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Subclasses of Noshiro-Type Starlike Harmonic Functions Involving \(q\)-Srivastava–Attiya Operator

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Abstract: In this paper, the harmonic function related to the \(q\)-Srivastava–Attiya operator is described to introduce a new class of complex harmonic functions that are orientation-preserving and univalent in the open-unit disk. We also cover some important aspects such as coefficient bounds, convolution conservation, and convexity constraints. Next, using sufficiency criteria, we calculate the sharp bounds of the real parts of the ratios of harmonic functions to their sequences of partial sums. In addition, for the first time some of the interesting implications of the \(q\)-Srivastava–Attiya operator in harmonic functions are also included.

Keywords: analytic functions; univalent; harmonic; harmonic starlike functions; convolution; \(q\)-differential operators; \(q\)-Srivastava–Attiya operator

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1. Introduction and Preliminaries

Minimal surfaces have been represented for a long time using planar harmonic univalent mappings. Such mappings, for instance, were employed in 1952 by Heinz [1] to investigate the Gaussian curvature of nonparametric minimum surfaces over the unit disc (see [2]). Applications for these mappings and associated functions can be found in a wide range of applied mathematical disciplines, including engineering, physics, electronics, medicine, operations research, aerodynamics, and other fields. Harmonic and meromorphic functions, for instance, are essential to the resolution of many physical issues, including the diffusion of salt through a channel, the flow of water through an underground aquifer, the steady-state temperature distribution, and the intensity of the electrostatic field. Harmonic univalent mappings are closely related to each other. Another significant distinction is the ability to create a harmonic univalent mapping on a border interval of the open unit disc. It is common knowledge that if \(f = u + iv\) has continuous partial derivatives, then \(f\) is only analytic if and when the Cauchy–Riemann equations \(u_x = v_y\) and \(u_y = -v_x\) are met. Every analytic function is, therefore, a complex-valued harmonic function. Not all complex-valued harmonic functions, however, are analytic since the Cauchy–Riemann equations must be used to link them, as no two solutions to the Laplace equation can be interpreted as the components \(u\) and \(v\) of an analytic function in \(D \subset \Omega\) of any simply connected domain. A harmonic function’s analytical function might not be harmonic. As an illustration, \(x\) is harmonic, but \(x^2\) is not. Furthermore, a harmonic function does not always have a harmonic inverse. The linear mapping \(w = a\theta + \beta\bar{\theta}\) with \(|\alpha| \neq |\beta|\) is the most basic example of a harmonic univalent function that need not be conformal. Another straightforward example is \(w = \theta + \frac{\theta^2}{2}\), which maps \(D\) harmonically onto a region inside a hypocycloid of three cusps.
Let \( h = \omega_1 + i\omega_2 \) be continuous and complex harmonic function in the \( \Omega \) complex domain whenever \( \omega_1 \) and \( \omega_2 \) real and harmonic in \( \Omega \). In \( D \subset \Omega \) of any simply connected domain, we uniquely represent \( h = \xi_1 + \overline{\xi_2} \), where \( \xi_1 \) and \( \xi_2 \) are analytic in \( D \). We say \( \xi_1 \) is an analytic part and \( \xi_2 \) a co-analytic part of \( h \). Also, \( h \) is locally univalent and makes sense in \( D \) only if \( |\xi_1^\prime(e)| > |\xi_2^\prime(e)| \) in \( D \) (see [3]). Symbolize by \( \mathcal{H} \) the family of functions of the form

\[
h = \xi_1 + \overline{\xi_2}
\]

which are harmonic, univalent and sense-preserving in the open unit disc \( U = \{ e : |e| < 1 \} \) so that \( h \) is normalized by \( h\pi(0) = h\pi^\prime(0) - 1 = 0 \). Then, for \( h = \xi_1 + \overline{\xi_2} \in \mathcal{H}, \xi_1 \) and \( \xi_2 \) are analytic functions (4) in \( U \) given by:

\[
\xi_1(e) = e + \sum_{n \geq 2} t_{1n} e^n, \quad \xi_2(e) = \sum_{n \geq 1} t_{2n} e^n; \quad (0 \leq t_{21} < 1),
\]

and \( h(e) \) is then written as:

\[
h(e) = e + \sum_{n \geq 2} t_{1n} e^n + \sum_{n \geq 1} t_{2n} e^n; \quad (0 \leq t_{21} < 1).
\]

We annotate the family \( \mathcal{H} \equiv \mathcal{S} \) if \( \xi_2 \equiv 0 \). Denote by \( \mathcal{H} \) the subfamily of \( \mathcal{H} \) consisting of harmonic functions defined by

\[
h(e) = e - \sum_{n \geq 2} t_{1n} e^n + \sum_{n \geq 1} t_{2n} e^n; \quad (0 \leq t_{21} < 1).
\]

For the class of harmonic functions with negative coefficients, see [4]. For \( h \in \mathcal{H} \) assumed as in (1) and \( H \in \mathcal{H} \) assumed by

\[
H(e) = F(e) + \overline{G(e)} = e + \sum_{n \geq 1} \omega_{1n} e^n + \sum_{n \geq 1} \omega_{2n} e^n,
\]

we evoke the Hadamard product (or convolution) of \( h \) and \( H \) by

\[
(h * H)(e) = e + \sum_{n \geq 2} t_{1n} \omega_{1n} e^n + \sum_{n \geq 1} t_{2n} \omega_{2n} e^n; \quad (e \in U).
\]

The subclass \( \mathcal{S}_H^0, \mathcal{S}_H \) includes all functions \( h \in \mathcal{S}_H \) with \( h_0(0) = 0 \), so \( \mathcal{S} \subset \mathcal{S}_H^0 \subset \mathcal{S}_H \). Clunie and Sheil-Small also considered starlike functions in \( \mathcal{S}_H \), denote by \( \mathcal{S}_H^* \). The subclass of all starlike functions in \( \mathcal{S}_H^0 \) can be denoted by \( \mathcal{S}_H^* \). Starlikeness is not a hereditary property for harmonic mappings, so the image of every subdisk \( |e| \leq r < 1 \) is not necessarily starlike with respect to the origin [5,6]. Thus, we need a property to explain the starlikeness of a map in a hereditary form. We have the following definition.

**Definition 1** ([5]). A harmonic mapping \( f \) with \( h(0) = 0 \) is said to be fully starlike if it maps every circle \( |e| = r \leq 1 \) in a one-to-one manner onto a curve that bounds a starlike domain with respect to the origin.

For \( h \in \mathcal{S}_H \), the family of fully starlike functions is denoted by \( \mathcal{FS}_H^* \). In 1980, Mocanu gave a relation between fully starlikeness and a differential operator of a non-analytic function [7]. Let

\[
Dh = \xi_1 e - \overline{\xi_2 e}
\]

and clearly

\[
D^2h = \xi_1 e + \overline{\xi_2 e} + \xi_1 e - \overline{\xi_2 e}.
\]
Let $h \in C(U)$ be a complex-valued function such that $h(0) = 0, h(\varrho) \neq 0$ for all $\varrho \in U - 0$, and $|h(\varrho)| > 0$ in $U$ and

$$\text{Re} \left( \frac{Dh(\varrho)}{h(\varrho)} \right) = \frac{\varrho \xi_1 - \overline{\varrho \xi_2}}{\overline{h(\varrho)}} > 0.$$  

Then, $f$ is univalent and fully starlike in $U$. However, a fully-starlike mapping need not be univalent [8]. We restrict our discussion to the $S_H$ class. The harmonic function $h(z) = \text{Re} \left( \frac{\varrho}{1-\varrho} \right) + i\text{Im} \left( \frac{\varrho}{1-\varrho} \right)$ is not fully starlike [5], thus $f \notin S_H$ Clunie and SheilSmall [3] posed the following harmonic analogues of the Bieberbach conjecture (see Conjecture 2.3) for the family $S_H$:

1. $|\varrho_{1n}| - |\varrho_{2n}| \leq 2n, (n = 2, 3, \ldots)$;
2. $|\varrho_{1n}| < \frac{2n^2 + 1}{3}, (n = 2, 3, \ldots)$;
3. $|\varrho_{2n}| < \frac{2n^2 + 1}{3}, (n = 2, 3, \ldots)$.

