

Article

Fuzzy Differential Subordination Associated with a General Linear Transformation

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Abstract: In this study, we investigate a possible relationship between fuzzy differential subordination and the theory of geometric functions. First, using the Al-Oboudi differential operator and the Babalola convolution operator, we establish the new operator $\mathcal{BS}_{\alpha,\lambda}^{m,t} : A_n \rightarrow A_n$ in the open unit disc U . The second step is to develop fuzzy differential subordination for the operator $\mathcal{BS}_{\alpha,\lambda}^{m,t}$. By considering linear transformations of the operator $\mathcal{BS}_{\alpha,\lambda}^{m,t}$, we define a new fuzzy class of analytic functions in U which we denote by $T_F^{\lambda,t}(m, \alpha, \delta)$. Several innovative results are found using the concept of fuzzy differential subordination and the operator $\mathcal{BS}_{\alpha,\lambda}^{m,t}$ for the function f in the class $T_F^{\lambda,t}(m, \alpha, \delta)$. In addition, we explore a number of examples and corollaries to illustrate the implications of our key findings. Finally, we highlight several established results to demonstrate the connections between our work and existing studies.

Keywords: linear transformation; fuzzy differential subordination; fuzzy set; analytic functions; Al-Oboudi differential operator; Babalola convolution operator

MSC: 30C45; 30C50



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1. Introduction and Definitions

The history of fuzzy sets theory began in 1965 with the publication of “Fuzzy Sets” [1] by Zadeh, which was first received with distrust but is now mentioned in more than 95,000 publications. Many links between fuzzy sets theory and other areas of mathematics have been developed due to the widespread interest in this topic among mathematicians. The excellent review article [2] from 2017 is a dedication to Zadeh’s work and explains how the fuzzy sets concept has developed over time and how it is connected to many various areas of mathematics, science, and technology. This issue celebrates the centennial of Zadeh birth with a number of excellent review articles, including one [3] that provides background on the evolution of fuzzy sets theory and shines a light on the work of Dzitac, a former student and colleague of Zadeh. In 2008, he collaborated on a book [4] with Zadeh, forever linking both of their names.

One of the most recent research techniques in the theory of single complex variable functions is the differential subordination method. It was investigated in [5] and introduced by Miller and Mocanu in [6,7]. This technique allows novel findings to be rapidly acquired

while simultaneously presenting certain well-established outcomes in the field. One of the more common results of the differential subordination approach is differential inequalities. Numerous papers and monographs on the theory of single functions of complex variables have been published as a direct consequence of the advancement of this method.

According to [8], “Knowing the properties of differential expression for a function, we can determine the properties of that function on a given interval.” This is the rationale behind the development of the differential subordination theory. In publishing their works [8,9], the authors intended to establish a new line of inquiry in mathematics by merging concepts from the domain of complex functions with those from fuzzy sets theory. As previously stated, the authors support their claim that a function’s characteristics can be ascertained on a certain fuzzy set by understanding the characteristics of a differential expression on that set. The case of actual functions has been left as an “open problem” by the authors, who only examined the case of a single complex function.

Fuzzy subordination was first mentioned in [8]. The concept of fuzzy differential subordination has been defined in [9]. The fuzzy differential subordination produced by the differential operator was studied in [10–12].

This kind of research is crucial for improving our comprehension of the relationships between various mathematical ideas and for creating new tools and approaches to solve mathematical difficulties.

Motivated by the studies of [8,9], our aim in this paper is to establish properties of differential subordination and fuzzy differential subordination associated with linear combinations of the Al-Oboudi differential operator and the Babalola convolution operator as defined in the open unit disc.

We refer to the set of all analytic functions (AFs) f in $U = \{\tau \in \mathbb{C} : |\tau| < 1\}$ as $\mathcal{H}(U)$ and to the class of all normalized analytic functions as A , ($A_1 = A$). The Taylor series for each $f \in A_n$ is of the following form:

$$f(\tau) = \tau + b_{n+1}\tau^{n+1} + \dots, \tau \in U.$$

When $b \in \mathbb{C}$ and $n \in \mathbb{N}^* = \mathbb{N} \cup \{0\}$, we write

$$\mathcal{H}[b, n] = \left\{ f \in \mathcal{H}(U) : f(\tau) = b + b_n\tau^n + b_{n+1}\tau^{n+1} + \dots, \tau \in U \right\}.$$

The family of all convex functions of order α for $0 \leq \alpha < 1$ is represented by $C(\alpha)$, and is defined as

$$C(\alpha) = \left\{ f \in A : \operatorname{Re} \left(1 + \frac{\tau f''(\tau)}{f'(\tau)} \right) > \alpha \right\}.$$

When $\alpha = 0$, then the class C of convex functions is obtained.

We subsequently discuss the background works that generate the notion of fuzzy differential subordinations and their corresponding definitions.

Definition 1 ([1]). Let Y be a non-empty set, let $F_L : Y \rightarrow [0, 1]$, and let

$$L = \{x \in Y : 0 < F_L(x) \leq 1\}.$$

Then, a pair (L, F_L) is a fuzzy subset of Y .

Remark 1. The function that determines membership in the fuzzy set (L, F_L) is termed F_L , and the set L is known as the support of the fuzzy set (L, F_L) . In addition, it is possible to indicate that

$$L = \operatorname{Supp}(L, F_L). \quad (1)$$

Remark 2. Suppose that $L \subset Y$; then,

$$F_L(x) = \begin{cases} 1, & \text{if } x \in L \\ 0, & \text{if } x \notin L. \end{cases}$$

Definition 2 ([13]). Let $U \subset \mathbb{C}$. For a fixed point, let $\tau_0 \in U$ and let the functions $f, g \in \mathcal{H}(U)$. Then, we can say that f is fuzzy subordinate to g and write

$$f \prec_{\mathcal{F}} g \text{ or } f(\tau) \prec_{\mathcal{F}} g(\tau) \quad (2)$$

if the following conditions are satisfied:

$$f(\tau_0) = g(\tau_0)$$

and

$$F_{f(U)}f(\tau) \leq F_{g(U)}g(\tau), \quad \tau \in U.$$

Definition 3 ([6]). Let us say that $\psi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ and that

$$\psi(b, 0; 0) = b.$$

Let h be univalent in U with $h(0) = b$. If φ is analytic in U with $\varphi(0) = b$ and satisfies the second-order fuzzy differential subordination

$$F_{\psi(\mathbb{C}^3 \times U)}\psi\left(\varphi(\tau), \tau\varphi'(\tau), \tau^2\varphi''(\tau); \tau\right) \leq F_{h(U)}h(\tau), \quad \tau \in U, \quad (3)$$

then φ is referred to as a fuzzy solution of the fuzzy differential subordination.

