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An Invariant of Riemannian Type for Legendrian Warped Product Submanifolds of Sasakian Space Forms

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Abstract: In the present paper, we investigate the geometry and topology of warped product Legendrian submanifolds in Sasakian space forms $\mathbb{D}^{2n+1}(\epsilon)$ and obtain the first Chen inequality that involves extrinsic invariants like the mean curvature and the length of the warping functions. This inequality also involves intrinsic invariants (δ -invariant and sectional curvature). In addition, an integral bound is provided for the Bochner operator formula of compact warped product submanifolds in terms of the gradient Ricci curvature. Some new results on mean curvature vanishing are presented as a partial solution to the well-known problem given by S.S. Chern.

Keywords: warped products; Legendrian; Sasakian space form; Ricci curvature; ordinary differential equations; Riemannian invariants; Bochner operator formula; eigenvalues

MSC: 53C20; 53C21; 53C40; 53C42; 53C80; 53Z05



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1. Introduction and Main Motivations

The geometry of warped product manifolds is rich and varied, and their properties depend crucially on the choice of the warping function. Understanding the behavior of this function is therefore of fundamental importance in the study of these objects. In recent years, there has been a surge of interest in the study of warped product manifolds, driven in part by their wide-ranging applications and their connections to other areas of mathematics. Therefore, the study of warped product manifolds has many important applications in geometry and physics. For example, in general relativity, warped product manifolds are used to model certain types of black hole spacetimes. In algebraic geometry, they arise in studying moduli spaces of vector bundles on algebraic varieties. In topology, they have been used to construct examples of exotic manifolds that do not admit a smooth structure [1–3]. On the other hand, the Chen delta invariant is a numerical invariant in algebraic topology that measures the extent to which a loop in space fails to be a boundary of a surface. More precisely, if a loop is the boundary of a surface, then the Chen delta invariant is zero. Otherwise, it gives a measure of how “far” the loop is from being a boundary. Applications of the delta-invariant can be found in various areas of mathematics, including topology, geometry, and algebraic geometry. For example, it has been used to study the topology of moduli spaces of algebraic curves, the geometry of the Kähler–Einstein metric on a complex manifold, and the topology of configuration spaces of particles in a Euclidean space. It has also found applications in physics, particularly in the study of topological field theories [4–6]. Numerous mathematicians have also investigated product manifolds and related submanifolds. To address the issues, new forms of Riemannian invariants, distinct

from classical invariants, must be introduced. Furthermore, general optimum links between the essential extrinsic invariants and the new intrinsic invariants for submanifolds must be established. This was the reason for Chen [7] to introduce a notion that delta-invariants on Riemannian manifolds and discussed in detail [4,8]. More specifically, they introduced a novel family of curvature functions on submanifolds in the 1990s. A good isometric immersion that creates the least amount of tension from the surrounding space at each point roughly describes the ideal immersion of a Riemannian manifold into a real space form [9]. Chen proposed that the submanifold satisfying the equality condition is known as the ideal submanifold and developed numerous inequalities in terms of invariants. Chen's submanifolds are a substitute for these submanifolds in [4]. Chen has described the ideal submanifolds in real space forms and complex space forms [6,7,9–11]. In addition, Dillen, Petrovic, Verstraelen, Mihai, and Tripathi investigated conformally flat, semisymmetric, and Ricci-semisymmetric submanifolds obeying Chen's inequality in real space forms [12–18] and also (see [10] and references therein) for more information about ideal submanifolds.

It should be noted that there are few studies on the δ -invariant for warped product structures other than the Chen-derived optimal inequality for CR-warped products in complex space form [19]. Recently, Mustafa et al. [20] constructed the first Chen invariant for warped product submanifolds in real space forms and discussed the minimality conditions on submanifolds. From this point of view, by using the Gauss equation instead of the Codazzi equation in the sense of [13], in the first part of this paper, we provide a sharp estimate of the squared norm of the mean curvature in terms of a warping function and the constant holomorphic sectional curvature in the spirit of [21–33], motivated by the historical development on the study of a warping function of a warped product submanifold [34]. As the main objective of our study, we present a novel method for establishing inequalities for δ -invariant curvature inequalities for warped product Legendrian submanifolds isometrically immersed in Sasakian space. This has been discussed in [20,21,35]. As a consequence of the main results discussed in this paper, we generalize a number of inequalities for areas on Euclidean spheres and Euclidean spaces. There is another significant group of Riemannian products in this family.

2. Preliminaries

A $(2m + 1)$ -dimensional manifold $\tilde{D}^{2m+1}$ endowed with an almost-contact structure (φ, ξ, η, g) is called an almost-contact metric manifold when it satisfies the following properties:

$$\begin{aligned}\varphi^2 &= -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0, \\ g(\varphi X_1, \varphi X_2) &= g(X_1, X_2) - \eta(X_1)\eta(X_2), \quad \text{and} \quad \eta(X_1) = g(X_1, \xi),\end{aligned}\quad (1)$$

for any $X_1, X_2 \in \mathfrak{X}(T\tilde{D}^{2m+1})$, where the Lie algebra of vector fields is on a manifold $\tilde{D}^{2m+1}$. In this case, φ , g , ξ , and η are called $(1, 1)$ -tensor fields, a structure vector field, and dual 1-form, respectively. Furthermore, an almost-contact metric manifold is known to be a *Sasakian manifold* (cf. [22,36,37]) if

$$(\tilde{\nabla}_{X_1} \varphi)X_2 = g(X_1, X_2)\xi - \eta(X_2)X_1, \quad \tilde{\nabla}_{X_1} \xi = -\varphi X_1, \quad (3)$$

for any vector fields X_1, X_2 on $\tilde{D}^{2m+1}$, where $\tilde{\nabla}$ denotes the Riemannian connection with respect to g . An n -dimensional Riemannian submanifold $\mathbb{D}^n$ of $\tilde{D}^{2m+1}$ is referred to as totally real if the standard almost-contact structure φ of $\tilde{D}^{2m+1}$ maps any tangent space of $\mathbb{D}^n$ into its corresponding normal space (see [22,35,38,39]). Now, let $\mathbb{D}^n$ be an isometric immersed submanifold of dimension n in $\tilde{D}^{2m+1}$, then $\mathbb{D}^n$ is referred to as a Legendrian submanifold if ξ is a normal vector field on $\mathbb{D}^n$ (i.e., $\mathbb{D}^n$ is a C -totally real submanifold) and $m = n$ [22,35,38]. Legendrian submanifolds play a substantial role in contact geometry. From the Riemannian geometric perspective, studying the Legendrian submanifolds of Sasakian manifolds was initiated in the 1970s.

Let $\mathbb{D}$ be an n -dimensional Riemannian submanifold of an m -dimensional Riemannian $\tilde{\mathcal{D}}^{2m+1}$ with induced metric g and if ∇ and $\nabla^\perp$ are induced connections on the tangent bundle TM and normal bundle $T^\perp\mathbb{D}$ of $\mathbb{D}^n$, respectively. Then, the Gauss and Weingarten formulas are given by

$$(i) \tilde{\nabla}_{\mathcal{X}_1} \mathcal{X}_2 = \nabla_{\mathcal{X}_1} \mathcal{X}_2 + \zeta(\mathcal{X}_1, \mathcal{X}_2), (ii) \tilde{\nabla}_{\mathcal{X}_1} N = -A_N \mathcal{X}_1 + \nabla_{\mathcal{X}_1}^\perp N, \quad (4)$$

for each $\mathcal{X}_1, \mathcal{X}_2 \in \mathfrak{X}(T\mathbb{D})$ and $N \in \mathfrak{X}(T^\perp\mathbb{D})$, where ζ and A_N are the second fundamental form and shape operator (corresponding to the normal vector field N), respectively, for the immersion of $\mathbb{D}^n$ into $\tilde{\mathcal{D}}^{2m+1}$, and they are related as follows:

$$g(\zeta(\mathcal{X}_1, \mathcal{X}_2), N) = g(A_N \mathcal{X}_1, \mathcal{X}_2). \quad (5)$$

