesis, one obtains the initial equivalence in (19). Now we have

$$\begin{aligned}I_{2k,\kappa}(\xi) &= \sigma^{(-2+2k)}(\xi)I_{0,1} + \varsigma^{(-2+2k)}(\xi)I_{2,1} + \xi\left[\sigma^{(2k-4)}(\xi)I_{0,1} + \varsigma^{(2k-4)}(\xi)I_{2,1}\right] \\ &:= \sigma^{(2k)}(\xi)I_{0,1} + \varsigma^{(2k)}(\xi)I_{2,1},\end{aligned}$$

where $\sigma^{(-2+2ks)}(\xi)$ and $\varsigma^{(-2+2ks)}(\xi)$ denote polynomials in ξ-th degree satisfying

$$deg\sigma^{(-2+2ks)}(\xi) \leq \left[\frac{k-s}{2}\right],\ deg\ \varsigma^{(-2+2ks)}(\xi) \leq \left[\frac{-1-s+k}{2}\right], s = 1,2.$$

Thus, $deg\sigma^{(2k)}(\xi) \leq \left[\frac{k}{2}\right]$, $deg\ \varsigma^{(k)}(\xi) \leq \left[\frac{k-1}{2}\right]$. This finishes the proof. □

Employing Lemma 1, one can attain the subsequent proposition directly.

Proposition 2. *For* $\xi \in (0, +\infty)$,

$$M(\xi) = \begin{cases} \sigma(\xi) I_{0,1}(\xi) + \varsigma(\xi) I_{2,1}(\xi) + \gamma(\xi) I_{1,1}(\xi), n \geq 2, \\ a_0 I_{0,1}(\xi) + a_1 I_{1,1}(\xi),\ n = 1, \\ a_0 I_{0,1}(\xi), n = 0, \end{cases} \tag{26}$$

where in a_0 *and* a_1 *denote fixed values and* $\sigma(\xi)$, $\varsigma(\xi)$ *and* $\gamma(\xi)$ *denote* ξ*-th degree polynomials having*

$$deg\sigma(\xi) \leq \left[\frac{n}{4}\right],\ deg\varsigma(\xi) \leq \left[\frac{n-2}{4}\right],\ deg\gamma(\xi) \leq \left[\frac{-1+n}{4}\right].$$

Proof. When $n \geq 3$ then the conclusion is derived directly by using Lemma 1. If $n = 2$, based on Equation (18) we obtain

$$M(\xi) = a_2 I_{2,1} + a_1 I_{1,1} + a_0 I_{0,1},$$

is attained, where the subscript ι, a_ι, taking 0, 1, 2 was a fixed value implying the conclusion be held. One can prove the conclusions for $n = 1, 2$ similarly. The proof is finished. □

Lemma 2. *Let* $(I_{0,1}, I_{2,1})^T$ *and* $I_{1,1}$ *satisfy the subsequent PF mappings*

$$\begin{pmatrix} I_{0,1} \\ I_{2,1} \end{pmatrix} = \begin{pmatrix} \frac{4}{3}\xi & -\frac{1}{3} \\ -\frac{4}{15}\xi & \frac{4}{5}\xi - \frac{4}{21} \end{pmatrix} \begin{pmatrix} I'_{0,1} \\ I'_{2,1} \end{pmatrix}, \tag{27}$$

and

$$I_{1,1} = \left(\xi + \frac{1}{4}\right) I'_{1,1}, \tag{28}$$

respectively.