Remark 3. Any univalent function q satisfying (3) is called fuzzy dominant with respect to the fuzzy solutions of the fuzzy differential subordination

$$F_{\varphi(U)}\varphi(\tau) \leq F_{q(U)}q(\tau), \quad \tau \in U.$$

Then, the fuzzy dominant $\tilde{q}$ that satisfies

$$F_{\tilde{q}(U)}\tilde{q}(\tau) \leq F_{q(U)}q(\tau), \quad \tau \in U$$

is referred as the fuzzy best dominant for all fuzzy dominants of (3).

Real and complex order integrals and derivatives have shown promise in mathematical modeling and analysis of practical issues in the sciences, and this work has made an impact on the study of geometric functions. A novel model of the human liver [14] and an examination of the dynamics of dengue transmission [15] are only two examples of the kind of research that can be considered part of the aforementioned field; and see [16–19]. The family of integral operators connected to the first-kind Lommel functions was introduced in [20], and has important applications in both pure and applied mathematics. As a consequence of the existence of differential and integral operators, functional analysis and operator theory can be used in the study of differential equations. Here, we employ the characteristics of differential operators to solve differential equations using the operator technique; such operators may be involved in the solution of partial differential equations, although this needs more study. The Babalola convolution operator is well recognized for its attractive results in geometric function theory. Its nature and several of its distinguishing characteristics are described below.

Definition 4 ([21]). Let f be an analytic function in A . The Babalola convolution operator, denoted as B_t^m , is defined by

$$B_t^m f(\tau) = (\Psi_{m,t} * \Psi_{m,t}^{(-1)} * f)(\tau),$$

where

$$\Psi_{m,t} = \frac{\tau}{(1-\tau)^{m-t+1}}, \quad m-t+1 > 0, \text{ and } m, t \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$$

and where

$$(\Psi_{m,t} * \Psi_{m,t}^{(-1)} * f)(\tau) = \frac{\tau}{1-\tau}.$$

Equivalently,

$$B_t^m f(\tau) = \tau + \frac{m+1}{m-t+1} b_2 \tau^2 + \frac{(m+1)(m+2)}{(m-t+1)(m-t+2)} b_3 \tau^3 + \dots \quad (4)$$

From (4), we have

$$B_t^m f(\tau) = \tau + \sum_{n=2}^{\infty} \left(\frac{[m+n-1]!}{m!} \right) \left(\frac{[m-t]!}{[m+n-t-1]!} \right) b_n \tau^n. \quad (5)$$

Remark 4. $B_0^0 f(\tau) = f(\tau)$, $B_1^1 f(\tau) = \tau f'(\tau)$; further, $B_m^m f(\tau) = \mathcal{R}^m f(\tau)$, as introduced by Ruscheweyh [22].

Remark 5. If $f \in A_n$ and if

$$f(\tau) = \tau + \sum_{j=n+1}^{\infty} b_j \tau^j,$$

then

$$\begin{aligned} B_t^m f(\tau) &= \tau + \sum_{j=n+1}^{\infty} \left(\frac{[m+j-1]!}{m!} \right) \left(\frac{[m-t]!}{[m+j-t-1]!} \right) b_j \tau^j \\ &= \tau + \sum_{j=n+1}^{\infty} C_{m+j-1,t}^m(n) b_n \tau^n, \end{aligned}$$

where

$$C_{m+j-1,t}^m(n) = \left(\frac{[m+j-1]!}{m!} \right) \left(\frac{[m-t]!}{[m+j-t-1]!} \right).$$

The Al-Oboudi differential operator, studied in [23], is a generalization of the Salagean differential operator.

Definition 5. For $\lambda \geq 0$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $f \in A$, the operator $S_\lambda^m: A \rightarrow A$, is defined by

$$\begin{aligned} S_\lambda^0 f(\tau) &= f(\tau), \\ S_\lambda^1 f(\tau) &= (1-\lambda)f(\tau) + \lambda\tau f'(\tau) = S_\lambda f(\tau) \\ &\dots \\ S_\lambda^m f(\tau) &= (1-\lambda)S_\lambda^{m-1} f(\tau) + \lambda\tau \left(S_\lambda^{m-1} f(\tau) \right)' = S_\lambda(S_\lambda^{m-1} f(\tau)). \end{aligned}$$

After a few simple calculations, we have

$$S_\lambda^m f(\tau) = \tau + \sum_{n=2}^{\infty} \{\lambda(n-1) + 1\}^m b_n \tau^n. \quad (6)$$

Remark 6. $S_\lambda^0 f(\tau) = f(\tau)$, $S_\lambda^1 f(\tau) = \tau f'(\tau)$, $S_\lambda^{m+1} f(\tau) = \tau (S_\lambda^m f(\tau))'$, $\tau \in U$.

Remark 7. If $f \in A_n$ and

$$f(\tau) = \tau + \sum_{j=n+1}^{\infty} b_j \tau^j,$$

then

$$S_{\lambda}^m f(\tau) = \tau + \sum_{j=n+1}^{\infty} \{\lambda(j-1) + 1\}^m b_j \tau^j.$$

The operator that is utilized to obtain the original results of this study is defined in the following.

Definition 6. Let $\alpha \geq 0$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $n \in \mathbb{N}$, and denote by $\mathcal{BS}_{\alpha, \lambda}^{m, t}$ the operator provided by $\mathcal{BS}_{\alpha, \lambda}^{m, t} : A_n \rightarrow A_n$:

$$\mathcal{BS}_{\alpha, \lambda}^{m, t} f(\tau) = (1 - \alpha) B_t^m f(\tau) + \alpha S_{\lambda}^m f(\tau).$$

Remark 8. When $t = m$ and $\lambda = 1$, then $\mathcal{BS}_{\alpha, \lambda}^m = L_{\alpha}^m$, as introduced in [24].