Similarly, the equations of Gauss and Codazzi are, respectively, given by

$$(i) R(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4) = \tilde{R}(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4) + g(\zeta(\mathcal{X}_1, \mathcal{X}_4), \zeta(\mathcal{X}_2, \mathcal{X}_3)) - g(\zeta(\mathcal{X}_1, \mathcal{X}_3), \zeta(\mathcal{X}_2, \mathcal{X}_4)). \quad (6)$$

$$(ii) (\tilde{R}(\mathcal{X}_1, \mathcal{X}_2) \mathcal{X}_3)^\perp = (\tilde{\nabla}_{\mathcal{X}_1} \zeta)(\mathcal{X}_2, \mathcal{X}_3) - (\tilde{\nabla}_{\mathcal{X}_2} \zeta)(\mathcal{X}_1, \mathcal{X}_3). \quad (7)$$

For all $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4 \in \mathfrak{X}(T\tilde{M})$, $\mathcal{R}$ and $\tilde{\mathcal{R}}$ are the curvature tensor of $\tilde{\mathcal{D}}^{2n+1}$ and $\mathbb{D}^n$, respectively. The mean curvature $\mathbb{H}$ of Riemannian submanifold $\mathbb{D}^n$ is given by

$$\mathbb{H} = \frac{1}{n} \text{trace}(\zeta). \quad (8)$$

A submanifold $\mathbb{D}^n$ of Riemannian manifold $\tilde{\mathcal{D}}^{2n+1}$ is said to be a *totally umbilical* if

$$\zeta(\mathcal{X}_1, \mathcal{X}_2) = g(\mathcal{X}_1, \mathcal{X}_2) \mathbb{H},$$

and *totally geodesic* if

$$\zeta(\mathcal{X}_1, \mathcal{X}_2) = 0,$$

for any $\mathcal{X}_1, \mathcal{X}_2 \in \mathfrak{X}(TM)$, respectively, where $\mathbb{H}$ is the mean curvature vector of $\mathbb{D}^n$. Furthermore, if $\mathbb{H} = 0$, then $\mathbb{D}^n$ is *minimal* in $\tilde{\mathcal{D}}^{2m+1}$. Moreover, the related null space or kernel of the second fundamental form of $\mathbb{D}^n$ at x is defined by

$$\mathbb{D}_x = \{ \mathcal{X}_1 \in T_x \mathbb{D} : \zeta(\mathcal{X}_1, \mathcal{X}_2) = 0, \text{ for all } \mathcal{X}_2 \in T_x \mathbb{D} \}. \quad (9)$$

In this context, we shall define another important Riemannian intrinsic invariant called the scalar curvature of $\tilde{\mathcal{D}}^{2m+1}$, denoted at $\tilde{\tau}(T_x \tilde{\mathcal{D}}^{2m+1})$, which, at some x in $\tilde{\mathcal{D}}^{2m+1}$, is given as

$$\tilde{\tau}(T_x \tilde{\mathcal{D}}^{2m+1}) = \sum_{1 \leq i < j \leq 2m+1} \tilde{\mathcal{K}}_{ij}, \quad (10)$$

where $\tilde{\mathcal{K}}_{ij} = \tilde{\mathcal{K}}(e_i \wedge e_j)$. It is clear that Equality (10) is congruent to the following equation, which will be frequently used in a subsequent proof:

$$2\tilde{\tau}(T_x \tilde{\mathcal{D}}^{2m+1}) = \sum_{1 \leq i < j \leq 2m+1} \tilde{\mathcal{K}}_{ij}, \quad (11)$$

Similarly, scalar curvature $\tilde{\tau}(L_x)$ of L -plan is given by

$$\tilde{\tau}(L_x) = \sum_{1 \leq i < j \leq m} \tilde{\mathcal{K}}_{ij}, \quad (12)$$

Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of the tangent space $T_x\mathbb{D}$ and $e_r = (e_{n+1}, \dots, e_{2m+1})$ belonging to an orthonormal basis of the normal space $T^\perp\mathbb{D}$, then we have

$$\zeta_{ij}^r = g(\zeta(e_i, e_j), e_r) \text{ and } \|\zeta\|^2 = \sum_{i,j=1}^n g(\zeta(e_i, e_j), \zeta(e_i, e_j)). \quad (13)$$

Let $\mathcal{K}_{ij}$ and $\tilde{\mathcal{K}}_{ij}$ denote the sectional curvature of the plane section spanned and e_i at x in the submanifold $\mathbb{D}^n$ and in the Riemannian space form $\tilde{\mathcal{D}}^{2n+1}(c)$, respectively. Thus, $\mathcal{K}_{ij}$ and $\tilde{\mathcal{K}}_{ij}$ are the intrinsic and extrinsic sectional curvatures of the span $\{e_i, e_j\}$ at x , thus from Gauss Equation (6)(i), we have

$$2\tilde{\tau}(T_x\tilde{\mathbb{D}}^n) = \mathcal{K}_{ij} = 2\tilde{\tau}(T_x\tilde{\mathcal{D}}^{2m+1}) = \tilde{\mathcal{K}}_{ij} + \sum_{r=n+1}^{2m+1} (\zeta_{ii}^r \zeta_{jj}^r - (\zeta_{ij}^r)^2). \quad (14)$$

The second invariant is called the *Chen first invariant*, which is defined as

$$\delta_{\tilde{\mathcal{D}}^{2m+1}}(x) = \tilde{\tau}(T_x\tilde{\mathcal{D}}^{2m+1}) - \inf \left\{ \tilde{\mathcal{K}}(\pi) : \pi \subset T_x\tilde{\mathcal{D}}^{2m+1}, x \in \tilde{\mathcal{D}}^{2m+1}, \dim \zeta = 2 \right\} \quad (15)$$

Assume that $\mathbb{D}_1^{s_1}$ and $\mathbb{D}_2^{s_2}$ are two Riemannian manifolds with their Riemannian metrics g_1 and g_2 , respectively. Let $f > 0$ be a smooth function defined on $\mathbb{D}_1^{s_1}$. Then, warped product manifold $\mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ is the manifold $\mathbb{D}_1^{s_1} \times \mathbb{D}_2^{s_2}$ furnished by the Riemannian metric $g = g_1 + f^2 g_2$ [1]. Assume that $\mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ is a warped product manifold, then for any $\mathcal{X}_1 \in \Gamma(T\mathbb{D}_1^{s_1})$ and $\mathcal{X}_3 \in \Gamma(T\mathbb{D}_2^{s_2})$, we find that

$$\nabla_{\mathcal{X}_3} \mathcal{X}_1 = \nabla_{\mathcal{X}_1} \mathcal{X}_3 = (\mathcal{X}_1 \ln f) \mathcal{X}_3. \quad (16)$$

Similarly, from unit vector fields, $\mathcal{X}_1$ and $\mathcal{X}_3$ are tangent to $\mathbb{D}_1^{s_1}$ and $\mathbb{D}_2^{s_2}$, respectively, thus deriving

$$\begin{aligned} \mathcal{K}(\mathcal{X}_1 \wedge \mathcal{X}_3) &= g(R(\mathcal{X}_1, \mathcal{X}_3)\mathcal{X}_1, \mathcal{X}_3) \\ &= (\nabla_{\mathcal{X}_1} \mathcal{X}_1) \ln f g(\mathcal{X}_3, \mathcal{X}_3) - g(\nabla_{\mathcal{X}_1} ((\mathcal{X}_1 \ln f) \mathcal{X}_3), \mathcal{X}_3) \\ &= (\nabla_{\mathcal{X}_1} \mathcal{X}_1) \ln f g(\mathcal{X}_3, \mathcal{X}_3) - g(\nabla_{\mathcal{X}_1} (\mathcal{X}_1 \ln f) \mathcal{X}_3 + (\mathcal{X}_1 \ln f) g(\nabla_{\mathcal{X}_1} \mathcal{X}_3, \mathcal{X}_3)) \\ &= (\nabla_{\mathcal{X}_1} \mathcal{X}_1) \ln f g(\mathcal{X}_3, \mathcal{X}_3) - (\mathcal{X}_1 \ln f)^2 - \mathcal{X}_1 (\mathcal{X}_1 \ln f). \end{aligned} \quad (17)$$