Also, $h(z) = \text{Re} \left( \frac{\varrho}{1-\varrho} \right) + i\text{Im} \left( \frac{\varrho}{1-\varrho} \right)$ maps $U$ onto the half plane; then, for $n = 1, 2, \ldots$, we have the bounds

$$||\varrho_{1n}| - |_{2n}||| \leq 1, \quad |\varrho_{1n}| \leq \frac{(n + 1)}{2}, \quad |\varrho_{2n}| \leq \frac{(n - 1)}{2},$$

which are sharp. The results of these types have been previously obtained only for functions in the special subclass of convex harmonic functions $C_H$ (see [9]). However, necessary coefficient conditions for functions in $C_H$ were also found in [3]. Another challenging area is the Riemann Mapping Theorem related to the harmonic univalent mappings. The best possible Riemann Mapping Theorem was obtained by Hengartner and Schober in [10]. However, the uniqueness problem of mappings in their theorem and also the radius of starlikeness for starlike mappings in $S_H$ are still open. Since it is difficult to directly prove several results or obtain sharp estimates for the families $S_H$ and $S^*_H$, one usually attempts to investigate them for various subclasses of these families. In this article, we also made an attempt to define new class based on the $q$-difference Hurwitz–Lerch operator.

$q$-Difference Hurwitz–Lerch Operator: We aptly evoke the concept of $q$-operators. The $q$-difference operator has fascinated and inspired many scientists due to its use in various areas of quantitative sciences. The application of $q$-calculus was initiated by Jackson [11] (see also [12–15]. Kanas and Răducanu [16] used fractional $q$-calculus operators when investigating certain classes of functions, which are analytic in $U$.

For $0 < q < 1$ the Jackson $q$-derivative function, $f \in S$ is given by the following definition [11]:

$$\nabla_q^1 \xi_1(\varrho) = \begin{cases} \frac{\varrho \xi_1(\varrho) - \varrho \xi_1(q \varrho)}{(1-q)\varrho} & \text{for } \varrho \neq 0, \\ \xi_1'(0) & \text{for } \varrho = 0, \end{cases}$$

and $\nabla_q^2 \xi_1(\varrho) = \nabla_q(\nabla_q \xi_1(\varrho))$. From (7), we have $\nabla_q \xi_1(\varrho) = 1 + \sum_{n \geq 2} [n]_{1, \varrho} \varrho^{n-1}$ where $[n] = \frac{1-q^n}{1-q}$, is sometimes called the basic number. If $q \to 1^-$, $[n] \to n$. For a function $\xi_1(\varrho) = \varrho^n$, we obtain $\nabla_q \xi_1(\varrho) = \nabla_q \varrho^n = \frac{1-q^n}{1-q} \varrho^{n-1} = [n]_q \varrho^{n-1}$ and

$$\lim_{q \to 1^-} \nabla_q \xi_1(\varrho) = \lim_{q \to 1^-} \left( [n]_q \varrho^{n-1} \right) = n\varrho^{n-1} = \xi_1'(\varrho),$$

where $\xi_1'$ is the ordinary derivative.

For the first time, a research paper was presented in conjunction with function theory and $q$-theory by Ismail et al. [17]. So far, only insignificant interest has been shown in this area, although it deserves more attention. Difference operator: $q$-related to the $q$-calculus
was introduced by Andrews et al. (see [18] Chapter 10), Srivastava [19] and references cited therein. Several interesting properties and characteristics of the Hurwitz–Lerch Zeta (HLZ) function, $\Phi(z, s, b)$, defined by (cf. e.g., [20], p. 121), can also be found in recent investigations by Choi and Srivastava [21], Ferreira and Lopez [22], Garg et al. [23], Lin and Srivastava [24], Lin et al. [25] and others. Furthermore, Srivastava and Attiya [26], Raducanu and Srivastava [27] and Prajapat and Goyal [28] and references cited therein have studied various subclasses of analytic functions based on HLZ functions.

In the following, we recall a general $q$-analogue of Hurwitz–Lerch Zeta function $\Phi_q(\varphi, \kappa, \omega)$ defined in [29],

$$
\Phi_q(\varphi, \kappa, \omega) := \sum_{n=0}^{\infty} \frac{q^n}{[n+\varphi]_q^\kappa} \tag{8}
$$

($\varphi \in \mathbb{C} \setminus \{Z_q^-\}; \omega \in \mathbb{C}; \mathbb{R}(\kappa) > 1$ and $|\varphi| = 1$) where, as usual, $Z_q^- := Z \setminus \{N\}$, $(Z := \{\pm 1, \pm 2, \pm 3, \ldots\}); N := \{1, 2, 3, \ldots\}$. Now we state the linear operator:

$$
J^{\kappa \omega}_{q \varphi} : A \to A
$$

defined, in terms of the Hadamard product (or convolution), by

$$
J^{\kappa \omega}_{q \varphi} \xi_1(\varphi) = G^{\kappa \omega}_{q \varphi}(\varphi) \tag{9}
$$

($\varphi \in \mathbb{C} \setminus \{Z_q^-\}; \omega \in \mathbb{C}; \xi_1 \in A$), where, for convenience,

$$
G^{\kappa \omega}_{q \varphi}(\varphi) := [1 + \varphi]_q^\omega [\Phi_q(\varphi, \kappa, \omega) - [\varphi]_q^{-\kappa}] \quad (\varphi \in \mathcal{U}). \tag{10}
$$

It is easy to observe from (9) and (10) that, for $\xi_1$ of the form (2), we have

$$
J^{\kappa \omega}_{q \varphi} \xi_1(\varphi) = \varphi + \sum_{n \geq 2} L^{\kappa \omega}_{q}(n, \varphi) \omega_1 n q^n \tag{11}
$$

where (and throughout this paper, unless otherwise mentioned) the parameters $\omega, \varphi$ and $L^{\kappa \omega}_{q}(n, \varphi)$ are constrained as follows:

($\varphi \in \mathbb{C} \setminus \{Z_q^-\}; \kappa \in \mathbb{C}$ and $L^{\kappa \omega}_{q}(n, \varphi) = \left[\frac{1 + \varphi}{n + \varphi}\right]^\omega$).

For $\xi_1(\varphi) \in A$ and $\varphi \in \mathcal{U}$

$$
J^{\kappa \omega}_{q \varphi} \xi_1(\varphi) = \varphi + \sum_{n \geq 2} \left[\frac{1 + \varphi}{n + \varphi}\right]^\omega \omega_1 n q^n \tag{12}
$$

For various choices of $\kappa$, we obtain different operators, which are listed below (see also [20,30,31]).
\[ J^0_q(\xi_1)(e) := \xi_1(e), \quad J^1_q(\xi_1)(e) := \int_0^e \frac{\xi_1(t)}{t} \, dt := A[\xi_1(e)], \quad (q - \text{Alexander operator}). \]

\[ J^{1.1}_q(\xi_1)(e) := \int_0^e \frac{\xi_1(t)}{t} \, dt := A[\xi_1(e)] \quad (q - \text{Libera operator}). \]

\[ J^{1.1}_q(\xi_1)(e) := \frac{[2]_q}{\varrho} \int_0^e \frac{\xi_1(t)}{t} \, dt := A[\xi_1(e)] \quad (\varrho > -1) (q - \text{Bernardi operator}). \]

\[ \xi = \psi = \frac{\varrho}{\varrho} \]

which is closely related to some multiplier transformation studied by Fleet [32]. Motivated by the study on harmonic univalent functions [4,33–41], for the determination of this article, we will first become acquainted with the new operator (the \( q \)-Srivastava–Attiya operator)

\[ z \nabla_q (J_q^{\psi_\varphi} h(e)) = e ^\varphi \left( \frac{e \nabla_q (J_q^{\psi_\varphi} \xi_1(e))}{(1 - \varphi)} (J_q^{\psi_\varphi} h(e)) + \varphi \rho \right) \geq \mathbb{R} \]

and describe a subclass of \( \mathcal{H} \) symbolized \( \text{HS}^{\psi_\varphi}_q(\varphi, \mathbb{R}) \), which contains convolution (6) and consists of all functions of the form (1) such that they satisfy inequality:

\[ \mathbb{R} \left( \frac{e \nabla_q (J_q^{\psi_\varphi} h(e))}{(1 - \varphi)} \right) \geq \mathbb{R} \left( \frac{e \nabla_q (J_q^{\psi_\varphi} \xi_1(e))}{(1 - \varphi)} \right) \geq \mathbb{R} \]

where \( q \in \mathbb{U}, 0 \leq \varphi \leq 1 \) and \( e ^\varphi = \frac{e \varphi}{\varphi} \varphi = re^{i\varphi} \) where \( 0 = \varphi < 2\pi \). Also denote \( \text{HS}^{\psi_\varphi}_q(\varphi, \mathbb{R}) = \text{HS}^{\psi_\varphi}_q(\varphi, \mathbb{R}) \cap \mathbb{R} \).