Proof. From (16) one gets $\frac{\partial \beta}{\partial \nu} = \frac{1}{\beta}$, which implies

$$I'_{\iota,\kappa} = \kappa \int\limits_{L^1_\xi} \nu^\iota \beta^{\kappa-2} d\nu. \tag{29}$$

Thus,

$$I_{\iota,\kappa} = \frac{1}{\kappa+2} I'_{\iota,\kappa+2}. \tag{30}$$

Multiplication of both sides of Equation (29) by ξ leads to

$$\xi I'_{\iota,\kappa} = \frac{\kappa}{2(\kappa+2)} I'_{\iota,\kappa+2} + \frac{1}{2} I'_{\iota+2,\kappa} + \frac{1}{4} I'_{\iota+4,\kappa}. \tag{31}$$

Alternatively, for $\kappa \geq 1$

$$\begin{gathered} I_{\iota,\kappa} = \int\limits_{L^1_\xi} \nu^\iota \beta^\kappa d\nu = \oint\limits_{L^1_\xi \cup \overrightarrow{BO} \cup \overrightarrow{OA}} \nu^\iota \beta^\kappa d\nu - \int\limits_{\overrightarrow{BO}} \nu^\iota \beta^\kappa d\nu - \int\limits_{\overrightarrow{OA}} \nu^\iota \beta^\kappa d\nu = \oint\limits_{L^1_\xi \cup \overrightarrow{BO} \cup \overrightarrow{OA}} \nu^\iota \beta^\kappa d\nu = \\ -\frac{\kappa}{\iota+1} \int\limits_{L^1_\xi} \nu^{\iota+1} \beta^{\kappa-1} d\beta = \frac{\kappa}{\iota+1} \int\limits_{L^1_\xi} \nu^{\iota+1} \beta^{\kappa-1} \frac{\nu + \nu^3}{\beta} d\nu = \frac{1}{\iota+1}\left(I'_{\iota+2,\kappa} + I'_{\iota+4,\kappa}\right) \end{gathered}. \tag{32}$$

is attained. By (30)–(32), for $\kappa \geq 1$

$$I_{\iota,\kappa} = \frac{1}{\iota + 2\kappa + 1}\left(4h I'_{\iota,\kappa} - I'_{\iota+2,\kappa}\right), \tag{33}$$

is attained. It is implied that

$$I_{0,1} = \frac{4}{3}\xi I'_{0,1} - \frac{1}{3}I'_{2,1}, I_{1,1} = \xi I'_{1,1} - \frac{1}{4}I'_{3,1}, I_{2,1} = \frac{4}{5}\xi I'_{2,1} - \frac{1}{5}I'_{4,1}. \tag{34}$$

Moreover, ${I'}_{0,1}(\xi) \neq 0$ for $\xi \in (0,+\infty)$. Afterward, $\omega(\xi)$ fulfills the subsequent Riccati mapping

$$G(\xi)\omega'(\xi) = -\frac{1}{4}\omega^2(\xi) + 2\left(\xi - \frac{2}{7}\right)\omega(\xi) + \xi, \tag{35}$$

where $G(\xi) = 7^{-1}(-9+28\xi)\xi$. Now by considering (27), we have

$$G(\xi)\begin{pmatrix} I''_{0,1} \\ I''_{2,1} \end{pmatrix} = \begin{pmatrix} -\xi + \frac{4}{7} & \frac{1}{4} \\ \xi & \xi \end{pmatrix}\begin{pmatrix} I'_{0,1} \\ I'_{2,1} \end{pmatrix}. \tag{36}$$

where $G(\xi) = 7^{-1}(28\xi - 9)\xi$. Based on (36), Relation (35) is attained, which finishes the proof. □

4. Establishing Theorem 1

Assume that $\#\{\varphi(\xi) = 0, \xi \in (\lambda_1, \lambda_2)\}$ denotes isolated zero numbers of $\varphi(\xi)$ on (λ_1, λ_2) reckoning with the multiplicity. From (28), $I_{1,1}(\xi) = c\left(\xi + \frac{1}{4}\right)$ is attained, where c denotes a real fixed value. Thus,

$$M(\xi) = \sigma(\xi)I_{0,1}(\xi) + \varsigma(\xi)I_{2,1}(\xi) + c\left(\xi + \frac{1}{4}\right)\gamma(\xi).$$