Remark 9. If $f \in A_n$ and

$$f(\tau) = \tau + \sum_{j=n+1}^{\infty} b_j \tau^j,$$

then

$$\mathcal{BS}_{\alpha, \lambda}^{m, t} f(\tau) = \tau + \sum_{j=n+1}^{\infty} \left(\alpha \{\lambda(j-1) + 1\}^m + (1 - \alpha) C_{m+j-1}^m \right) b_j \tau^j, \quad \tau \in U. \quad (7)$$

Remark 10. If $\alpha = 0$, then $\mathcal{BS}_{0, \lambda}^{m, t} f(\tau) = B_t^m f(\tau)$, while for $\alpha = 1$ we have $\mathcal{BS}_{1, \lambda}^{m, t} f(\tau) = S_{\lambda}^m f(\tau)$.

Remark 11. For $\lambda = t = m = 0$, then $\mathcal{BS}_{\alpha, 0}^{0, 0} f(\tau) = (1 - \alpha) B_0^0 f(\tau) + \alpha S_0^0 f(\tau) = f(\tau) = B_0^0 f(\tau) = S_0^0 f(\tau)$.

Remark 12. For $t = m = 1$ and $\lambda = 1$ in (7), we have

$$\begin{aligned} \mathcal{BS}_{\alpha, 1}^{1, 1} f(\tau) &= (1 - \alpha) B_1^1 f(\tau) + \alpha S_1^1 f(\tau) \\ &= (1 - \alpha) \tau f'(\tau) + \alpha \tau f'(\tau) \\ &= \tau f'(\tau) = B_1^1 f(\tau) = S_1^1 f(\tau), \quad \tau \in U. \end{aligned}$$

Definition 7 ([25]). Let

$$\begin{aligned} f(U) &= \sup \left(f(U), F_{f(U)} \right) \\ &= \left\{ \tau \in U : 0 < F_{f(U)} f(\tau) \leq 1 \right\}, \end{aligned}$$

where $f(U)$ is the membership function for the fuzzy set $F_{f(U)}$, and is connected to the function f . The membership function of the fuzzy set $(f + g)(U)$ connected to the function $f + g$ coincides with the half of the sum of the membership functions of the fuzzy set $f(U)$, that is,

$$F_{(f+g)(U)}(f + g)(\tau) = \frac{F_{f(U)} f(\tau) + F_{g(U)} g(\tau)}{2}, \quad \tau \in U.$$

Remark 13. Let $0 < F_{f(U)} f(\tau) \leq 1$ and let $0 < F_{g(U)} g(\tau) \leq 1$; then, it is obvious that $0 < F_{(f+g)(U)}((f + g)(\tau)) \leq 1$, $\tau \in U$.

First, using the operator provided by the definition above, a novel class of fuzzy analytic functions is defined.

Definition 8. Let the function $f \in A_n$ be contained in the class $T_F^{\lambda,t}(m, \alpha, \delta)$ if

$$F_{(\mathcal{BS}_{\alpha,\lambda}^{m,t}f)'(U)} \left(\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau) \right)' > \delta, \quad \tau \in U, \quad (8)$$

where $\delta \in (0, 1]$, $\alpha \geq 0$, $m \in \mathbb{N}_0$, and $n \in \mathbb{N}$.

This study follows a notable current trend in the study of fuzzy differential subordination, namely, the creation and study of new fuzzy classes of functions using new operators. Based on the recently discovered linear differential operator $\mathcal{BS}_{\alpha,\lambda}^{m,t}$, a novel class $T_F^{\lambda,t}(m, \alpha, \delta)$ of fuzzy differential subordinations is generated in Section 1. In Section 2, we provide the known lemmas that establish our main results. The main results of the paper are presented in Section 3. In this section, we prove the convexity of the newly formed class and obtain fuzzy differential subordination via the operator $\mathcal{BS}_{\alpha,\lambda}^{m,t}$. These primary findings provide interesting corollaries, including the fuzzy best dominants for the investigated fuzzy differential subordination. We provide several examples to illustrate the value of these new results. In the last portion, we provide our final remarks.

2. Preliminaries

To prove our main results, we apply the following lemmas.

Lemma 1 ([6]). Suppose that $h \in A_n$; then,

$$\begin{aligned} L[f](\tau) &= F(\tau) \\ &= \frac{1}{n\tau^{\frac{1}{n}}} \int_0^\tau h(t)t^{\frac{1}{n}-1}dt, \quad \tau \in U. \end{aligned}$$

If

$$\operatorname{Re} \left(\frac{\tau h''(\tau)}{h'(\tau)} + 1 \right) > -\frac{1}{2}, \quad \tau \in U,$$

then $L(f) = F \in C$.

Lemma 2 ([26]). Suppose that $\gamma \in \mathbb{C}^*$ is a complex number, $\operatorname{Re} \gamma \geq 0$, and h is a convex function with $h(0) = b$; then, if $\varphi \in \mathcal{H}[b, n]$ with $\varphi(0) = b$, $\psi : \mathbb{C}^2 \times U \rightarrow \mathbb{C}$,

$$\psi(\varphi(\tau), \tau\varphi'(\tau); \tau) = \varphi(\tau) + \frac{1}{\gamma}\tau\varphi'(\tau),$$

an analytic function in U , and

$$\begin{aligned} F_{\psi(\mathbb{C}^2 \times U)} \left(\varphi(\tau) + \frac{1}{\gamma}\tau\varphi'(\tau) \right) &\leq F_{h(U)}h(\tau), \\ \text{i.e., } \varphi(\tau) + \frac{1}{\gamma}\tau\varphi'(\tau) &\prec_{\mathcal{F}} h(\tau), \quad \tau \in U, \end{aligned} \quad (9)$$

then

$$\begin{aligned} F_{\varphi(U)}\varphi(\tau) &\leq F_{g(U)}g(\tau) \leq F_{h(U)}h(\tau), \\ \text{i.e., } \varphi(\tau) &\prec_{\mathcal{F}} g(\tau) \prec_{\mathcal{F}} h(\tau), \quad \tau \in U, \end{aligned}$$

meaning that

$$g(\tau) = \frac{\gamma}{n\tau^{\gamma/n}} \int_0^\tau h(t)t^{\gamma/n-1}dt, \quad \tau \in U$$

is the fuzzy best dominant and is convex.