We deem it appropriate to comment underneath some of the function classes that transpire from the function class \( \text{HS}^{\psi_\varphi}_q(\varphi, \mathbb{R}) \) defined above. Indeed, we observe that if we fix the parameters \( \varphi \) suitably, \( q \to 1 ^- \). We denote the reliable reducible new classes of \( \text{HS}^{\psi_\varphi}_q(\varphi, \mathbb{R}) \), which have not been studied so far in association with the \( q \)-Srivastava–Attiya operator, as illustrated below:

**Remark 1.** (i) If \( \varphi = 0 \), we let \( \text{HS}^{\psi_\varphi}_q(0, \mathbb{R}) = \text{HS}^{\psi_\varphi}_q(\varphi, \mathbb{R}) \) which satisfies

\[ \mathbb{R} \left( \frac{e \nabla_q (J_q^{\psi_\varphi} h(e))}{(1 - \varphi)} \right) \geq \mathbb{R} \]

(ii) If \( \varphi = 1 \), we let \( \text{HS}^{\psi_\varphi}_q(1, \mathbb{R}) = \text{NH}^{\psi_\varphi}_q(\varphi, \mathbb{R}) \) satisfying the criteria

\[ \mathbb{R} \left( D_q (J_q^{\psi_\varphi} h(e)) \right) \geq \mathbb{R} \]

(iii) When \( q \to 1 \), let \( \mathcal{M} \text{H}^{\psi_\varphi}_q(\varphi, \mathbb{R}) \) which satisfies

\[ \mathbb{R} \left( \frac{e (J_q^{\psi_\varphi} h(e))'}{(1 - \varphi)} \right) \geq \mathbb{R} \]

(iv) When \( q \to 1 \), we let \( \mathcal{M} \text{H}^{\psi_\varphi}_q(0, \mathbb{R}) = \text{HS}^{\psi_\varphi}_q(\varphi, \mathbb{R}) \), satisfying the criteria

\[ \mathbb{R} \left( \frac{e (J_q^{\psi_\varphi} h(e))'}{J_q^{\psi_\varphi} h(e)} \right) \geq \mathbb{R} \]

(v) When \( q \to 1 \) and taking \( \varphi = 1 \), we let \( \text{HS}^{\psi_\varphi}_q(0, \mathbb{R}) = \text{RH}^{\psi_\varphi}_q(\varphi, \mathbb{R}) \), satisfying the criteria

\[ \mathbb{R} (J_q^{\psi_\varphi} h(e))' \geq \mathbb{R} \]
Started by prior papers (see [3,4,34–41]) on the subject of harmonic functions, in this study, we obtain a sufficiency criterion for functions \( h \) given by (3) to be in the class \( \mathcal{H}S_{q}^{\kappa,\nu}(\omega, R) \). It is shown that this criterion is also necessary for \( h \in \overline{\mathcal{H}S_{q}^{\kappa,\nu}(\omega, R)} \). Furthermore, distortion limits and convexity conditions, extreme points, and partial sum problems \( h \in \overline{\mathcal{H}S_{q}^{\kappa,\nu}(\omega, R)} \) are also obtained. The special cases of our results yield the corresponding results for the function classes given in Remark 1.

2. The Coefficient Bounds

We will denote
\[
L_{q}^{\kappa}(n, \nu) = \left( \frac{1 + \nu}{n + \nu} \right)^{n} \quad (20)
\]
throughout our study unless otherwise stated.

In the following theorem, we obtain a sufficient criterion for \( h \in \mathcal{H}S_{q}^{\kappa,\nu}(\omega, R) \).

**Theorem 1.** Let \( h = \xi_{1} + \xi_{2} \) be given by (3). If
\[
\sum_{n \geq 1} \left[ \frac{|n|_{q} - (1 - \omega)R}{1 - R} |j_{1,n}| + \frac{|n|_{q} + (1 - \omega)R}{1 - R} |j_{2,n}| \right] L_{q}^{\kappa}(n, \nu) \leq 2 \quad (21)
\]
where \( r_{1} = 1 \) and \( 0 \leq \nu < 1 \), then \( h \in \mathcal{H}S_{q}^{\kappa,\nu}(\omega, R) \).

**Proof.** In order to achieve the result, it is sufficient to determine whether \( h \in \mathcal{H}S_{q}^{\kappa,\nu}(\omega, R) \) validates the relationship (21). From (19), we can write
\[
\mathbb{R} \left( \frac{\partial \nu_{q}(J_{q}^{\kappa,\nu,\omega}(\psi)) - \partial \nu_{q}(J_{q}^{\kappa,\nu,\omega}(\psi))}{(1 - \omega)(J_{q}^{\kappa,\nu,\omega}(\psi)) + \omega \psi'} \right) \geq \mathbb{R}
\]
\[
= \mathbb{R} \left( \frac{A(\psi)}{B(\psi)} \right) \geq \mathbb{R}
\]
where
\[
A(\psi) = \partial \nu_{q}(J_{q}^{\kappa,\nu,\omega}(\psi)) - \partial \nu_{q}(J_{q}^{\kappa,\nu,\omega}(\psi)) = \nu + \sum_{n \geq 2} |n|_{q} L_{q}^{\kappa}(n, \nu) s_{1,n} \psi - \sum_{n \geq 1} |n|_{q} L_{q}^{\kappa}(n, \nu) s_{2,n} \psi
\]
\[
B(\psi) = \omega \psi' + (1 - \omega)(J_{q}^{\kappa,\nu,\omega}(\psi)) = \nu + \sum_{n \geq 2} (1 - \omega) L_{q}^{\kappa}(n, \nu) s_{1,n} \psi + \sum_{n \geq 1} (1 - \omega) L_{q}^{\kappa}(n, \nu) s_{2,n} \psi.
\]

Considering the fact that \( \text{Re} \{ \psi \} \geq R \) if and only if \( |1 - R + \psi| \geq |1 + R - \psi| \), it suffices to show that
\[
|A(\psi) + (1 - R)B(\psi)| - |A(\psi) - (1 + R)B(\psi)| \geq 0. \quad (22)
\]
Substituting for \( A(\rho) \) and \( B(\rho) \) in (22), we have