Suppose that $\{e_1, \dots, e_n\}$ is an orthonormal frame for $\mathbb{D}^n$, then sum up over the vector fields such that

$$\sum_{i=1}^{s_1} \sum_{j=1}^{s_2} \mathcal{K}(e_i \wedge e_j) = \sum_{i=1}^{s_1} \sum_{j=1}^{s_2} \left((\nabla_{e_i} e_i) \ln f - e_i (e_i \ln f) - (e_i \ln f)^2 \right),$$

which implies that

$$\sum_{i=1}^{s_1} \sum_{j=1}^{s_2} \mathcal{K}(e_i \wedge e_j) = s_2 \left(\Delta(\ln f) - \|\nabla(\ln f)\|^2 \right). \quad (18)$$

But, it was proved [9] that for arbitrary warped product submanifolds,

$$\sum_{i=1}^{s_1} \sum_{j=1}^{s_2} \mathcal{K}(e_i \wedge e_j) = \frac{s_2 \Delta f}{f}. \quad (19)$$

Thus, from (18) and (19), we obtain

$$\frac{\Delta f}{f} = \Delta(\ln f) - \|\nabla(\ln f)\|^2. \quad (20)$$

The following remarks are consequences of warped product submanifolds:

Remark 1. A warped product manifold $\mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ is said to be trivial if the warping function f is constant or simply a Riemannian product manifold.

Remark 2. If $\mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ is a warped product manifold, then $\mathbb{D}_1$ is a totally geodesic and $\mathbb{D}_2$ is a totally umbilical submanifold of $\mathbb{D}^n$, respectively.

A Sasakian manifold is said to be Sasakian space form with a constant φ -sectional curvature ϵ if and only if the Riemannian curvature tensor $\tilde{R}$ can be written as (see [22,38]):

$$\begin{aligned} \tilde{R}(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4) = & \left(\frac{\epsilon + 3}{4} \right) \left\{ g(\mathcal{X}_2, \mathcal{X}_3)g(\mathcal{X}_1, \mathcal{X}_4) - g(\mathcal{X}_1, \mathcal{X}_3)g(\mathcal{X}_2, \mathcal{X}_4) \right\} \\ & + \left(\frac{\epsilon - 1}{4} \right) \left\{ \eta(\mathcal{X}_1)\eta(\mathcal{X}_3)g(\mathcal{X}_2, \mathcal{X}_4) + \eta(\mathcal{X}_4)\eta(\mathcal{X}_2)g(\mathcal{X}_1, \mathcal{X}_3) \right. \\ & \quad - \eta(\mathcal{X}_2)\eta(\mathcal{X}_3)g(\mathcal{X}_1, \mathcal{X}_4) - \eta(\mathcal{X}_1)g(\mathcal{X}_2, \mathcal{X}_3)\eta(\mathcal{X}_4) \\ & \quad + g(\varphi\mathcal{X}_2, \mathcal{X}_3)g(\varphi\mathcal{X}_1, \mathcal{X}_4) - g(\varphi\mathcal{X}_1, \mathcal{X}_3)g(\varphi\mathcal{X}_2, \mathcal{X}_4) \\ & \quad \left. + 2g(\mathcal{X}_1, \varphi\mathcal{X}_2)g(\varphi\mathcal{X}_3, \mathcal{X}_4) \right\}, \end{aligned} \quad (21)$$

where $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4 \in \mathfrak{X}(T\tilde{\mathcal{D}}^{2m+1})$. Moreover, $\mathbb{R}^{2m+1}$ and $\mathbb{S}^{2m+1}$ with standard Sasakian structures can be given as typical examples of Sasakian space forms. Many geometers have drawn significant attention to minimal Legendrian submanifolds in particular. We recall the following important algebraic lemma.

Lemma 1. Let $t_1, t_2, \dots, t_n, s$ ($n + 1$) ($n \geq 2$) be a real number such that

$$\sum_{i=1}^n (t_i)^2 = (n - 1) \left(\sum_{i=1}^n t_1^2 + s \right). \quad (22)$$

Then, $2t_1t_2 \geq s$ with an equality holds if and only if $t_1 + t_2 = t_3 = \dots = t_n$.

Theorem 1. Let $\phi : \mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ be an isometric immersion of a warped product Legendrian submanifold $\mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ into a Sasakian space form $\tilde{\mathcal{D}}^{2n+1}(\epsilon)$. Then, for each point $x \in \mathbb{D}^n$ and each plane section $\pi_i \subset T_x\mathbb{D}_i^{n_i}$, for $i = 1, 2$, we obtain

(1) Let $\pi_1 \subset T_x\mathbb{D}_1^{s_1}$, then

$$\begin{aligned} \delta_{\mathbb{D}^{s_1}}(x) \leq & \frac{n^2}{2} \|\mathbb{H}\|^2 + s_2 \|\nabla(\ln f)\|^2 - s_2 \Delta(\ln f) \\ & + \left\{ \frac{s_1}{2} (s_1 + 2s_2 - 1) - 1 \right\} \left(\frac{\epsilon + 3}{4} \right). \end{aligned} \quad (23)$$

The equality of the above inequality holds at $x \in \mathbb{D}^n$ if and only if there exists an orthonormal basis $\{e_1 \cdots e_n\}$ of $T_x \mathbb{D}^n$ and orthonormal basis $\{e_n + 1 \cdots e_{2n+1}\}$ of $T_x^\perp$ such that (a) $\pi = \text{Span}\{e_1, e_2\}$ and (b) shape operators take the following form

$$(i) A_{e_{n+1}} = \left(\begin{array}{ccccc|ccc} \mu_1 & \zeta_{12}^{n+1} & 0 & \cdots & 0_{1s_1} & 0_{1s_1+1} & \cdots & 0_{1n} \\ \zeta_{12}^{n+1} & \mu_2 & 0 & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \mu & \cdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0_{s_1 1} & 0 & 0 & \cdots & \mu & 0_{s_1 s_1+1} & \cdots & 0_{s_1 n} \\ \hline 0_{s_1+1 1} & \cdots & \cdots & \cdots & 0_{s_1+1 s_1} & \zeta_{s_1+1 s_1+1}^{n+1} & \cdots & \zeta_{s_1+1 n}^{n+1} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0_{n1} & \cdots & \cdots & \cdots & 0_{ns_1} & \zeta_{ns_1+1}^{n+1} & \cdots & \zeta_{nn}^{n+1} \end{array} \right),$$

where $\mu = \mu_1 + \mu_2$. If $r \in \{n+2, \cdots m\}$, then we have the matrix

$$(ii) A_{e_r} = \left(\begin{array}{ccccc|ccc} \zeta_{11}^r & \zeta_{12}^r & 0 & \cdots & 0_{1s_1} & 0_{1s_1+1} & \cdots & 0_{1n} \\ \zeta_{21}^r & -\zeta_{11}^r & 0 & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0_{33} & \cdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0_{s_1 1} & 0 & 0 & \cdots & 0_{s_1 s_1} & 0_{s_1 s_1+1} & \cdots & 0_{s_1 n} \\ \hline 0_{s_1+1 1} & \cdots & \cdots & \cdots & 0_{s_1+1 s_1} & \zeta_{s_1+1 s_1+1}^{n+1} & \cdots & \zeta_{s_1+1 n}^{n+1} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0_{n1} & \cdots & \cdots & \cdots & 0_{ns_1} & \zeta_{ns_1+1}^{n+1} & \cdots & \zeta_{nn}^{n+1} \end{array} \right),$$

