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Stability Conditions for Linear Semi-Autonomous Delay Differential Equations

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Abstract: We present a new method for obtaining stability conditions for certain classes of delay differential equations. The method is based on the transition from an individual equation to a family of equations, and next the selection of a representative of this family, the test equation, asymptotic properties of which determine those of all equations in the family. This approach allows us to obtain the conditions that are the criteria for the stability of all equations of a given family. These conditions are formulated in terms of the parameters of the class of equations being studied, and are effectively verifiable. The main difference of the proposed method from the known general methods (using Lyapunov–Krasovsky functionals, Razumikhin functions, and Azbelev W -substitution) is the emphasis on the exactness of the result; the difference from the known exact methods is a significant expansion of the range of applicability. The method provides an algorithm for checking stability conditions, which is carried out in a finite number of operations and allows the use of numerical methods.

Keywords: delay differential equation; stability; Cauchy function; effective stability condition

MSC: 34K20; 34K25



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1. Introduction

Since the time of A.M. Lyapunov, the problem of stability of solutions remains one of the most important problems in the study of any dynamical systems. In this respect, delay differential equations are no exception. The basic concepts and definitions of the classical stability theory for ordinary differential equations (ODEs) were actually transferred without changes to functional differential equations (FDEs), for which ODEs are the most studied special case. The study of stability of FDEs initially followed the schemes developed for ODEs. Analogues of the Lyapunov function method were found, the place of characteristic polynomials was taken by quasipolynomials, and generalizations of theorems on differential inequalities appeared. However, the direct transfer of known approaches to FDEs, first, was never a simple task, and always led to the appearance of additional restrictions, and second, it was rarely possible to achieve such beautiful exactness that could be achieved when working with ODEs. Thus, the new object required new ideas and methods that would take into account the features of equations with deviating argument.

Consider the known methods for studying the stability of FDEs. There are general methods which can be successfully applied to a very wide class of equations. Apparently, the most famous of them is the method of Lyapunov–Krasovsky functionals, the foundations of which were laid in monograph [1]. Over several decades of development of this method, many new ideas have been proposed (see [2,3]). Nowadays, the method continues to be applied to different classes of equations, and these classes are expanded (see, for example, works [4–8]). Another well-known method is the Razumikhin function method, the foundations of which are outlined in [9–11]. Although this method is not so widely

used today, new modifications periodically appear, including its use in combination with the Lyapunov–Krasovsky method [12,13]. We should also note the younger Azbelev W-method, the foundations of which were developed in the 1980s–1990s [14–19], and which continues to be actively developed in the works of a number of researchers today [20,21].

However, these methods are “semi-effective”: the stability conditions obtained on their basis depend on some auxiliary function, functional, or model equation, the good choice of which requires a special skill. Thus, the listed methods cannot guarantee the exactness of the obtained stability conditions.

On the other hand, there are methods that make it possible to obtain necessary and sufficient conditions for stability: for example, the study of an autonomous FDE can be replaced by the study of zeros of its characteristic function (as shown in the classic works [22–26]), and the study of FDEs with periodic parameters can be reduced to the problem of the spectrum of the monodromy operator (ideas from classic works [27–29] continue to be developed in the 21st century [30,31]). These methods provide exact stability tests; however, they are applicable to a relatively narrow classes of equations.

Thus, the choice of a research method is always the choice between generality and exactness; there are no methods that provide both at the same time. However, in this work, we try to combine some advantages of the both approaches: on the one hand, to cover a sufficiently wide class of equations, and on the other, to achieve the exactness of stability conditions comparable to that of stability tests for autonomous and periodic equations. Our approach may be corresponded to the so-called “Lyapunov’s first method”, that is estimating the growth or decay exponent of solutions, but within the framework of this formulation of the question, we propose a specific algorithm for obtaining the required estimates. To do this, we identify the “worst-behavior equation” in the class of equations under study. This idea in itself is not new: Myshkis used it in an implicit form when, using their examples, he showed the sharpness of the famous constant $3/2$ in the stability conditions he found. However, we are apparently the first to systematically develop this idea as applied to certain classes of equations.

Since we set the goal to obtain *exact* stability criteria for FDEs with parameters of a general form (in the study of which the construction of effective necessary and sufficient conditions is apparently impossible), we come to the question of giving a clear meaning to the very concept of *exactness*. In the absence of a definition, this concept can be interpreted too broadly; as experience shows, for almost any stability test one can find grounds to claim its “exactness”, or “sharpness”.

We consider, instead of an individual equations, families of equations defined by a certain finite set of numerical parameters. *We will call sufficient stability conditions of an equation exact, if they provide a stability criterion for some prescribed family of equations.* If exact conditions are satisfied, then all equations of the family are stable; if they are broken, then there is a representative of the family that is not stable. At first glance, checking the conditions not for one, but for many equations seems to be a much more difficult task, but it is possible that among the equations of the family, there are a small number of test equations, the stability of which guarantees the stability of the entire family.

Thus, in this work we present the method for studying the stability of FDEs, the *test method*, based on the approach described above. For specific classes of FDEs, we indicate a set of parameters characterizing the family, and for this family we construct a *test equation* by studying the asymptotic properties, from which we obtain information about the stability of the entire family. The test equation has a quite simple structure and can be studied analytically; checking the stability of the test equation is also possible using computer methods. For some families of delay differential equations with small number of parameters, we construct the stability region in the parameter space.

The test method made it possible to find a simple proof of the famous Myshkis “ $3/2$ -theorem”. The example constructed by Myshkis to show the unimprovability of the constant $3/2$ turns out to be nothing else but a (brilliantly guessed) test equation! This example suggested the form of test equations for FDEs with arbitrary number of

concentrated and distributed delays. Of course, as the original equation becomes more complex, the construction of the test equation becomes more complicated, and its study turns into a separate task. Solving it may not be easy, but the result is justified: the obtained stability test, if interpreted as a definition of a set in the parameter space, gives the region of stability with the boundary sharp in *each point*.

The paper is organized as follows. In Section 2, we formulate the problem. In Section 3, we define the main object of our study, a *semi-autonomous equation*, and obtain general results that justify our research method. On the basis of these results, combined in the form of several theorems in Section 3.4, we obtain, in Section 4, a number of new stability tests expressed in terms of parameters of a given equation, for several classes of linear delay differential equation with a small number of parameters; these tests are exact in some precisely defined sense.

2. Formulation of the Problem

Define some notation: $\mathbb{R}_+ = [0, +\infty)$, $\Delta = \{(t, s) \mid 0 \leq s \leq t\}$; the symbol $\triangleq$ means equality by definition; the statement $k = 1, n$ means that k runs from 1 to n .

2.1. Semi-Autonomous Equations, the Representation of Solutions

Consider the delay differential equation

$$\dot{x}(t) = ax(t) - \sum_{k=1}^n (T_k x)(t) + f(t), \quad t \in \mathbb{R}_+, \quad (1)$$

where

$$(T_k x)(t) = \int_{t-\omega_k}^t x(s) d_s r_k(t, s), \quad \int_{t-\omega_k}^t d_s r_k(t, s) = \rho_k,$$

ω_k and ρ_k are positive constants, the functions $r_k(t, \cdot)$ are nondecreasing, the functions $r_k(\cdot, s)$ are locally integrable, $r_k(t, t - \omega_k) = 0$, the integrals are understood in the Reimann–Stieltjes sense, and the function $f: \mathbb{R}_+ \rightarrow \mathbb{R}$ is supposed to be locally integrable. We understand a *solution* of Equation (1) as a locally absolutely continuous function $x: [s, +\infty) \rightarrow \mathbb{R}$ satisfying the equation almost everywhere on $[s, +\infty)$.

The most known representative of the class of equations of the form (1) is the equation with constant coefficients and concentrated variable delays

$$\dot{x}(t) = ax(t) - \sum_{k=1}^n \rho_k x(t - r_k(t)) + f(t), \quad t \in \mathbb{R}_+, \quad (2)$$

where $0 \leq r_k(t) \leq \omega_k$, $a \in \mathbb{R}$, $\rho_k \geq 0$. It is natural to call Equation (2) a *semi-autonomous equation*. We expand the meaning of this term.

Definition 1. We call the equations of the form (1) the *semi-autonomous equations*.

The definition of a solution x_s to Equation (1) formally requires an additional definition of values $x_s(\xi)$ for $\xi < s$. The classical approach to the study of delay differential equations suggests to set $x_s(\xi) = \varphi(\xi)$, where φ is a given *initial function*. However, note that when studying the asymptotics of solutions, we can set the initial function equal to zero without loss of generality; therefore, considering arbitrary integrable initial functions $\varphi: [s - \max_k \omega_k, s] \rightarrow \mathbb{R}$ does not extend the class of Equations (1) [14] (Sections 1.1 and 1.2).

It is known ([14,32] (Section 5.1)) that Equation (1), with a given initial value $x_s(s)$ and zero initial function, is uniquely solvable, and its solution is represented in the form

$$x_s(t) = C(t, s)x_s(s) + \int_s^t C(t, \tau)f(\tau) d\tau, \quad t \geq s, \quad (3)$$

where the function $C: \Delta \rightarrow \mathbb{R}$, which is called the *Cauchy function* of Equation (1), is locally absolutely continuous in the first argument. Thus, the solution to the initial value problem, or the *Cauchy problem*, for Equation (1) has the same integral representation as the solution to the Cauchy problem for ODE.

For all $(t, s) \in \Delta$ the following estimate is valid [19] (Section 3.1, Property 2): there is a real constant α such that

$$|C(t, s)| \leq \exp \alpha(t - s). \quad (4)$$

The Cauchy function does not depend on an initial function. Therefore, we do not further indicate how solutions to Equation (1) are defined for negative values of the argument, except in those cases when some specific initial function is significantly used.

In fact, all types of stability can be formulated as properties of the Cauchy function. However, we first give classical definitions of stability in terms of the initial function [33].

Definition 2. Equation (1) is called:

- *Stable (or Lyapunov stable)*, if for each $s \in \mathbb{R}_+$ and $\varepsilon > 0$ there exists $\delta_s > 0$ such that for any initial function φ such that $\|\varphi\| < \delta_s$, the estimate $\sup_{t \geq 0} |x_s(t)| < \varepsilon$ holds;
- *Asymptotically stable*, if for each $s \in \mathbb{R}_+$ and initial function φ the corresponding solution x_s has the property $\lim_{t \rightarrow +\infty} x_s(t) = 0$;
- *Uniformly stable*, if the number $\delta_s > 0$ in the definition of Lyapunov stability can be chosen independently of s ;
- *Exponentially stable*, if there are $M, \gamma > 0$ such that for all $(t, s) \in \Delta$ the estimate $|x_s(t)| \leq Ne^{-\gamma(t-s)}\|\varphi\|$ holds.

Representation (3) shows that the definitions of the uniform and exponential stabilities of (1) can be reformulated as follows. Equation (1) is called:

- *Uniformly stable*, if there exists $M > 0$ such that $|C(t, s)| \leq M$ for all $(t, s) \in \Delta$;
- *Exponentially stable*, if there exist $M, \gamma > 0$ such that for all $(t, s) \in \Delta$, the estimate $|C(t, s)| \leq M \exp(-\gamma(t - s))$ holds.

Below, we show that only these two types of stability are, in fact, realized for Equation (1).

2.2. Family of Equations; Stability of a Family

We study stability conditions for Equation (1), expressed through the values of coefficients a and ρ_k , and the maximum permissible values of delays ω_k . We are interested in stability criteria applied to equations with all possible delay functions r_k .

Our goal is to obtain *exact* stability conditions, and here we are faced with the need to define what an “exact condition” is. The fact is that the concept of *exactness* does not have a clear interpretation, and researchers sometimes succumb to the temptation to subordinate its meaning to their results; that is, to explicitly or implicitly interpret exactness so that the conditions established by them turn out to be exact. We consider it necessary to make the reasons for our understanding of exactness as clear as possible.

Set $\pi \triangleq \{a, \rho_1, \dots, \rho_n, \omega_1, \dots, \omega_n\} \in \mathbb{R}^{2n+1}$. Below, we speak about Equation (1), when we mean some fixed numbers a and ρ_k and functions r_k , and we use the term *family (1)*, when we speak about a *set* of Equation (1) with some fixed parameter set π and *all possible* functions r_k .

Definition 3. We call family (1) corresponding to a parameter set π (*asymptotically, uniformly, exponentially*) *stable*, if all equations of this family are stable in the corresponding sense of Definition 2.

Now, the exactness of a stability criterion is defined by the following: *sufficient* stability conditions for Equation (1) are exact if they are *necessary and sufficient* stability conditions for the corresponding family (1). The interpretation of the term *exactness* is determined by the choice of families into which the class of equations under study is divided.

Thus, the purpose of this research is to obtain *necessary and sufficient conditions for the uniform and exponential stability of family (1)*, where $a \in \mathbb{R}$, $\rho_k, \omega_k > 0$, and $r_k(t, \cdot)$ are not decreasing, $k = \overline{1, n}$.

3. Stability Criteria for (1) in Terms of Parameters of the Test Problem

First, we establish a simple necessary condition for the stability of family (1). Set $b = \sum_{k=1}^n \rho_k$.

Theorem 1. *If $a - b > 0$, then family (1) is not stable; if $a - b \geq 0$, then family (1) is not asymptotically stable.*

Proof. Equation (2) belongs to family (1). Setting $r_k(t) = 0, k = \overline{1, n}$, in (2), we obtain the ODE, whose solution is unbounded if $a - b > 0$; that is, this ODE is not stable. Hence, family (1) is not stable, as it contains an equation that is not stable.

Analogously, if $a - b \geq 0$, then each solution of the ODE does not tend to zero as $t \rightarrow +\infty$; that is, family (1) contains an equation which is not asymptotically stable; hence, family (1) is not asymptotically stable. $\square$

Consider the Cauchy problem for the following special case of Equation (2):

$$\begin{cases} \dot{y}(t) = ax(t) - \sum_{k=1}^n \rho_k y(t - r_k^*(t)), & t \in \mathbb{R}_+, \\ y(0) = 1, \end{cases} \quad (5)$$

where for $k = \overline{1, n}$ we put

$$r_k^*(t) = \begin{cases} t, & t \leq \omega_k, \\ \omega_k, & t > \omega_k. \end{cases} \quad (6)$$

Theorem 1 shows that when studying the stability of family (1), we can set $a \leq b$. Then, for $a < b$, the solution $y = y(t)$ to problem (5) is decreasing on some nonempty interval $(0, t_0)$; for $a = b$, the solution of problem (5) is $y(t) \equiv 1$.

Define a value $l = l(\pi)$ as follows. If the function y is *nonincreasing on the semiaxis* $(0, +\infty)$, set $l = \infty$. Otherwise define l equal to the maximum length of the starting at zero interval of the decrease of the function y . Explicitly, set

$$l \triangleq \inf\{t \geq 0 : \forall \varepsilon > 0 \exists \delta \in (0, \varepsilon) (y(t + \delta) \geq y(t))\}.$$

We say that the *test problem* for family (1) is Problem (5), where:

- If $l = \infty$, then the functions r_k^* are defined by equality (6) for all $t \in \mathbb{R}_+$;
- If $l < \infty$, then the functions r_k^* are defined by (6) on the set $[0, l]$ and then extended onto $\mathbb{R}_+$ by the rule $r_k^*(t + l) = r_k^*(t)$.

The solution to the test problem will interest us mainly on the interval $[0, l]$, on which the test problem can be given the equivalent form

$$\begin{cases} \dot{y}(t) = ax(t) - \sum_{k=1}^n \rho_k y(t - \omega_k), & t \geq 0, \\ y(\xi) = 1, & \xi \leq 0. \end{cases} \quad (7)$$

Further, we successively consider the cases of the oscillating solution to Problem (5) (in this case $l < \infty$) and its monotonic solution ($l = \infty$).

3.1. Oscillating Solutions to Test Problem

Set $\omega \triangleq \max_k \omega_k$. For each $s \in \mathbb{R}$, define the function $y_s: [s - \omega, +\infty) \rightarrow \mathbb{R}$ by the equality $y_s(t) = y(t - s)$, where y is the solution to the test problem (5). We call the functions of the set $\{y_s\}_{s \in \mathbb{R}}$ the *test functions* of family (1).

Everywhere in this section, we suppose that for a given set π we have $l < \infty$. This implies that $a < b$.

From the given above definition, we obtain the following properties of test functions:

- (a) If $t \in [s - \omega, s]$, then $y_s(t) = 1$;
- (b) For each number $t \in \mathbb{R}$ and $x \in [y(l), 1)$, there exists a unique number $s \in [t - l, t)$, such that $y_s(t) = x$;
- (c) If $t \in [s_1, s_1 + l] \cap [s_2, s_2 + l]$, then the conditions $s_1 < s_2$ and $y_{s_1}(t) < y_{s_2}(t)$ are equivalent;
- (d) For each $\varepsilon > 0$, there exists $\delta \in (0, \varepsilon)$ such that $y_s(s + l + \delta) \geq y_s(s + l)$.

The following result describes the connection between the test functions y_s and family (1), and lays the foundation for the *test method* of stability investigation.

Lemma 1. Suppose x is the solution of the Cauchy problem

$$\begin{cases} \dot{x}(t) = ax(t) - \sum_{k=1}^n (T_k x)(t), & t \geq s; \\ x(s) = x_0; \quad x(\zeta) = 0, & \zeta < s; \end{cases} \quad (8)$$

and there are given constants $\tau \in [0, +\infty)$, $s \in [\tau - l, \tau)$ and $\alpha < 0$. Then, if $x(\tau) = \alpha y_s(\tau)$ and for all $t \in [\tau - \omega, \tau]$ the inequality $x(t) \geq \alpha y_s(t)$ holds, then there exists $\lambda > 0$ such that for all $t \in [\tau, \tau + \lambda]$ the inequality $x(t) \leq \alpha y_s(t)$ holds.

Proof. By virtue of the linearity of problems (7) and (8), we can without loss of generality put $\alpha = -1$. Denote $\eta = -y_s$. Since the function η is stationary on the segment $[s - \omega, s]$ and is increasing on the interval (s, τ) , we have $x(\tau) = \eta(\tau) > \eta(\tau - \min_k \omega_k)$. Therefore, by virtue of the continuity of the function x , there exists a number $\delta > 0$ such that for all $t \in [\tau, \tau + \delta]$, we have $x(t) \geq \eta(\tau - \min_k \omega_k) \geq \eta(t - \omega_k)$.

For arbitrary $t \in [\tau, \tau + \delta]$ and $k \in \{1, \dots, n\}$, we have

$$\begin{aligned} (T_k x)(t) &= \int_{t-\omega_k}^{\tau} x(s) d_s r_k(t, s) + \int_{\tau}^t x(s) d_s r_k(t, s) \\ &\geq \int_{t-\omega_k}^{\tau} \eta(s) d_s r_k(t, s) + \int_{\tau}^t \eta(t - \omega_k) d_s r_k(t, s) \\ &\geq \eta(t - \omega_k) \int_{t-\omega_k}^{\tau} d_s r_k(t, s) = \rho_k \eta(t - \omega_k). \end{aligned}$$

By virtue of the established inequality and Equations (1) and (7), for $t \in [\tau, \tau + \delta]$, we obtain

$$\dot{x}(t) - \dot{\eta}(t) = a(x(t) - \eta(t)) - \sum_{k=1}^n ((T_k x)(t) - \rho_k \eta(t - \omega_k)) \leq a(x(t) - \eta(t)).$$

Consider the function $z = x - \eta$ on the segment $[\tau, \tau + \delta]$. We have

$$\begin{cases} \dot{z}(t) \leq az(t), & t \in [\tau, \tau + \delta]; \\ z(\tau) = 0. \end{cases}$$

In the case $a \geq 0$, we have $z(t) \leq z(\tau)e^{a(t-\tau)} = 0$.