\[
|A(\rho) + (1 - \lambda)B(\rho)| - |A(\rho) - (1 + \lambda)B(\rho)| = |(2 - \lambda)e + \sum_{n \geq 2} |[n|q - (1 - \alpha)|L_q^n(n, \rho)|_{|t_1n|}e^n| - | - \lambda e + \sum_{n \geq 2} |[n|q - (1 + \alpha)|L_q^n(n, \rho)|_{|t_1n|}e^n|
- \sum_{n \geq 1} |[n|q - (1 - \alpha)|L_q^n(n, \rho)|_{|t_{2n}|}|e^n| + \sum_{n \geq 1} |[n|q - (1 + \alpha)|L_q^n(n, \rho)|_{|t_{1n}|}|e^n|
\geq (2 - \lambda)|e| - \sum_{n \geq 2} |[n|q - (1 - \alpha)|L_q^n(n, \rho)|_{|t_{1n}|}|e^n| - \sum_{n \geq 2} |[n|q - (1 - \alpha)|L_q^n(n, \rho)|_{|t_{2n}|}|e^n|
- \sum_{n \geq 1} |[n|q - (1 - \alpha)|L_q^n(n, \rho)|_{|t_{1n}|}|e^n| - \sum_{n \geq 1} |[n|q - (1 + \alpha)|L_q^n(n, \rho)|_{|t_{2n}|}|e^n|
= 2(1 - \lambda)|e| \left\{ 2 - \sum_{n \geq 1} \left[ \frac{[n|q - (1 - \alpha)|R}{1 - \lambda}|_{|t_{1n}|} + \frac{[n|q + (1 - \alpha)|R}{1 - \lambda}|_{|t_{2n}|} \right] L_q^n(n, \rho)|e|^{n-1} \right\}
= 2(1 - \lambda) \left\{ 2 - \sum_{n \geq 1} \left[ \frac{[n|q - (1 - \alpha)|R}{1 - \lambda}|_{|t_{1n}|} + \frac{[n|q + (1 - \alpha)|R}{1 - \lambda}|_{|t_{2n}|} \right] L_q^n(n, \rho) \right\}.
\]

The above condition is non-negative by (21), and so \( \xi_1 \in HS^{\kappa, \lambda}_q(\alpha, \lambda) \). \( \square \)

The harmonic function

\[
h(\rho) = \rho + \sum_{n \geq 2} \left( \frac{1 - \lambda}{[n|q - \lambda(1 - \alpha)]} L_q^n(n, \rho) \right) x_n e^n + \sum_{n \geq 1} \left( \frac{1 - \lambda}{[n|q + \lambda(1 - \alpha)]} L_q^n(n, \rho) \right) y_n \varepsilon^n \]

(23)

where \( \sum_{n \geq 2} |x_n| + \sum_{n \geq 1} |y_n| = 1 \), shows that the coefficient bound in (21) is sharp. Then, \( h(\rho) \) as in (23) and \( h(\rho) \in HS^{\kappa, \lambda}_q(\alpha, \lambda) \) because

\[
\sum_{n \geq 1} \left( \frac{[n|q - (1 - \alpha)|R}{1 - \lambda}|_{|t_{1n}|} + \frac{[n|q + (1 - \alpha)|R}{1 - \lambda}|_{|t_{2n}|} \right) L_q^n(n, \rho) = 1 + \sum_{n \geq 2} |x_n| + \sum_{n \geq 1} |y_n| = 2.
\]

The following theorem states that such coefficient restrictions cannot be further improved.

**Theorem 2.** For \( n_1 = 1 \) and \( 0 \leq \lambda < 1 \), \( h = \xi_1 + \xi_2 \in HS^{\kappa, \lambda}_q(\alpha, \lambda) \) if and only if

\[
\sum_{n \geq 1} \left[ \frac{[n|q - (1 - \alpha)|R}{1 - \lambda}|_{|t_{1n}|} + \frac{[n|q + (1 - \alpha)|R}{1 - \lambda}|_{|t_{2n}|} \right] L_q^n(n, \rho) \leq 2.
\]

**Proof.** Since \( HS^{\kappa, \lambda}_q(\alpha, \lambda) \subset HS^{\kappa, \lambda}_q(\alpha, \lambda) \), we only need to prove the “only if” part of the theorem. To this end, for \( h \in \xi_1 \) of the form (4), we state that the condition

\[
\Re \left( \frac{z \nabla_q(j^{\kappa, \lambda}_q h(\rho)) - \alpha \nabla_q(j^{\kappa, \lambda}_q g(\rho))}{(1 - \alpha)(j^{\kappa, \lambda}_q h(\rho)) + \alpha q'} \right) \geq \lambda
\]

Equivalently,
(1 - \kappa)q - \sum_{n \geq 2} (|n|q - (1 - \omega)N) L_q^r(n, \varphi) t_1n \bar{q}^n + \sum_{n \geq 1} |n|q + (1 - \omega)N \sum_{n \geq 1} |n|q + (1 - \omega)N \bar{q}^n \\
\geq 0.

The above mandatory condition must hold for all values of \( z \) in \( U \). Upon taking the values of \( q \) on the positive real axis where \( 0 \leq q < 1 \), we must have

\[
(1 - \kappa) - \sum_{n \geq 2} (|n|q - (1 - \omega)N) L_q^r(n, \varphi) t_1n \bar{q}^n + \sum_{n \geq 1} |n|q + (1 - \omega)N \bar{q}^n \\
1 - \sum_{n \geq 2} (|n|q - (1 - \omega)N) L_q^r(n, \varphi) t_2n \bar{q}^n + \sum_{n \geq 1} |n|q + (1 - \omega)N \bar{q}^n \geq 0. \tag{25}
\]

If the condition (24) does not hold, then the numerator in (25) is negative for \( r \) close enough to 1. Thus, there exists \( q_0 = r_0 \) in \((0, 1)\) for which the proportion (25) is negative. This is contrary to a necessary condition for \( h \in \overline{HS}_p^{\omega}(\alpha, N) \). This completes the proof of the theorem.

3. Distortion Bounds and Extreme Points

The subsequent theorem provides the distortion limits for functions in \( \overline{HS}_p^{\omega}(\alpha, N) \), which yields a covering result for the class \( \overline{HS}_p^{\omega}(\alpha, N) \).

**Theorem 3.** Let \( h \in \overline{HS}_p^{\omega}(\alpha, N) \). Then, for \( |q| = r < 1 \), we have

\[
(1 - \iota_1) r - L_q^r(2, \varphi) \left( \frac{1 - \kappa}{2|q| - (1 - \omega)N} \right) \leq |h(q)| \leq (1 + \iota_2) r + L_q^r(2, \varphi) \left( \frac{1 + \kappa}{2|q| - (1 - \omega)N} \right)
\]

when, for \( n = 2 \) in (20), we obtain

\[
L_q^r(2, \varphi) = \left( \frac{1 + \varphi}{2 + \varphi} \right)^k q_n \tag{26}
\]

**Proof.** We will show the right-hand inequality only by taking the absolute value \( h(q) \),

\[
|h(q)| = \left| q + \sum_{n \geq 2} t_1n \bar{q}^n + \sum_{n \geq 1} t_2n \bar{q}^n \right|
\leq (1 + \iota_1) |q| + \sum_{n \geq 2} (t_1n + t_2n) |q|^n
= (1 + \iota_1) r + \sum_{n \geq 2} t_1n + t_2n r^2
= (1 + \iota_2) r + L_q^r(2, \varphi) \times \sum_{n \geq 2} \left( \frac{1 - \kappa}{2|q| - (1 - \omega)N} t_1n + \frac{1 + \kappa}{1 - \kappa} t_2n \right)^2
= (1 + \iota_2) r + \frac{(1 - \kappa)}{2|q| - (1 - \omega)N} \left( 1 + \frac{1 + \kappa}{1 - \kappa} \iota_2n \right)^2
\leq (1 + \iota_2) r + L_q^r(2, \varphi) \left( \frac{1 - \kappa}{2|q| - (1 - \omega)N} \right) \left( 1 + \frac{1 + \kappa}{2|q| - (1 - \omega)N} \iota_2n \right)^2.
\]

The proof of the left-hand inequality follows on lines similar to that of the right-hand side’s inequality. □

The covering result follows from the left hand inequality given in Theorem 3.
Theorem 4. If $h \in \overline{HS}_q^{\kappa, \psi}(\omega, \mathbb{R})$, then
$$
\left\{ w : |w| < \frac{([2]_q - (1 - \omega)\mathbb{R})L_q^\kappa(2, \psi) - (1 - \mathbb{R})}{([2]_q - (1 - \omega)\mathbb{R})L_q^\kappa(2, \psi)} \frac{([2]_q - (1 - \omega)\mathbb{R})L_q^\kappa(2, \psi) - (1 + \mathbb{R})}{([2]_q - (1 - \omega)\mathbb{R})L_q^\kappa(2, \psi)} |_{t_21} \right\} \subset h(U)
$$
where $L_q^\kappa(2, \psi)$ given in (26).