(2) If $\pi_2 \subset T_x \mathbb{D}_2^{s_2}$, then

$$\begin{aligned} \delta_{\mathbb{D}^{s_2}}(x) &\leq \frac{n^2}{2} \|\mathbb{H}\|^2 + s_2 \|\nabla(\ln f)\|^2 - s_2 \Delta(\ln f) \\ &\quad + \left\{ \frac{s_2}{2} (s_2 + 2s_1 - 1) - 1 \right\} \left(\frac{\epsilon + 3}{4} \right). \end{aligned} \quad (24)$$

Equalities of the above equation hold if and only if

$$(iii) A_{e_{n+1}} = \left(\begin{array}{cccc|cccc} \zeta_{11}^{n+1} & \cdots & \cdots & \zeta_{1s_1}^{n+1} & 0_{1s_1+1} & \cdots & \cdots & \cdots & 0_{1n} \\ \vdots & \ddots & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \zeta_{s_1 11}^{n+1} & \cdots & \cdots & \zeta_{s_1 s_1}^{n+1} & 0_{s_1 s_1+1} & \cdots & \cdots & \cdots & 0_{s_1 n} \\ \hline 0_{s_1+1 1} & \cdots & \cdots & 0_{s_1+1 s_1} & \mu_1 & \zeta_{s_1+1 s_1+2}^{n+1} & 0 & \cdots & 0_{s_1+1 n} \\ \vdots & \ddots & \ddots & \vdots & \zeta_{s_1+2 s_1+1}^{n+1} & \mu_2 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & 0 & 0 & \mu & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ 0_{n1} & \cdots & \cdots & 0_{ns_1} & 0_{ns_1+1} & 0 & \cdots & 0 & \mu \end{array} \right),$$

where $\mu = \mu_1 + \mu_2$. If $r \in \{n+2, \dots, m\}$, thus we have

$$(iv) A_{e_r} = \left(\begin{array}{cccc|cccccc} \zeta_{11}^r & \cdots & \cdots & \zeta_{1s_1}^r & 0_{1s_1+1} & \cdots & \cdots & \cdots & 0_{1n} \\ \vdots & \ddots & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \zeta_{s_1 11}^r & \cdots & \cdots & \zeta_{s_1 s_1}^r & 0_{s_1 s_1+1} & \cdots & \cdots & \cdots & 0_{s_1 n} \\ \hline 0_{s_1+11} & \cdots & \cdots & 0_{s_1+1s_1} & \zeta_{s_1+1s_1+1}^r & \zeta_{s_1+1s_1+2}^{n+1} & 0 & \cdots & 0_{s_1+1n} \\ \vdots & \ddots & \ddots & \vdots & \zeta_{s_1+2s_1+1}^{n+1} & -\zeta_{s_1+1s_1+1}^r & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ 0_{n1} & \cdots & \cdots & 0_{ns_1} & 0_{ns_1+1} & 0 & \cdots & 0 & 0 \end{array} \right),$$

(v) If the equality holds in (1) or (2), then $\mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_1}$ is mixed totally geodesic in $\mathbb{D}_\epsilon^{2n+1}$. Moreover, $\mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_1}$ is both $\mathbb{D}_1^{s_1}$ -minimal and $\mathbb{D}_2^{s_2}$ -minimal. Thus, $\mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_1}$ is a minimal warped product submanifold in $\mathbb{D}_\epsilon^{2n+1}$.

Proof. Let $\pi_1 \subset T_x \mathbb{D}_1$ be 2-plane for $x \in \mathbb{D}^n$, then we consider the orthonormal basis $\{e_1 \cdots e_{s_1}, e_{s_1+1}, \dots, e_n\}$ of $T_x \mathbb{D}^n$ such that $\{e_1, \dots, e_{s_1}\}$ is an orthonormal basis for $T_x \mathbb{D}_1$ and $\{e_{s_1+1}, \dots, e_n\}$ is for $T_x \mathbb{D}_2$. Similarly, $\{e_{n+1}, \dots, e_{2n+1}\}$ is an orthonormal basis for $T_x^\perp \mathbb{D}^n$. Assume that $\pi = \text{Span}\{e_1, e_2\}$ such that the normal vector e_{n+1} is in the direction of mean curvature vector $\mathbb{H}$, thus from (21) and Gauss Equation (6), we obtain

$$n^2 \|\mathbb{H}\|^2 = 2\tau(T_x \mathbb{D}^n) + \|\zeta\|^2 - n(n-1) \left(\frac{\epsilon+3}{4} \right). \quad (25)$$

which implies that

$$\begin{aligned} \left(\sum_{i=1}^{s_1} \zeta_{ii}^{n+1} \right)^2 &= 2\tau(T_x \mathbb{D}^n) + \|\zeta\|^2 - n(n-1) \left(\frac{\epsilon+3}{4} \right) \\ &\quad - \left(\sum_{j=s_1+1}^n \zeta_{jj}^{n+1} \right)^2 - 2 \sum_{A=s_1+1}^{s_1} \sum_{B=s_1+1}^n \zeta_{AA}^{n+1} \zeta_{BB}^{n+1}. \end{aligned} \quad (26)$$

Let us consider the following:

$$\begin{aligned} \Omega &= 2\tau(T_x \mathbb{D}^n) - n(n-1) \left(\frac{\epsilon+3}{4} \right) \\ &\quad - \frac{(s_1-2)}{(s_1-1)} \left(\sum_{i=1}^{s_1} \zeta_{ii}^{n+1} \right)^2 - \left(\sum_{j=s_1+1}^n \zeta_{jj}^{n+1} \right)^2 \\ &\quad - 2 \sum_{A=s_1+1}^{s_1} \sum_{B=s_1+1}^n \zeta_{AA}^{n+1} \zeta_{BB}^{n+1}. \end{aligned} \quad (27)$$

It follows from (26) and (27), and we find that

$$\left(\sum_{i=1}^{s_1} \zeta_{ii}^{n+1} \right)^2 = (s_1-1) \left(\Omega + \|\zeta\|^2 \right). \quad (28)$$

The above equation can be expressed as

$$\left(\sum_{i=1}^{s_1} \zeta_{ii}^{n+1}\right)^2 = (s_1 - 1) \left\{ \Omega + \sum_{i=1}^{s_1} (\zeta_{ii}^{n+1})^2 + \sum_{j=s_1+1}^n (\zeta_{jj}^{n+1})^2 + \sum_{i \neq j=1}^n (\zeta_{ij}^{n+1})^2 + \sum_{r=n+2}^{2n+1} \sum_{i,j=1}^n (\zeta_{ij}^{n+1})^2 \right\} \quad (29)$$

Therefore, we shall apply Lemma 1 on the above equation, i.e.,

$$t_\alpha = \zeta_{\alpha\alpha}^{n+1}, \forall t_\alpha \in \{1 \cdots, s_1\}$$

and

$$s = \Omega + \sum_{j=s_1+1}^n (\zeta_{jj}^{n+1})^2 + \sum_{i \neq j=1}^n (\zeta_{ij}^{n+1})^2 + \sum_{r=n+2}^{2n+1} \sum_{i,j=1}^n (\zeta_{ij}^{n+1})^2$$

Thus, we obtain that

$$\zeta_{11}^{n+1} \zeta_{22}^{n+1} \geq \frac{1}{2} \left\{ \Omega + \sum_{j=s_1+1}^n (\zeta_{jj}^{n+1})^2 + \sum_{i \neq j=1}^n (\zeta_{ij}^{n+1})^2 + \sum_{r=n+2}^{2n+1} \sum_{i,j=1}^n (\zeta_{ij}^{n+1})^2 \right\} \quad (30)$$

Then, from (21) and (14), we derive

$$K(\pi_1) = \left(\frac{\epsilon + 3}{4}\right) + \sum_{r=n+1}^{2n+1} (\zeta_{11}^r \zeta_{22}^r - (\zeta_{12}^r)^2). \quad (31)$$