Put $a < 0$. Assume that there exists a number $t_1 \in [\tau, \tau + \delta]$ such that $z(t_1) > 0$. Then, there exists $t_0 = \sup\{t \in [\tau, t_1] : z(t) \leq 0\}$, where by virtue of the continuity of z , we have $z(t_0) = 0$ and $z(t) > 0$ for all $t \in (t_0, t_1)$. But then $z(t_1) = \int_{t_0}^{t_1} \dot{z}(t) dt \leq a \int_{t_0}^{t_1} z(t) dt < 0$, which contradicts our assumption.

Thus, $z \leq 0$ for $[\tau, \tau + \delta]$, and we can set $\lambda = \delta$. $\square$

Lemma 2. For the solution y of problem (7) the inequality $y(l) < 0$ is valid.

Proof. Since Problem (7) is autonomous, it is obvious that the function y is continuously differentiable on $[0, +\infty)$. Hence, $\dot{y}(l) = 0$. Moreover, the function y is stationary on the interval $[-\omega, 0)$ and strictly decreasing on $(0, l)$, hence $y(l - \omega_k) > y(l)$.

Assume that $y(l) \geq 0$. Then, since $\rho_k > 0$, $k = \overline{1, n}$, and $b > a$, we obtain from (7) that

$$0 = \dot{y}(l) = ay(l) - \sum_{k=1}^n \rho_k y(l - \omega_k) < ay(l) - \sum_{k=1}^n \rho_k y(l) = y(l)(a - b) \leq 0.$$

We have arrived at a contradiction. $\square$

Denote $h = -y(l)$. It follows from Lemma 2 that $h > 0$.

Theorem 2. Suppose x is the solution of the Cauchy problem (8). If there exists $t_1, t_2 \in \mathbb{R}$ such that $l + \omega \leq t_1 < t_2$ and $|x(t_1)| < |x(t_2)|$, then there exists $t_0 \in [t_1 - l - \omega, t_1)$ such that $x(t_0) = -\frac{1}{h}x(t_1)$.

Proof. 1. Since the function x is continuous, we can suppose that either $0 < x(t_1) < x(t_2)$, or $0 > x(t_1) > x(t_2)$. Consider the first case, since the second one is reduced to it by replacing the sign of the initial value x_0 . Let $\tau = \sup\{t \in [t_1, t_2] : x(t) \leq x(t_1)\}$. We have $\tau \in [t_1, t_2)$, $x(\tau) = x(t_1)$ and $x(t) > x(\tau)$ for all $t \in (\tau, t_2)$.

2. For each $s \in \mathbb{R}$, define the function $\eta_s : [s - \omega, +\infty) \rightarrow \mathbb{R}$ by the following: $\eta_s = -\frac{x(\tau)}{h}y_s$, where y_s is a test function. By this definition and Lemma 2, we have $\eta_{\tau-l}(\tau) = x(\tau) > 0$. By virtue of property (d) of test functions for each $\varepsilon > 0$, there exists $\delta \in (0, \varepsilon)$ such that $\eta_{\tau-l}(\tau + \delta) \leq \eta_{\tau-l}(\tau)$. From here, with regard to item 1, we obtain: for each $\varepsilon > 0$, there exists a point $\tau_1 \in (\tau, \tau + \varepsilon)$ such that $\eta_{\tau-l}(\tau_1) < x(\tau_1)$.

3. Assume that the conclusion of the theorem is false. Then, by virtue of the continuity of the function x , we have $x(t) > -\frac{x(\tau)}{h}$ for all $t \in [\tau - l - \omega, \tau]$. By virtue of property (b) of test functions, for each $t \in [\tau - l, \tau]$ such that $x(t) \leq x(\tau)$, there exists a unique number $s(t) \in [t - l, t]$ such that $\eta_{s(t)}(t) = x(t)$; for the remaining $t \in [\tau - l, \tau]$, put $s(t) = \tau - l$. Denote $\sigma = \sup\{s(t) : t \in [\tau - l, \tau]\}$. Taking account of $s(\tau) = \tau - l$, we have $\sigma \in [\tau - l, \tau]$. For all $t \in [\sigma - \omega, \sigma]$, by virtue of the assumption made and property (a) of test functions, we have $\eta_\sigma(t) = -\frac{x(\tau)}{h} < x(t)$. Assume that $t \in [\sigma, \tau]$ and $\eta_\sigma(t) > x(t)$. Then, $\eta_\sigma(t) > \eta_{s(t)}(t)$; therefore, by virtue of property (c) of test functions, $s(t) > \sigma$, which contradicts to the definition of the point σ . Thus, for all $t \in [\sigma - \omega, \tau]$ we have $\eta_\sigma(t) \leq x(t)$.

4. In the case $\sigma = \tau - l$, by item 3, the conditions of Lemma 1 are satisfied in the point τ (for $s = \tau - l$, $\alpha = -\frac{x(\tau)}{h} < 0$). Therefore, there exists a number $\lambda > 0$ such that $\eta_{\tau-l}(t) \geq x(t)$ for all $t \in [\tau, \tau + \lambda]$, which contradicts item 2.

5. Consider the case $\sigma > \tau - l$. For each $\varepsilon > 0$, there exists a number $s_\varepsilon \in [\sigma - \varepsilon, \sigma]$ such that $\eta_{s_\varepsilon}(t) = x(t)$ for some $t \in [\tau - l, \tau]$. Since test functions are uniformly continuous on the compact $[\sigma, \tau]$ we have $\sup_{t \in [\sigma, \tau]} |\eta_\sigma(t) - \eta_{s_\varepsilon}(t)| \rightarrow 0$ for $\varepsilon \rightarrow 0$. Hence, the set $T = \{t \in [\sigma, \tau] : \eta_\sigma(t) = x(t)\}$ is not empty. Denote $\tau_0 = \sup T$. By virtue of property (c) of test functions, $\eta_\sigma(\tau) < \eta_{\tau-l}(\tau) = x(\tau)$; therefore, $\tau_0 < \tau$. For all $t \in (\tau_0, \tau)$, we have $\eta_\sigma(t) < x(t)$. On the other hand, by item 3 and Lemma 1, there exists a number $\lambda > 0$ such that $\eta_\sigma(t) \leq x(t)$ for all $t \in [\tau_0, \tau_0 + \lambda]$. We have arrived at a contradiction. $\square$

The following stability criteria for family (1) are consequences of Theorem 2.

Theorem 3. For the exponential (uniform) stability of family (1), it is necessary and sufficient that the inequality $h < 1$ ($h \leq 1$) holds.

Proof. *Necessity.* Consider the Cauchy function $C(t, s)$ of the equation from the test problem of family (1), which is problem (5), where we suppose that the delay functions r_k^* are defined by (6) on the set $[0, l)$ and extended onto $\mathbb{R}_+$ by the rule $r_k^*(t + l) = r_k^*(t)$.

For all $t \in [ml, (m+1)l]$ we have $C(t+s, s) = (-h)^m y_{ml}(t)$; in particular, $C(ml+s, s) = (-h)^m$. Thus, if $h \geq 1$, then $C(\cdot, s)$ does not tend to zero; if $h > 1$, then the Cauchy function is unbounded.

Sufficiency. 1. Suppose $h \leq 1$. Consider an arbitrary solution x of Equation (1). Let us prove that there exists a number $L > 0$ such that, for each $\tau_0 \geq l + \omega$, if

$$\sup_{t \in [\tau_0 - l - \omega, \tau_0]} |x(t)| = S < \infty, \quad (9)$$

then $\sup_{t \in [\tau_0, \tau_0 + L]} |x(t)| < Sh$.

Suppose that for a given τ_0 , inequality (9) holds. If there exist points $t_1, t_2 \geq \tau_0$ such that $t_1 < t_2$ and $Sh < |x(t_1)| < |x(t_2)|$, then by Theorem 2, we have

$$\sup_{t \in [\tau_0 - l - \omega, \tau_0]} |x(t)| \geq \frac{|x(t_1)|}{h} > S,$$

which contradicts (9). Therefore, if for some $\tau_1 > \tau_0$ we have $|x(t)| \geq Sh$ for all $t \in [\tau_0, \tau_1]$, then the function $|x(\cdot)|$ is not increasing on the interval (τ_0, τ_1) . Put $\tau_1 = \tau_0 + \omega + \delta$, where $\delta > 0$. Suppose that for all $t \in [\tau_0, \tau_1]$, we have $x(t) \geq Sh$ (the case $x(t) \leq -Sh$ is analogous). Then, for almost all $t \in [\tau_0 + \omega, \tau_1]$, we have

$$\dot{x}(t) = ax(t) - \sum_{k=1}^n (T_k x)(t) \leq \left(a - \sum_{k=1}^n \rho_k \right) x(t) \leq (a-b)Sh,$$

from which

$$x(\tau_1) \leq S + \delta(a-b)Sh = Sh \left(1 + \left(\delta(b-a) - \frac{1-h}{h} \right) \right).$$

Hence, the value δ cannot exceed the value $\frac{1-h}{h(b-a)}$. Thus, the described above number L exists: the condition $L > \omega + \frac{1-h}{h(b-a)}$ is sufficient.

2. Denote $M = \sup_{t \in [0, l + \omega]} |x(t)|$, $T = L + l + \omega$, and $d(t) = Mh^n$ for all $t \in [nT, (n+1)T]$, $n = 0, 1, 2, \dots$. Prove that

$$|x(t)| \leq d(t), \quad t \geq 0. \quad (10)$$

Assume that this is not valid. Denote $\tau = \inf\{t \geq 0 : |x(t)| > d(t)\}$. Let us show that $\tau \neq nT$, $n = 1, 2, \dots$. Indeed, suppose $\tau = n_0T$, then $|x(\tau)| \geq Mh^{n_0+1}$. Now, since by virtue of item 1 $\inf_{t \in [\tau-L, \tau]} |x(t)| < Mh^{n_0+1}$, there exist points $t_1, t_2 \in [\tau-L, \tau]$, such that $t_1 < t_2$ and $Mh^{n_0+1} < |x(t_1)| < |x(t_2)|$. Therefore, by Theorem 2, there exists a point $t_0 \in [\tau-T, \tau]$ such that $|x(t_0)| > Mh^{n_0}$, which contradicts the definition of the point τ .

Thus, $\tau \in (nT, (n+1)T)$ for some n . Then, there exist points $t_1, t_2 \in (nT, (n+1)T)$ such that $t_1 < t_2$ and $Mh^n < |x(t_1)| < |x(t_2)|$. But by Theorem 2, this implies that there exists $t_0 > (n-1)T$ such that $|x(t_0)| > Mh^{n-1}$, which again contradicts the definition of the point τ .

3. In Theorem 2, and items 1 and 2, one may consider the function $C(\cdot, s)$, with an arbitrary initial point s , in the capacity of the function x . By virtue of property (4) of the Cauchy function, for $t \in [s, s+T]$ the values $C(t, s)$ are bounded uniformly in s . Therefore, if one redefines $M = \sup_{t \in [s, s+l+\omega]} |C(t, s)|$, then the inequality (10) turns into the inequality

$$\sup_{t-s \in [nT, (n+1)T]} |C(t, s)| \leq Mh^n,$$

from which the theorem follows in obvious way. $\square$

3.2. Monotone Solutions to Test Problem

In Section 3.1, we obtained effective criteria of the exponential and uniform stabilities of family (1) in the case where the solution of autonomous problem (7) reaches the first minimum in a finite interval. In this subsection, we consider the opposite case: the solution of problem (7) remains decreasing on the semiaxis.

3.2.1. Auxiliary Problem

Associate problem (7) with the problem

$$\begin{cases} \dot{u}(t) + \sum_{k=1}^n \rho_k e^{-a\omega_k} u(t - \omega_k) = 0, & t \geq 0, \\ u(0) = 1; \quad u(\xi) = 0, & \xi < 0. \end{cases} \quad (11)$$

Show that the study of the case that the solution to Problem (7) is decreasing on the semiaxis $(0, +\infty)$; that is, the case $l = \infty$ can be replaced by the study of the positiveness of the solution to problem (11).

Lemma 3. Suppose $a - b < 0$. Then

- $l = \infty$ if and only if the solution to problem (11) is positive on the semiaxis $[0, +\infty)$;
- In the case $l < \infty$, the first minimum of the solution to problem (7) coincides with the first zero of the solution to problem (11).

Proof. Make the following change of variables in Problem (7):

$$y(t) = 1 + (a - b) \int_0^t e^{as} u(s) ds. \quad (12)$$

Substituting expressions for $y(t)$ and $\dot{y}(t)$ into the equation from problem (7), we obtain

$$\begin{aligned} \dot{y}(t) = e^{at}(a - b)u(t) &= a \left(1 + (a - b) \int_0^t e^{as} u(s) ds \right) \\ &\quad - \sum_{k=1}^n \rho_k \left(1 + (a - b) \int_0^{t-\omega_k} e^{as} u(s) ds \right), \end{aligned}$$

from which

$$e^{at}u(t) = 1 + a \int_0^t e^{as} u(s) ds - \sum_{k=1}^n \left(\rho_k \int_0^{t-\omega_k} e^{as} u(s) ds \right).$$

Differentiating this equality and dividing the result by e^{at} , we obtain that the function u is the solution to problem (11). Thus, formula (12) establishes a one-to-one correspondence between the solutions to problems (7) and (11). Since $\dot{y}(t) = e^{at}(a - b)u(t)$, the inequalities $\dot{y}(t) < 0$ and $u(t) > 0$ are equivalent for all $t \in [0, +\infty)$. $\square$

Define a function $P: \mathbb{R} \rightarrow \mathbb{R}$ by the equality

$$P(\zeta) = -\zeta - a + \sum_{k=1}^n \rho_k e^{\zeta \omega_k}.$$

The following Lemma connects zeros of the function P and the positiveness of solutions to problem (11). Such a connection, in various terms, was previously noted by a number of authors [34–38].

Lemma 4. The solution of problem (11) is positive on the semiaxis $[0, +\infty)$ if and only if the function P has at least one zero.

Proof. *Necessity.* Assume that the solution of problem (11) is positive on $[0, +\infty)$, while the function P has no zeros. As is known [37], the characteristic function of the equation in Problem (11) is

$$F(p) = p + \sum_{k=1}^n \rho_k e^{-\omega_k(p+a)}, \quad p \in \mathbb{C}.$$

Note that for $\zeta \in \mathbb{R}$, we have $F(-\zeta - a) = P(\zeta)$. Hence, the function F has no real roots. But then from Theorem 2.1.1 of monograph [37] (see also [34,35,39]), it follows that the solution to problem (11) oscillates, which contradicts the above assumption.

Sufficiency. The solution u of problem (11) defines the Cauchy function C of the equation from this problem by the equality $C(t, s) = u(t - s)$.

Define an operator $\mathcal{L}$ on the class of absolutely continuous on the set $\mathbb{R}$ functions,

$$(\mathcal{L}x)(t) = \dot{x}(t) + \sum_{k=1}^n \rho_k e^{-a\omega_k} x(t - \omega_k).$$

Assume that $P(\zeta_0) = 0$ for some $\zeta_0 \in \mathbb{R}$. Set $v(t) = e^{-(a+\zeta_0)t}$, $t \in \mathbb{R}$. We have

$$(\mathcal{L}v)(t) = e^{-(a+\zeta_0)t}(-\zeta_0 - a + \sum_{k=1}^n \rho_k e^{\zeta_0 \omega_k}) = e^{-(a+\zeta_0)t} P(\zeta_0) = 0.$$

Hence, with regard to the fact that the equation from Problem (11) has the form $(\mathcal{L}x)(t) = 0$, by (Lemma 2.4.3) [19] (the lemma on a differential inequality; see also [40]), the Cauchy function of this equation is positive. $\square$

Let us pay attention to some properties of the function P . We have

$$P'(\zeta) = -1 + \sum_{k=1}^n \rho_k \omega_k e^{\zeta \omega_k}, \quad P''(\zeta) = \sum_{k=1}^n \rho_k \omega_k^2 e^{\zeta \omega_k}.$$

Thus, $P''(\zeta) > 0$ for all $\zeta \in \mathbb{R}$, the function P' is increasing on the whole axis, with $\lim_{\zeta \rightarrow -\infty} P'(\zeta) = -1$ and $\lim_{\zeta \rightarrow +\infty} P'(\zeta) = +\infty$. Hence, the function P has a unique minimum point ζ^* . If $P(\zeta^*) > 0$, then the function P has no zeros; if $P(\zeta^*) = 0$, then ζ^* is the only zero of P ; if $P(\zeta^*) < 0$, then P has exactly two zeros, to the right and to the left of ζ^* .

Let the set of parameters of family (1) define a set of points in the space $\mathbb{R}^{n+1}$ as follows. Denote by F the set of points $(u_0, u_1, \dots, u_n) \in \mathbb{R}^{n+1}$ defined by the following system of equations in parametric form:

$$-\zeta - u_0 + \sum_{k=1}^n u_k e^{\zeta \omega_k} = 0, \quad -1 + \sum_{k=1}^n u_k \omega_k e^{\zeta \omega_k} = 0, \quad \zeta \in \mathbb{R}. \quad (13)$$

Further, set $A = (a, 0, \dots, 0) \in \mathbb{R}^{n+1}$ and $M = (a, \rho_1, \dots, \rho_n) \in \mathbb{R}^{n+1}$. The parametric equations

$$u_0 = a, \quad u_k = \rho_k \tau, \quad k = \overline{1, n}, \quad \tau \geq 0, \quad (14)$$

define the ray AM . Substituting (14) into (13), we obtain that the ray AM has an intersection point with the set F if and only if the equation

$$-(\zeta + a) \sum_{k=1}^n \rho_k \omega_k e^{\zeta \omega_k} + \sum_{k=1}^n \rho_k e^{\zeta \omega_k} = 0. \quad (15)$$

is solvable with respect to ζ .

The left-hand side of Equation (15) has a unique maximum point $\zeta = -a$, since its derivative with respect to ζ , which is equal to $-(\zeta + a) \sum_{k=1}^n \rho_k \omega_k^2 e^{\zeta \omega_k}$, has its unique zero at this point. In the case $\zeta < -a$ the left-hand side of Equation (15) is increasing from 0

to $\sum_{k=1}^n \rho_k e^{\zeta \omega_k}$; in the case $\zeta > -a$, it is decreasing to $-\infty$. Therefore, Equation (15) has a unique root; denote it by ζ_0 . Thus, the ray AM has a unique common point with the set F , corresponding to the values $\zeta = \zeta_0$ and $\tau = \tau_0 = (\sum_{k=1}^n \rho_k \omega_k e^{\zeta_0 \omega_k})^{-1}$ of parameters of Equations (13) and (14).

In connection with the above, later on we call the set F a *surface* and say that the point M lies *below* the surface F , if $\tau_0 > 1$, *on* the surface F if $\tau_0 = 1$, and *above* the surface F if $\tau_0 < 1$.

Lemma 5. *The function P has at least one zero if and only if the point M does not lie above the surface F .*

Proof. *Necessity.* Suppose the point M lies above the surface F . This implies that there exist parameters ζ_0 and $\tau_0 < 1$ such that

$$-\zeta_0 - a + \tau_0 \sum_{k=1}^n \rho_k e^{\zeta_0 \omega_k} = 0.$$

Consider the following auxiliary function $Q: \mathbb{R} \rightarrow \mathbb{R}$ and its derivatives:

$$Q(\zeta) = -\zeta - a + \tau_0 \sum_{k=1}^n \rho_k e^{\zeta \omega_k},$$

$$Q'(\zeta) = -1 + \tau_0 \sum_{k=1}^n \rho_k \omega_k e^{\zeta \omega_k}, \quad Q''(\zeta) = \tau_0 \sum_{k=1}^n \rho_k \omega_k^2 e^{\zeta \omega_k}.$$

Since $Q''(\zeta) > 0$ for all $\zeta \in \mathbb{R}$, the function Q' is increasing. Since $Q'(\zeta_0) = 0$ and Q' changes sign from minus to plus when passing through the point ζ_0 , it follows that ζ_0 is the unique minimum point of the function Q . Suppose ζ^* is a minimum point of the function P . Then, by virtue of the inequality $\tau_0 < 1$, we have

$$P(\zeta^*) > Q(\zeta^*) \geq Q(\zeta_0) = 0.$$

Thus, the continuous function P is positive at the point where it takes its smallest value. Hence, it has no zeros.