If $n \geq 2$, then based on Equations (26) and (27)

$$M'(\xi) = P_{[\frac{n}{4}]}(\xi)I'_{0,1}(\xi) + P_{[\frac{n-2}{4}]}(\xi)I'_{2,1}(\xi) + P_{[\frac{n-1}{4}]}(\xi),$$

is attained, where $P_k(\xi)$ represents *a* ξ-th degree polynomial having at most k. Note that Equation (36) and some computations lead to $\xi \in (0,+\infty)$ and $\xi \neq \frac{9}{28}$

$$M^{([\frac{n-1}{4}]+2)}(\xi) = \frac{1}{G^{([\frac{n-1}{4}]+1)}(\xi)}\left[\Phi_{[\frac{n}{4}]+[\frac{n-1}{4}]+1}(\xi)I'_{0,1}(\xi) + \Psi_{[\frac{n-1}{4}]+[\frac{n-2}{4}]+1}(\xi)I'_{2,1}(\xi)\right].$$

where $\Phi_{[\frac{n}{4}]+[\frac{n-1}{4}]+1}(\xi)$ (resp. $\Psi_{[\frac{n-1}{4}]+[\frac{n-2}{4}]+1}(\xi)$) represents an ξ-th degree polynomial with at most $[\frac{n}{4}] + \left[\frac{n-1}{4}\right] + 1$ (resp. $\left[\frac{n-1}{4}\right] + [\frac{n-2}{4}] + 1$). Thus, for $\xi \in (0,+\infty)$ and $\xi \neq \frac{9}{28}$,

$$M^{([\frac{n-1}{4}]+2)}(\xi) = \frac{I'_{0,1}(\xi)}{G^{([\frac{n-1}{4}]+1)}(\xi)}\left[\Phi_{[\frac{n}{4}]+[\frac{n-1}{4}]+1}(\xi) + \Psi_{[\frac{n-1}{4}]+[\frac{n-2}{4}]+1}(\xi)\omega(\xi)\right].$$

Hence, for $\xi \in (0,+\infty)$ the following relation

$$\#\left\{M^{([\frac{n-1}{4}]+2)}(\xi) = 0\right\} = \#\left\{\Phi_{[\frac{n}{4}]+[\frac{n-1}{4}]+1}(\xi) + \Psi_{[\frac{n-1}{4}]+[\frac{n-2}{4}]+1}(\xi)\omega(\xi) = 0\right\} + \left[\frac{n-1}{4}\right] + 1. \tag{37}$$

is attained. Subsequently, the number of zeros of $\Phi_{[\frac{n}{4}]+[\frac{n-1}{4}]+1}(\xi) + \Psi_{[\frac{n-1}{4}]+[\frac{n-2}{4}]+1}(\xi)\omega(\xi)$ on $(0,+\infty)$ will be estimated.

Let $\chi(\xi) = \Phi_{[\frac{n}{4}]+[\frac{n-1}{4}]+1}(\xi) + \Psi_{[\frac{n-1}{4}]+[\frac{n-2}{4}]+1}(\xi)\omega(\xi)$. Based on Equation (35), the following relation is obtained:

$$G(\xi)\Psi_{[\frac{n-1}{4}]+[\frac{n-2}{4}]+1}(\xi)\chi'(\xi) = -\frac{1}{4}\chi^2(\xi) + P_{[\frac{n-1}{4}]+[\frac{n-2}{4}]+2}(\xi)\,\chi(\xi) + P_{[\frac{n}{4}]+2[\frac{n-1}{4}]+[\frac{n-2}{4}]+3}(\xi).$$

Now, considering Lemma 2 and a similar reasoning as in [38] for $\xi \in (0, +\infty)$, we obtain

$$\#\{\chi(\xi) = 0\} \leq \#\left\{\Psi_{[\frac{n-1}{4}]+[\frac{n-2}{4}]+1}(\xi) = 0\right\} + \#\left\{P_{[\frac{n}{4}]+2[\frac{n-1}{4}]+[\frac{n-2}{4}]+3}(\xi) = 0\right\} + 1 \leq \left[\frac{n}{4}\right] + 3\left[\frac{n-1}{4}\right] + 2\left[\frac{n-2}{4}\right] + 5. \quad (38)$$