If we combine Equations (30) and (31), we obtain

$$K(\pi_1) \geq \left(\frac{\epsilon + 3}{4}\right) + \frac{1}{2} \Omega + \frac{1}{2} \sum_{j=s_1+1}^n (\zeta_{jj}^{n+1})^2 + \sum_{r=n+1}^{2n+1} (\zeta_{11}^r \zeta_{22}^r - (\zeta_{12}^r)^2) + \frac{1}{2} \sum_{i \neq j=1}^n (\zeta_{ij}^{n+1})^2 + \frac{1}{2} \sum_{r=n+2}^{2n+1} \sum_{i,j=1}^n (\zeta_{ij}^{n+1})^2. \quad (32)$$

We choose the last two terms of the above equation, and we derive

$$\begin{aligned} \sum_{\substack{i,j=1 \\ i \neq j}}^n (\zeta_{ij}^{n+1})^2 &= \sum_{\substack{i,j=3 \\ i \neq j}}^n (\zeta_{ij}^{n+1})^2 + 2 \sum_{j=3}^n (\zeta_{1j}^{n+1})^2 \\ &\quad + 2(\zeta_{12}^{n+1})^2 + 2 \sum_{j=3}^n (\zeta_{2j}^{n+1})^2. \end{aligned} \quad (33)$$

Moreover, for the last term, we obtain

$$\begin{aligned} \sum_{r=n+2}^{2n+1} \sum_{i,j=1}^n (\zeta_{ij}^{n+1})^2 &= \sum_{r=n+2}^{2n+1} \sum_{i,j=3}^n (\zeta_{ij}^{n+1})^2 + 2 \sum_{r=n+2}^{2n+1} \sum_{j=3}^n (\zeta_{1j}^{n+1})^2 \\ &\quad + 2 \sum_{r=n+2}^{2n+1} \sum_{j=3}^n (\zeta_{2j}^{n+1})^2 + 2(\zeta_{12}^{n+1})^2 \\ &\quad + \sum_{r=n+2}^{2n+1} \left((\zeta_{11})^2 + (\zeta_{22})^2 \right). \end{aligned}$$

Furthermore, we have

$$\sum_{r=n+2}^{2n+1} \zeta_{11}^r \zeta_{22}^2 + \frac{1}{2} \sum_{r=n+2}^{2n+1} \left((\zeta_{11}^r)^2 + (\zeta_{22}^r)^2 \right) = \frac{1}{2} \sum_{r=n+2}^{2n+1} \left(\zeta_{11}^r + \zeta_{22}^r \right)^2, \quad (34)$$

$$\begin{aligned} \sum_{j=3}^n \left((\zeta_{1j}^{n+1})^2 + (\zeta_{2j}^{n+1})^2 \right) + \sum_{r=n+2}^{2n+1} \sum_{j=3}^n (\zeta_{1j}^{n+1})^2 + \sum_{r=n+2}^{2n+1} \sum_{j=3}^n (\zeta_{2j}^{n+1})^2 \\ = \sum_{r=n+2}^{2n+1} \sum_{j=3}^n \left\{ (\zeta_{2j}^{n+1})^2 + (\zeta_{2j}^{n+1})^2 \right\} \end{aligned} \quad (35)$$

After adding (33) and (59), then using (34) and (35), and taking into account that

$$(\zeta_{12}^{n+1})^2 + \sum_{r=n+2}^{2n+1} (\zeta_{12}^{n+1})^2 = \sum_{r=n+1}^{2n+1} (\zeta_{12}^{n+1})^2.$$

We obtain

$$\begin{aligned} \sum_{\substack{i,j=1 \\ i \neq j}}^n (\zeta_{ij}^{n+1})^2 + \sum_{r=n+2}^{2n+1} \sum_{i,j=1}^n (\zeta_{ij}^{n+1})^2 = 2 \sum_{r=n+2}^{2n+1} \sum_{j=3}^n \left\{ (\zeta_{2j}^{n+1})^2 + (\zeta_{2j}^{n+1})^2 \right\} \\ + \sum_{\substack{i,j=3 \\ i \neq j}}^n (\zeta_{ij}^{n+1})^2 + \sum_{r=n+2}^{2n+1} \sum_{i,j=3}^n (\zeta_{ij}^{n+1})^2 \\ - 2 \sum_{r=n+2}^{2n+1} \left\{ \zeta_{11}^r \zeta_{22}^r - (\zeta_{12}^r)^2 \right\} \\ + \sum_{r=n+2}^{2n+1} \left(\zeta_{11}^r + \zeta_{22}^r \right)^2. \end{aligned} \quad (36)$$

It follows from (32) and (36) that one derives

$$\begin{aligned} K(\pi_1) \geq \left(\frac{\epsilon + 3}{4} \right) + \frac{1}{2} \Omega + \frac{1}{2} \sum_{\beta=s_1+1}^n (\zeta_{\beta\beta}^{n+1})^2 \\ + \sum_{r=n+2}^{2n+1} \sum_{j=3}^n \left\{ (\zeta_{2j}^{n+1})^2 + (\zeta_{2j}^{n+1})^2 \right\} \\ + \frac{1}{2} \left\{ \sum_{\substack{i,j=3 \\ i \neq j}}^n (\zeta_{ij}^{n+1})^2 + \sum_{r=n+2}^{2n+1} \sum_{i,j=3}^n (\zeta_{ij}^{n+1})^2 \right\} + \frac{1}{2} \sum_{r=n+2}^{2n+1} \left(\zeta_{11}^r + \zeta_{22}^r \right)^2, \end{aligned}$$

which implies that

$$\begin{aligned} K(\pi_1) \geq \left(\frac{\epsilon + 3}{4} \right) \\ + \frac{1}{2} \left\{ \Omega + \sum_{\substack{i,j=3 \\ i \neq j}}^n (\zeta_{ij}^{n+1})^2 + \sum_{r=n+2}^{2n+1} \sum_{i,j=3}^n (\zeta_{ij}^{n+1})^2 + \sum_{\beta=s_1+1}^n (\zeta_{\beta\beta}^{n+1})^2 \right\}. \end{aligned}$$

From (27), we arrive at

$$K(\pi_1) \geq \left(\frac{\epsilon + 3}{4} \right) + \tau(T_x \mathbb{D}^n) + \frac{1}{2(s_1 - 1)} \left(\sum_{\alpha=s_1+1}^n \zeta_{\alpha\alpha}^{n+1} \right)^2$$

$$\begin{aligned}
& -\frac{n^2}{2} \|\mathbb{H}\|^2 - \frac{n(n-1)}{2} \left(\frac{\epsilon+3}{4} \right) \\
& + \frac{1}{2} \left\{ \sum_{\substack{i,j=3 \\ i \neq j}}^n (\zeta_{ij}^{n+1})^2 + \sum_{r=n+2}^{2n+1} \sum_{i,j=3}^n (\zeta_{ij}^{n+1})^2 + \sum_{\beta=s_1+1}^n (\zeta_{\beta\beta}^{n+1})^2 \right\}. \quad (37)
\end{aligned}$$