Sufficiency. Suppose the point M does not lie above the surface F . Then, there exist $\zeta_0 > 0$ and $\tau_0 \geq 1$ such that

$$-\zeta_0 - a + \tau_0 \sum_{k=1}^n \rho_k \omega_k e^{\zeta_0 \omega_k} = 0.$$

Hence,

$$P(\zeta_0) = -\zeta_0 - a + \sum_{k=1}^n \rho_k e^{\zeta_0 \omega_k} \leq -\zeta_0 - a + \tau_0 \sum_{k=1}^n \rho_k \omega_k e^{\zeta_0 \omega_k} = 0,$$

that is, $P(\zeta_0) \leq 0$. Since, moreover, $\lim_{\zeta \rightarrow +\infty} P(\zeta) = +\infty$, the function P has at least one zero. $\square$

Theorem 4. *The following statements are equivalent:*

- (a) *The solution of problem (11) is positive on the semiaxis $[0, +\infty)$;*
- (b) *The function P has at least one zero;*
- (c) *The point M lies on or below the surface F .
If $P(0) > 0$, then any of the above statements is equivalent to the following:*
- (d) *The solution of problem (7) is decreasing on the semiaxis $[0, +\infty)$.*

Proof. It follows from Lemma 4 that (a) $\Leftrightarrow$ (b).

It follows from Lemma 5 that (a) $\Leftrightarrow$ (c).

To prove the second part of the theorem, take into account that $P(0) = b - a$, and it follows from Lemma 3 that statement (a) is equivalent to the condition $l = \infty$. By virtue of the definition of l , this means that the solution to problem (7) is decreasing monotonically. $\square$

3.2.2. Necessary Stability Conditions

By Theorem 1, the validity of the inequality $P(0) \geq 0$ is a necessary condition for the stability of family (1), while the validity of $P(0) > 0$ is that for the asymptotic stability. Below we obtain more exact conditions.

Lemma 6. *If the function P has a zero in the set $(-\infty, 0]$, then family (1) is not asymptotically stable; if P has a zero in $(-\infty, 0)$, then family (1) is not stable.*

Proof. Suppose $P(\zeta_0) = 0$, where $\zeta_0 \leq 0$. Put in Equation (2), which is a special case of (1), $r_k(t) = \omega_k$, $k = \overline{1, n}$, and supply (2) with the initial condition $x(\xi) = e^{-\zeta_0 \xi}$, $\xi \in (-\infty, 0]$. The solution to the obtained problem is the function

$$x(t) = e^{-\zeta_0 t}, \quad t \in [0, +\infty).$$

If $\zeta_0 \leq 0$, then $x(t)$ does not tend to zero as $t \rightarrow +\infty$, therefore family (1) with the corresponding parameter set π is not asymptotically stable. If $\zeta_0 < 0$, then $x(t)$ is unbounded at $t \in [0, +\infty)$, therefore family (1) is not stable. $\square$

Lemma 7. *Suppose the function P have zeros. Then, all of them are in the set $(0, +\infty)$ if and only if $P(0) > 0$ and $P'(0) < 0$.*

Proof. *Necessity.* Consider the cases when at least one of the inequalities $P(0) > 0$ and $P'(0) < 0$ is not satisfied.

If $P(0) \leq 0$, then, since $\lim_{\zeta \rightarrow -\infty} P(\zeta) = +\infty$, the function P turns to 0 on the set $(-\infty, 0]$.

Let $P(0) > 0$ and $P'(0) \geq 0$. Consider the minimum point ζ^* of the function P . The function P' is increasing on $\mathbb{R}$, hence $\zeta^* \leq 0$. If $P(\zeta^*) \leq 0$, then P has zeros on the set $(-\infty, 0]$; if $P(\zeta^*) > 0$, then P has no zeros, which contradicts the condition of the lemma.

Sufficiency. Since the function P' is increasing on the set $(-\infty, 0]$, the inequality $P'(0) < 0$ implies that P' is negative on this set; that is, the function P is decreasing on it. But then, it follows from the inequality $P(0) > 0$ that P is positive on $(-\infty, 0]$, hence all zeros of P are in $(0, +\infty)$. $\square$

3.2.3. Sufficient Conditions for Asymptotic Stability

In this subsection, we show that if the conditions of Theorem 4 are satisfied, then the necessary conditions for the asymptotic stability obtained in the previous subsection are in fact sufficient, and, moreover, the asymptotic stability implies positiveness and exponential estimate for the Cauchy functions of all equations of family (1).

Lemma 8. *Suppose the function P has zeros in $(0, +\infty)$. Then, the Cauchy function of Equation (1) is positive on Δ .*

Proof. By the change of variables $x(t) = e^{at}u(t)$, transform Equation (1) to the form

$$(\mathcal{L}u)(t) \triangleq \dot{u}(t) + \sum_{k=1}^n e^{-at} \int_{t-\omega_k}^t e^{a\tau} u(\tau) d_\tau r_k(t, \tau) = 0, \quad t \geq 0. \quad (16)$$

It is easy to see that the Cauchy functions C and C_0 of Equations (1) and (16), respectively, are related by the equality

$$C_0(t, s) = C(t, s)e^{-a(t-s)}, \quad (t, s) \in \Delta.$$

By the conditions of the lemma, there exists $\zeta_0 > 0$ such that $P(\zeta_0) = 0$. Set

$$v(t) = e^{-(\zeta_0+a)t}, \quad t \in \mathbb{R}.$$

We have

$$\begin{aligned} (\mathcal{L}v)(t) &= -(\zeta_0 + a)e^{-(\zeta_0+a)t} + e^{-at} \sum_{k=1}^n \int_{t-\omega_k}^t e^{a\tau} e^{-(\zeta_0+a)\tau} d\tau r_k(t, \tau) \\ &\leq e^{-(\zeta_0+a)t} \left(-\zeta_0 + a + \sum_{k=1}^n e^{\zeta_0\omega_k} \rho_k \right) = e^{-(\zeta_0+a)t} P(\zeta_0) = 0. \end{aligned}$$

From here, by the lemma on a differential inequality (Lemma 2.4.3) [19], we obtain that the function C_0 is positive on the set Δ , and hence the function C is also positive. $\square$

Lemma 9. Suppose the function P has zeros, and all of them are in $(0, +\infty)$. Then, there is a number $T > 0$ such that for the Cauchy function C of any equation of family (1), and for all $s \in [0, +\infty)$, the function $C(\cdot, s)$ is not increasing on the semiaxis $(T + s, +\infty)$.

Proof. 1. Denote by O the origin of the space $\mathbb{R}^{n+1}$. Applying the concept of location with respect to the surface F , introduced in Section 3.2.1 for the point M , to an arbitrary point of the ray OM , we obtain the following: a point $M_\tau(a\tau, \rho_1\tau, \dots, \rho_n\tau)$ lies above the surface F if $\tau'_0 = \left(\tau \sum_{k=1}^n \rho_k \omega_k e^{\zeta'_0 \omega_k} \right)^{-1} < 1$, where $\zeta = \zeta'_0$ is the unique root of the equation

$$(-a\tau - \zeta) \sum_{k=1}^n \rho_k \omega_k e^{\zeta \omega_k} + \sum_{k=1}^n \rho_k e^{\zeta \omega_k} = 0.$$

In subsequent items 2–4 of the proof, we establish that if the conditions of the theorem are satisfied, then there is a number $\tau > 1$ such that the point M_τ lies above the surface F . This fact is used in item 5, which completes the proof by applying results of Section 3.1.

2. The ray OM is determined by the parametric equations

$$u_k = \rho_k \tau, \quad k = \overline{0, n}, \quad \tau \geq 0. \quad (17)$$

Substituting (17) into (13), we obtain that the ray OM has an intersection point with the set F if and only if the equation

$$\sum_{k=1}^n \rho_k e^{\zeta \omega_k} - \zeta \sum_{k=1}^n \rho_k \omega_k e^{\zeta \omega_k} = a \quad (18)$$

is solvable with respect to ζ . By Lemma 7, the inequality $\sum_{k=1}^n \rho_k > a$ holds. Therefore, Equation (18) is solvable; in the case $a \leq 0$, it has a unique root, which is positive, while in the case $a > 0$, it has two roots, positive and negative. Denote the positive root by ζ_1 . Thus, the ray OM has a common point $M_1(a\tau_1, \rho_1\tau_1, \dots, \rho_n\tau_1)$ with the surface F ; this point corresponds to the values $\zeta = \zeta_1$ and $\tau = \tau_1 = \left(\sum_{k=1}^n \rho_k \omega_k e^{\zeta_1 \omega_k} \right)^{-1}$ of parameters of Equations (13) and (17).

In the following items, we consider the two cases corresponding to different signs of the parameter a .

3. Suppose $a \leq 0$. First, prove that $\tau_1 \geq 1$. By the conditions of the theorem, the function P has zeros. Therefore, by Theorem 4, the point $M(a, \rho_1, \dots, \rho_n)$ does not lie above

the surface F , which is equivalent to the fact that there exists a root ζ_0 of Equation (15) such that $\tau_0 = (\sum_{k=1}^n \rho_k \omega_k e^{\zeta_0 \omega_k})^{-1} \geq 1$. Substituting ζ_0 in place of ζ in the left-hand side of (18), we see that

$$\begin{aligned} \sum_{k=1}^n \rho_k e^{\zeta_0 \omega_k} - \zeta_0 \sum_{k=1}^n \rho_k \omega_k e^{\zeta_0 \omega_k} &= (a + \zeta_0) \sum_{k=1}^n \rho_k \omega_k e^{\zeta_0 \omega_k} \\ &\quad - \zeta_0 \sum_{k=1}^n \rho_k \omega_k e^{\zeta_0 \omega_k} = a \sum_{k=1}^n \rho_k \omega_k e^{\zeta_0 \omega_k} = \frac{a}{\tau_0} \leq a. \end{aligned}$$

Hence, taking account of the fact that the left-hand side of (18) decreases as ζ increases, we obtain that $\zeta_0 \geq \zeta_1$; that is

$$\tau_1 = \frac{1}{\sum_{k=1}^n \rho_k \omega_k e^{\zeta_1 \omega_k}} \geq \frac{1}{\sum_{k=1}^n \rho_k \omega_k e^{\zeta_0 \omega_k}} = \tau_0 \geq 1.$$

Now, we show that for any $\tau > \tau_1$, the point $M_\tau(a\tau, \rho_1\tau, \dots, \rho_n\tau)$ of the ray OM lies above the surface F . The ray defined by equations

$$u_0 = a\tau, \quad u_k = \rho_k \tau s, \quad k = \overline{1, n}, \quad s \geq 0,$$

has a unique common point with the surface F , $M_{0\tau}(a\tau, \rho_1\tau s_\tau, \dots, \rho_n\tau s_\tau)$, where $s_\tau = (\tau \sum_{k=1}^n \rho_k \omega_k e^{\zeta_\tau \omega_k})^{-1}$, and ζ_τ is the unique root of the equation

$$(-a\tau - \zeta) \sum_{k=1}^n \rho_k \omega_k e^{\zeta \omega_k} + \sum_{k=1}^n \rho_k e^{\zeta \omega_k} = 0. \quad (19)$$

Substituting the root ζ_1 of Equation (18) into the left-hand side of (19) in place of ζ , we obtain:

$$\begin{aligned} (-a\tau - \zeta_1) \sum_{k=1}^n \rho_k \omega_k e^{\zeta_1 \omega_k} + \sum_{k=1}^n \rho_k e^{\zeta_1 \omega_k} &\geq (-a\tau_1 - \zeta_1) \sum_{k=1}^n \rho_k \omega_k e^{\zeta_1 \omega_k} + \sum_{k=1}^n \rho_k e^{\zeta_1 \omega_k} \\ &= -a\tau_1 \sum_{k=1}^n \rho_k \omega_k e^{\zeta_1 \omega_k} + a - \sum_{k=1}^n a_k e^{\zeta_1 \omega_k} + \sum_{k=1}^n \rho_k e^{\zeta_1 \omega_k} \\ &= -a \left(\tau_1 \sum_{k=1}^n \rho_k \omega_k e^{\zeta_1 \omega_k} - 1 \right) = 0. \end{aligned}$$

From here, with regard to the fact that the left-hand side of (19) is positive for $\zeta < \zeta_\tau$ and negative for $\zeta > \zeta_\tau$, we obtain that $\zeta_1 \leq \zeta_\tau$. Hence, taking account of $\tau > \tau_1$, we obtain

$$s_\tau = \frac{1}{\tau \sum_{k=1}^n \rho_k \omega_k e^{\zeta_\tau \omega_k}} < \frac{1}{\tau_1 \sum_{k=1}^n \rho_k \omega_k e^{\zeta_1 \omega_k}} = 1,$$

that is the point M_τ lies above the surface F .

4. Suppose $a > 0$. Set $\tau = (\sum_{k=1}^n \rho_k) \cdot (a \sum_{k=1}^n \rho_k \omega_k)^{-1}$. From Lemma 7, we have $\tau > 1$. Show that the point $M_\tau(a\tau, \rho_1\tau, \dots, \rho_n\tau)$ lies above the surface F . Draw a ray

$$u_0 = a\tau, \quad u_k = \rho_k \tau s, \quad k = \overline{1, n}, \quad s \geq 0.$$

It has a unique common point with the surface F , $M_{0\tau}(a\tau, \rho_1\tau s_\tau, \dots, \rho_n\tau s_\tau)$, where $s_\tau = (\tau \sum_{k=1}^n \rho_k \omega_k e^{\zeta_\tau \omega_k})^{-1}$, and ζ_τ is the unique root of equation

$$(-a\tau - \zeta) \sum_{k=1}^n \rho_k \omega_k e^{\zeta \omega_k} + \sum_{k=1}^n \rho_k e^{\zeta \omega_k} = 0.$$

For the indicated above value of τ , we have $\zeta_\tau = 0$. Hence,

$$s_\tau = \frac{1}{\sum_{k=1}^n \rho_k \omega_k} = \frac{a}{\sum_{k=1}^n \rho_k} < 1,$$

and the point M_τ lies above F , by the definition.

5. Substituting $a\tau$ in place of a and $\rho_k\tau$ in place of ρ_k , $k = \overline{1, n}$, in the definitions of the point M and the function $P(\zeta)$, we obtain, respectively, the point M_τ and the function

$$Q(\zeta) = -\zeta - a\tau + \sum_{k=1}^n \rho_k \tau e^{\zeta \omega_k}.$$

Note that the surface F does not depend on the parameters ρ_k . Taking this into account, by virtue of Theorem 4, from the fact that the point M_τ lies above the surface F , we obtain that the function Q has no real roots. Now, replace the parameters ω_k in the definition of the function P with the parameters $\tau\omega_k$, $k = \overline{1, n}$. We obtain the function

$$P_\tau(\zeta) = -\zeta - a + \tau \sum_{k=1}^n \rho_k e^{\zeta \tau \omega_k}.$$

Since $P_\tau(\zeta) = \frac{1}{\tau} Q(\zeta\tau)$, the function P_τ has no real roots. Hence, having made the same replacement in the definition of problem (7), by Theorem 4, we obtain that the solution $y: [0, +\infty) \rightarrow \mathbb{R}$ of the problem

$$\begin{cases} \dot{y}(t) = ay(t) - \sum_{k=1}^n \rho_k y(t - \omega_k \tau), & t \geq 0; \\ y(\xi) = 1, & \xi \leq 0; \end{cases}$$

is not decreasing on the semiaxis $[0, +\infty)$.

Set $l_\tau = \inf\{t \geq 0 : \dot{y}(t) > 0\}$, $\omega = \max_k \omega_k$, and $T = l_\tau + \tau\omega$. By Lemma 1, $h_\tau = -y(l_\tau) > 0$. Consider the Cauchy function C of Equation (1). Assume that there are points $s \geq 0$, $t_1 \geq s + T$, and $t_2 > t_1$, such that $C(t_1, s) < C(t_2, s)$. Since $\tau > 1$, we have $0 \leq r_k(t) \leq \tau\omega_k$, $k = \overline{1, n}$, $t \geq 0$. Therefore, Theorem 2 is applicable to the function C , where we put $l = l_\tau$, $h = h_\tau$. Thus, there is a point $t_0 \in (s, t_1)$ such that $-h_\tau C(t_0, s) = C(t_1, s)$. But, by Lemma 8, the Cauchy function is positive for all $(t, s) \in \Delta$. We have arrived at a contradiction; therefore, the function $C(\cdot, s)$ is not increasing on the semiaxis $(T + s, \infty)$. $\square$

Lemma 10. Suppose the function P has zeros, and all of them are on the semiaxis $(0, +\infty)$. Then, there exists a number $N > 0$ such that for the Cauchy function C of any equation of family (1) for all $(t, s) \in \Delta$ the following estimate is valid:

$$0 < C(t, s) \leq N e^{-\alpha(t-s)}, \quad (20)$$

where $\alpha = -a + \sum_{k=1}^n \rho_k > 0$.

Proof. The Cauchy function as the function $C(\cdot, s)$ of the first argument is a solution to the equation

$$\frac{\partial}{\partial t} C(t, s) - aC(t, s) = - \sum_{k=1}^n \int_{t-\omega_k}^t C(\eta, s) d_\eta r(t, \eta), \quad t \geq s, \quad (21)$$

supplemented with the initial conditions $C(\xi, s) = 0$, $\xi < s$, and $C(s, s) = 1$.

Adding the same term $\sum_{k=1}^n \rho_k C(t, s)$ to both sides of Equation (21), we obtain

$$\frac{\partial}{\partial t} C(t, s) + \alpha C(t, s) = \sum_{k=1}^n \int_{t-\omega_k}^t (C(t, s) - C(\eta, s)) d_\eta r(t, \eta), \quad t \geq s.$$

Therefore, for any fixed $T, s > 0$ the function $C(\cdot, s)$ is a solution to the equation

$$C(t, s) = e^{-\alpha(t-s-T)}C(s+T, s) + \int_{s+T}^t e^{-\alpha(t-\tau)} \sum_{k=1}^n \int_{\tau-\omega_k}^{\tau} (C(\tau, s) - C(\eta, s)) d_{\eta} r(t, \eta) d\tau, \quad t \geq s+T,$$

Moreover, by virtue of Lemma 9, the number $T > 0$ can be chosen such that the function $C(\cdot, s)$ is decreasing on the semiaxis $(T+s, \infty)$. Then, for $\tau > s+T$ we have $C(\tau, s) \leq C(\eta, s)$. Additionally, from Estimate (4), we obtain

$$\sup_{s \geq 0} |C(s+T, s)| < \infty.$$

Hence,

$$C(t, s) \leq e^{-\alpha(t-s)} e^{-\alpha T} \sup_{s \geq 0} |C(s+T, s)| = Ne^{-\alpha(t-s)}.$$

The positiveness of the Cauchy function is guaranteed by Lemma 8. $\square$

From Lemmas 6, 7 and 10, taking into account that the validity of the inequality $P(0) > 0$ is a necessary condition for stability (Theorem 1), we obtain the following result.

Theorem 5. Suppose one of conditions (a)–(c) of Theorem 4 is satisfied. Then, the following statements are equivalent:

- (a) Family (1) is asymptotically stable;
- (b) All zeros of the function P are on the semiaxis $(0, \infty)$;
- (c) The condition $P'(0) < 0$ is satisfied.

We can make the obtained result more precise using Lemmas 8–10.

Theorem 6. Suppose one of conditions (a)–(c) of Theorem 4 is satisfied and family (1) is asymptotically stable. Then

- (a) The Cauchy functions of all equations in family (1) are positive on the set Δ ;
- (b) There exists $T > 0$ such that the Cauchy function of any equation in family (1) for each $s \in [0, \infty)$ is decreasing with respect to the first argument on the semiaxis $(T+s, \infty)$;
- (c) For the Cauchy function of any equation in family (1), the estimate (20) is valid.

Note that condition (c) of Theorem 6 means that the asymptotic stability of family (1) is equivalent to its exponential stability.