Proof. Using the left inequality of Theorem 3 and letting $r \to 1$, we prove that
$$
(1 - t_{21}) - \frac{1}{L_q^\kappa(2, \psi)} \left( \frac{1 - \mathbb{R}}{[2]_q - (1 - \omega)\mathbb{R}} - \frac{1 + \mathbb{R}}{[2]_q - (1 - \omega)\mathbb{R}} \right) t_{21}
= (1 - t_{21}) - \frac{1}{L_q^\kappa(2, \psi)([2]_q - (1 - \omega)\mathbb{R})} [1 - \mathbb{R} - (1 + \mathbb{R}) t_{21}]
= (1 - t_{21})L_q^\kappa(2, \psi)(([2]_q - (1 - \omega)\mathbb{R}) - (1 - \mathbb{R}) + (1 + \mathbb{R}) t_{21})
= \left\{ \frac{([2]_q - (1 - \omega)\mathbb{R})L_q^\kappa(2, \psi) - (1 - \mathbb{R})}{([2]_q - (1 - \omega)\mathbb{R})L_q^\kappa(2, \psi)} \frac{([2]_q - (1 - \omega)\mathbb{R})L_q^\kappa(2, \psi) - (1 + \mathbb{R})}{([2]_q - (1 - \omega)\mathbb{R})L_q^\kappa(2, \psi)} |_{t_{21}} \right\} \subset h(U).
$$

For any compact family, the maximum or minimum of the real part of any continuous linear functional occurs at one of the extreme points of the closed convex hull. Since $\overline{HS}_q^{\kappa, \psi}(\omega, \mathbb{R})$ convex families, we will use the necessary and sufficient coefficient inequalities of Theorems 1 and 2 to determine their extreme points. Next, we regulate the extreme points of closed convex hulls of $\overline{HS}_q^{\kappa, \psi}(\omega, \mathbb{R})$ symbolized by $cH\overline{HS}_q^{\kappa, \psi}(\omega, \mathbb{R})$.

Theorem 5. A function $h \in \overline{HS}_q^{\kappa, \psi}(\omega, \mathbb{R})$ if and only if
$$
h(q) = \sum_{n \geq 1} \left( X_n h_n(q) + Y_n g_n(q) \right)
$$
where
$$
f_1(q) = q, \quad \xi_{1n}(q) = q - \frac{1 - \mathbb{R}}{([n]_q - (1 - \omega)\mathbb{R})L_q^\kappa(n, \psi)} q^n; \quad (n \geq 2), \quad (27)
$$
$$
\xi_{2n}(q) = q + \frac{1 - \mathbb{R}}{([n]_q + (1 - \omega)\mathbb{R})L_q^\kappa(n, \psi)} q^n; \quad (n \geq 2), \quad (28)
$$
$$
\sum_{n \geq 1} (X_n + Y_n) = 1, \quad X_n \geq 0 \text{ and } Y_n \geq 0. \text{ In particular, the extreme points of } \overline{HS}_q^{\kappa, \psi}(\omega, \mathbb{R}) \text{ are } \{ \xi_{1n} \} \text{ and } \{ \xi_{2n} \}.
$$

Proof. We annotate that for $h$, as in the above theorem, we may state
$$
h(q) = \sum_{n \geq 1} \left( X_n \xi_{1n}(q) + Y_n \xi_{2n}(q) \right)
= \sum_{n \geq 1} (X_n + Y_n) q - \sum_{n \geq 2} \frac{1 - \mathbb{R}}{([n]_q - (1 - \omega)\mathbb{R})L_q^\kappa(n, \psi)} X_n q^n
+ \sum_{n \geq 1} \frac{1 - \mathbb{R}}{([n]_q + (1 - \omega)\mathbb{R})L_q^\kappa(n, \psi)} Y_n q^n.
$$
Theorem 6. The family \( \mathcal{HS}_{q}^{s,\varphi} (\omega, \kappa) \) is closed under convex combinations.

Proof. For \( i = 1, 2, \ldots \), suppose that \( h_i \in \mathcal{HS}_{q}^{s,\varphi} (\omega, \kappa) \) where

\[
h_i(e) = e - \sum_{n \geq 2} t_{i,n} e^n + \sum_{n \geq 2} \sum_{n \geq 1} (\xi_{1,n}(e) - e) X_n + \sum_{n \geq 1} \xi_{2,n}(e) Y_n.
\]

Then, by Theorem 2

\[
\sum_{n \geq 2} \frac{([n]_q - (1 - \omega)\kappa) \text{L}_q^s(n, \varphi)}{1 - \kappa} |t_{1,n}| + \sum_{n \geq 1} \frac{([n]_q + (1 - \omega)\kappa) \text{L}_q^s(n, \varphi)}{1 - \kappa} |t_{2,n}| \leq 1.
\]
Theorem 8. For \(0 \leq \delta \leq \kappa < 1\), let \(h \in \mathcal{HS}^{\phi, \psi}_{q}(\omega, \kappa)\) and \(H \in \mathcal{HS}^{\phi, \psi}_{q}(\omega, \kappa)\). Then, \(h(\cdot) \ast H(\cdot) \in \mathcal{HS}^{\phi, \psi}_{q}(\omega, \kappa) \subset \mathcal{HS}^{\phi, \psi}_{q}(\omega, \delta)\).

Proof. Let \(h \in \mathcal{HS}^{\phi, \psi}_{q}(\omega, \kappa)\) and \(H(\cdot) \in \mathcal{HS}^{\phi, \psi}_{q}(\omega, \delta)\). Then, \(h(\cdot) \ast H(\cdot)\) is given by (6).
For $h(q) \ast H(q) \in \mathcal{HS}^{\kappa,\nu}_{f}(\varphi, \delta)$, we note that $|\varphi_{1m}| \leq 1$ and $|\varphi_{2m}| \leq 1$. Now, by Theorem 2, we have

$$\sum_{n \geq 2} \frac{|n|_{q} - (1 - \omega)\delta}{1 - \delta} L_{q}^{\kappa}(n, \varphi)|\varphi_{1n}| + \sum_{n \geq 1} \frac{|n|_{q} - (1 - \omega)\delta}{1 - \delta} L_{q}^{\kappa}(n, \varphi)|\varphi_{2n}| \leq 1,$$

and since $0 \leq \delta \leq \kappa < 1$

$$\sum_{n \geq 2} \frac{|n|_{q} - (1 - \omega)\kappa}{1 - \kappa} L_{q}^{\kappa}(n, \varphi)|\varphi_{1n}| + \sum_{n \geq 1} \frac{|n|_{q} - (1 - \omega)\kappa}{1 - \kappa} L_{q}^{\kappa}(n, \varphi)|\varphi_{2n}| \leq 1,$$

and by Theorem 2, we obtain the desired result. \(\square\)

5. Partial Sums Results

Many researchers have studied and distinguished partial sum results for different classes of analytic functions based on the results provided by Silvia [42]. Silverman [43] determined that the lower bounds on ratios like $\{ \Re \frac{f(z)}{f_{n}(z)} \}$ or $\{ \Re \frac{f_{n}(z)}{f(z)} \}$ have been found to be sharp only when $n = 1$. The lower bounds in question are strictly increasing functions of $n$. Analogous results on harmonic functions have not yet been explored in the literature. Recently, in [44], Porwal filled this gap by checking exciting results on partial sums of star harmonic univalent functions (see [45]). In this section, we examine partial sum results for $h \in \mathcal{HS}^{\kappa,\nu}_{f}(\varphi, \kappa)$.