Lemma 3 ([26]). Suppose that g represents a convex function in U ; moreover, suppose that

$$h(\tau) = g(\tau) + n\alpha\tau g'(\tau), \tau \in U,$$

where $\alpha > 0$ and $n \in \mathbb{Z}^+$.

Let

$$\varphi(\tau) = g(0) + \varphi_n\tau^n + \varphi_{n+1}\tau^{n+1} + \dots, \quad \tau \in U,$$

be analytic in U , and

$$F_{\varphi(U)}(\varphi(\tau) + \alpha\tau\varphi'(\tau)) \leq F_{h(U)}h(\tau),$$

that is,

$$\varphi(\tau) + \alpha\tau\varphi'(\tau) \prec \mathcal{F}h(\tau), \quad \tau \in U.$$

Then,

$$F_{\varphi(U)}\varphi(\tau) \leq F_{g(U)}g(\tau),$$

that is,

$$\varphi(\tau) \prec_{\mathcal{F}} g(\tau), \quad \tau \in U,$$

and this result is sharp.

3. Main Results

Theorem 1. The set $T_F^{\lambda,t}(m, \alpha, \delta)$ is convex.

Proof. Consider the functions

$$f_j(\tau) = \tau + \sum_{j=n+1}^{\infty} b_{jk}\tau^j \in T_F^{\lambda,t}(m, \alpha, \delta).$$

To approach the necessary conclusion, the function

$$h(\tau) = \mu_1 f_1(\tau) + \mu_2 f_2(\tau) \tag{10}$$

must belong to the class $T_F^{\lambda,t}(m, \alpha, \delta)$ with $\mu_1, \mu_2 \in \mathbb{Z}^+$ such that $\mu_1 + \mu_2 = 1$. Next, we show that $h \in T_F^{\lambda,t}(m, \alpha, \delta)$. Taking the derivative of (10), we have

$$h'(\tau) = (\mu_1 f_1(\tau) + \mu_2 f_2(\tau))'(\tau) = \mu_1 f_1'(\tau) + \mu_2 f_2'(\tau)$$

and

$$\begin{aligned} & \left(\mathcal{BS}_{\alpha,\lambda}^{m,t} h(\tau) \right)' \\ &= \left(\mathcal{BS}_{\alpha,\lambda}^{m,t} (\mu_1 f_1(\tau) + \mu_2 f_2(\tau)) \right)'(\tau) = \mu_1 \left(\mathcal{BS}_{\alpha,\lambda}^{m,t} f_1(\tau) \right)' + \mu_2 \left(\mathcal{BS}_{\alpha,\lambda}^{m,t} f_2(\tau) \right)'. \end{aligned}$$

From Definition 7, we have

$$\begin{aligned}
 & F_{(\mathcal{BS}_{\alpha,\lambda}^{m,t}h)'(U)} \left(\mathcal{BS}_{\alpha,\lambda}^{m,t}h(\tau) \right)' \\
 = & F_{((\mathcal{BS}_{\alpha,\lambda}^{m,t}(\mu_1 f_1 + \mu_2 f_2))'(U))} \left(\mathcal{BS}_{\alpha,\lambda}^{m,t}(\mu_1 f_1 + \mu_2 f_2)(\tau) \right)' \\
 = & F_{((\mathcal{BS}_{\alpha,\lambda}^{m,t}(\mu_1 f_1 + \mu_2 f_2))'(U))} \left(\mu_1 \left(\mathcal{BS}_{\alpha,\lambda}^{m,t}f_1(\tau) \right)' + \mu_2 \left(\mathcal{BS}_{\alpha,\lambda}^{m,t}f_2(\tau) \right)' \right) \\
 = & \frac{F_{(\mu_1 (\mathcal{BS}_{\alpha,\lambda}^{m,t}f_1(\tau))'(U))} \left(\mu_1 \left(\mathcal{BS}_{\alpha,\lambda}^{m,t}f_1(\tau) \right)' \right) + F_{(\mu_2 (\mathcal{BS}_{\alpha,\lambda}^{m,t}f_2(\tau))'(U))} \left(\mu_2 \left(\mathcal{BS}_{\alpha,\lambda}^{m,t}f_2(\tau) \right)' \right)}{2} \\
 = & \frac{F_{(\mathcal{BS}_{\alpha,\lambda}^{m,t}f_1(\tau))'(U)} \left(\left(\mathcal{BS}_{\alpha,\lambda}^{m,t}f_1(\tau) \right)' \right) + F_{(\mathcal{BS}_{\alpha,\lambda}^{m,t}f_2(\tau))'(U)} \left(\mathcal{BS}_{\alpha,\lambda}^{m,t}f_2(\tau) \right)'}{2}.
 \end{aligned}$$

If $f_1, f_2 \in T_F^{\lambda,t}(m, \alpha, \delta)$, then

$$\delta < F_{(\mathcal{BS}_{\alpha,\lambda}^{m,t}f_1)'(U)} \left(\mathcal{BS}_{\alpha,\lambda}^{m,t}f_1(\tau) \right)' \leq 1.$$

Furthermore,

$$\delta < F_{(\mathcal{BS}_{\alpha,\lambda}^{m,t}f_2)'(U)} \left(\mathcal{BS}_{\alpha,\lambda}^{m,t}f_2(\tau) \right)' \leq 1, \quad \tau \in U.$$

Therefore,

$$\delta < \frac{F_{(\mathcal{BS}_{\alpha,\lambda}^{m,t}f_1)'(U)} \left(\mathcal{BS}_{\alpha,\lambda}^{m,t}f_1(\tau) \right)' + F_{(\mathcal{BS}_{\alpha,\lambda}^{m,t}f_2)'(U)} \left(\mathcal{BS}_{\alpha,\lambda}^{m,t}f_2(\tau) \right)'}{2} \leq 1.$$