3.2.4. Sufficient Conditions for Uniform Stability

Here, we find the conditions for the uniform stability of family (1) in the case that conditions (a)–(c) of Theorem 4 are satisfied. By condition (b), the function P has real zeros. If the function P has a zero on the semiaxis $(-\infty, 0)$, then by Lemma 6, family (1) is not stable. If all zeros of the function P are on the semiaxis $(0, +\infty)$, then by Lemma 10, family (1) is asymptotically stable. Thus, the only case that needs additional study is $P(0) = 0$ or, in the notation of Section 3, $b = a$.

Lemma 11. If $P(0) = 0$ and $P'(0) > 0$, then family (1) is not stable.

Proof. Consider the function P on the set $(-\infty, 0]$. At the point $\xi = 0$ it vanishes, while its derivative is positive, which implies that in some left neighborhood of this point the function P takes negative values. Since $\lim_{\xi \rightarrow -\infty} P(\xi) = +\infty$, there is a point $\xi_0 \in (-\infty, 0)$ at which $P(\xi_0) = 0$. Set in Equation (2) $r_k(t) = \omega_k$, $k = \overline{1, n}$, and supplement the equation with the initial condition

$$x(\xi) = e^{-\xi_0 \xi}, \quad \xi \in (-\infty, 0].$$

Substituting $x(t) = e^{-\zeta_0 t}$ into the equation for $t \geq 0$, we obtain:

$$-\zeta_0 e^{-\zeta_0 t} - a e^{-\zeta_0 t} + \sum_{k=1}^n \rho_k e^{-\zeta_0(t-\omega_k)} = e^{-\zeta_0 t} P(\zeta_0) = 0.$$

Thus, the function x is a solution to Equation (1), and x increases to infinity as $t \rightarrow +\infty$. Therefore, family (1) is not stable (Section 3.3) [19]. $\square$

Lemma 12. *If $P(0) = 0$ and $P'(0) < 0$, then family (1) is uniformly stable.*

Proof. Let us show that the solution x of the inhomogeneous problem corresponding to Equation (1),

$$\begin{cases} \dot{x}(t) - ax(t) + \sum_{k=1}^n \int_{t-\omega_k}^t x(\tau) d_\tau r_k(t, \tau) = f(t), & t \geq 0, \\ x(\xi) = 0, & \xi \leq 0, \end{cases} \quad (22)$$

for any right-hand side $f \in L_1$ belongs to the space C_∞ .

Since $a = \sum_{k=1}^n \rho_k$, the equation from problem (22) can be rewritten in the form

$$\dot{x}(t) = \sum_{k=1}^n \int_{t-\omega_k}^t \left(\int_{\tau}^t \dot{x}(\zeta) d\zeta \right) d_\tau r_k(t, \tau) + f(t).$$

Estimate the norm of the integral operator K ,

$$(Ky)(t) \triangleq \sum_{k=1}^n \int_{t-\omega_k}^t \left(\int_{\tau}^t y(\zeta) d\zeta \right) d_\tau r_k(t, \tau).$$

Suppose $y \in L_1$. Show that $Ky \in L_1$. Changing the order of integration and taking into account that the delay is bounded, we obtain:

$$\begin{aligned} \|Ky\|_1 &\leq \int_0^\infty \sum_{k=1}^n \int_{t-\omega_k}^t \left(\int_s^t |y(\zeta)| d\zeta \right) d_\tau r_k(t, \tau) dt \\ &\leq \int_0^\infty |y(\zeta)| \sum_{k=1}^n \int_{\zeta}^{\zeta+\omega_k} \int_{t-\omega_k}^t d_\tau r_k(t, \tau) d\zeta dt = \sum_{k=1}^n \rho_k \omega_k \|y\|_1. \end{aligned}$$

Since, by the conditions of the lemma, $P'(0) < 0$, it follows that $\sum_{k=1}^n \rho_k \omega_k < 1$, the operator K acts from L_1 to L_1 , and $\|K\|_{L_1 \rightarrow L_1} < 1$.

So, the operator $I - K$ is continuously invertible, and for any $f \in L_1$, we have

$$\dot{x} = (I - K)^{-1} f \in L_1.$$

Since $x(t) = x(t) - x(0) = \int_0^t \dot{x}(\tau) d\tau$, the function x is bounded on $\mathbb{R}_+$. By the Bohl–Perron theorem (Theorem 3.3.3) [19], it follows that the Cauchy function of any equation of family (1) is bounded; that is, family (1) is uniformly stable. $\square$

Lemma 13. *If $P(0) = 0$ and $P'(0) = 0$, then family (1) is not stable.*

Proof. In Equation (2), set $r_k(t) = \omega_k$, $k = \overline{1, n}$, and supplement the equation with the initial condition $x(\xi) = \xi$, $\xi \in (-\infty, 0]$. Substituting $x(t) = t$ into the equation, we obtain

$$1 - at + \sum_{k=1}^n \rho_k(t - \omega_k) = t \left(a - \sum_{k=1}^n \rho_k \right) + \left(1 - \sum_{k=1}^n \rho_k \omega_k \right) = tP(0) + P'(0) = 0,$$

that is, the unbounded function x is a solution to Equation (2). Therefore, family (1) is not stable. $\square$

From Theorem 5 and Lemmas 11–13, we obtain the following stability criterion.

Theorem 7. Suppose one of conditions (a)–(c) of Theorem 4 is satisfied. Then, the family of Equations (1) is uniformly stable if and only if the inequalities $P(0) \geq 0$ and $P'(0) < 0$ hold.

Note that Lemmas 11–13 establish that the stability of family (1) is equivalent to its uniform stability.

3.3. Theorem on 5/3

In this section, we consider the case $a \leq 0$ and observe the relationship between the values l and $y(l)$. Recall the notation $h = -y(l)$, where y is the solution to Problem (7).

Theorem 8. If $a \leq 0$ and $h \geq \frac{1}{2}$, then $l \leq \frac{2\omega}{2h+1} + \omega$.

Proof. Let y_0 be the solution to the problem

$$\begin{cases} \dot{y}_0(t) = -\frac{2h+1}{2\omega} y_0(t-\omega), & t \geq 0, \\ y_0(\xi) = 1, & \xi \leq 0. \end{cases} \quad (23)$$

It is easily seen that

$$y_0(t) = \begin{cases} 1 - \frac{2h+1}{2\omega} t, & t \in [0, \omega], \\ \frac{1}{2} - h - \frac{2h+1}{2\omega} (t - \omega) + \frac{(2h+1)^2}{8\omega^2} (t - \omega)^2, & t \in [\omega, 2\omega], \dots \end{cases}$$

Thus, if $h \geq \frac{1}{2}$, then the function y_0 reaches its first minimum $y_0\left(\frac{2\omega}{2h+1} + \omega\right) = -h$ in the segment $[0, 2\omega]$. Note that y_0 is linear on $[0, \omega]$, $y_0(0) = 1$, and $y_0\left(\frac{2\omega}{2h+1}\right) = 0$.

Define the family of functions $\{\eta_s\}_{s \in \mathbb{R}}$, $\eta_s: \mathbb{R} \rightarrow \mathbb{R}$, as follows:

$$\eta_s(t) = \begin{cases} 1, & t < s; \\ y_0(t-s), & t \in [s, s + \frac{2\omega}{2h+1} + \omega]; \\ -h, & t > s + \frac{2\omega}{2h+1} + \omega. \end{cases}$$

Associate the behavior of functions $\{\eta_s\}$ with that of the solution y to test problem (7).

There is $s_0 = \inf\{s : y(t) \leq \eta_s(t), t \in [0, l]\} \in [0, l]$. For brevity, denote $\eta = \eta_{s_0}$. We have: $y(0) = \eta(s_0) = 1$, $y(l) = \eta(s_0 + \frac{2\omega}{2h+1} + \omega) = -h$, and functions y and η are continuously differentiable and decreasing in the intervals $(0, l)$ and $(s_0, s_0 + \frac{2\omega}{2h+1} + \omega)$, respectively. Here, $l \leq s_0 + \frac{2\omega}{2h+1} + \omega$, since otherwise $y(s_0 + \frac{2\omega}{2h+1} + \omega) > -h = \eta(s_0 + \frac{2\omega}{2h+1} + \omega)$, which contradicts the definition of the function η .

There is a point $t_0 \in [0, l]$ such that

$$y(t_0) = \eta(t_0). \quad (24)$$

Indeed, with regard to the definition of the point s_0 and the uniform continuity of the function $\delta = \eta - y$ on the interval $[0, l]$ we have $\inf\{\delta(t) : t \in [0, l]\} = 0$, which implies that the function δ reaches zero.

Thus, if $t_0 = 0$, then $s_0 = 0$, and therefore $l \leq \frac{2\omega}{2h+1} + \omega$. Let us show that the case $t_0 > 0$ is impossible; thus, the theorem will be proved.

Note that if $t_0 > 0$, then

$$\dot{y}(t_0) = \dot{\eta}(t_0). \quad (25)$$

Indeed, if $l = s_0 + \frac{2\omega}{2h+1} + \omega$, then $\dot{y}(t_0) = \dot{\eta}(t_0) = 0$, but if $l < s_0 + \frac{2\omega}{2h+1} + \omega$ and $\dot{y}(t_0) \neq \dot{\eta}(t_0)$, then in any neighborhood of the point t_0 , there is a point t_1 such that $y(t_1) > \eta(t_1)$, which contradicts the definition of η .

Further, from (7) under the condition $a \leq 0$, it follows that

$$\forall t \in (0, l) : \dot{y}(t) \geq 0. \quad (26)$$

In the case that $t_0 \in (0, s_0 + \omega]$, we have $\dot{\eta}(t_0) = 0$, which, taking account of (24)–(26), contradicts the definition of the function η . This means that this case is impossible.

Consider the remaining case $t_0 > s_0 + \omega$.

If $t_0 = l$, then $\eta(t_0) = y(l) = -h$, hence $t_0 = \frac{2\omega}{2h+1} + \omega$. But then, for $k = \overline{1, n}$ we have $y(l - \omega_k) \leq y(l - \omega) < \eta(l - \omega) = 0$, hence $\dot{y}(l) < 0$, which is incorrect. Therefore, $t_0 < l \leq \frac{2\omega}{2h+1} + \omega$ and $\eta(t_0 - \omega) > \eta(l - \omega) \geq 0$.

Compare the values of $y(l)$ and $\eta(l)$. From Equations (7) and (23), we have

$$\eta(t_0) - \eta(l) = - \int_{t_0}^l \dot{\eta}(s) ds = \int_{t_0}^l \frac{2h+1}{2\omega} \eta(s - \omega) ds = \frac{2h+1}{2\omega} \int_{t_0-\omega}^{l-\omega} \eta(s) ds; \quad (27)$$

$$\begin{aligned} y(t_0) - y(l) &= - \int_{t_0}^l \dot{y}(s) ds = - \int_{t_0}^l \left(ay(s) - \sum_{k=1}^n a_k y(s - \omega_k) \right) ds \\ &= -a \int_{t_0}^l y(s) ds + \sum_{k=1}^n a_k \int_{t_0-\omega_k}^{l-\omega_k} y(s) ds. \end{aligned} \quad (28)$$

On the other hand, from (25), we have

$$\frac{2h+1}{2\omega} \eta(t_0 - \omega) = -ay(t_0) + \sum_{k=1}^n a_k y(t_0 - \omega_k). \quad (29)$$

Multiply both sides of equality (29) by $\frac{l-t_0}{2} \left[1 + \frac{\eta(l-\omega)}{\eta(t_0-\omega)} \right]$. In the left-hand side, we obtain the product of $\frac{2h+1}{2\omega}$ by the area of the trapezoid with the vertices at the points $(t_0 - \omega, 0)$, $(t_0 - \omega, \eta(t_0 - \omega))$, $(l - \omega, \eta(l - \omega))$, and $(l - \omega, 0)$, and therefore the value (27). In the right-hand side, we obtain a value greater than (28). Indeed, for $k = \overline{1, n}$, we have

- If $y(t_0 - \omega_k) = 0$, then $y(t_0 - \omega_k) \frac{l-t_0}{2} \left[1 + \frac{\eta(l-\omega)}{\eta(t_0-\omega)} \right] = 0 > \int_{t_0-\omega_k}^{l-\omega_k} y(s) ds$;
- If $y(t_0 - \omega_k) > 0$, then, taking into account $\dot{\eta}(t) = 0$ for $t \in [t_0 - \omega, l - \omega]$, (26) and $y\left(\frac{2\omega}{2h+1}\right) < 0$, for $t \in [t_0 - \omega_k, l - \omega_k]$ we have $\dot{y}(t) < \frac{y(t_0-\omega_k)}{\eta(t_0-\omega_k)} \dot{\eta}(t)$; hence, $y(l - \omega_k) < \frac{\eta(l-\omega_k)}{\eta(t_0-\omega_k)} y(t_0 - \omega_k) = \frac{\eta(l-\omega)}{\eta(t_0-\omega)} y(t_0 - \omega_k)$, from which

$$\begin{aligned} y(t_0 - \omega_k) \frac{l-t_0}{2} \left[1 + \frac{\eta(l-\omega)}{\eta(t_0-\omega)} \right] &> \frac{l-t_0}{2} [y(t_0 - \omega_k) + y(l - \omega_k)] \\ &\geq \int_{t_0-\omega_k}^{l-\omega_k} y(s) ds; \end{aligned} \quad (30)$$

- If $y(t_0 - \omega_k) < 0$, then, taking into account $y(l - \omega_k) < y(t_0 - \omega_k)$ and (26), we obtain the chain of inequalities (30).

Since $y(t_0) < 0$, similar to the latter case $\frac{l-t_0}{2} y(t_0) \left[1 + \frac{\eta(l-\omega)}{\eta(t_0-\omega)} \right] > \int_{t_0}^l y(s) ds$.

Thus, $\eta(t_0) - \eta(l) > y(t_0) - y(l)$. From here, with regard to (24), we obtain $y(l) > \eta(l)$, which contradicts the definition of the function η . The case $t_0 > s_0 + \omega$ is impossible. $\square$

Corollary 1. Suppose $a \leq 0$. Then,

- If $l \geq 5\omega/3$, then $y(l) \geq -1$;
- If $l > 5\omega/3$, then $y(l) > -1$.

3.4. Conclusions

Let us combine the basic results of Section 2 and formulate them in the form of several theorems. Denote $M = M(a, \rho_1, \dots, \rho_n)$.

Theorem 9. Family (1) is

- Exponentially stable if and only if one of the following conditions is satisfied:
 - (e1) Point M does not lie above the surface F , $a < \sum_{k=1}^n \rho_k$ and $\sum_{k=1}^n \rho_k \omega_k < 1$;
 - (e2) Point M lies above the surface F and $y(l) > -1$;
- Uniformly stable if and only if one of the following conditions is satisfied:
 - (b1) Point M does not lie above the surface F , $a \leq \sum_{k=1}^n \rho_k$ and $\sum_{k=1}^n \rho_k \omega_k < 1$;
 - (b2) Point M lies above the surface F and $y(l) \geq -1$.

Proof. Taking account of

$$P(0) = \sum_{k=1}^n \rho_k - a, \quad P'(0) = \sum_{k=1}^n \rho_k \omega_k,$$

and Theorem 4, item (e1) follows from Theorems 5 and 6, and item (b1) coincides with Theorem 7.

Prove items (e2) and (b2). If the point M lies above the surface F , then by Theorem 4, the function $P(\zeta)$ has no real roots. Since $P(\zeta) \rightarrow +\infty$ as $\zeta \rightarrow +\infty$, it follows that $P(0) > 0$, hence $l < \infty$. Taking this into account, the result follows from Theorem 3. $\square$

If the parameters of Equation (1) are given numbers, then Theorem 9 can be considered as a stability testing algorithm suitable for computer implementation. The first items of these theorems correspond to the condition $l = \infty$, the verification of which, by virtue of Theorem 4, can be replaced by checking that the roots of the equation $P(\zeta) = 0$ are real and the solution to problem (11) is positive.

The items (e2) and (b2) of Theorem 9 correspond to the case $l < \infty$; this means that the first minimum of the solution to problem (11) is negative and is reached at the end point. Since problem (11) is a differential-difference equation with a given initial function, the construction of the solution to the problem on each finite segment is carried out by integration “step by step”. The only problem is estimating the number l , which is the length of the segment on which it is necessary to construct the solution to the equation of problem (11).

In the case $a \leq 0$, stability conditions for family (1) can be expressed exclusively in terms of the values of l and $y(l)$.

Theorem 10. Suppose $a \leq 0$. Then, family (1) is

- Exponentially stable if and only if one of the following conditions is satisfied:
 - $\frac{5\omega}{3} < l \leq +\infty$;
 - $l \leq \frac{5\omega}{3}$ and $y(l) > -1$;
- Uniformly stable if and only if one of the following conditions is satisfied:
 - $\frac{5\omega}{3} < l \leq +\infty$;
 - $l \leq \frac{5\omega}{3}$ and $y(l) \geq -1$.

Proof. If $a \leq 0$, then $P(0) = -a + \sum_{k=1}^n \rho_k > 0$.

Suppose $l = \infty$. Then, by Theorem 4, the function P has zeros. Since $a \leq 0$, they cannot be in $(-\infty, 0]$. This means that the exponential (and also uniform) stability of family (1) follows from Theorem 5.

Suppose now $\frac{5\omega}{3} < l < +\infty$. Then, by Corollary 1, we have $y(l) > -1$; therefore, by Theorem 3, the family is exponentially stable.

Finally, in the case $l \leq \frac{5\omega}{3}$, it follows from Theorem 3 that family (1) is exponentially stable if $y(l) > -1$, and uniformly stable if $y(l) \geq -1$. $\square$

Theorem 10 significantly simplifies checking the stability of family (1): it is sufficient to construct the solution to problem (11) on the segment $[0, \frac{5\omega}{3}]$. If the solution does not take values less than -1 on this segment, then family (1) is stable. The case $l = \infty$ does not need to be considered separately.

The condition $y(l) = -1$, which defines the boundary of the stability region, can be expressed in an equivalent form in terms of the solution u of problem (11).

Theorem 11. *If $l < \infty$, then the inequality $y(l) > -1$ is equivalent to the inequality*

$$(b-a) \int_0^l e^{as} u(s) ds < 2,$$

and the inequality $y(l) \geq -1$ is equivalent to the inequality

$$(b-a) \int_0^l e^{as} u(s) ds \leq 2.$$

Proof. The theorem follows from Formula (12). $\square$

Theorem 12. *Suppose $l < \infty$. If $a = 0$, then*

- *The following statements are equivalent:*
 - (a1) $y(l) > -1$;
 - (b1) $\sum_{k=1}^n \rho_k \int_0^l u(s) ds < 2$;
 - (c1) $\sum_{k=1}^n \left(\rho_k \int_{l-\omega_k}^l u(s) ds \right) < 1$;
- *The following statements are equivalent:*
 - (a2) $y(l) \geq -1$;
 - (b2) $\sum_{k=1}^n \rho_k \int_0^l u(s) ds \leq 2$;
 - (c2) $\sum_{k=1}^n \left(\rho_k \int_{l-\omega_k}^l u(s) ds \right) \leq 1$.

Proof. The equivalence of inequalities (ax) and (bx) follows from Theorem 11.

By Lemma 3, we have $u(l) = 0$. Using Formula (12), we obtain

$$1 = u(0) - u(l) = - \int_0^l \dot{u}(s) ds = \sum_{k=1}^n \left(\rho_k \int_0^l u(s - \omega_k) ds \right) = \sum_{k=1}^n \left(\rho_k \int_0^{l-\omega_k} u(s) ds \right).$$

This implies the equivalence of inequalities (bx) and (cx). $\square$

4. Effective Stability Tests

In this section, on the basis of the results of Section 2, we obtain effective stability conditions for several classes of equations described by a small number of parameters.

Section 4.1 is dedicated to the case $n = 1$, $a = 0$, in which family (1) turns into a family of equations having a one-dimensional stability region. Next, we study in detail the cases when the stability regions of family (1) are two- and three-dimensional.