Using (11) and (19) together in (37), we obtain

$$\begin{aligned}
K(\pi_1) & \geq \tau(T_x \mathbb{D}_1^{s_1}) + \tau(T_x \mathbb{D}_2^{s_2}) + \frac{s_2 \nabla f}{f} - \frac{n^2}{2} \|\mathbb{H}\|^2 \\
& + \left(1 - \frac{n(n-1)}{2} \right) \left(\frac{\epsilon+3}{4} \right) \\
& + \frac{1}{2} \left\{ \sum_{\substack{i,j=3 \\ i \neq j}}^n (\zeta_{ij}^{n+1})^2 + \sum_{r=n+2}^{2n+1} \sum_{i,j=3}^n (\zeta_{ij}^{n+1})^2 + \sum_{\beta=s_1+1}^n (\zeta_{\beta\beta}^{n+1})^2 \right\}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\tau(T_x \mathbb{D}_1^{s_1}) - K(\pi_1) & \leq \frac{n^2}{2} \|\mathbb{H}\|^2 - \tau(T_x \mathbb{D}_2^{s_2}) - \frac{s_2 \nabla f}{f} \\
& + \left(\frac{n(n-1)}{2} - 1 \right) \left(\frac{\epsilon+3}{4} \right) \\
& - \frac{1}{2} \left\{ \sum_{\substack{i,j=3 \\ i \neq j}}^n (\zeta_{ij}^{n+1})^2 + \sum_{r=n+2}^{2n+1} \sum_{i,j=3}^n (\zeta_{ij}^{n+1})^2 + \sum_{\beta=s_1+1}^n (\zeta_{\beta\beta}^{n+1})^2 \right\}. \quad (38)
\end{aligned}$$

The Gauss Equation (6)(i) for $\tau(T_x \mathbb{D}_2^{s_2})$ gives us

$$\begin{aligned}
\tau(T_x \mathbb{D}_2^{s_2}) & = \frac{s_2(s_2-1)}{2} \left(\frac{\epsilon+3}{4} \right) \\
& - \frac{1}{2} \sum_{r=n+1}^{2n+1} \sum_{A,B=s_1+1}^n (\zeta_{AB}^{n+1})^2 \\
& - \frac{1}{2} \sum_{r=n+1}^{2n+1} (\zeta_{s_1+1s_1+1} + \cdots + \zeta_{nn}^r). \quad (39)
\end{aligned}$$

In view of Equations (38) and (39), we find that

$$\begin{aligned}
\tau(T_x \mathbb{D}_1^{s_1}) - K(\pi_1) & \leq \frac{n^2}{2} \|\mathbb{H}\|^2 - \frac{s_2(s_2-1)}{2} \left(\frac{\epsilon+3}{4} \right) \\
& - \frac{1}{2} \left\{ \sum_{\substack{i,j=3 \\ i \neq j}}^n (\zeta_{ij}^{n+1})^2 + \sum_{r=n+2}^{2n+1} \sum_{i,j=3}^n (\zeta_{ij}^r)^2 \right. \\
& \quad \left. + \sum_{\beta=s_1+1}^n (\zeta_{\beta\beta}^{n+1})^2 - \sum_{r=n+1}^{2n+1} \sum_{A,B=s_1+1}^n (\zeta_{AB}^{n+1})^2 \right\} \\
& + \left(\frac{n(n-1)}{2} - 1 \right) \left(\frac{\epsilon+3}{4} \right) - \frac{s_2 \nabla f}{f}. \quad (40)
\end{aligned}$$

Then, the last relation turns into

$$\begin{aligned}
\tau(T_x \mathbb{D}_1^{s_1}) - K(\pi_1) &\leq \frac{n^2}{2} \|\mathbb{H}\|^2 - \frac{s_2(s_2-1)}{2} \left(\frac{\epsilon+3}{4} \right) + \left(\frac{n(n-1)}{2} - 1 \right) \left(\frac{\epsilon+3}{4} \right) \\
&\quad - \frac{1}{2} \left\{ \sum_{\substack{k,l=3 \\ k \neq l}}^{s_1} (\zeta_{kl}^{n+1})^2 + 2 \sum_{k=3}^{2n+1} \sum_{l=s_1+1}^n (\zeta_{kl}^{n+1})^2 + \sum_{\substack{A,B=s_1+1 \\ A \neq B}}^{s_1} (\zeta_{kl}^{n+1})^2 \right. \\
&\quad + \sum_{r=n+2}^{2n+1} \sum_{k,l=3}^{s_1} (\zeta_{kl}^{n+1})^2 + 2 \sum_{r=n+2}^{2n+1} \sum_{k=3}^{s_1} \sum_{A=s_1+1}^n (\zeta_{kl}^r)^2 \\
&\quad + \sum_{r=n+2}^{2n+1} \sum_{A,B=s_1+1}^n (\zeta_{AB}^r)^2 + \sum_{\beta=s_1+1}^n (\zeta_{\beta\beta}^{n+1})^2 \\
&\quad \left. - \sum_{r=n+1}^{2n+1} \sum_{A,B=s_1+1}^n (\zeta_{AB}^r)^2 \right\} - \frac{s_2 \nabla f}{f}.
\end{aligned} \tag{41}$$

With the preceding above equation and the help of the following two relations:

$$\sum_{A=s_1+1}^n (\zeta_{AA}^2)^2 + \sum_{\substack{A,B=3 \\ A \neq B}}^n (\zeta_{AB}^{n+1})^2 = \sum_{A,B=s_1+1}^n (\zeta_{AB}^{n+1})^2.$$

and

$$\sum_{A,B=s_1+1}^n (\zeta_{AB}^{n+1})^2 + \sum_{r=n+2}^{2n+1} \sum_{A,B=s_1+1}^n (\zeta_{AB}^r)^2 = \sum_{r=n+1}^{2n+1} \sum_{A,B=s_1+1}^n (\zeta_{AB}^r)^2.$$

Assertion (41) is as follows:

$$\begin{aligned}
\tau(T_x \mathbb{D}_1^{s_1}) - K(\pi_1) &\leq \left\{ \frac{s_1}{2} (s_1 + 2s_2 - 1) - 1 \right\} \left(\frac{\epsilon+3}{4} \right) \\
&\quad - \frac{1}{2} \left\{ \sum_{\substack{k,l=3 \\ k \neq l}}^{s_1} (\zeta_{kl}^{n+1})^2 + \sum_{r=n+2}^{2n+1} \sum_{k,l=3}^{s_1} (\zeta_{kl}^{n+1})^2 \right. \\
&\quad + 2 \sum_{\alpha=3}^{s_1} \sum_{\beta=s_1+1}^n (\zeta_{\alpha\beta}^{n+1})^2 + 2 \sum_{r=n+2}^{2n+1} \sum_{A=3}^{s_1} \sum_{B=s_1+1}^n (\zeta_{AB}^r)^2 \left. \right\} \\
&\quad + \frac{n^2}{2} \|\mathbb{H}\|^2 - \frac{s_2 \nabla f}{f}.
\end{aligned} \tag{42}$$

The first inequality of Theorem 1 holds from the above equation and (15). For the second case, if $\pi \subset T_x \mathbb{D}_2^{s_2}$, we consider $\pi_2 = \text{Span}\{e_{s_1+1}, e_{s_1+1}\}$, following same methodology as first case as:

$$\begin{aligned}
\left(\sum_{\alpha=s_1+1}^n \zeta_{\alpha\alpha}^{n+1} \right)^2 &= 2\tau(T_x \mathbb{D}^n) + \|\zeta\|^2 - n(n-1) \left(\frac{\epsilon+3}{4} \right) - \left(\sum_{\beta=s_1+1}^n \zeta_{\beta\beta}^{s_1} \right)^2 \\
&\quad - 2 \sum_{\alpha=1}^{s_1} \sum_{\beta=s_1+1}^n \zeta_{\alpha\alpha}^{n+1} \zeta_{\beta\beta}^{n+1}.
\end{aligned}$$

Considering the following:

$$\Psi = 2\tau(T_x \mathbb{D}^n) - n(n-1) \left(\frac{\epsilon+3}{4} \right)$$

$$\begin{aligned}
& - \frac{(s_1 - 2)}{(s_1 - 1)} \left(\sum_{\alpha=s_1+1}^{s_1} \zeta_{\alpha\alpha}^{n+1} \right)^2 - \left(\sum_{\beta=s_1+1}^n \zeta_{\beta\beta}^{s_1} \right)^2 \\
& - 2 \sum_{\alpha=1}^{s_1} \sum_{\beta=s_1+1}^n \zeta_{\alpha\alpha}^{n+1} \zeta_{\beta\beta}^{n+1}.
\end{aligned}$$