Thus, we obtain

$$\delta < F_{(\mathcal{BS}_{\alpha,\lambda}^{m,t}h)'(U)} \left(\mathcal{BS}_{\alpha,\lambda}^{m,t}h(\tau) \right)' \leq 1,$$

which means that $h \in T_F^{\lambda,t}(m, \alpha, \delta)$ and $T_F^{\lambda,t}(m, \alpha, \delta)$ is convex. $\square$

Theorem 2. Suppose that g is a convex function in U and is defined as

$$h(\tau) = g(\tau) + \frac{1}{c+2} \tau g'(\tau)$$

with $c > 0, \tau \in U$. Moreover, let $f \in T_F^{\lambda,t}(m, \alpha, \delta)$ and

$$G(\tau) = I_c(f)(\tau) = \frac{c+2}{\tau^{c+1}} \int_0^\tau t^c f(t) dt, \quad \tau \in U.$$

Then, the fuzzy differential subordination

$$\begin{aligned}
 F_{(\mathcal{BS}_{\alpha,\lambda}^{m,t}f)'(U)} \left(\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau) \right)' & \leq F_{h(U)} h(\tau), \\
 \text{i.e., } \left(\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau) \right)' & \prec_{\mathcal{F}} h(\tau), \quad \tau \in U,
 \end{aligned} \tag{11}$$

implies that

$$\begin{aligned}
 F_{(\mathcal{BS}_{\alpha,\lambda}^{m,t}G)'(U)} \left(\mathcal{BS}_{\alpha,\lambda}^{m,t}G(\tau) \right)' & \leq F_{g(U)} g(\tau), \\
 \text{i.e., } \left(\mathcal{BS}_{\alpha,\lambda}^{m,t}G(\tau) \right)' & \prec_{\mathcal{F}} g(\tau), \quad \tau \in U,
 \end{aligned} \tag{12}$$

and this result is sharp.

Proof. As a consequence of our definition of the function $G(\tau)$, we have

$$\tau^{c+1}G(\tau) = (c+2) \int_0^\tau t^c f(t) dt. \quad (13)$$

Differentiating Equation (13) with respect to τ , we obtain

$$(c+1)G(\tau) + \tau G'(\tau) = (c+2)f(\tau)$$

and have

$$\begin{aligned} & (c+1)\mathcal{BS}_{\alpha,\lambda}^{m,t}G(\tau) + \tau \left(\mathcal{BS}_{\alpha,\lambda}^{m,t}G(\tau) \right)' \\ &= (c+2)\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau), \tau \in U. \end{aligned} \quad (14)$$

Differentiating (14), we have

$$\left(\mathcal{BS}_{\alpha,\lambda}^{m,t}G(\tau) \right)' + \frac{1}{c+2} \tau \left(\mathcal{BS}_{\alpha,\lambda}^{m,t}G(\tau) \right)'' = \left(\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau) \right)', \tau \in U. \quad (15)$$

From Equation (15), the fuzzy differential subordination is

$$\begin{aligned} & F_{\mathcal{BS}_{\alpha,\lambda}^{m,t}G(U)} \left(\left(\mathcal{BS}_{\alpha,\lambda}^{m,t}G(\tau) \right)' + \frac{1}{c+2} \tau \left(\mathcal{BS}_{\alpha,\lambda}^{m,t}G(\tau) \right)'' \right) \\ & \leq F_{g(U)} \left(g(\tau) + \frac{1}{c+2} \tau g'(\tau) \right). \end{aligned} \quad (16)$$

Let

$$\varphi(\tau) = \left(\mathcal{BS}_{\alpha,\lambda}^{m,t}G(\tau) \right)', \tau \in U \quad (17)$$

and let $\varphi \in \mathcal{H}[1, n]$. By substituting (17) into (16), we obtain

$$F_{\varphi(U)} \left(\varphi(\tau) + \frac{1}{c+2} \tau \varphi'(\tau) \right) \leq F_{g(U)} \left(g(\tau) + \frac{1}{c+2} \tau g'(\tau) \right), \tau \in U.$$

Lemma 3 allows us to have

$$\begin{aligned} & F_{\varphi(U)} \varphi(\tau) \leq F_{g(U)} g(\tau), \\ \text{i.e., } & F_{(\mathcal{BS}_{\alpha,\lambda}^{m,t}G)'(U)} \left(\mathcal{BS}_{\alpha,\lambda}^{m,t}G(\tau) \right)' \leq F_{g(U)} g(\tau), \tau \in U. \end{aligned}$$

The most effective best dominant is g , meaning that we have

$$\left(\mathcal{BS}_{\alpha,\lambda}^{m,t}G(\tau) \right)' \prec_{\mathcal{F}} g(\tau), \tau \in U.$$

□

Example 1. Let $f \in T_F^{1,1}(1, \frac{1}{2}, 1)$; then,

$$f'(\tau) + \tau f''(\tau) \prec_{\mathcal{F}} \frac{3-2\tau}{3(1-\tau)^2}$$

and

$$G'(\tau) + \tau G''(\tau) \prec_{\mathcal{F}} \frac{1}{1-\tau}$$

with

$$G(\tau) = \frac{3}{\tau^2} \int_0^\tau t f(t) dt.$$

Theorem 3. Suppose that

$$h(\tau) = \frac{1 + (2\beta - 1)\tau}{1 + \tau}, \quad \beta \in [0, 1]$$

and let $m - t > -1$, $c > 0$ and

$$I_c(f)(\tau) = \frac{c+2}{\tau^{c+1}} \int_0^\tau t^c f(t) dt, \quad \tau \in U.$$

Then,

$$I_c \left[T_F^{\lambda, t}(m, \alpha, \beta) \right] \subset T_F^{\lambda, t}(m, \alpha, \beta^*), \quad (18)$$

where

$$\beta^* = 2\delta - 1 + \frac{(c+2)(2-2\delta)}{n} \int_0^1 \frac{t^{\frac{c+2}{n}} - 1}{1+t} dt.$$