The boundaries of the regions are represented as a union of a finite or countable set of curves or surfaces defined analytically. Stability regions are constructed in coordinate systems defined by parameters of equations.

4.1. One Term, One Delay

Set $a = 0$ and $n = 1$ in Equation (1), and denote $\omega_1 = \omega$. Thus, we consider a family of equations of the form

$$\dot{x}(t) + b(Tx)(t) = f(t), \quad t \geq 0, \quad (31)$$

where

$$(Tx)(t) = \int_{t-\omega}^t x(\tau) d_\tau r(t, \tau), \quad \int_{t-\omega}^t d_\tau r(t, \tau) = 1, \quad (32)$$

$b \in \mathbb{R}$, $r(t, \cdot)$ is nondecreasing, $r(t - \omega, t) = 0$, $r(\cdot, \tau)$ is locally integrable.

If $b < 0$, then family (31) is unstable, since the equation $\dot{x}(t) + bx(t) = 0$, which is a special case of (31), is unstable.

In case $b = 0$, we have $C(t, s) \equiv 1$, hence family (31) is uniformly (but not asymptotically) stable.

Investigate the case $b > 0$ on the basis of Theorem 10.

On the segment $[0, 2\omega]$, the solution of the test problem of family (31)

$$\begin{cases} \dot{y}(t) + by(t - \omega) = 0, & t \in [0, l]; \\ y(\xi) = 1, & \xi \leq 0; \end{cases} \quad (33)$$

has the form

$$y(t) = \begin{cases} 1 - bt, & t \in [0, \omega]; \\ 1 - bt + \frac{1}{2}b^2(t - \omega)^2, & t \in [\omega, 2\omega]. \end{cases}$$

Suppose l , which is the point of the first minimum of the function y , is on the segment $[0, 2\omega]$; then $l = \omega + \frac{1}{b}$ and $y(l) = \frac{1}{2} - b\omega$.

If $b\omega < \frac{3}{2}$, then $l > \frac{5\omega}{3}$. By Theorem 10, family (31) is exponentially stable.

If $b\omega = \frac{3}{2}$, then $l = \frac{5\omega}{3}$ and $y(l) = -1$. By Theorem 10, family (31) is uniformly stable, but is not asymptotically stable.

If $b\omega > \frac{3}{2}$, then $l \in [0, \frac{5\omega}{3})$ and $y(l) < -1$. By Theorem 10, family (31) is not uniformly stable.

Suppose now that $l > 2\omega$. Then, $l > \frac{5\omega}{3}$, and by Theorem 10, family (31) is exponentially stable.

Combining the results in a single statement, we obtain the following theorem.

Theorem 13. Family (31) is

- Exponentially stable, if and only if $0 < b\omega < 3/2$;
- Uniformly stable, if and only if $0 \leq b\omega \leq 3/2$.

Consider a special case of family (31) that is the family of equations with one concentrated delay

$$\dot{x}(t) + bx(t - r(t)) = f(t), \quad t > 0, \quad (34)$$

where $b \in \mathbb{R}$ and $0 \leq r(t) \leq \omega$. Since the equation in test problem (33) is a special case of Equation (34), where

$$r(t) = \begin{cases} t, & t \in [0, \omega], \\ \omega, & t \in (\omega, l], \end{cases}$$

it follows that Theorem 13 remains valid if family (31) is replaced by family (34); thus, it presents a generalization of the famous “3/2-theorem” by Myshkis [41]. Note that Theorem 13 was also obtained by a different method in [42].

4.2. Properties of the Solution to the Simplest Initial Value Problem

In this section, we obtain a number of results that is used later to describe stability regions of some equations explicitly.

Consider the problem

$$\begin{cases} \dot{z}(t) = -pz(t-1), & t \geq 0, \\ z(\xi) = 0, & \xi < 0, \\ z(0) = 1. \end{cases} \quad (35)$$

Denote the solution to problem (35), with coefficient p , by z_p .

The equation in problem (35) belongs to the class of equations whose solution can be obtained explicitly by integrating “step by step”, sequentially on the intervals $[0, 1]$, $[1, 2]$, $\dots$, $[n, n+1]$, $\dots$

Proposition 1 ([43]). *The solution z_p to problem (35) is determined by the formula*

$$z_p(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} p^k (t-k)^k \chi(t-k),$$

where χ is the characteristic function of the set $[0, +\infty)$.

Solutions of scalar FDEs, unlike ODEs, do not necessarily preserve sign on $\mathbb{R}_+$. The existence of zeros of solutions to Equation (35) is determined by the value of p .

Proposition 2 ([41] (Th. 39, p. 190) and [44]). *If $p \leq 1/e$, then the function z_p is positive on $[0, +\infty)$; if $p > 1/e$, then it is oscillating (that is it has an unbounded on the right sequence of zeros).*

Consider the case $p > 1/e$. Denote the smallest of zeros of the function z_p by t_p .

Lemma 14. *If $z_p(t) > 0$ for all $t \in [0, T]$ and $p > q$, then $z_q(t) > 0$ for all $t \in [0, T]$.*

Proof. By virtue of the Cauchy formula, Equation (35) with the coefficient q is equivalent to the equation

$$z_q(t) = z_p(t) + (p-q) \int_0^t z_p(t-\tau) z_q(\tau-1) d\tau, \quad t \in [0, T]. \quad (36)$$

It follows from the conditions of the lemma that $z_p(t-\tau) > 0$ for all $\tau \in [0, T]$.

Suppose $z_q(t) > 0$ for all $t \in [0, n]$, where n is a natural number such that $[0, n] \subseteq [0, T]$. Then, it follows from (36) that $z_q(t) > 0$ for all $t \in [0, n+1] \cap [0, T]$.

It remains to note that $z_q(t) = 1 > 0$ for $t \in [0, 1]$. $\square$

Lemma 15. *If $z_p(t) > 0$ for all $t \in [0, T]$ and $p > q$, then $z_q(t) > z_p(t)$ for all $t \in [0, T]$.*

Proof. It follows from (36) and Lemma 14. $\square$

Lemma 16. *Suppose the functions z_p and z_q have zeros on $\mathbb{R}_+$. Then, the inequality $p > q$ holds if and only if $t_q > t_p$.*

Proof. Assume that $p > q$ and $t_q \leq t_p$. Then, for $t < t_q$, the inequalities $z_p(t) > 0$ and $z_q(t) > 0$ are true. From Equality (36) we obtain:

$$0 = z_q(t_q) = z_p(t_q) + (p-q) \int_0^{t_q} z_p(t_q-\tau) z_q(\tau-1) d\tau > 0,$$

that is, we have arrived at a contradiction.

Conversely, assume that $t_q > t_p$ and $p \leq q$. Then, if $p = q$, then $t_q = t_p$; if $p < q$, then, by what was proved above, $t_q < t_p$, that is again we have arrived at a contradiction. $\square$

Lemma 17. Suppose $p > q$, the functions z_p and z_q have zeros on $\mathbb{R}_+$, and $s \in (q, p)$. Then, the function z_s also has zeros on $\mathbb{R}_+$, and $t_s \in (t_p, t_q)$.

Proof. Suppose that $z_s(t) \neq 0$ for all $t \in \mathbb{R}_+$. Then, $z_s(t_q) > 0$. Since $s > q$, then by Lemma 15, we have $0 = z_q(t_q) > z_s(t_q)$, and we arrive at a contradiction. The second statement of the lemma follows from Lemma 16. $\square$

Lemma 18. Suppose the function z_p has zeros on $\mathbb{R}_+$, and $\tau > t_p$. Then, there exists a unique $q < p$ such that $\tau = t_q$.

Proof. Divide the proof of the existence of the number q into two stages.

1. Assume that $\tau \in (t_p, t_p + 1]$. Denote $\varphi(s) = z_s(\tau)$ and consider the function φ on the interval $[0, p]$. Obviously, $\varphi(0) = z_0(\tau) = 1 > 0$. Let us show that $\varphi(p) < 0$. To do this, consider the function z_p on the interval $[t_p, \tau]$. Since $z_p(t) > 0$ for $t < t_p$, and $\dot{z}_p(t) = -pz_p(t-1)$, then z_p is decreasing monotonically on $[t_p, \tau]$. But $z_p(t_p) = 0$, therefore $z_p(\tau) = \varphi(p) < 0$. It follows from Proposition 1 that the function φ is continuous on the set $[0, p]$. Using this, find $q \in (0, p)$ for which $\varphi(q) = z_q(\tau) = 0$. Make sure that $\tau = t_q$. Indeed, if $\tau > t_q$, then by Rolle's theorem, there is a point $\xi \in (t_p, \tau)$ such that $\dot{z}_q(\xi) = 0$, and then it follows from Equation (35) that $z_q(\xi - 1) = 0$. But $\xi - 1 < t_p$, and this contradicts the definition of t_q .

2. Assume that $\tau \in (t_p + n, t_p + n + 1]$, where $n \in \mathbb{N}$. First, set $\tau = t_p + 1$ and find, taking into account the statement proved at the first stage, a number $p_1 < p$ such that $\tau = t_p + 1 = t_{p_1}$. Further, set $\tau = t_p + 2 = t_{p_1} + 1$ and find, by the same statement, a number $p_2 < p_1$ such that $\tau = t_{p_1} + 1 = t_{p_2}$. Repeating this procedure n times, we reach the set

$$(t_p + n, t_p + n + 1] = (t_{p_n}, t_{p_n} + 1],$$

to which it remains to apply once again the statement proved at the first stage and find

$$q < p_n < \dots < p_1 < p$$

such that $\tau = t_q$.

The uniqueness of q follows from Lemma 16. $\square$

We see from Proposition 1 that for each fixed $n \in \mathbb{N}_0$, the quantity $z_p(n+1)$ as a function of the argument p defined on the set $[0, +\infty)$ is a polynomial of n th degree in the variable $p \in \mathbb{R}$:

$$z_p(n+1) = 1 + \sum_{k=1}^n (-1)^k \frac{p^k}{k!} (n+1-k)^k.$$

Denote $z_p(n+1) = H_n(p)$. Note that $H_0(p) \equiv 1$.

The following statements associates roots of the polynomials $H_n(p)$ with roots of the functions z_p .

Theorem 14. For any $n \in \mathbb{N}$:

- The polynomial H_n has at least one positive real root;
- If p is the smallest positive root of the polynomial H_n , then $t_p = n + 1$.

Proof. It is obvious that all polynomials of odd degrees have at least one root on $\mathbb{R}_+$. Let p_0 denote the smallest root of a polynomial H_{2k-1} . Assume that $t_{p_0} < 2k$. Then, by Lemma 18 there is $q < p_0$ such that $t_q = 2k$, i.e., $0 = z_q(2k) = H_{2k-1}(q)$, contrary to the definition of p_0 . The statement is proven for polynomials of odd degrees.

Now consider a polynomial H_{2k} . Since $2k < 2k + 1$, by Lemma 18 there is $p_1 < p_0$ such that $t_{p_1} = 2k + 1$, which implies, by virtue of $H_{2k}(p_1) = u_{p_1}(2k + 1) = 0$, that the

polynomial H_{2k} has roots on $\mathbb{R}_+$. Let p_2 be its smallest root. If $p_2 < p_1$, then from Lemma 18, we obtain $t_{p_2} > t_{p_1} = 2k + 1$, therefore, $u_{p_1}(t_{p_1}) = u_{p_1}(2k + 1) > 0$. But then $H_{2k}(p_1) > 0$, which leads to a contradiction. Therefore, $p_2 = p_1$, that is the statement is also true for polynomials of even degrees. $\square$

Denote by p_n the smallest positive root of the polynomial H_n and establish some properties of the sequence $\{p_n\}_{n=1}^{\infty}$.

The numbers p_n can be found with any degree of accuracy; Table 1 shows the first 12 of them (accurate to 10^{-3}).

Table 1. Values $p_n, n = \overline{1, 12}$.

p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}
1	0.586	0.482	0.439	0.417	0.404	0.396	0.389	0.385	0.382	0.380	0.378

Theorem 15. The sequence $\{p_n\}_{n=1}^{\infty}$ decreases monotonically; $\lim_{n \rightarrow \infty} p_n = 1/e$.

Proof. By Theorem 14 we have $t_{p_{n-1}} = n$, $t_{p_n} = n + 1$. Since $n < n + 1$, it follows that $t_{p_{n-1}} < t_{p_n}$, and then $p_n < p_{n-1}$ by Lemma 18. Moreover, the sequence $\{p_n\}$ is bounded below. Indeed, the polynomials H_n do not have zeros for $p \leq 0$, hence $p_n > 0$.

From the above, it follows that the sequence $\{p_n\}_{n=1}^{\infty}$ converges to a limit, which is its infimum. Next, prove that $\alpha = \inf_n p_n = 1/e$.

Assume that $p_n < 1/e$ for some $n \in \mathbb{N}$. Obviously, the first zero of the polynomial $H_1(p) = 1 - p$ is $p_1 = 1$. Thus, $p_n < 1/e < p_1$; therefore, by virtue of Lemma 17 and Theorem 14, the solution to problem (35) for $p = 1/e$ has a root on the interval $(2, n + 1)$, which contradicts Proposition 2. Therefore, for any $n \in \mathbb{N}$, the estimate $p_n \geq 1/e$ is valid; that is, $\alpha \geq 1/e$.

Assume now that $\alpha > 1/e$. By virtue of Proposition 2, the solution to problem (35) for $p = \alpha$ has an infinite number of zeros on $\mathbb{R}_+$. If $t_\alpha \in [n, n + 1]$, then by Lemma 18 and Theorem 14, we obtain $p_{n+1} < p_n \leq \alpha$, which is not true. Therefore, $\alpha = 1/e$. $\square$

Let us combine results obtained above in a single statement.

Theorem 16. Suppose z_p is the solution to problem (35) with coefficient p , and t_p is the first zero of the function z_p . Then:

- If $0 \leq p \leq 1/e$, then $z_p(t) > 0$ for all $t \in \mathbb{R}_+$;
- If $p = p_n$, then $t_p = n + 1$ for all $n \in \mathbb{N}$;
- If $p_1 = 1 < p < \infty$, then $t_p \in (1, 2)$;
- If $p_{n+1} < p < p_n$, then $t_p \in (n + 1, n + 2)$ for all $n \in \mathbb{N}$.

Now, we describe t_p for p belonging sequentially to the intervals $[p_n, p_{n-1})$, $n \in \mathbb{N}$, setting $p_0 = +\infty$.

Let us start with the first, semi-infinite, interval: suppose $p_1 \leq p \leq p_0$. Since $H_1(p) = z_p(2) = 1 - p$, then $p_1 \geq 1$. By virtue of (35), for $t \in [1, 2)$ we have $z_p(t) = 1 - p(t - 1)$, therefore, $t_p = 1 + 1/p$.

Consider the second interval: suppose $p_2 \leq p < p_1$. The number p_2 can also be found exactly as the smaller root of the equation

$$H_2(p) \triangleq 1 - 2p + \frac{p^2}{2} = 0.$$

We have $p_2 = 2 - \sqrt{2}$. From Theorem 16, it follows that $t_p \in (2, 3]$,

$$z_p(t) = 1 - p(t - 1) + \frac{p^2(t - 2)^2}{2},$$

whence $t_p = 2 + \frac{1-\sqrt{2p-1}}{p}$.

For intervals $p_n \leq p < p_{n-1}$ with $n \geq 3$, there is no point in representing numbers p_n through radicals: it is more convenient to use numerical methods to determine the boundaries of intervals with any degree of accuracy. An explicit analytical representation of t_p as a function of p also becomes impossible, so we replace it with an implicit one.

Lemma 19. For any $n \geq 2$, the equality

$$z_p(t) \triangleq 1 + \sum_{k=1}^n (-1)^k \frac{p^k}{k!} (t-k)^k = 0, \quad t \in [n, n+1], \quad (37)$$

defines on the set $[p_n, p_{n-1}]$ a unique, continuous, monotonically decreasing function $t = t(p)$ mapping the segment $[p_n, p_{n-1}]$ onto the segment $[n, n+1]$.

Proof. Suppose that for a given $p \in [p_{n-1}, p_n]$, there exist $\tau_1, \tau_2 \in [n, n+1]$ for which $z_p(\tau_1) = z_p(\tau_2) = 0$. Then, by Rolle's theorem, there is a point $\zeta \in (\tau_1, \tau_2)$ at which $\dot{z}_p(\zeta) = 0$, and therefore $z_p(\zeta - 1) = 0$. But $\zeta - 1 < n$, which contradicts Theorem 16. Consequently, on the interval $[n, n+1]$, the equation $z_p(t) = 0$ has a unique root, and it is equal to t_p . Thus, $t(p) = t_p$. The remaining statements of the lemma follow from Lemma 17 and Theorem 14. $\square$

From Theorem 16 and Lemma 19, we obtain the following statement.

Theorem 17. The function $t = t(p)$ is a continuous monotonically decreasing mapping of the set $(1/e, +\infty)$ onto the set $(1, +\infty)$.

Figure 1 shows the graph of the function $t = t_p$.

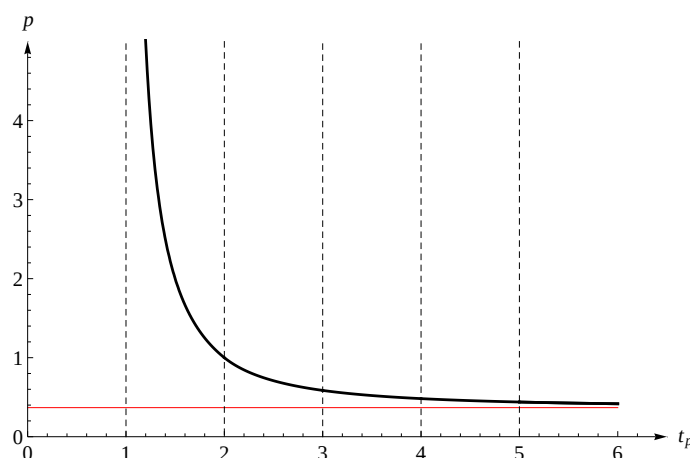


Figure 1. Graph of the function $t = t(p)$.

4.3. Two Terms, One Delay

Set in Equation (1) $n = 1$, denote $\omega_1 = \omega$ and consider a family of equations of the form

$$\dot{x}(t) = ax(t) - b(Tx)(t) + f(t), \quad t \geq 0, \quad (38)$$

where $a \in \mathbb{R}$ and the operator T is determined by formula (32). The most common case of Equation (38) is the equation with concentrated delay

$$\dot{x}(t) = ax(t) - bx(t - r(t)) + f(t), \quad t \geq 0, \quad (39)$$

where $a, b \in \mathbb{R}, 0 \leq r(t) \leq \omega$.

In this subsection, we obtain a complete description of the stability region of family (38) for each $a, b \in \mathbb{R}$ and $\omega > 0$.

4.3.1. The Test Problem and its Properties

We start with the case $b > 0$.

The test problem for the family (38) is the problem

$$\begin{cases} \dot{y}(t) = ay(t) - by(t - \omega), & t \in (0, l], \\ y(\xi) = 1, & \xi \leq 0. \end{cases} \quad (40)$$

As above, denote the solution of problem (40) by $y = y(t)$, and the first minimum of the function y by l .

Make a change of variables in (40), which is similar to the change (12) used in the proof of Lemma 3 and makes delay equal to 1:

$$y(t) = 1 + (a - b)\omega \int_0^{t/\omega} e^{a\omega\tau} z(\tau) d\tau. \quad (41)$$

Substituting (41) into (40), we see that z is the solution to problem (35), where $p = b\omega e^{-a\omega}$. Since

$$\dot{y}(t) = (a - b)e^{at} z(t/\omega), \quad (42)$$

the problem of finding l is equivalent to that of determining the first zero of the solution $z = z_p$ of problem (35) for a given p . Obviously, $l = \omega t_p$.

To find the stability region of family (38), use Theorem 9. First, construct the surface F .