The last two equation implies that

$$\left(\sum_{\alpha=s_1+1}^n \zeta_{\alpha\alpha}^{n+1} \right)^2 = (s_2 - 1) \left(\Psi + \|\zeta\|^2 \right),$$

which implies that

$$\begin{aligned}
\left(\sum_{\alpha=s_1+1}^n \zeta_{\alpha\alpha}^{n+1} \right)^2 &= (s_2 - 1) \left\{ \Psi + \left(\sum_{\alpha=1}^{s_1} \zeta_{\alpha\alpha}^{n+1} \right)^2 + \left(\sum_{\beta=s_1+1}^n \zeta_{\beta\beta}^{n+1} \right)^2 \right. \\
&\quad \left. + \sum_{\substack{\alpha, \beta=1 \\ \alpha \neq \beta}}^n (\zeta_{\alpha\beta}^{n+1})^2 + \sum_{r=n+2}^{2n+1} \sum_{\alpha=\beta=1}^n (\zeta_{\alpha\beta}^r)^2 \right\}. \quad (43)
\end{aligned}$$

Similarly, applying Lemma 1 in the above equation, we obtain

$$\begin{aligned}
\zeta_{s_1+1s_1+1}^{n+1} \zeta_{s_1+2s_1+2}^{n+1} &\geq \frac{1}{2} \left\{ \Psi + \left(\sum_{\alpha=1}^{s_1} \zeta_{\alpha\alpha}^{n+1} \right)^2 + \sum_{\substack{\alpha, \beta=1 \\ \alpha \neq \beta}}^n (\zeta_{\alpha\beta}^{n+1})^2 \right. \\
&\quad \left. + \sum_{r=n+2}^{2n+1} \sum_{\alpha=\beta=1}^n (\zeta_{\alpha\beta}^r)^2 \right\} \quad (44)
\end{aligned}$$

From (21) and (14), we find that

$$K(\pi_2) = \left(\frac{\epsilon + 3}{4} \right) + \sum_{r=n+1}^{2n+1} \left(\zeta_{s_1+1s_1+1}^r \zeta_{s_1+2s_1+2}^r - (\zeta_{s_1+1s_1+2}^r)^2 \right) \quad (45)$$

Equations (44) and (45) are implied such that

$$\begin{aligned}
K(\pi_2) &\geq \left(\frac{\epsilon + 3}{4} \right) + \sum_{r=n+1}^{2n+1} \left(\zeta_{s_1+1s_1+1}^r \zeta_{s_1+2s_1+2}^r - (\zeta_{s_1+1s_1+2}^r)^2 \right) \\
&\quad \frac{1}{2} \left\{ \Psi + \left(\sum_{\alpha=1}^{s_1} \zeta_{\alpha\alpha}^{n+1} \right)^2 + \sum_{\substack{\alpha, \beta=1 \\ \alpha \neq \beta}}^n (\zeta_{\alpha\beta}^{n+1})^2 + \sum_{r=n+2}^{2n+1} \sum_{\alpha=\beta=1}^n (\zeta_{\alpha\beta}^r)^2 \right\}. \quad (46)
\end{aligned}$$

Following the method from (27) and (42), we obtain the second inequality of Theorem 1. On the other hand, for the equality condition, we define two different cases whether the 2-plane π_i is tangent to the first factor or to the second factor. In the first case, we consider $\pi_1 \subset T_x \mathbb{D}_1^{s_1}$, then the equality holds if and only if equalities hold in (30), (32), (38), (39) and (42), and we obtain the following condition:

$$\zeta_{11}^{n+1} + \zeta_{22}^{n+1} = \zeta_{33}^{n+1} = \dots = \zeta_{s_1 s_1}^{n+1} \quad (47)$$

$$\sum_{r=n+2}^{2n+1} \sum_{j=3}^n \left((\zeta_{2j}^{n+1})^2 + (\zeta_{2j}^{n+1})^2 \right) + \sum_{r=n+2}^{2n+1} \left(\zeta_{11}^r + \zeta_{22}^r \right)^2 = 0, \quad (48)$$

$$\sum_{r=n+1}^{2n+1} \left(\zeta_{s_1+1s_1+1}^r + \dots + \zeta_{nn}^r \right) = \left(\sum_{\alpha=1}^{s_1} \zeta_{\alpha\alpha}^{n+1} \right)^2 = 0, \quad (49)$$

$$\sum_{\substack{k,l=3 \\ k \neq l}}^{s_1} (\zeta_{kl}^{n+1})^2 + \sum_{r=n+2}^{2n+1} \sum_{k,l=3}^{s_1} (\zeta_{kl}^{n+1})^2 + \sum_{\alpha=3}^{s_1} \sum_{\beta=s_1+1}^n (\zeta_{\alpha\beta}^{n+1})^2 + \sum_{r=n+2}^{2n+1} \sum_{A=3}^{s_1} \sum_{B=s_1+1}^n (\zeta_{AB}^r)^2 = 0. \quad (50)$$

Equation (49) clearly indicates that the warped product $\mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_1}$ is both a $\mathbb{D}_1^{s_1}$ -minimal and $\mathbb{D}_2^{s_2}$ -minimal warped product Legendrian submanifold in $\mathbb{D}_e^{2n+1}$. It can be concluded that the warped product Legendrian submanifold $\mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_1}$ is minimal in $\mathbb{D}_e^{2n+1}$. Moreover, we shall classify the other case in two techniques, as they depend on the vector fields r . Assuming that $r = n + 1$, we define the following:

$$\zeta_{11}^{n+1} + \zeta_{22}^{n+1} = \zeta_{33}^{n+1} = \dots = \zeta_{s_1 s_1}^{n+1}$$

and

$$\sum_{j=3}^n \zeta_{1j}^{n+1} = \sum_{j=3}^n \zeta_{2j}^{n+1} = \sum_{\substack{k,l=3 \\ k \neq l}}^{s_1} (\zeta_{kl}^{n+1})^2 = \sum_{\alpha=3}^{s_1} \sum_{\beta=s_1+1}^n (\zeta_{\alpha\beta}^{n+1})^2 = 0.$$

Thus, the above condition is equivalent to the following matrices:

$$(i) A_{e_{n+1}} = \left(\begin{array}{ccccc|ccc} \mu_1 & \zeta_{12}^{n+1} & 0 & \dots & 0_{1s_1} & 0_{1s_1+1} & \dots & 0_{1n} \\ \zeta_{12}^{n+1} & \mu_2 & 0 & \dots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \mu & \dots & \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\ 0_{s_1 1} & 0 & 0 & \dots & \mu & 0_{s_1 s_1+1} & \dots & 0_{s_1 n} \\ \hline 0_{s_1+1 1} & \dots & \dots & \dots & 0_{s_1+1 s_1} & \zeta_{s_1+1 s_1+1}^{n+1} & \dots & \zeta_{s_1+1 n}^{n+1} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0_{n1} & \dots & \dots & \dots & 0_{ns_1} & \zeta_{ns_1+1}^{n+1} & \dots & \zeta_{nn}^{n+1} \end{array} \right),$$

where $\mu = \mu_1 + \mu_2$ gives the (i) theorem. Similarly, if $r \in \{n + 2, \dots, m\}$, then the above condition implies that

$$\zeta_{11}^{n+1} + \zeta_{22}^{n+1} = \sum_{j=3}^n \zeta_{1j}^{n+1} = \sum_{j=3}^n \zeta_{2j}^{n+1} = \sum_{\substack{k,l=3 \\ k \neq l}}^{s_1} (\zeta_{kl}^{n+1})^2 = \sum_{\alpha=3}^{s_1} \sum_{\beta=s_1+1}^n (\zeta_{\alpha\beta}^{n+1})^2 = 0.$$