Let $\mathcal{HS}^{\kappa,\nu}_{f}(\varphi, \kappa) \subseteq \mathcal{H}$ consisting of functions $h = \xi_{1} + \xi_{2}$, as assumed as in (3) with

$$\sum_{n \geq 2} \mathbf{M}_{n}|\varphi_{1n}| + \sum_{n \geq 1} \mathbf{Q}_{n}|\varphi_{2n}| \leq 1$$

(30)

where

$$\mathbf{M}_{n} = \frac{|n|_{q} - (1 - \omega)\kappa}{1 - \kappa} \text{ and } \mathbf{Q}_{n} = \frac{|n|_{q} + (1 - \omega)\kappa}{1 - \kappa}$$

unless otherwise stated. Now, we discuss the ratio of $h$, as assumed as in (3) with $\varphi_{21} = 0$, where

$$h_{\ell}(q) = q + \sum_{n = 2}^{\ell} t_{1n}q^{n} + \sum_{n = 2}^{\ell} t_{2n}q^{n},$$

$$h_{k}(q) = q + \sum_{n = 2}^{k} t_{1n}q^{n} + \sum_{n = 2}^{k} t_{2n}q^{n},$$

$$h_{\ell k}(q) = q + \sum_{n = 2}^{\ell} t_{1n}q^{n} + \sum_{n = 2}^{k} t_{2n}q^{n}.$$}

We begin by obtaining the sharp bounds for $\Re \{ \frac{h(q)}{h_{\ell}(q)} \}$.

Theorem 9. If $h$ of the form (3) with $\varphi_{21} = 0$ and holds (30), then

$$\Re \left\{ \frac{h(q)}{h_{\ell}(q)} \right\} \geq \frac{\mathbf{M}_{\ell+1} - (1 - \kappa)}{\mathbf{M}_{\ell+1}}, \quad (q \in \mathbb{U})$$

(31)

where

$$\mathbf{M}_{n} = \begin{cases} 1 - \kappa, & \text{if } n = 2, 3, \ldots, \ell \\ \mathbf{M}_{\ell+1}, & \text{if } n = \ell + 1, \ell + 2, \ldots. \end{cases}$$
$$Q_n \geq 1 - \Re, \quad \text{if} \quad n = 2, 3, \ldots$$

The result (31) is sharp for

$$h(\varphi) = \varphi + \frac{1 - \Re}{M_{\ell+1}} e^{\ell+1}. \quad (32)$$

**Proof.** To prove (31) we set

$$1 + \frac{w(\varphi)}{1 - w(\varphi)} = \frac{M_{\ell+1}}{1 - \Re} \left[ \frac{h(re^{i\varphi})}{h(re^{i\varphi})} - \frac{M_{\ell+1} - (1 - \Re)}{M_{\ell+1}} \right]$$

$$= 1 + \sum_{n=2}^{\ell} t_n r^{n-1} e^{i(n-1)\varphi} + \sum_{n=2}^{\ell} \frac{r^{n-1}}{n-1} e^{-i(n-1)\varphi} - \frac{M_{\ell+1} - (1 - \Re)}{M_{\ell+1}} \left( \sum_{n=\ell+1}^{\infty} t_n r^{n-1} e^{i(n-1)\varphi} \right). \quad (33)$$

It suffices to show that $|w(\varphi)| \leq 1$. Now, from (33) we can write

$$w(\varphi) = \frac{M_{\ell+1}}{1 - \Re} \left( \sum_{n=\ell+1}^{\infty} t_n r^{n-1} e^{i(n-1)\varphi} \right) \left( 2 + 2 \left( \sum_{n=2}^{\ell} |t_n| + \sum_{n=2}^{\ell} |\varphi_n| - \frac{M_{\ell+1}}{1 - \Re} \sum_{n=\ell+1}^{\infty} |t_n| \right) \right) - \frac{M_{\ell+1}}{1 - \Re} \sum_{n=\ell+1}^{\infty} |t_1 n|.$$ 

Hence, we obtain

$$|w(\varphi)| \leq \frac{M_{\ell+1}}{1 - \Re} \left( \sum_{n=\ell+1}^{\infty} |t_1 n| \right) \left( 2 + \frac{\sum_{n=2}^{\ell} |t_1 n| + \sum_{n=2}^{\ell} |\varphi_n|}{1 - \Re} - \frac{M_{\ell+1}}{1 - \Re} \sum_{n=\ell+1}^{\infty} |t_1 n| \right).$$

Now, $|w(\varphi)| \leq 1$ if

$$\sum_{n=2}^{\ell} |t_1 n| + \sum_{n=2}^{\ell} |\varphi_n| + \frac{M_{\ell+1}}{1 - \Re} \sum_{n=\ell+1}^{\infty} |t_1 n| \leq 1.$$ 

From the condition (30), it is sufficient to show that

$$\sum_{n=2}^{\ell} |t_1 n| + \sum_{n=2}^{\ell} |\varphi_n| + \frac{M_{\ell+1}}{1 - \Re} \sum_{n=\ell+1}^{\infty} |t_1 n| \leq \sum_{n=2}^{\ell} |t_1 n| + \sum_{n=2}^{\ell} |\varphi_n| + \frac{Q_n}{1 - \Re} |t_1 n|.$$

which is equivalent to

$$\sum_{n=2}^{\ell} \left( \frac{M_n - (1 - \Re)}{1 - \Re} \right) |t_1 n| + \sum_{n=2}^{\ell} \left( \frac{Q_n - (1 - \Re)}{1 - \Re} \right) |\varphi_n| + \sum_{n=\ell+1}^{\infty} \left( \frac{M_n - M_{n+1}}{1 - \Re} \right) |\varphi_n| \geq 0.$$ 

To see that $h(\varphi)$ as in (32) gives the sharp result, we observe that for $\varphi = re^{i\pi/n}$

$$h(\varphi) = \frac{h(e^{i\varphi})}{h(e^{i\varphi})} = 1 + \frac{1 - \Re}{M_{\ell+1}} e^{\ell+1} \to 1 - \frac{1 - \Re}{M_{\ell+1}}$$

when $r \to 1$. 

□
We next determine bounds for $\Re \{ h(q)/q(q) \}$.

**Theorem 10.** If $h$ of the form (3) with $\tau_1 = 0$ and holds (30), then

$$\Re \left\{ \frac{h(q)}{\tilde{h}(q)} \right\} \geq \frac{M_{\ell+1}}{M_{\ell+1} + 1 - R} \quad (q \in \mathbb{U}) \tag{34}$$

where

$$M_n \geq \begin{cases} 1 - R, & \text{if } n = 2, 3, \ldots, \ell \\ M_{\ell+1}, & \text{if } n = \ell + 1, \ell + 2, \ldots. \end{cases} \tag{35}$$

The result (34) is sharp for

$$h(q) = q + \frac{1 - R}{M_{\ell+1}} q^{\ell+1}. \tag{36}$$

**Proof.** To prove (34) we let

$$\frac{1 + \tilde{w}(q)}{1 - \tilde{w}(q)} = \frac{M_{\ell+1} + 1 - R}{1 - R} \left[ \frac{h(re^{i\varphi})}{h(re^{i\varphi})} - \frac{M_{\ell+1}}{M_{\ell+1} + 1 - R} \right]$$

$$= \frac{1 + \sum_{n=2}^{\ell} t_1 r^{|n-1| e^{i(n-1)\varphi}} + \sum_{n \geq 2} \frac{M_{\ell+1}}{1 - R} \left( \sum_{n=\ell+1}^{\infty} t_1 r^{|n-1| e^{i(n-1)\varphi}} \right)}{1 + \sum_{n=2}^{\ell} |t_{1n}| + \sum_{n \geq 2} \frac{M_{\ell+1}}{1 - R} \left( \sum_{n=\ell+1}^{\infty} |t_{1n}| \right)}.$$

Hence, we obtain

$$|\tilde{w}(q)| \leq \frac{M_{\ell+1} + 1 - R}{1 - R} \left[ \sum_{n=\ell+1}^{\infty} |t_{1n}| \right] \leq 1.$$

The last inequality is equivalent to

$$\sum_{n=2}^{\ell} |t_{1n}| + \sum_{n \geq 2} \frac{M_{\ell+1}}{1 - R} \sum_{n=\ell+1}^{\infty} |t_{1n}| \leq 1.$$

Making use of (30) and (35), we obtain (5). Finally, equality (34) holds for $h(q)$ as in (36). \(\square\)

We next turn to ratios for the for $\Re \left\{ h'(q)/h(q) \right\}$ and $\Re \left\{ h'(q)/h'(q) \right\}$.