According to formula (13), the parametric equation of the surface F for family (38) has the form

$$F = \left\{ -\zeta - u_0 = u_1 e^{\zeta\omega} = 0, \quad -1 + u_1 \omega e^{\zeta\omega} = 0, \quad \zeta \in \mathbb{R} \right\}.$$

Eliminating the parameter ζ and passing to the coordinates $u = u_0\omega$, $v = u_1\omega$, we obtain that F is the curve $v = e^{u-1}$. Now the concepts *above* and *below* used in Theorem 9 are geometrically obvious for F .

Express the curve $y(l) = -1$ in terms of parameters of family (38). Consider the sequence $\{p_n\}_{n \in \mathbb{N}}$ of roots of the polynomials $H_n(p)$. From Theorem 16 and formula (42), taking account of Lemma 3, we obtain the following result.

Theorem 18. *Let y be the solution to problem (40), and l , the point of its first minimum. Then:*

- *If $0 \leq b\omega \leq e^{a\omega-1}$, then the function $y = y(t)$ is positive and monotonically decreases for all $t \in \mathbb{R}_+$;*
- *If $b\omega = p_n e^{a\omega}$, then y has the first minimum at the point $l = (n + 1)\omega$ for all $n \in \mathbb{N}$;*
- *If $e^{a\omega} < b\omega < \infty$, then $l \in (\omega, 2\omega)$;*
- *If $p_{n+1} e^{a\omega} < b\omega < p_n e^{a\omega}$, then $l \in ((n + 1)\omega, (n + 2)\omega)$ for all $n \in \mathbb{N}$.*

Let us give a graphical interpretation of Theorem 18. Divide the region lying above F with curves

$$v = p_1 e^u, v = p_2 e^u, \dots, v = p_n e^u, \dots$$

to an infinite number of areas. It follows from Theorem 15 that the sequence of curves converges to the curve $v = e^{u-1}$, which is F .

In Figure 2, on the left, the curves $v = p_n e^u$, $n \in \mathbb{N}$, are indicated by dashed lines, and the "limit" curve $v = e^{u-1}$ by the solid thin line. When moving from one region to another, the curve $y(l) = -1$ change the form; therefore, it also consists of a countable set of links.

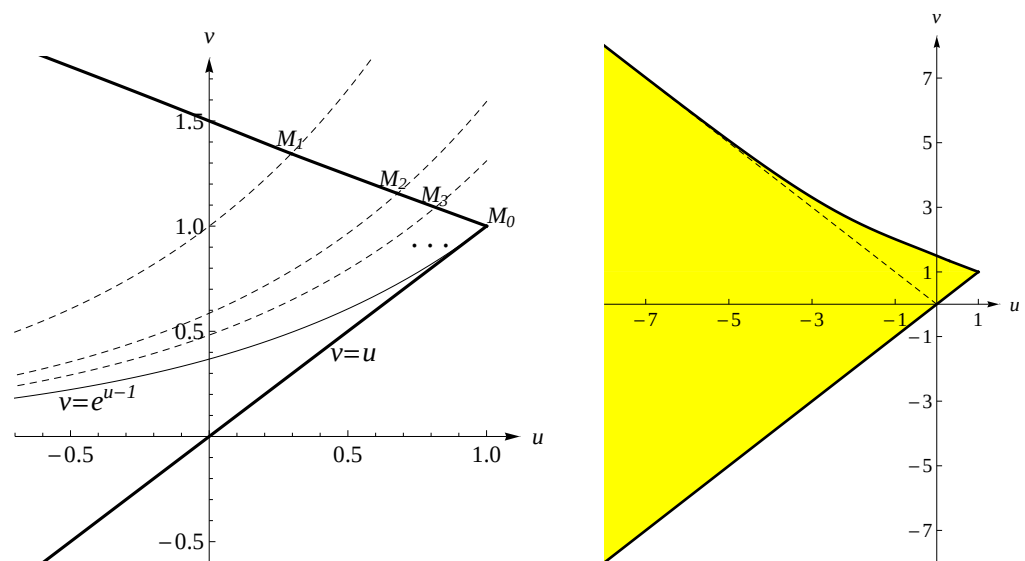


Figure 2. Curves $v = p_n e^u$, $n \in \mathbb{N}$, and $v = e^{u-1}$.

Let us describe the curve $y(l) = -1$ sequentially in each of the regions shown in Figure 2 on the left.

Lemma 20. Suppose $a \neq 0$. Then, for $l \in [n\omega, (n+1)\omega]$, the following equality holds:

$$y(l) = \left(\frac{b}{a}\right)^{n+1} + \left(1 - \frac{b}{a}\right) \sum_{k=1}^n \left(\frac{b}{a}\right)^k z(t_p - k) e^{a\omega(t_p - k)}. \quad (43)$$

Proof. Transforming the right-hand side of the equality (41), with regard to Equation (35), we obtain

$$\begin{aligned} 1 + (a-b)\omega \int_0^{t/\omega} e^{a\omega\tau} z(\tau) d\tau \\ = \left(1 - \frac{b}{a}\right) z\left(\frac{t}{\omega}\right) e^{at} + \frac{b}{a} \left(1 + \omega(a-b) \int_0^{\frac{t-\omega}{\omega}} e^{a\omega\tau} z(\tau) d\tau\right); \end{aligned}$$

therefore, for $t \geq \omega$ we have

$$y(t) = \left(1 - \frac{b}{a}\right) z\left(\frac{t}{\omega}\right) e^{at} + \frac{b}{a} y(t - \omega). \quad (44)$$

It is easy to see that for $t \in [0, \omega]$, the solution to Equation (40) has the form

$$y(t) = \left(1 - \frac{b}{a}\right) e^{at} + \frac{b}{a};$$

applying formula (44) n times, we obtain that for $t \in [n\omega, (n+1)\omega]$

$$y(t) = \left(\frac{b}{a}\right)^{n+1} + \left(1 - \frac{b}{a}\right) \sum_{k=0}^n \left(\frac{b}{a}\right)^k z\left(\frac{t}{\omega} - k\right) e^{a(t-k\omega)}.$$

If $l \in [n\omega, (n+1)\omega]$, then, setting $t = l$ in the last equality and taking into account that $z(l/\omega) = z_p(t_p) = 0$, we obtain the required formula (43). $\square$

Construct the curve $y(l) = -1$ in the case that $b\omega > e^{a\omega}$. We have $t_p \in (1, 2]$, and from (43), for $a \neq 0$, we obtain the boundary equation

$$y(l) \triangleq \left(\frac{b}{a}\right)^2 + \left(1 - \frac{b}{a}\right) \frac{b}{a} \exp\left(\frac{a}{b} e^{a\omega}\right) = -1.$$

Equation (40) for $a = 0$ was considered in Section 4.1, where it was shown that $y(l) = \frac{1}{2} - b\omega$.

Use the coordinate system Ouv . Define the function

$$\Phi_1(u, v) = \begin{cases} \left(\frac{v}{u}\right)^2 + \left(1 - \frac{v}{u}\right) \frac{v}{u} \exp\left(\frac{u}{v} e^u\right), & u \neq 0; \\ \frac{1}{2} - v, & u = 0; \end{cases}$$

in the domain $\{(u, v) : v > e^u\}$. By direct calculation, we can establish that

$$\lim_{u \rightarrow 0} \Phi_1(u, v) = \frac{1}{2} - v = \Phi_1(0, v),$$

therefore, Φ_1 is continuous in the considered domain. Further, it is easy to verify that for $v > e^u$ the equality $\Phi_1(u, v) = -1$ defines a unique, continuous and monotonically decreasing function $v = \varphi_1(u)$.

Consider the remaining part of the plane (u, v) . For each $n = 2, 3, \dots$, define the function

$$\Phi_n(u, v) = \left(\frac{v}{u}\right)^{n+1} + \left(1 - \frac{v}{u}\right) \sum_{k=0}^n \left(\frac{v}{u}\right)^k z(t_p - k) e^{u(t_p - k)}.$$

in the domain $\{(u, v) : p_n e^u < v < p_{n-1} e^u\}$. Recall that here $z = z_p(t)$ is the fundamental solution to problem (35) for $p = v e^{-u}$, and the properties of the sequence $\{p_n\}$ and the functions $t = t_p$ are defined by Theorems 15–17.

By Lemma 19, equality (37) for each $n \in \mathbb{N}$ uniquely defines the function t_p on $[p_n, p_{n-1}]$, from which it follows that the equation $\Phi_n(u, v) = -1$ defines for $p_n e^u < v < p_{n-1} e^u$ a single-valued, continuous, and monotonically decreasing function $v = \varphi_n(u)$. Moreover, the curves $v = \varphi_n(u)$ and $v = \varphi_{n-1}(u)$ have a common point lying on the curve $v = p_{n-1} e^u$; denote this point $M_n(u_n, v_n)$. Since the functions φ_n are monotonically decreasing, then the sequence $\{u_n\}$ increases monotonically and the sequence $\{v_n\}$ decreases as n increases. Further, since the points M_n belong to the region of the uniform stability of Equation (40), it follows from Theorem 9 that $u_n < 1$. On the other hand, for $n \geq 2$, we have $v_n > 0$. This means that both the sequences $\{u_n\}$ and $\{v_n\}$ of coordinates converge; therefore, the sequence $\{M_n\}$ also converges. Let us find its limit.

Theorem 19. The sequence of points $\{M_n\}_{n=1}^\infty$ converges to the point $M_0(1, 1)$ with respect to the norm of the space $\mathbb{R}^2$.

Proof. Denote the coordinates of the limit point M_0 by (u_0, v_0) . Since by Theorem 15, $\lim_{n \rightarrow \infty} p_n = 1/e$, then the sequence of curves $v = p_n e^u$ converges to the curve $v = e^{u-1}$; therefore, point M_0 lies on this curve.

Correspond the points $M_n(u_n, v_n)$ and $M_0(u_0, v_0)$ in the following test equations:

$$\dot{y}(t) = u_n y(t) - v_n y(t - r_n(t)), \quad t \geq 0, \quad (45)$$

where

$$r_n(t) = \begin{cases} t, & t \in [0, 1], \\ 1, & t \in [1, n+1), \end{cases} \quad r_n(t+n+1) = r_n(t), \quad n \in \mathbb{N};$$

and

$$\dot{y}(t) = u_0 y(t) - v_0 y(t - r_0(t)), \quad t \geq 0, \quad (46)$$

where

$$r_0(t) = \begin{cases} t, & t \in [0, 1], \\ 1, & t \in [1, \infty). \end{cases}$$

Denote by y_n and y_0 the solutions to Equations (45) and (46), respectively, with initial values $y_n(0) = y_0(0) = 1$. As noted above, $u_n < 1$, which means $u_0 \leq 1$.

Suppose that $u_0 < 1$. Then, it follows from Lemma 10 that Equation (46) is exponentially stable and its Cauchy function $C_0(t, s)$ satisfies the estimate

$$\int_0^t |C_0(t, \tau)| d\tau \leq \int_0^t K e^{-\gamma(t-\tau)} d\tau \leq K \gamma^{-1} \triangleq \sigma < \infty.$$

It follows from properties of the solution to the test equation and the choice of points M_n that $|y_n(t)| \leq 1$ for all $t \geq 0$ and $n \in \mathbb{N}$. Using the fact that $u_n \rightarrow u_0$ and $v_n \rightarrow v_0$, we choose $m \in \mathbb{N}$ such that

$$|u_m - u_0| < \frac{1}{4\sigma}, \quad |v_m - v_0| < \frac{1}{4\sigma},$$

and consider Equation (45) for $n = m$ on the interval $[0, m + 1]$. By virtue of the Cauchy formula (3), the function y_m is a solution to the following integral equation

$$y(t) = y_0(t) + \int_0^t C_0(t, \tau) ((u_m - u_0)y(\tau) + (v_m - v_0)y(\tau - r_0(\tau))) d\tau, \quad t \in [0, m + 1].$$

It follows from properties of the solution to the test equation and the definition of the points M_n that $|y_m(\tau)| \leq 1$ for all $t \geq 0$. By virtue of the choice of m and σ , we obtain

$$|y_m(m + 1) - y_0(m + 1)| \leq \sigma(|u_m - u_0| + |v_m - v_0|) < 1/2.$$

But by Lemma 10, $y_0(m + 1) > 0$, and by the choice of point $M_m(u_m, v_m)$, we have $y_m(m + 1) = -1$. The resulting contradiction means that $u_0 = 1$, and since $v_0 = e^{u_0-1}$, it follows that $v_0 = 1$. $\square$

Denote

$$\Phi(u, v) = \begin{cases} \Phi_1(u, v), & v \geq e^u, \\ \Phi_n(u, v), & p_n e^u \leq v \leq p_{n-1} e^u. \end{cases}$$

For $u \in (-\infty, 1)$, the equation $\Phi(u, v) = -1$ defines a single-valued monotonically decreasing function, the graph of which lies entirely above F . This graph has an asymptote $v = -u$ as $u \rightarrow -\infty$ (see Figure 2, on the right). At points M_n , the links of the curve are continuously joined, and $M_n \rightarrow M_0(1, 1)$, which means that the required curve is defined at the point $u = 1$ by continuity. For $u > 1$, it is not determined.

4.3.2. Stability Tests

First, prove a simple stability test, which is valid for all delays $\omega < \infty$.

Lemma 21. Suppose that $a < 0$ and $|b| < -a$. Then, Equation (38) is exponentially stable for all $\omega < \infty$.

Proof. Consider Equation (38) with the initial condition $x(0) = 0$ and a right-hand side $f \in L_\infty$. Using the Cauchy formula (3) for the ODE, rewrite the equation in the equivalent form

$$x(t) + (Kx)(t) = g(t), \quad t \geq 0,$$

where

$$(Ky)(t) = b \int_0^t e^{a(t-\tau)} (Ty)(\tau) d\tau, \quad g(t) = \int_0^t e^{a(t-\tau)} f(\tau) d\tau.$$

Since $a < 0$, we have $g \in L_\infty$, the operator K acts from L_∞ to C_∞ , and

$$\|Ky\|_\infty = \sup_{t \geq 0} |b| \int_0^t e^{a(t-\tau)} |(Ty)(\tau)| d\tau \leq |b| \int_0^t e^{a(t-\tau)} \int_{\tau-\omega}^\tau d_\zeta r(\tau, \zeta) d\tau \|y\|_\infty \leq \frac{|b|}{-a} \|y\|_\infty.$$

Therefore, with regard to the conditions of the lemma, $\|K\|_{L_\infty \rightarrow C_\infty} \leq \frac{|b|}{-a} < 1$, hence the operator $I + K$ is boundedly invertible, and the solution to Equation (38) belongs to C_∞ . To complete the proof, it remains to refer to the Bohl–Perron theorem [19] (Theorem 3.3.1). $\square$

Theorem 20. Family (38) is exponentially stable if and only if $b > a$, $a\omega < 1$ and $\Phi(a\omega, b\omega) > -1$.

Proof. Suppose $b > 0$. Applying Theorem 9, compose the stability region of two parts. If $b\omega \leq e^{a\omega-1}$, then the point $(a\omega, b\omega)$ does not lie above the surface F . From the conditions of the theorem, it follows that the conditions $P(0) = b - a > 0$ and $P'(0) = a\omega - 1 < 0$ are satisfied for the point $(a\omega, b\omega)$. Hence, by virtue of item (e1) of Theorem 9, the point $(a\omega, b\omega)$ belongs to the region of exponential stability. If $b\omega > e^{a\omega-1}$, then the point $(a\omega, b\omega)$ lies above F , and, by virtue of item (e2) of Theorem 9, the criterion of exponential stability is the condition $y(I) > -1$, which is equivalent to the condition $\Phi(a\omega, b\omega) > -1$.

Suppose now $b \leq 0$. Lemma 21 implies that if the point $(a\omega, b\omega)$ belongs to the interior of the angle $\{(u, v) \in \mathbb{R}^2 : u < 0, v > u\}$, then it belongs to the region of the exponential stability of family (38). However, if $b \leq a$, then there is a representative of family (38), $\dot{x}(t) = (a - b)x(t)$, that is not uniformly exponentially stable. $\square$

Thus, the region of the exponential stability of family (38) is the interior of the infinite “angle”, whose boundaries are the straight line $u = v$ and the curve $\Phi(u, v) = -1$ (see Figure 2).

To prove the criterion for the uniform stability of family (38), we prove one more auxiliary statement.

Lemma 22. For the Cauchy function of the equation

$$\dot{x}(t) = \int_0^t x(\tau) d_\tau r(t, \tau) - x(t) + f(t), \quad t \geq 0, \quad (47)$$

where the function $r(t, \cdot)$ is nondecreasing, $r(t, 0) = 0$, the function $r(\cdot, \tau)$ is measurable and $\int_0^t d_s r(t, s) = 1$, for all $(t, s) \in \Delta$, the estimate $0 < C(t, s) \leq 1$ is valid.

Proof. The positiveness of the Cauchy function follows from Lemma 3. Prove the required estimate. Denote by $E(s, \mu)$ the set of $t \geq s$ such that $C(\xi, s) < \mu$ for all $\xi \leq t$. Since $C(s, s) = 1$, the set $E(s, \mu)$ is not empty for all $s \geq 0$ and $\mu > 1$. If for all $s \geq 0$ and $\mu > 1$, we have $\sup E(s, \mu) = \infty$, then the estimate $C(t, s) \leq 1$ is valid. Assume that there exists $s_0 \geq 0$ and $\mu_0 > 1$ such that $\sup E(s_0, \mu_0) = \tau_0 < \infty$. Set $x(\cdot) = C(\cdot, s_0)$. For all $t \in [s_0, \tau_0)$, we have $x(t) < x(\tau_0) = \mu_0$. Choose $\varepsilon \in (0, 1)$ such that $\tau_0 - \varepsilon \in (s_0, \tau_0)$, and find

$$\tau = \sup\{t \in [\tau_0 - \varepsilon, \tau_0) : x(t) \leq x(\tau_0 - \varepsilon)\}.$$

It is obvious that $0 < \tau_0 - \tau \leq \varepsilon < 1$. Since the function x is absolutely continuous, there is a point $t_0 \in (\tau, \tau_0)$ such that

$$\dot{x}(t_0) \geq \frac{x(\tau_0) - x(\tau)}{\tau_0 - \tau} \geq x(\tau_0) - x(\tau).$$

On the other hand, the Cauchy function of Equation (47) is a solution to a homogeneous equation; therefore, it follows from the choice of the points τ and τ_0 that

$$\begin{aligned}\dot{x}(t_0) &= \int_{s_0}^{t_0} x(s) d_{\zeta} r(t_0, \zeta) - x(t_0) \leq x(\tau_0) \int_{s_0}^{t_0} d_{\zeta} r(t_0, \zeta) - x(t_0) \\ &\leq x(\tau_0) - x(t_0) < x(\tau_0) - x(\tau).\end{aligned}$$

Contradiction. $\square$

Corollary 2. *If $a = b < 0$, then family (38) is uniformly stable for all $\omega \leq 0$.*

Proof. If $a = b < 0$, then Equation (38) is reduced to Equation (47) by the change of variables $t \mapsto -at$. By virtue of Lemma 22, this proves the uniform stability of family (38). If $a = b = 0$, the uniform stability is obvious. $\square$

Theorem 21. *Family (38) is uniformly stable if and only if $b \geq a$, $a\omega < 1$ and $\Phi(a\omega, b\omega) \geq -1$.*

Proof. Suppose $b > 0$. If $b\omega \leq e^{a\omega-1}$, then, by item (b1) of Theorem 9, the criterion for uniform stability is that $P(0) = b - a \geq 0$ and $P'(0) = a\omega - 1 < 0$ hold, which is true according to the conditions of the theorem. If $b\omega > e^{a\omega-1}$, then by virtue of item (b2) of Theorem 9, the criterion for uniform stability is the condition $y(l) \geq -1$, which is equivalent to $\Phi(a\omega, b\omega) \geq -1$.