Hence, from (37) and (38), for $\xi \in (0, +\infty)$, the following relation:

$$\#\{M(\xi) = 0\} \leq \left[\frac{n}{4}\right] + 5\left[\frac{n-1}{4}\right] + 2\left[\frac{n-2}{4}\right] + 8,$$

is attained. If $n = 1$, then from (18), we attain

$$M(\xi) = a_0 I_{0,1}(\xi) + a_1 I_{1,1}(\xi) = a_0 I_{0,1}(\xi) + a_1 c\left(\xi + \frac{1}{4}\right).$$

Based on Equation (23), $I_{0,1}(\xi) = \frac{4}{3}hI'_{0,1}(\xi)$ is attained. Hence, $M''(\xi) = a_0 I''_{0,1}(\xi) = -\frac{a_0}{4\xi}I'_{0,1}(\xi)$. Noting that $I'_{0,1}(\xi) \neq 0$ for $\xi \in (0, +\infty)$, $M(\xi)$ contains at most two 0s in $(0, +\infty)$. When $n = 0$, an easy computation gives $M(\xi) = a_0 I_{0,1}(\xi)$. Because

$$I_{0,1}(\xi) = \int_{L^1_\xi} \beta dv = \oint_{L^1_\xi \cup \overrightarrow{BO} \cup \overrightarrow{OA}} \beta dv - \int_{\overrightarrow{BO}} \beta dv - \int_{\overrightarrow{OA}} \beta dv,$$
$$= \oint_{L^1_\xi \cup \overrightarrow{BO} \cup \overrightarrow{OA}} \beta dv = \iint_{\Omega} dv d\beta \neq 0,$$

where Ω is the interior of $L^1_\xi \cup \overrightarrow{BO} \cup \overrightarrow{OA}$. Therefore, $M(\xi)$ does not contain zero in $(0, +\infty)$. The proof finishes now.

We investigated LC bifurcations in systems composed of PSD mappings, specifically focusing on LCNs within a PSDS consisting of four regions. We employed a Melnikov mapping of order one to analyze the system's behavior when perturbed by nth-degree real polynomials. This approach provides valuable insights into the bifurcation behavior of PSDS systems with a global center. The practical implications of this research extend to various applications where understanding LCs is critical. For instance, in the study of dynamical systems, these findings can help predict and control oscillatory behavior, which is prevalent in diverse fields, including engineering, biology, and physics. By determining the LCNs around critical points in such systems, engineers can make informed decisions to optimize performance and stability. The mathematical representation of the Melnikov function and the application of recurrence formulas further contribute to a comprehensive understanding of the system's behavior. It is important to note that the higher boundary derived from Theorem 1 is not always optimal. This suggests that there may be room for further refinement and exploration, particularly in determining the lower boundary for LCNs.

5. Conclusions

In the article, a system composed of planar third-order Liénard mappings with a global epicenter having at most

$$\left[\frac{n}{4}\right] + 5\left[\frac{n-1}{4}\right] + 2\left[\frac{n-2}{4}\right] + 8,$$

LCs around the critical point by utilizing a first-order Melnikov function containing four zones is shown. Recurrence formulas have been applied to get the comprehensive mathematical representation of a MM whose order is 1 that could be stated as combinations of $I_{0,1}(\xi)$, $I_{1,1}(\xi)$ and $I_{2,1}(\xi)$ with polynomial coefficients. Afterward, bifurcating LCNs near the critical point by employing PF and the Riccati equation is determined. The higher boundary given by Theorem 1 is not optimal generally. The lesser boundary for LCNs could not be given because verifying the coefficient independence for the polynomials of $I_{0,1}(\xi)$, $I_{1,1}(\xi)$ and $I_{2,1}(\xi)$ has been difficult, which will be the future direction of the conducted research.

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