**Theorem 11.** If $h$ of the form (3) with $b_1 = 0$ satisfies the condition (30), then

$$\Re \left\{ \frac{h'(q)}{h'(q)} \right\} \geq \frac{M_{\ell+1} - (\ell + 1)(1 - R)}{M_{\ell+1}} \quad (q \in \mathbb{U}) \tag{37}$$

where

$$M_n \geq \begin{cases} 1 - R, & \text{if } n = 2, 3, \ldots, \ell \\ M_{\ell+1}, & \text{if } n = \ell + 1, \ell + 2, \ldots. \end{cases} \tag{35}$$

$$Q_n \geq 1 - R, \quad \text{if } n = 2, 3, \ldots$$
The result (37) is sharp for \( h(q) = q + \frac{1-h}{M_{\ell+1}} q^{\ell+1}. \)

**Proof.** To prove (37) we define

\[
\frac{1+w(q)}{1-w(q)} = \frac{M_{\ell+1}}{(\ell+1)1-N} \left[ \frac{\xi_{1}(q)}{\xi_{1}(q)} - \frac{M_{\ell+1} - (\ell+1)(1-N)}{M_{\ell+1}} \right] \]

\[
1 + \sum_{n=2}^{\ell} nt_{1}n^{n-1}e^{i(n-1)\varphi} + \sum_{n=2}^{\ell} nt_{2}n^{n-1}e^{-i(n+1)\varphi} = \frac{M_{\ell+1}}{(\ell+1)1-N} \left[ \sum_{n=2}^{\ell} nt_{1}n^{n-1}e^{i(n-1)\varphi} - \sum_{n=2}^{\ell} nt_{2}n^{n-1}e^{-i(n+1)\varphi} \right].
\]

The result (37) follows by using the techniques used in Theorem 9. \( \square \)

Proceeding exactly as in the proof of Theorem 10, we can prove the following theorem.

**Corollary 1.** If \( h \) of the form (3) with \( t_{21} = 0 \) and satisfies (30), then

\[
\Re \left\{ \frac{h_{q}(q)}{h(q)} \right\} \geq \frac{Q_{k+1} - (1-N)}{Q_{k+1}}, \quad (q \in \mathbb{U}).
\]  

The result is sharp for \( h(q) = q + \frac{1-N}{Q_{k+1}} q^{k+1}. \)

We next determine bounds for \( \Re \{ h(q)/h_{q}(q) \} \) and \( \Re \{ h_{q}(q)/h(q) \}. \)

**Corollary 2.** If \( h \) of the form (3) with \( t_{21} = 0 \) and satisfies (30), then

\[
\Re \left\{ \frac{h(q)}{h_{k}(q)} \right\} \geq \frac{Q_{k+1} - (1-N)}{Q_{k+1}}, \quad (q \in \mathbb{U})
\]

where

\[
Q_{n} \geq \begin{cases} 
1-N, & \text{if } n = 2, 3, \ldots, k \\
Q_{k+1}, & \text{if } n = k+1, k+2, \ldots.
\end{cases}
\]

\[
M_{n} \geq 1-N, \quad \text{if } n = 2, 3, \ldots.
\]

The result (39) is sharp for \( h(q) = q + \frac{1-N}{Q_{k+1}} q^{k+1}. \)

**Theorem 12.** If \( h \) of the form (3) with \( t_{21} = 0 \) and satisfies (30), then

\[
\Re \left\{ \frac{h_{k}(q)}{h(q)} \right\} \geq \frac{Q_{k+1}}{Q_{k+1} + 1-N}, \quad (q \in \mathbb{U})
\]

where

\[
Q_{n} \geq \begin{cases} 
1-N, & \text{if } n = 2, 3, \ldots, k \\
Q_{k+1}, & \text{if } n = k+1, k+2, \ldots.
\end{cases}
\]

\[
M_{n} \geq 1-N, \quad \text{if } n = 2, 3, \ldots.
\]

The result (40) is sharp for \( h(q) = q + \frac{1-N}{Q_{k+1}} q^{k+1}. \)

**Proof.** To prove (40) we set

\[
\frac{1+w(q)}{1-w(q)} = \frac{Q_{k+1} + 1-N}{1-N} \left[ \frac{\xi_{1}(q)}{\xi_{1}(q)} - \frac{Q_{k+1}}{Q_{k+1} + 1-N} \right].
\]
where

\[
1 + \sum_{n=2}^{k} \frac{t_1 n^{n-1} \varphi(n) - i(n+1) \varphi}{\sum_{n=2}^{k} \frac{t_2 n^{n-1} e^{-i(n+1) \varphi}}{M_{q+1}}} \geq \frac{Q_{k+1} - (1 - \mathbb{R})}{Q_{k+1}}.
\]

We omit the details of this proof because it runs parallel to that from Theorem 10. □

**Corollary 3.** If \( \mathbf{h} \) of the form (3) with \( t_{21} = 0 \) and holds (30), then

\[
\Re \left\{ \frac{h(\zeta)}{R_{\ell,k}(\zeta)} \right\} \geq \frac{M_{\ell+1} - (1 - \mathbb{R})}{M_{\ell+1}}, \quad (\zeta \in \mathbb{U})
\]

where

\[
M_n \geq \begin{cases} 1 - \mathbb{R}, & \text{if } n = 2, 3, \ldots, \ell, \ell + 1 \\ M_{\ell+1}, & \text{if } n = \ell + 1, \ell + 2, \ldots. \end{cases}
\]

\[
Q_n \geq \begin{cases} 1 - \mathbb{R}, & \text{if } n = 2, 3, \ldots, k \\ Q_{k+1}, & \text{if } n = k + 1, k + 2, \ldots. \end{cases}
\]

The result (41) is sharp for \( \mathbf{h}(\zeta) = \zeta + \frac{1 - \mathbb{R}}{M_{\ell+1}} \zeta^{\ell+1} \).

**Corollary 4.** If \( \mathbf{h} \) of the form (3) with \( t_{21} = 0 \) and holds (30), then

\[
\Re \left\{ \frac{h'(\zeta)}{h(\zeta)} \right\} \geq \frac{Q_{k+1} - (1 - \mathbb{R})}{Q_{k+1}}, \quad (\zeta \in \mathbb{U})
\]

where

\[
Q_n \geq \begin{cases} 1 - \mathbb{R}, & \text{if } n = 2, 3, \ldots, k \\ Q_{k+1}, & \text{if } n = k + 1, k + 2, \ldots. \end{cases}
\]

\[
M_n \geq \begin{cases} 1 - \mathbb{R}, & \text{if } n = 2, 3, \ldots, k \\ M_{\ell+1}, & \text{if } n = k + 1, k + 2, \ldots. \end{cases}
\]

**Corollary 5.** If \( \mathbf{h} \) of the form (3) with \( t_{21} = 0 \) and satisfies (30), then

\[
\Re \left\{ \frac{h_{\ell,k}'}{h_{\ell,k}} \right\} \geq \frac{M_{\ell+1} - (1 - \mathbb{R})}{M_{\ell+1}} \quad (\zeta \in \mathbb{U}).
\]

**Corollary 6.** If \( \mathbf{h} \) of the form (3) with \( t_{21} = 0 \) and holds (30), then

\[
\Re \left\{ \frac{h_{\ell,k}'}{h_{\ell,k}} \right\} \geq \frac{Q_{k+1} - (1 - \mathbb{R})}{Q_{k+1}} \quad (\zeta \in \mathbb{U}).
\]

The result (44) is sharp for \( \mathbf{h}(\zeta) = \zeta + \frac{1 - \mathbb{R}}{Q_{k+1}} \zeta^{k+1} \).