Suppose now $b \leq 0$. If $b > a$, then uniform stability follows from Theorem 20. If $b < a$, then the family is not uniformly stable, since there is a representative of the family (38), $\dot{x}(t) = (a - b)x(t)$, that is not uniformly stable. It remains to consider the case $b = a$. If $a < 0$, then Equation (38) is reduced to Equation (47) by the change of variables $t \mapsto -at$. This, by Lemma 22, proves the uniform stability of family (38). If $a = b = 0$, uniform stability is obvious. $\square$

In Figure 2, the stability region of family (38) is shaded on the right. The region of exponential stability is the interior of the infinite curvilinear angle bounded by the straight line $u = v$ and the curve $\Phi(u, v) = -1$, the region of uniform stability is the closure of this angle excluding the point $M_0(1, 1)$.

For family (39), Theorems 20 and 21 are valid, as well as all their corollaries, since Equation (39) is a special case of Equation (38), and, on the other hand, the test Equation (40) is a special case of Equation (39).

As stated above, the boundary of the stability region of family (38), which is defined by the function Φ , consists of an infinite number of links; however, for $u < 0$, it is given by the single equality $\Phi_1(u, v) = -1$, and this makes it possible to set the boundaries of the stability region in a more convenient form.

Introduce the function ψ as follows,

$$\psi(s) = \begin{cases} 0, & s \in [-1, 0], \\ s \ln \frac{1+s^2}{s(s-1)}, & s \in (-\infty, -1]. \end{cases}$$

Corollary 3. *Suppose $a < 0$, $b > 0$. Then:*

- Family (38) is exponentially stable if and only if $e^{a\omega} > \psi(b/a)$;
- Family (38) is uniformly stable if and only if $e^{a\omega} \geq \psi(b/a)$.

Proof. When the replacement $v/u = s$, $e^u = \xi$ is made in the inequality $\Phi_1(u, v) > -1$, it becomes equivalent to the inequality $e^{\xi/s} < \frac{1+s^2}{s(s-1)}$. Suppose $s \in [-1, 0)$. Then, since

$$e^{\xi/s} < 1 < \frac{1+s^2}{s(s-1)}$$

for all $\xi > 0$, the required inequality is satisfied automatically. Suppose $s \in (-\infty, -1]$. Then, the required inequality is equivalent to

$$\xi > s \ln \frac{1+s^2}{s(s-1)} = \psi(s),$$

from which the first statement of the corollary follows. The reasoning is similar for the case of nonstrict inequality. $\square$

Figure 3 shows the graph of the function $\xi = \psi(s)$. The stability region is shaded: for family (38) to be stable in the case that $a < 0$ and $b > 0$, it is necessary and sufficient that the point with coordinates $(b/a, e^{a\omega})$ belongs to this region.

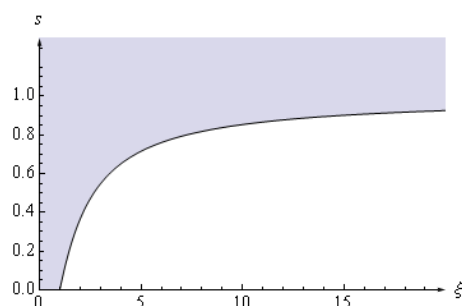


Figure 3. Graph of the function $\xi = \psi(s)$.

Below, we present some other convenient sufficient criteria for the stability of family (38), which are easy to obtain using the form of the stability domain.

Corollary 4. If $a < b$ and $0 < b\omega \leq 1$, then family (38) is exponentially stable.

One may find the abscissa of the point $M_1(u_1, v_1)$ from the equation $\Phi_1(u, e^u) = -1$. Numerical methods give $u_1 \approx 0.296$.

Corollary 5. If $a < b$, $a\omega \leq u_1$ and $\Phi_1(a\omega, b\omega) > -1$, then family (38) is exponentially stable.

Corollary 6. If $a < b$, $a\omega \leq u_1$ and $\Phi_1(a\omega, b\omega) \geq -1$, then family (38) is uniformly stable.

The above obtained results can be compared with the best known results on stability regions for Equation (38).

In paper [45], for the equation

$$\dot{x}(t) = ax(t) - bx(t - r(t)), \quad t \geq 0, \quad (48)$$

in the case $a > 0$, $b > 0$, and $0 \leq r(t) \leq \omega$, the following stability conditions were obtained.

Suppose D_1 is a domain on the plane Ouv for $u > 0$, $v > 0$, bounded by the line $u = v$ and the curve $v = g_1(u)$, consisting of two links:

$$\begin{aligned} \frac{1}{2}(u+v)^2 &= 1-u & \text{for } v \leq 1, \\ \frac{u+v}{2(v-u)}(2v-u^2-1) &= 1-u & \text{for } v \geq 1. \end{aligned}$$

In Figure 4, on the left, the curve g_1 is shown in red. The boundaries of the domain D_1 are not included in it.

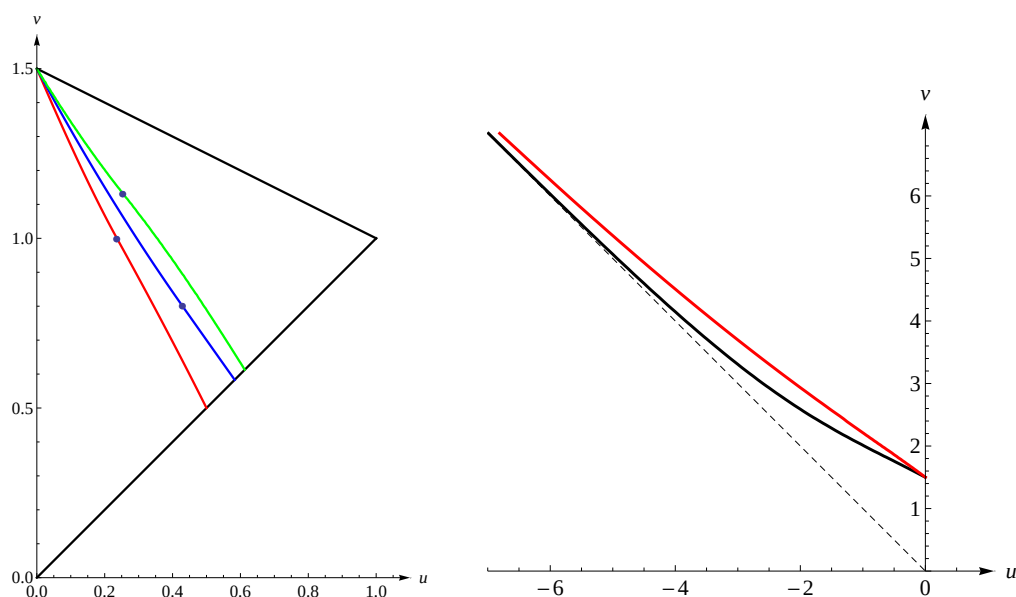


Figure 4. On the left, the curves $v = g_1(u)$, $v = g_2(u)$, $v = w_2(u)$ and the exact boundary of the stability region of Equation (48). On the right, the exact and erroneous boundary.

Proposition 3 ([45]). Equation (48) is exponentially stable if $(a\omega, b\omega) \in D_1$.

This result was refined in paper [46], where the boundary of the region $v = g_1(u)$ was replaced by the more exact boundary $v = g_2(u)$, also consisting of two links:

$$\begin{aligned} \frac{v(v+u)}{u^2}(e^u - 1 - u)^2 &= 1 - u & \text{for } \frac{v}{u}(e^u - 1) \leq 1, \\ \frac{v}{u} \left(e^{-\frac{v+u}{u}} \ln \frac{v+u}{v} \right) &= 1 - u & \text{for } \frac{v}{u}(e^u - 1) \geq 1. \end{aligned}$$

In Figure 4, on the left, the curve g_2 is shown in blue. Together with the line $u = v$, it bounds the set $D_2 \supset D_1$; in [46], it was proven that for $(a\omega, b\omega) \in D_2$, Equation (48) is uniformly stable.

In Figure 4, on the left, the boundary of the stability region of Equation (48), which is given by Theorems 20 and 21 is shown in black; it is obvious that the sets D_1 and D_2 are included in it.

In [46], Equation (48) was considered for $a < 0, b > 0$, and the following proposition was given.

Proposition 4. Let $a < 0, b > 0$. If the point $(a\omega, b\omega)$ lies either not above the line $u + v = 0$, or not above the curve

$$\frac{v^2}{u^2} \left\{ 1 - \frac{v+u}{v} \left(1 - \frac{2uv}{(u+v)^2} e^u \right)^{1/2} \right\} = 1, \quad (49)$$

then Equation (48) is uniformly stable. If $(a\omega, b\omega)$ lies below the curve (49), then Equation (48) is asymptotically stable.

The exponential stability of Equation (48) in the case $a + b \leq 0$ is obvious, therefore we apply Proposition 4 to the case $a + b > 0$. Then, equality (49) should be considered for $u + v > 0$; it can be simplified and takes the form

$$\left(2 - \frac{u}{v}\right)\left(1 + \frac{u}{v}\right)^2 = 2e^u.$$

In Figure 4 on the right, this curve is shown in red, the exact stability boundary of family (48) is indicated in black, and the line $u + v = 0$ is indicated by the dashed line. It is clear from Figure 4 that the red line goes above the black one, therefore, it does not belong to the stability region of Equation (48). Therefore, Proposition 4 is false.

In paper [47], the stability of nonlinear delay equations was studied; the linear case of equations have the form

$$\dot{x}(t) = -p(t)(cx(t) + x(t - \tau)), \quad t \geq 0, \quad (50)$$

where $p(t) \geq 0$, $c \in \mathbb{R}$, $\tau > 0$.

By a known change of variables [48], Equation (50) is reduced to the form (48). The result of paper [47] takes the following form.

Proposition 5. Equation (50) is asymptotically stable, if

- $a < 0$, $a\omega < b\omega < w_1(a\omega)$;
- $a = 0$, $0 < b\omega < 3/2$;
- $a > 0$, $a\omega < b\omega < w_2(a\omega)$.

For $a \leq 0$, the boundary $v = w_2(u)$ coincides with the boundary of the domain D ; for $a > 0$, the curve $v = w_2(u)$ consists of two links and is defined implicitly by the following equations:

$$\begin{aligned} e^u + \frac{v^2}{v} + \frac{u-v}{u} \ln\left(\frac{v-u}{v}\right) &= 2 + \frac{u}{v}, & \text{if } 0 < u \leq u_0, \\ \frac{u}{v}e^u + e^{-u} + u\left(1 - \frac{u}{v} + \frac{u^2}{v^2}\right) &= 1 + \frac{u}{v} + \frac{u^2}{v^2}, & \text{if } u_0 \leq u < 1, \end{aligned}$$

where $u_0 \approx 0.2536$. The curve $v = w_2(u)$ goes below the boundary of the domain D (in Figure 4 on the right, it is shown in green).

Theorems 20 and 21 cover a wider class of equations: the delay in (38) can be variable and distributed, the stability region is wider, and the estimate of the Cauchy function provides more information than about asymptotic stability. The advantage of the result of [47] is that it is applicable to nonlinear equations whose coefficients satisfy the conditions of the Yorke type [49].

4.4. Two and Three Delays

Consider the family of equations with three delays

$$\dot{x}(t) + ax(t - r_1(t)) + bx(t - r_2(t)) + cx(t - r_3(t)) = 0, \quad (51)$$

where $a, b, c \geq 0$ and the integrable functions r_1, r_2, r_3 are subjected to the inequalities

$$0 \leq r_1(t) \leq 1, \quad 0 \leq r_2(t) \leq 1.2, \quad 0 \leq r_3(t) \leq 1.5;$$

construct a stability region for this family in the three-dimensional coefficient space (a, b, c) .

It follows from Theorem 13 that the stability region of family (51) is situated in the first octant not above the plane $a + b + c = 3/2$.

The test problem (5) for family (51) has the form

$$\begin{cases} \dot{u}(t) - au(t-1) + bu(t-1.2) + cu(t-1.5) = 0, & t > 0; \\ u(0) = 1; & u(\xi) = 0, & \xi < 0. \end{cases} \quad (52)$$

Let us find the point $l = l(a, b, c)$ of the first zero of problem (52). In the considered example, we have $\omega = 3/2$, therefore, by virtue of Theorem 10, we are interested only in the case $0 < l \leq 5\omega/3 = 5/2$.

Construct the solution of problem (52) in the segment $[0, 2.5]$:

$$\begin{aligned} t \in [0, 1]: & u(t) \equiv 1, u(1) = 1; \\ t \in [1, 1.2]: & u(t) = 1 - a(t-1), u(1.2) = 1 - 0.2a; \\ t \in [1.2, 1.5]: & u(t) = u(1.2) - (a+b)(t-1.2), u(1.5) = 1 - 0.5a - 0.3b; \\ t \in [1.5, 2]: & u(t) = u(1.5) - (a+b+c)(t-1.5), u(2) = 1 - a - 0.8b - 0.5; \\ t \in [2, 2.2]: & u(t) = u(2) - (a+b+c)(t-2) + 0.5a^2(t-2)^2, \\ & u(2.2) = 1 - 1.2a - b - 0.7c + 0.02a^2; \\ t \in [2.2, 2.4]: & u(t) = u(2.2) - (a+b+c-0.2a^2)(t-2.2) + 0.5(a^2+2ab)(t-2.2)^2, \\ & u(2.4) = 1 - 1.4a - 1.2b - 0.9c + 0.08a^2 + 0.04ab; \\ t \in [2.4, 2.5]: & u(t) = u(2.4) - (a+b+c-0.4a^2-0.4ab)(t-2.4) + 0.5(a+b)^2(t-2.4)^2, \\ & u(2.5) = 1 - 1.5a - 1.3b - c + 0.125a^2 + 0.09ab + 0.005b^2. \end{aligned}$$

Since $a + b + c \leq 3/2$, the solution does not have zeros on $[0, 1.5]$; hence, $l \in (1.5, 2.5]$.

Construct four surfaces A_1 – A_4 that determine intervals contained in $(1.5, 2.5]$, to which l can belong.

The surface A_1 is defined by the equation $u(2) = 0$, which is

$$1 - a - 0.8b - 0.5c = 0.$$

The surface A_2 is defined by the equation $u(2.2) = 0$, which is

$$1 - 1.2a - b - 0.7c + 0.02a^2 = 0.$$

The surface A_3 is defined by the equation $u(2.4) = 0$, which is

$$1 - 1.4a - 1.2b - 0.9c + 0.08a^2 + 0.04ab = 0.$$

The surface A_4 is defined by the equation $u(2.5) = 0$, which is

$$1 - 1.5a - 1.3b - c + 0.125a^2 + 0.09ab + 0.005b^2 = 0.$$

The surfaces A_1 – A_4 are represented in Figure 5.

The points (a, b, c) that are not above A_1 correspond to those coefficients of Equation (52), for which $l \in (1.5, 2]$; the points (a, b, c) between A_1 and A_2 (including A_2) correspond to the case $l \in (2, 2.2]$; the points between A_2 and A_3 (including A_3), to the case $l \in (2.2, 2.4]$; the points between A_3 and A_4 (including A_4), to the case $l \in (2.4, 2.5]$; and the points below A_4 correspond to the situation $l \in (2.5, \infty)$ and, by Theorem 10, belong to the stability region.

Thus, we should construct the boundaries of the stability region in the four domains bounded by the surfaces A_1 – A_4 . Denote these boundaries by B_1, B_2, B_3, B_4 . All of them are defined by the two equalities

$$\begin{cases} u(l) = 0, \\ \int_0^l u(s) ds = \frac{2}{a+b+c}; \end{cases} \quad (53)$$

their specific form depends on intervals to which l belongs.

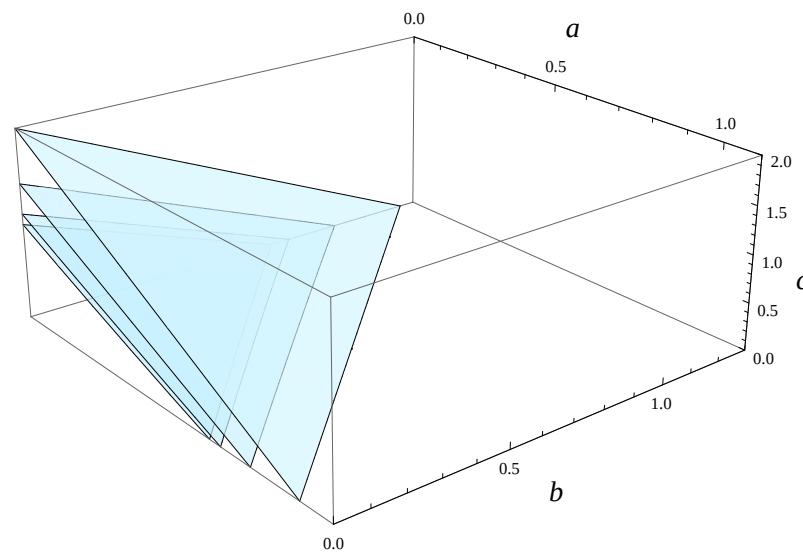


Figure 5. The surfaces A_1, A_2, A_3, A_4 .

Denote $I(\tau) = \int_0^\tau u(s) ds$. Obviously, for all $\tau \in [0, 2.5]$, the function $I(\tau)$ is expressed through a, b, c .

Let $l \in (1.5, 2]$. Then, the first of equalities (53) has the form

$$u(1.5) - (a + b + c)(l - 1.5) = 0.$$

Since

$$\int_0^l u(s) ds = I(1.5) + \int_{1.5}^l u(s) ds = I(1.5) + u(1.5)(l - 1.5) - 0.5(a + b + c)(l - 1.5)^2,$$

the second equality in (53) is written in the form

$$I(1.5) + u(1.5)(l - 1.5) - 0.5(a + b + c)(l - 1.5)^2 = \frac{2}{a + b + c}.$$

Denote $l - 1.5 = \zeta \in [0, 0.5]$. We see that the surface B_1 is defined by the equation

$$\begin{cases} u(1.5) - (a + b + c)\zeta = 0, \\ I(1.5) + u(1.5)\zeta - 0.5(a + b + c)\zeta^2 = \frac{2}{a + b + c}, \quad \zeta \in [0, 0.5]. \end{cases}$$

Let $l \in (2, 2.2]$. Arguing similarly, we obtain the surface B_2 :

$$\begin{cases} u(2) - (a + b + c)\zeta + 0.5a^2\zeta^2 = 0, \\ I(2) + u(2)\zeta - 0.5(a + b + c)\zeta^2 + \frac{a^2}{6}\zeta^3 = \frac{2}{a + b + c}, \quad \zeta \in [0, 0.2]. \end{cases}$$

Let $l \in (2.2, 2.4]$. Obtain the equation of the surface B_3 :

$$\begin{cases} u(2.2) - (a + b + c - 0.2a^2)\zeta + 0.5a(a + 2b)\zeta^2 = 0, \\ I(2.2) + u(2.2)\zeta - 0.5(a + b + c - 0.2a^2)\zeta^2 + \frac{1}{6}a(a + 2b)\zeta^3 = \frac{2}{a + b + c}, \quad \zeta \in [0, 0.2]. \end{cases}$$

Finally, for $l \in (2.4, 2.5]$ we obtain the equation of the surface B_4 :

$$\begin{cases} u(2.4) - (a + b + c - 0.4a(a + b))\zeta + 0.5(a + b)^2\zeta^2 = 0, \\ I(2.4) + u(2.4)\zeta - 0.5(a + b + c - 0.4a(a + b))\zeta^2 + \frac{1}{6}(a + b)^2\zeta^3 = \frac{2}{a + b + c}, \quad \zeta \in [0, 0.1]. \end{cases}$$

The surfaces B_1 – B_4 , forming the boundary of the stability region of family (51), are represented in Figure 6. The points placed in the first octant not above the boundary compose the region of uniform stability. By excluding the surfaces B_1 – B_4 and the point $(0,0,0)$ from it, we obtain the region of exponential stability.