Proof. We can use the same justifications as in the proof of Theorem 2, as the function h presented in the theorem is convex. When we interpret the premise of Theorem 3, we can see that

$$F_{\varphi(U)} \left(\varphi(\tau) + \frac{1}{c+2} \tau \varphi'(\tau) \right) \leq f_{h(U)} h(\tau),$$

where $\varphi(\tau)$ is provided by (17). By applying Lemma 2, the following fuzzy inequality is obtained:

$$F_{\varphi(U)} \varphi(\tau) \leq F_{g(U)} g(\tau) \leq F_{h(U)} h(\tau),$$

i.e.,

$$F_{(\mathcal{BS}_{\alpha, \lambda}^{m, t} G)'(U)} \left(\mathcal{BS}_{\alpha, \lambda}^{m, t} G \right)' \leq F_{g(U)} g(\tau) \leq F_{h(U)} h(\tau),$$

where

$$\begin{aligned} g(\tau) &= \frac{c+2}{n\tau^{\frac{c+2}{n}}} \int_0^\tau t^{\frac{c+2}{n}-1} \frac{1 + (2\delta - 1)}{1+t} dt \\ &= 2\delta - 1 + \frac{(c+2)(2-2\delta)}{n\tau^{\frac{c+2}{n}}} \int_0^1 \frac{t^{\frac{c+2}{n}} - 1}{1+t} dt. \end{aligned}$$

It is understood that $g(U)$ is symmetric with regard to the real axis using the notion of convexity for function g , and we can write

$$\begin{aligned} &F_{\mathcal{BS}_{\alpha, \lambda}^{m, t} G(U)} \left(\mathcal{BS}_{\alpha, \lambda}^{m, t} G(\tau) \right)' \\ &\geq \min_{|\tau|=1} F_{g(U)} g(\tau) = F_{g(U)} g(1) \end{aligned} \quad (19)$$

and

$$\beta^* = g(1) = 2\delta - 1 + \frac{(c+2)(2-2\delta)}{n} \int_0^1 \frac{t^{\frac{c+2}{n}} - 1}{1+t} dt.$$

From (19), it is possible to deduce inclusion (18). $\square$

Theorem 4. Let the function g be a convex function with $g(0) = 1$ and

$$h(\tau) = g(\tau) + \tau g'(\tau), \quad \tau \in U,$$

let $f \in \mathcal{A}_n$ satisfy

$$\begin{aligned} F_{(\mathcal{BS}_{\alpha,\lambda}^{m,t}f)'(U)} \left(\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau) \right)' &\leq F_{h(U)}h(\tau), \\ \text{i.e., } \left(\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau) \right)' &\prec_{\mathcal{F}} h(\tau), \tau \in U. \end{aligned} \quad (20)$$

and let $m - t > -1$. Then, we obtain the following fuzzy differential subordination:

$$\begin{aligned} F_{\mathcal{BS}_{\alpha,\lambda}^{m,t}f(U)} \frac{\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau)}{\tau} &\leq F_{g(U)}g(\tau), \\ \text{i.e., } \frac{\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau)}{\tau} &\prec_{\mathcal{F}} g(\tau), \tau \in U. \end{aligned}$$

and the result is sharp.

Proof. Using Equation (7) about the operator $\mathcal{BS}_{\alpha,\lambda}^{m,t}$, we can write

$$\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau) = \tau + \sum_{j=n+1}^{\infty} \left[\alpha \{ \lambda(j-1) + 1 \}^m + (1-\alpha) C_{m+j-1,t}^m \right] b_j \tau^j, \tau \in U.$$

Considering

$$\begin{aligned} \varphi(\tau) &= \frac{\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau)}{\tau} \\ &= \frac{\tau + \sum_{j=n+1}^{\infty} \left[\alpha \{ \lambda(j-1) + 1 \}^m + (1-\alpha) C_{m+j-1,t}^m \right] b_j \tau^j}{\tau} \\ &= 1 + \varphi_n \tau + \varphi_{n+1} \tau^{n+1} + \dots \end{aligned}$$

we can deduce that $\varphi \in \mathcal{H}[1, n]$.

Let $\tau\varphi(\tau) = \mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau)$, for $\tau \in U$. Taking the derivative, we obtain

$$\left(\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau) \right)' = \varphi(\tau) + \tau\varphi'(\tau). \quad (21)$$

Using (21) in (20), we can then write

$$\begin{aligned} F_{\varphi(U)} \left(\varphi(\tau) + \tau\varphi'(\tau) \right) &\leq F_{h(U)}h(\tau) \\ &= F_{g(U)} \left(g(\tau) + \tau g'(\tau) \right). \end{aligned}$$

Using Lemma 3, we obtain

$$F_{\varphi(U)}\varphi(\tau) \leq F_{g(U)}g(\tau),$$

that is,

$$F_{(\mathcal{BS}_{\alpha,\lambda}^{m,t}f)'(U)} \frac{\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau)}{\tau} \leq F_{g(U)}g(\tau), \tau \in U.$$

Therefore,

$$\frac{\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau)}{\tau} \prec_{\mathcal{F}} g(\tau), \tau \in U,$$

and this result is sharp. $\square$

Theorem 5. Suppose that h denotes a convex function of order $-\frac{1}{2}$ with $h(0) = 1$. Let $f \in A_n$ satisfy

$$F_{(\mathcal{BS}_{\alpha,\lambda}^{m,t}f)'(U)} \left(\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau) \right)' \leq F_{h(U)} h(\tau),$$

i.e.,

$$\left(\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau) \right)' \prec \mathcal{F}_{h(\tau)}, \tau \in U,$$

and let $m - t > -1$. Then,

$$\begin{aligned} F_{\mathcal{BS}_{\alpha,\lambda}^{m,t}f(U)} \frac{\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau)}{\tau} &\leq F_{q(U)} q(\tau), \\ \text{i.e., } \frac{\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau)}{\tau} &\prec \mathcal{F}q(\tau), \tau \in U, \end{aligned} \quad (22)$$

where

$$q(\tau) = \frac{1}{n\tau^{\frac{1}{n}}} \int_0^\tau h(t) t^{\frac{1}{n}-1} dt$$

is both convex and fuzzy best dominant.