**Corollary 7.** If \( \mathbf{h} \) of the form (3) with \( t_{21} = 0 \) and satisfies (30), then

\[
\Re \left\{ \frac{h'(\zeta)}{h(\zeta)} \right\} \geq \frac{M_{\ell+1} - (\ell + 1)(1 - \mathbb{R})}{M_{\ell+1}} \quad (\zeta \in \mathbb{U})
\]

where

\[
M_n \geq \begin{cases} 1 - \mathbb{R}, & \text{if } n = 2, 3, \ldots, \ell \\ M_{\ell+1}, & \text{if } n = \ell + 1, \ell + 2, \ldots. \end{cases}
\]

\[
Q_n \geq \begin{cases} 1 - \mathbb{R}, & \text{if } n = 2, 3, \ldots, \ell \\ Q_{k+1}, & \text{if } n = k + 1, k + 2, \ldots. \end{cases}
\]

The result (45) is sharp for \( \mathbf{h}(\zeta) = \zeta + \frac{1 - \mathbb{R}}{M_{\ell+1}} \zeta^{\ell+1} \).
Corollary 8. If \( h \) of the form (3) with \( \iota_2 = 0 \) satisfies (30), then

\[
\mathfrak{R} \left\{ \frac{h'(z)}{h(z)} \right\} \geq \frac{M_{\ell+1}}{M_{\ell+1} + (\ell + 1)(1 - \eta)} \quad (q \in \mathbb{U}).
\]

(46)

The result (46) is sharp for \( h(q) = q + \frac{1 - \eta}{M_{\ell+1}} q^{\ell+1} \).

6. Integral Means Inequalities

An analytic function \( h \) is subordinate to an analytic function \( g \), written \( h(z) < g(z) \), provided there is an analytic function \( w \) defined on \( \mathbb{D} \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) sustaining \( h(z) = g(w(z)) \). Using the principle of subordination and the following Lemma 1, we obtain integral means inequalities for the functions in the family \( \mathbf{HS}_{q}^{\psi}(\omega, \mathbb{N}) \) due to Dziok [36] and Silverman [46].

Lemma 1 ([47]). If the functions \( \phi \) and \( \psi \) are analytic in \( U \) with \( \psi < \phi \), ( \( \psi \) is subordinate to \( \phi \)), then for \( \eta > 0 \), and \( 0 < r < 1 \),

\[
\int_{0}^{2\pi} \left| \frac{\phi(re^{i\varphi})}{\psi(re^{i\varphi})} \right|^{\eta} d\varphi \leq \int_{0}^{2\pi} \left| \frac{\phi(re^{i\varphi})}{\psi(re^{i\varphi})} \right|^{\eta} d\varphi.
\]

(47)

Due to the recent work of Dziok [36], we suppose \( h \in \mathbf{HS}_{q}^{\psi}(\omega, \mathbb{N}) \) \( \eta > 0 \), \( 0 \leq \omega < 1 \), \( 0 \leq \kappa \leq 1 \), and \( \xi_{12}(q) \) is defined by

\[
\xi_{12}(q) = q - \frac{1 - \kappa}{([2\eta] - (1 - \omega)\mathbb{N})L_{q}(2, \varphi)^{2}} \quad (n \geq 2),
\]

\[
\xi_{22}(q) = q + \frac{1 - \kappa}{([2\eta] + (1 - \omega)\mathbb{N})L_{q}(2, \varphi)^{2}} \quad (n \geq 2).
\]

where \( L_{q}(2, \varphi) \) is given by (26). Since

\[
\frac{\xi_{1n}(q)}{q} \prec \frac{\xi_{12}(q)}{q} \quad \text{and} \quad \frac{\xi_{2}(q)}{q} \prec \frac{\xi_{12}(q)}{q},
\]

by Lemma 1, we have

\[
\int_{0}^{2\pi} \left| \frac{\xi_{1n}(q)}{q} \right|^{\eta} d\varphi \leq \int_{0}^{2\pi} \left| \frac{\xi_{12}(q)}{q} \right|^{\eta} d\varphi, \quad (q = re^{i\varphi})
\]

\[
\int_{0}^{2\pi} \left| \frac{\xi_{2n}(q)}{q} \right|^{\eta} d\varphi = \int_{0}^{2\pi} \left| \frac{\xi_{22}(q)}{q} \right|^{\eta} d\varphi \leq \int_{0}^{2\pi} \left| \frac{\xi_{12}(q)}{q} \right|^{\eta} d\varphi, \quad (q = re^{i\varphi}).
\]

Thus, we have the following result:

Lemma 2. Let \( 0 < r < 1 \), \( \eta > 0 \). Then,

\[
\frac{1}{2\pi} \int_{0}^{2\pi} \left| \frac{\xi_{1n}(q)re^{i\varphi}}{q} \right|^{\eta} d\varphi \leq \frac{1}{2\pi} \int_{0}^{2\pi} \left| \frac{\xi_{12}(q)re^{i\varphi}}{q} \right|^{\eta} d\varphi, \quad (n = 1, 2, 3, \cdots)
\]

\[
\frac{1}{2\pi} \int_{0}^{2\pi} \left| \frac{\xi_{2n}(q)re^{i\varphi}}{q} \right|^{\eta} d\varphi = \frac{1}{2\pi} \int_{0}^{2\pi} \left| \frac{\xi_{12}(q)re^{i\varphi}}{q} \right|^{\eta} d\varphi, \quad (n = 2, 3, 4, \cdots).
\]

where \( \xi_{1n}^{*} \) and \( \xi_{2n}^{*} \) are defined by (27) and (28).
By Lemma 13 and Theorem 5, we have the following:

**Theorem 13.** Let $0 < r < 1, \eta > 0$. Then,

$$
\frac{1}{2\pi} \int_0^{2\pi} \left| h(re^{i\phi}) \right|^\eta d\phi \leq \frac{1}{2\pi} \int_0^{2\pi} \left| \xi_{12}^n(re^{i\phi}) \right|^\eta d\phi, \quad (n = 1, 2, 3, \ldots)
$$

$$
\frac{1}{2\pi} \int_0^{2\pi} \left| \nabla_q \left( J_q^\varphi \xi^*_n(re^{i\phi}) \right) \right|^\eta d\phi = \frac{1}{2\pi} \int_0^{2\pi} \left| \nabla_q \left( J_q^\varphi \xi^*_n(re^{i\phi}) \right) \right|^\eta d\phi, \quad (n = 2, 3, 4, \ldots)
$$

where $\xi^*_n$ and $\xi^*_n$ are defined by (27) and (28).

7. Conclusions

For a suitable choice of $\varphi$, when we break through $HS_q^{\varphi}(\varphi, \kappa)$ with $\varphi = 0$ and $q \to 1^-$, the many results which exist in this paper motivate the expansion and simplification of the earlier simpler classes of harmonic functions (see [39–41]) associated with the $q$-Srivastava–Attiya operator. Correspondingly, setting $\varphi = 1$ can provide interesting results for Noshiro-type harmonic functions based on the $q$-Srivastava–Attiya operator. The facts convoluted in the beginnings of such a specialization of meaning (see (14)–(17)), as were obtained in this article, are relatively straightforward and therefore omitted. Utilizing the principles of quantum calculus and connecting Hurwitz–Lech zeta functions of certain meromorphic and harmonic functions, the study undertaken in this article can be extended to investigate necessary and sufficient conditions, problems for partial sums, distortion limits, convexity conditions, and convolution preservation and its implications. Finally, many problem remain open, and it may be interesting to extend the obtained results in this article. Furthermore, it may be interesting to characterize the domain as open and bounded to define a deferential operator applicable to differential equations, or a partial deferential in finite or infinite Banach and Hilbert spaces or in general in Sobolev space, for example. Or the Srivastava–Attiya operator can be extended to the domain of control system analysis where the field of vector space is complex in $C^2(U)$ in some cases for future works, such as in [48]. The operator introduced in this article can also be applied to extend the study on various subclasses of bi-univalent functions, meromorphic functions and symmetric functions [49–52]. By using the Miller–Ross-type Poisson distribution series (see [53] and references cited therein), one can also study certain inclusion results for $HS_q^{\varphi}(\varphi, \kappa)$.


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