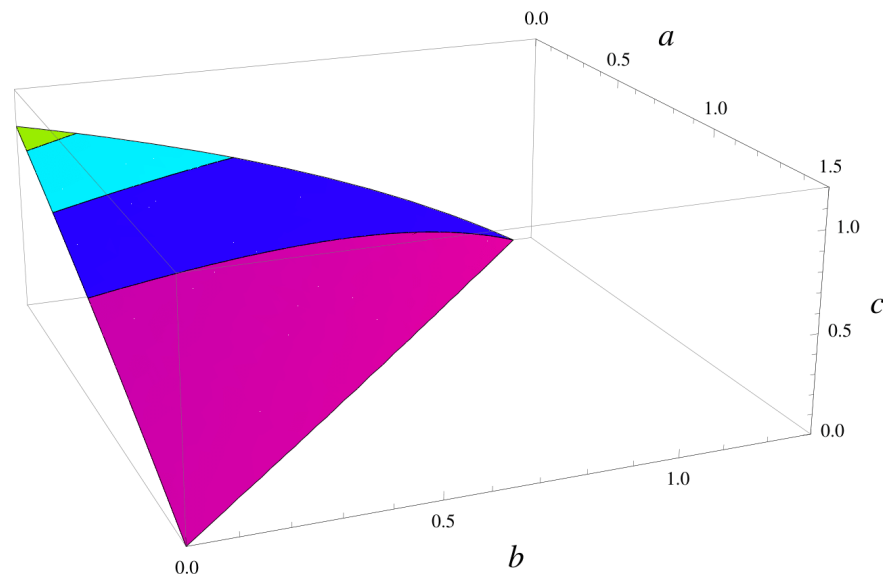


Figure 6. The surfaces B_1, B_2, B_3, B_4 .

In Figures 7–9, the sections of the stability region by the planes $a = 0$, $b = 0$ and $c = 0$ are represented, which are the stability regions for the three families of equations with two delays. The stability of the family

$$\dot{x}(t) + a_1 x(t - r_1(t)) + a_2 x(t - r_2(t)) = 0,$$

where $0 \leq r_1(t) \leq \omega_1$ and $0 \leq r_2(t) \leq \omega_2$, and the form of its stability region, are studied in detail in paper [50].

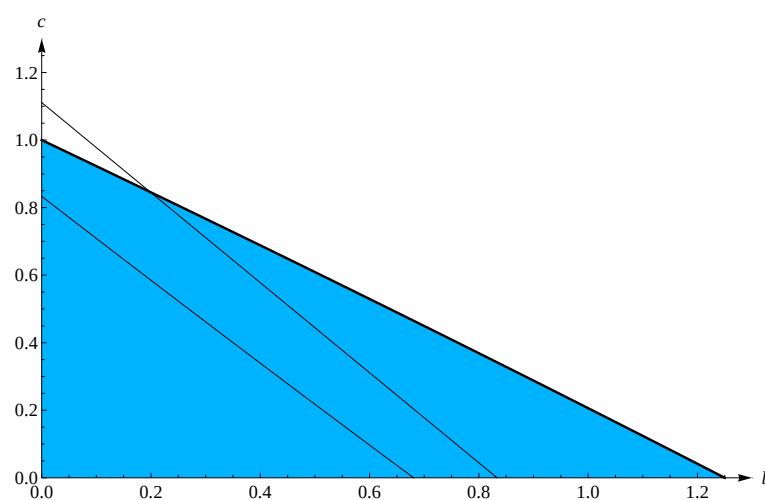


Figure 7. The section of the stability region of family (51) by the plane $a = 0$.

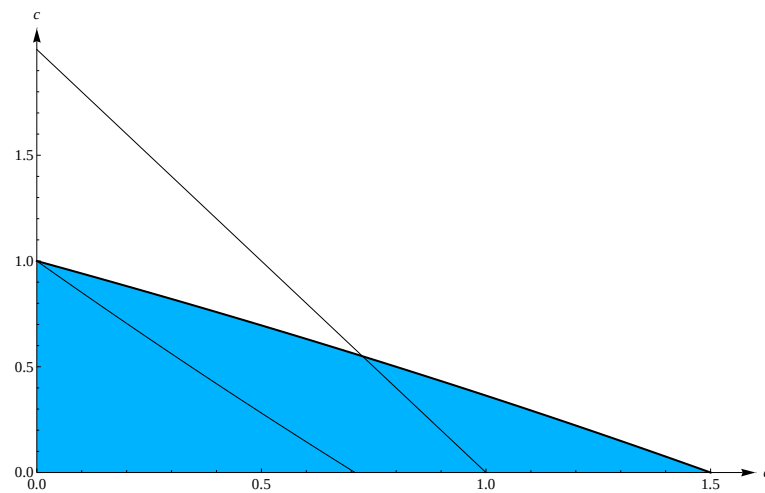


Figure 8. The section of the stability region of family (51) by the plane $b = 0$.

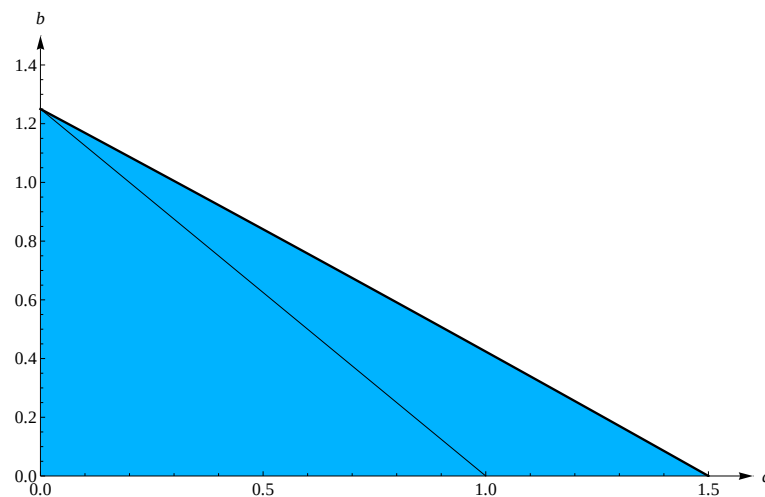


Figure 9. The section of the stability region of family (51) by the plane $c = 0$.

5. Discussion

The method presented in the article for studying the stability of solutions to FDEs, in contrast to the known ones, makes it possible to obtain exact stability criteria for a given family of equations. Such exactness was previously achieved only in classical Myshkis stability conditions and their refinements. The method is based on the construction and study of a test equation, which is the “worst-behavior” equation of a given family.

The applications of the method give effectively verifiable necessary and sufficient conditions for uniform and exponential stability of families of linear semi-autonomous delay differential equations. For equations specified by a small number of parameters, exact and effectively verifiable stability criteria are obtained, presented in analytical and geometric form. The new approach made it possible

- To obtain a simple proof of the Myshkis theorem on $3/2$, as well as to clarify it and generalize it to a wider class of Equations (31);
- To find the stability region with the sharp boundary for equations of the form (38); thereby to improve the best known stability regions found in works [47,51–53] (see Section 4.3.2);
- To construct the sharp three-dimensional stability region for Equation (51) with three non-zero delays; similar results have not yet been obtained in the literature.

Theorem 9 can be viewed as an algorithm suitable for computer implementation. It becomes especially simple when all the coefficients of Equation (51) have the same sign.

In this case, as follows from Theorem 10, it is enough to construct the solution to the test equation on the segment, $[0, 5\omega/3]$, where ω is the largest delay. If the solution graph does not take values less than (-1) on this segment, then family (51) is uniformly stable; if the graph does not reach the value (-1) , then family (51) is exponentially stable.

The test method allows to find the exact boundary of the stability region for Equation (1) with the coefficient a of arbitrary sign. The most difficult case to study is $a > 0$: since the ODE $\dot{x} = ax$ is unstable, stability can be achieved due to the delay term. The obtained result complements works that study FDEs with coefficients of different signs [45,54,55], and makes it possible to effectively calculate the exponent in the estimate (20) [56,57].

The authors admit that the idea of the test method can be implemented for equations of neutral type: one can try to construct an equation of the worst behavior, without inverting the operator at the derivative, using the methods of differential inequalities.

There is also an interesting and difficult problem to find a multidimensional analogue of the test equation for FDE systems. For a semi-autonomous system $\dot{x}(t) + Ax(h(t)) = 0$, where the matrix A has a real spectrum, the result is obvious. Fundamental difficulties arise if A has complex eigenvalues.

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References

1. Krasovskii, N.N. *Stability of Motion. Applications of Lyapunov's Second Method to Differential Systems and Equations with Delay*; Brenner, J.L., Translator; Stanford University Press: Stanford, CA, USA, 1963.
2. Repin, I.M. Quadratic Liapunov functionals for systems with delay. *J. Appl. Math. Mech.* **1965**, *29*, 669–672. [\[CrossRef\]](#)
3. Dolgiĭ, Y.F.; Kim, A.V. On the method of Lyapunov functionals for systems with aftereffect. *Differ. Uravn.* **1991**, *27*, 1313–1318.
4. Demidenko, G.V.; Matveeva, I.I. Stability of solutions of differential equations with retarded argument and periodic coefficients in the linear terms. *Sib. Mat. Zhurnal* **2007**, *48*, 1025–1040. [\[CrossRef\]](#)
5. Aleksandrov, A.Y. Construction of Lyapunov-Krasovskii functionals for some classes of positive delay systems. *Sib. Mat. Zhurnal* **2018**, *59*, 957–969. [\[CrossRef\]](#)
6. Yskak, T. Stability of solutions to systems of differential equations with distributed delay. *Funct. Differ. Equ.* **2018**, *25*, 97–108.
7. Alexandrova, I.V.; Zhabko, A.P. Lyapunov-Krasovskii functionals for homogeneous systems with multiple delays. *Vestn. St.-Peterbg. Univ. Prikl. Mat. Inform. Protessy Upr.* **2021**, *17*, 183–195. [\[CrossRef\]](#)
8. Demidenko, G.V.; Matveeva, I.I. The second Lyapunov method for time-delay systems. In *Functional Differential Equations and Applications*; Springer Proceedings in Mathematics & Statistics; Springer: Singapore, 2021; Volume 379, pp. 145–167. [\[CrossRef\]](#)
9. Razumihin, B.S. A method of studying the stability of systems with an aftereffect. *Dokl. Akad. Nauk SSSR* **1966**, *167*, 1234–1237.
10. Kato, J. On Liapunov-Razumikhin type theorems for functional differential equations. *Funkcial. Ekvac.* **1973**, *16*, 225–239.
11. Seifert, G. Liapunov-Razumikhin conditions for stability and boundedness of functional differential equations of Volterra type. *J. Differ. Equ.* **1973**, *14*, 424–430. [\[CrossRef\]](#)
12. Medvedeva, I.V.; Zhabko, A.P. Synthesis of Razumikhin and Lyapunov-Krasovskii approaches to stability analysis of time-delay systems. *Autom. J. IFAC* **2015**, *51*, 372–377. [\[CrossRef\]](#)
13. Egorov, A.V. Razumikhin approach in the generalized Myshkis problem for systems with distributed delay. *Vestn. St.-Peterbg. Univ. Prikl. Mat. Inform. Protessy Upr.* **2023**, *19*, 148–161. [\[CrossRef\]](#)
14. Azbelev, N.; Maksimov, V.; Rakhmatullina, L. *Introduction to the Theory of Linear Functional-Differential Equations*; Advanced Series in Mathematical Science and Engineering; World Federation Publishers Company: Atlanta, GA, USA, 1995; Volume 3.
15. Azbelev, N.V.; Berezhanskii, L.M.; Simonov, P.M.; Chistyakov, A.V. The stability of linear systems with aftereffect. I. *Differ. Uravn.* **1987**, *23*, 745–754.

16. Azbelev, N.V.; Berezanskiĭ, L.M.; Simonov, P.M.; Chistyakov, A.V. The stability of linear systems with aftereffect. II. *Differ. Uravn.* **1991**, *27*, 555–562.
17. Azbelev, N.V.; Berezanskiĭ, L.M.; Simonov, P.M.; Chistyakov, A.V. The stability of linear systems with aftereffect. III. *Differ. Nye Uravn.* **1991**, *27*, 1659–1668.
18. Azbelev, N.V.; Berezanskiĭ, L.M.; Simonov, P.M.; Chistyakov, A.V. The stability of linear systems with aftereffect. IV. *Differ. Uravn.* **1993**, *29*, 196–204.
19. Azbelev, N.V.; Simonov, P.M. Stability of differential equations with aftereffect. In *Stability and Control: Theory, Methods and Applications*; Taylor & Francis: London, UK, 2003; Volume 20.
20. Domoshnitsky, A.; Maghakyan, A.; Berezansky, L. W-transform for exponential stability of second order delay differential equations without damping terms. *J. Inequal. Appl.* **2017**, *2017*, 20. [[CrossRef](#)] [[PubMed](#)]
21. Berezansky, L.; Domoshnitsky, A.; Koplatadze, R. *Oscillation, Nonoscillation, Stability and Asymptotic Properties for Second and Higher Order Functional Differential Equations*; CRC Press: Boca Raton, FL, USA, 2020.
22. Pontrjagin, L. On zeros of some transcendental functions. *Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR]* **1942**, *6*, 115–134.
23. Ćebotarev, N.G.; Meĭman, N.N. The Routh-Hurwitz problem for polynomials and entire functions. Real quasipolynomials with $r = 3, s = 1$. *Trudy Mat. Inst. Steklov.* **1949**, *26*, 331.
24. Neĭmark, Y.I. D-decomposition of the space of quasipolynomials. (On the stability of linearized distributive systems). *Akad. Nauk SSSR. Prikl. Mat. Meh.* **1949**, *13*, 349–380.
25. Zubov, V.I. On the theory of linear stationary systems with lagging arguments. *Izv. Vysš. Učebn. Zaved. Mat.* **1958**, 86–95.
26. Èl'sgol'ts, L.E.; Norkin, S.B. *Vvedenie v Teoriyu Differentsial'nykh Uravneniĭ s Otklonyayushchimsya Argumentom*, 2nd ed.; Izdat. "Nauka": Moscow, Russia, 1971.
27. Zverkin, A.M. On the theory of linear differential equations with a lagging argument and periodic coefficients. *Dokl. Akad. Nauk SSSR* **1959**, *128*, 882–885.
28. Zverkin, A.M. On the theory of differential-difference equations with delay that is commensurable with the period of the coefficients. *Differ. Uravn.* **1988**, *24*, 1481–1492, 1651.
29. Shimanov, S.N. On the theory of linear differential equations with periodic coefficients and time lag. *Prikl. Mat. Meh.* **1963**, *27*, 450–458. [[CrossRef](#)]
30. Dolgii, Y.F. Characteristic equation in the problem of asymptotic stability in periodic systems with aftereffect. *Proc. Steklov Inst. Math.* **2005**, *11* (Suppl. S1), S82–S94.
31. Dolgii, Y.F. Application of self-adjoint boundary value problems to investigation of stability of periodic delay systems. *Proc. Steklov Inst. Math.* **2006**, *255* (Suppl. S2), S16–S25. [[CrossRef](#)]
32. Maksimov, V.P. The Cauchy formula for a functional-differential equation. *Differ. Uravn.* **1977**, *13*, 601–606, 770–771.
33. Hale, J. *Theory of Functional Differential Equations*, 2nd ed.; Applied Mathematical Sciences; Springer: Berlin/Heidelberg, Germany, 1977; Volume 3.
34. Tramov, M.I. Conditions for the oscillation of the solutions of first order differential equations with retarded argument. *Izv. Vysš. Učebn. Zaved. Mat.* **1975**, *3*, 92–96.
35. Ladas, G.; Sficas, Y.G.; Stavroulakis, I.P. Necessary and sufficient conditions for oscillations. *Am. Math. Mon.* **1983**, *90*, 637–640. [[CrossRef](#)]
36. Erbe, L.H.; Kong, Q.; Zhang, B.G. *Oscillation Theory for Functional-Differential Equations*; Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, Inc.: New York, NY, USA, 1995; Volume 190.
37. Györi, I.; Ladas, G. *Oscillation Theory of Delay Differential Equations*; With applications, Oxford Science Publications; Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press: New York, NY, USA, 1991.
38. Gopalsamy, K. *Stability and Oscillations in Delay Differential Equations of Population Dynamics*; Mathematics and its Applications, Kluwer Academic Publishers Group: Dordrecht, The Netherlands, 1992; Volume 74. [[CrossRef](#)]
39. Agarwal, R.P.; Berezansky, L.; Braverman, E.; Domoshnitsky, A. *Nonoscillation Theory of Functional Differential Equations with Applications*; Springer: New York, NY, USA, 2012. [[CrossRef](#)]
40. Berezansky, L.; Braverman, E. On oscillation of equations with distributed delay. *Z. Anal. Anwendungen* **2001**, *20*, 489–504. [[CrossRef](#)]
41. Myshkis, A.D. *Lineĭnye Differentsial'nye Uravneniya s Zapazdyvayushchim Argumentom*, 2nd ed.; Izdat. "Nauka": Moscow, Russia, 1972; p. 352.
42. Krisztin, T. On stability properties for one-dimensional functional-differential equations. *Funkcial. Ekvac.* **1991**, *34*, 241–256.
43. Myškis, A.D. On solutions of linear homogeneous differential equations of the first order of stable type with a retarded argument. *Mat. Sbornik N. S.* **1951**, *28*, 641–658.
44. Koplatadze, R.G.; Chanturiya, T.A. Oscillating and monotone solutions of first-order differential equations with deviating argument. *Differ. Uravn.* **1982**, *18*, 1463–1465.
45. Yoneyama, T.; Sugie, J. On the stability region of scalar delay-differential equations. *J. Math. Anal. Appl.* **1988**, *134*, 408–425. [[CrossRef](#)]
46. Yoneyama, T. Uniform stability for one-dimensional delay-differential equations with dominant delayed term. *Tohoku Math. J.* **1989**, *41*, 217–236. [[CrossRef](#)]

47. Tang, X.H. Asymptotic behavior of delay differential equations with instantaneously terms. *J. Math. Anal. Appl.* **2005**, *302*, 342–359. [[CrossRef](#)]
48. Ladas, G.; Sficas, Y.G.; Stavroulakis, I.P. Asymptotic behavior of solutions of retarded differential equations. *Proc. Am. Math. Soc.* **1983**, *88*, 247–253. [[CrossRef](#)]
49. Yorke, J.A. Asymptotic stability for one dimensional differential-delay equations. *J. Differ. Equ.* **1970**, *7*, 189–202. [[CrossRef](#)]
50. Malygina, V.V.; Chudinov, K.M. Stability of solutions to differential equations with several variable delays. III. *Russ. Math. (Iz. VUZ)* **2013**, *57*, 37–48; Translation of *Izv. Vyssh. Uchebn. Zaved. Mat.* **2013**, *8*, 44–56. [[CrossRef](#)]
51. Amemiya, T. On the delay-independent stability of a delayed differential equation of 1st order. *J. Math. Anal. Appl.* **1989**, *142*, 13–25. [[CrossRef](#)]
52. Yoneyama, T. On the $\frac{3}{2}$ stability theorem for one-dimensional delay-differential equations. *J. Math. Anal. Appl.* **1987**, *125*, 161–173. [[CrossRef](#)]
53. Yoneyama, T. The $3/2$ stability theorem for one-dimensional delay-differential equations with unbounded delay. *J. Math. Anal. Appl.* **1992**, *165*, 133–143. [[CrossRef](#)]
54. Berezansky, L.; Braverman, E. New stability conditions for linear differential equations with several delays. *Abstr. Appl. Anal.* **2011**, *2011*, 178568. [[CrossRef](#)]
55. Berezansky, L.; Braverman, E. On stability of delay equations with positive and negative coefficients with applications. *Z. Anal. Anwend.* **2019**, *38*, 157–189. [[CrossRef](#)]
56. Malygina, V.V. Exponent estimation for stable solutions of a certain class of differential-difference equations. *Izv. Vyss. Uchebnykh Zaved. Mat.* **2021**, *65*, 67–79. [[CrossRef](#)]
57. Malygina, V.V.; Chudinov, K.M. About exact two-sided estimates for stable solutions to autonomous functional differential equations. *Sib. Math. J.* **2022**, *63*, 299–315; Translation of *Sibirsk. Mat. Zh.* **2022**, *63*, 360–378. [[CrossRef](#)]

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