Proof. Let

$$\begin{aligned} \varphi(\tau) &= \frac{\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau)}{\tau} = \frac{\tau + \sum_{j=n+1}^{\infty} \left[\alpha \{ \lambda(j-1) + 1 \}^m + (1-\alpha) C_{m+j-1,t}^m \right] b_j \tau^j}{\tau} \\ &= 1 + \sum_{j=n+1}^{\infty} \left[\alpha \{ \lambda(j-1) + 1 \}^m + (1-\alpha) C_{m+j-1,t}^m \right] b_j \tau^{j-1} \\ &= 1 + \sum_{j=n+1}^{\infty} \varphi_j b_j \tau^{j-1}, \tau \in U, \varphi \in [1, n]. \end{aligned}$$

as

$$\operatorname{Re} \left(1 + \frac{\tau h''(\tau)}{h'(\tau)} \right) > \frac{-1}{2}, \tau \in U.$$

From Lemma 1, we know that

$$q(\tau) = \frac{1}{n\tau^{\frac{1}{n}}} \int_0^\tau h(t) t^{\frac{1}{n}-1} dt$$

is a convex function and verifies the differential equation related to the following fuzzy differential subordination (22):

$$q(\tau) + \tau q'(\tau) = h(\tau).$$

Therefore, it is the fuzzy best dominant. Taking the derivative, we obtain

$$\left(\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau) \right)' = \varphi(\tau) + \tau \varphi'(\tau), \tau \in U$$

and

$$F_{\varphi(U)} (\varphi(\tau) + \tau \varphi'(\tau)) \leq F_{h(U)} h(\tau), \tau \in U.$$

From Lemma 3, we have

$$\begin{aligned} F_{\varphi(U)} \varphi(\tau) &\leq F_{q(U)} q(\tau), \tau \in U, \\ \text{i.e., } F_{\mathcal{BS}_{\alpha,\lambda}^{m,t}f(U)} \frac{\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau)}{\tau} &\leq F_{q(U)} q(\tau), \tau \in U. \end{aligned}$$

Thus, we obtain

$$\frac{\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau)}{\tau} \prec_{\mathcal{F}} q(\tau), \tau \in U.$$

□

Corollary 1. Suppose that

$$h(\tau) = \frac{1 + (2\beta - 1)\tau}{1 + \tau}$$

is a convex function in U , $0 \leq \beta < 1$. Let $m - t > -1$, $\lambda \geq 0$, $\alpha \geq 0$, $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, $f \in \mathcal{A}_n$ and verify the fuzzy differential subordination

$$F_{(\mathcal{BS}_{\alpha,\lambda}^{m,t}f)'(U)} \left((\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau))' \right) \leq F_{h(U)} h(\tau), \quad (23)$$

that is,

$$(\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau))' \prec_{\mathcal{F}} h(\tau), \tau \in U.$$

Then,

$$\begin{aligned} F_{\mathcal{BS}_{\alpha,\lambda}^{m,t}f(U)} \frac{\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau)}{\tau} &\leq F_{q(U)} q(\tau), \\ \text{i.e., } \frac{\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau)}{\tau} &\prec_{\mathcal{F}} q(\tau), \tau \in U, \end{aligned}$$

and

$$q(\tau) = 2\beta - 1 + \frac{2(1-\beta)}{n\tau^{\frac{1}{n}}} \int_0^\tau \frac{t^{\frac{1}{n}-1}}{1+t} dt, \tau \in U$$

is convex and fuzzy best dominant.

Proof. We have

$$h(\tau) = \frac{1 + (2\beta - 1)\tau}{1 + \tau}$$

with

$$h(0) = 1 \text{ and } h'(\tau) = \frac{-2(1-\beta)}{(1+\tau)^2}$$

and

$$h''(\tau) = \frac{4(1-\beta)}{(1+\tau)^3}$$

along with

$$\begin{aligned} & \operatorname{Re} \left(\frac{\tau h''(\tau)}{h'(\tau)} + 1 \right) \\ &= \operatorname{Re} \left(\frac{1-\tau}{1+\tau} \right) \\ &= \operatorname{Re} \left(\frac{1-\phi \cos \theta - i\phi \sin \theta}{1+\phi \cos \theta + i\phi \sin \theta} \right) \\ &= \frac{1-\phi^2}{1+2\phi \cos \theta + \phi^2} > 0 > -\frac{1}{2}. \end{aligned}$$

Following the same steps as in the proof of Theorem 5 and considering

$$\varphi(\tau) = \frac{\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau)}{\tau},$$

the fuzzy differential subordination (23) becomes

$$F_{\mathcal{BS}_{\alpha,\lambda}^{m,t}f(U)}(\varphi(\tau) + \tau\varphi'(\tau)) \leq F_{h(U)}h(\tau), \quad \tau \in U.$$

According to Lemma 2, for $\gamma = 1$ we have

$$F_{\varphi(U)}\varphi(\tau) \leq F_{q(U)}q(\tau),$$

$$F_{\mathcal{BS}_{\alpha,\lambda}^{m,t}f(U)} \frac{\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau)}{\tau} \leq F_{q(U)}q(\tau).$$

Thus,

$$\begin{aligned} q(\tau) &= \frac{1}{n\tau^{\frac{1}{n}}} \int_0^\tau h(t)t^{\frac{1}{n}-1}dt, \quad \tau \in U \\ &= \frac{1}{n\tau^{\frac{1}{n}}} \left(\int_0^\tau t^{\frac{1}{n}-1} \frac{1+(2\beta-1)t}{1+t} dt, \quad \tau \in U \right) \\ &= 2\beta - 1 + \frac{2(1-\beta)}{n\tau^{\frac{1}{n}}} \int_0^\tau \frac{t^{\frac{1}{n}-1}}{1+t} dt, \quad \tau \in U. \end{aligned}$$

□

Example 2. Suppose that

$$h(\tau) = \frac{1-\tau}{1+\tau}$$

with

$$h(0) = 1, \quad h'(\tau) = \frac{-2}{(1+\tau)^2}$$

and

$$h''(\tau) = \frac{4}{(1+\tau)^3}.$$

Furthermore, if

$$\begin{aligned} \operatorname{Re} \left(\frac{\tau h''(\tau)}{h'(\tau)} + 1 \right) &= \operatorname{Re} \left(\frac{1-\tau}{1+\tau} \right) \\ &= \operatorname{Re} \left(\frac{1-\phi \cos \theta - i\phi \sin \theta}{1+\phi \cos \theta + i\phi \sin \theta} \right) \\ &= \frac{1-\phi^2}{1+2\phi \cos \theta + \phi^2} > 0 > -\frac{1}{2}, \end{aligned}$$

then the function h is convex in U .

Suppose that

$$f(\tau) = \tau + \tau^2, \quad \tau \in U.$$

For $n = 1, \lambda = 1, \alpha = 2, m = t = 1$, we obtain

$$\begin{aligned} \mathcal{BS}_{2,1}^{1,1}f(\tau) &= -B_1^1f(\tau) + 2S_1^1f(\tau) \\ &= -\tau f'(\tau) + 2\tau f'(\tau) \\ &= \tau f'(\tau) \\ &= \tau + 2\tau^2. \end{aligned}$$

Then,

$$\left(\mathcal{BS}_{2,1}^{1,1}f(\tau) \right)' = 1 + 4\tau$$

and

$$\frac{\mathcal{BS}_{2,1}^{1,1}f(\tau)}{\tau} = 1 + 2\tau$$

Because

$$q(\tau) = \frac{1}{\tau} \int_0^\tau \frac{1-t}{1+t} dt = -1 + \frac{2\ln(1+\tau)}{\tau}.$$

From Theorem 5, we have

$$1 + 4\tau \prec \mathcal{F} \frac{1-\tau}{1+\tau}, \tau \in U,$$

which induces

$$1 + 2\tau \prec \mathcal{F} - 1 + \frac{2\ln(1+\tau)}{\tau}, \tau \in U.$$

Theorem 6. Let $h(\tau) = g(\tau) + \tau g'(\tau)$, $\tau \in U$ and let g be a convex function in U with $g(0) = 1$; furthermore, let $f \in A_n$ satisfy

$$F_{(\mathcal{BS}_{\alpha,\lambda}^{m,t}f(U))} \left(\frac{\tau \mathcal{BS}_{\alpha,\lambda}^{m+1,t}f(\tau)}{\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau)} \right)' \leq F_{h(U)} h(\tau), \text{ i.e., } \left(\frac{\tau \mathcal{BS}_{\alpha,\lambda}^{m+1,t}f(\tau)}{\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau)} \right)' \prec \mathcal{F} h(\tau), \tau \in U, \quad (24)$$

with $\alpha \geq 0$, $m-t > -1$, $m \in \mathbb{N}_0$, $n \in \mathbb{N}$. Then, we obtain the sharp fuzzy differential subordination

$$\begin{aligned} F_{\mathcal{BS}_{\alpha,\lambda}^{m,t}f(U)} \frac{\mathcal{BS}_{\alpha,\lambda}^{m+1,t}f(\tau)}{\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau)} &\leq F_{g(U)} g(\tau), \\ \text{i.e., } \frac{\mathcal{BS}_{\alpha,\lambda}^{m+1,t}f(\tau)}{\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau)} &\prec \mathcal{F} g(\tau), \tau \in U. \end{aligned}$$

Proof. Because

$$f \in A_n \text{ and } f(\tau) = \tau + \sum_{j=n+1}^{\infty} b_j \tau^j,$$

we have

$$\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau) = \tau + \sum_{j=n+1}^{\infty} \left[\alpha \{ \lambda(j-1) + 1 \}^m + (1-\alpha) C_{m+j-1,t}^m \right] b_j \tau^j, \tau \in U.$$

Considering

$$\varphi(\tau) = \frac{\mathcal{BS}_{\alpha,\lambda}^{m+1,t}f(\tau)}{\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau)} = \frac{\tau + \sum_{j=n+1}^{\infty} \left[\alpha \{ \lambda(j-1) + 1 \}^{m+1} + (1-\alpha) C_{m+j,t}^{m+1} \right] b_j \tau^j}{\tau + \sum_{j=n+1}^{\infty} \left[\alpha \{ \lambda(j-1) + 1 \}^m + (1-\alpha) C_{m+j-1,t}^m \right] b_j \tau^j},$$

we have

$$\varphi'(\tau) = \frac{\left(\mathcal{BS}_{\alpha,\lambda}^{m+1,t}f(\tau) \right)'}{\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau)} - \varphi(\tau) \cdot \frac{\left(\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau) \right)'}{\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau)}$$

and we obtain

$$\varphi(\tau) + \tau \varphi'(\tau) = \left(\frac{\tau \mathcal{BS}_{\alpha,\lambda}^{m+1,t}f(\tau)}{\mathcal{BS}_{\alpha,\lambda}^{m,t}f(\tau)} \right)'.$$

Thus, the relation from (24) becomes

$$F_{\varphi(U)} (\varphi(\tau) + \tau \varphi'(\tau)) \leq F_{h(U)} h(\tau) = F_{g(U)} (g(\tau) + \tau g'(\tau)), \tau \in U.$$

Following the application of Lemma 3, we have the required result. $\square$

4. Conclusions

In this article, fuzzy differential subordination is studied in relation to geometric function theory. First, we develop a new operator $\mathcal{BS}_{\alpha,\lambda}^{m,t}:A_n \rightarrow A_n$ in the open unit disc U . Then, taking this operator into consideration, we create fuzzy differential subordination. Next, we define a particular fuzzy class of analytic functions in U , which we call $T_F^{\lambda,t}(m, \alpha, \delta)$. Using the idea of fuzzy differential subordination and the operator $\mathcal{BS}_{\alpha,\lambda}^{m,t}$ for the function f in the class $T_F^{\lambda,t}(m, \alpha, \delta)$, many novel results can be proved. When $\lambda = 1$ and $t = m$, all the results provided in this article reduce to known results proved previously in [11].

For conclusions that offer coefficient estimates, distortion theorems, or closure theorems, as is typical in geometric function theory, further research on the newly introduced class may be needed. Additionally, the introduction of this class can serve as an inspiration for future research that introduces and characterizes additional intriguing fuzzy classes. In order to identify additional feasible values of δ for accurate definitions of fuzzy classes, the constraint placed on $\delta \in (0, 1]$ should be further examined.

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