This is equivalent to the second metric:

$$(ii) A_{e_r} = \left(\begin{array}{ccccc|ccc} \zeta_{11}^r & \zeta_{12}^r & 0 & \dots & 0_{1s_1} & 0_{1s_1+1} & \dots & 0_{1n} \\ \zeta_{21}^r & -\zeta_{11}^r & 0 & \dots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0_{33} & \dots & \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\ 0_{s_1 1} & 0 & 0 & \dots & 0_{s_1 s_1} & 0_{s_1 s_1+1} & \dots & 0_{s_1 n} \\ \hline 0_{s_1+1 1} & \dots & \dots & \dots & 0_{s_1+1 s_1} & \zeta_{s_1+1 s_1+1}^{n+1} & \dots & \zeta_{s_1+1 n}^{n+1} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0_{n1} & \dots & \dots & \dots & 0_{ns_1} & \zeta_{ns_1+1}^{n+1} & \dots & \zeta_{nn}^{n+1} \end{array} \right),$$

It is clear that the above two conditions show that $\mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_1}$ is a mixed totally geodesic warped product Legendrian submanifold in $\mathbb{D}_\epsilon^{2n+1}$. Furthermore, the equality sign in (ii) holds if and only if the following two matrices are satisfied:

$$(iii) A_{e_{n+1}} = \left(\begin{array}{cccc|cccc} \zeta_{11}^{n+1} & \cdots & \cdots & \zeta_{1s_1}^{n+1} & 0_{1s_1+1} & \cdots & \cdots & \cdots & 0_{1n} \\ \vdots & \ddots & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \zeta_{s_1 11}^{n+1} & \cdots & \cdots & \zeta_{s_1 s_1}^{n+1} & 0_{s_1 s_1 + 1} & \cdots & \cdots & \cdots & 0_{s_1 n} \\ \hline 0_{s_1+11} & \cdots & \cdots & 0_{s_1+1s_1} & \mu_1 & \zeta_{s_1+1s_1+2}^{n+1} & 0 & \cdots & 0_{s_1+1n} \\ \vdots & \ddots & \ddots & \vdots & \zeta_{s_1+2s_1+1}^{n+1} & \mu_2 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & 0 & 0 & \mu & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ 0_{n1} & \cdots & \cdots & 0_{ns_1} & 0_{ns_1+1} & 0 & \cdots & 0 & \mu \end{array} \right),$$

where $\mu = \mu_1 + \mu_2$. If $r \in \{n+2, \dots, 2n+1\}$, thus we have

$$(iv) A_{e_r} = \left(\begin{array}{cccc|cccc} \zeta_{11}^r & \cdots & \cdots & \zeta_{1s_1}^r & 0_{1s_1+1} & \cdots & \cdots & \cdots & 0_{1n} \\ \vdots & \ddots & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \zeta_{s_1 11}^r & \cdots & \cdots & \zeta_{s_1 s_1}^r & 0_{s_1 s_1 + 1} & \cdots & \cdots & \cdots & 0_{s_1 n} \\ \hline 0_{s_1+11} & \cdots & \cdots & 0_{s_1+1s_1} & \zeta_{s_1+1s_1+1}^r & \zeta_{s_1+1s_1+2}^{n+1} & 0 & \cdots & 0_{s_1+1n} \\ \vdots & \ddots & \ddots & \vdots & \zeta_{s_1+2s_1+1}^{n+1} & -\zeta_{s_1+1s_1+1}^r & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ 0_{n1} & \cdots & \cdots & 0_{ns_1} & 0_{ns_1+1} & 0 & \cdots & 0 & 0 \end{array} \right),$$

From the above, it is also clear that $\mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_1}$ is both a $\mathbb{D}_1^{s_1}$ -minimal and $\mathbb{D}_2^{s_2}$ -minimal warped product Legendrian submanifold in $\mathbb{D}_\epsilon^{2n+1} \times \mathbb{R}$, which implies the minimality of the warped product Legendrian submanifold $\mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_1}$ in $\mathbb{D}_\epsilon^{2n+1}$. This completes the proof of the theorem. $\square$

Warped product manifolds have studied themselves to be a profitable ambient space to obtain a wide range of distinct geometrical properties for immersion. We now find the inequalities for the Riemannian manifold that has constant sectional curvature $\epsilon \in \{1, -3\}$ and can be expressed as a product manifold of $\mathbb{D}_\epsilon^{2n+1}$. We find the following result as follows.

2.1. An Application for Warped Product Legendrian Submanifold in $\mathbb{S}^{2n+1}$ with $\epsilon = 1$

Theorem 2. Assume that $\phi : \mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ is an isometric immersion of a warped product submanifold $\mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ into a Euclidean sphere $\mathbb{S}^{2n+1}$. Then, for each point $x \in \mathbb{D}^n$ and each plane section $\pi_i \subset T_x \mathbb{D}_i^{n_i}$, for $i = 1, 2$, we obtain the following for

(a) $\pi_1 \subset T_x \mathbb{D}_1^{s_1}$

$$\delta_{\tilde{\mathbb{D}}^{s_1}}(x) \leq \frac{n^2}{2} \|\mathbb{H}\|^2 + s_2 \|\nabla(\ln f)\|^2 - s_2 \Delta(\ln f) + \left\{ \frac{s_1}{2} (s_1 + 2s_2 - 1) - 1 \right\}.$$

The equality of the above inequality holds at $x \in \mathbb{D}^n$ if and only if there exists an orthonormal basis $\{e_1 \cdots e_n\}$ of $T_x \mathbb{D}^n$ and orthonormal basis $\{e_{n+1} \cdots e_{2n+1}\}$ of $T_x^\perp$ such that (a) $\pi = \text{Span}\{e_1, e_2\}$ and (b) shape operators take the following form

$$(i) A_{e_{n+1}} = \left(\begin{array}{ccccc|ccc} \mu_1 & \zeta_{12}^{n+1} & 0 & \cdots & 0_{1s_1} & 0_{1s_1+1} & \cdots & 0_{1n} \\ \zeta_{12}^{n+1} & \mu_2 & 0 & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \mu & \cdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0_{s_1 1} & 0 & 0 & \cdots & \mu & 0_{s_1 s_1+1} & \cdots & 0_{s_1 n} \\ \hline 0_{s_1+1 1} & \cdots & \cdots & \cdots & 0_{s_1+1 s_1} & \zeta_{s_1+1 s_1+1}^{n+1} & \cdots & \zeta_{s_1+1 n}^{n+1} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0_{n1} & \cdots & \cdots & \cdots & 0_{ns_1} & \zeta_{ns_1+1}^{n+1} & \cdots & \zeta_{nn}^{n+1} \end{array} \right),$$

where $\mu = \mu_1 + \mu_2$. If $r \in \{n+2, \cdots m\}$, then we have the matrix

$$(ii) A_{e_r} = \left(\begin{array}{ccccc|ccc} \zeta_{11}^r & \zeta_{12}^r & 0 & \cdots & 0_{1s_1} & 0_{1s_1+1} & \cdots & 0_{1n} \\ \zeta_{21}^r & -\zeta_{11}^r & 0 & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0_{33} & \cdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0_{s_1 1} & 0 & 0 & \cdots & 0_{s_1 s_1} & 0_{s_1 s_1+1} & \cdots & 0_{s_1 n} \\ \hline 0_{s_1+1 1} & \cdots & \cdots & \cdots & 0_{s_1+1 s_1} & \zeta_{s_1+1 s_1+1}^{n+1} & \cdots & \zeta_{s_1+1 n}^{n+1} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0_{n1} & \cdots & \cdots & \cdots & 0_{ns_1} & \zeta_{ns_1+1}^{n+1} & \cdots & \zeta_{nn}^{n+1} \end{array} \right),$$

(b) for $\pi_2 \subset T_x \mathbb{D}_2^{s_2}$

$$\delta_{\tilde{\mathbb{D}}^{s_2}}(x) \leq \frac{n^2}{2} \|\mathbb{H}\|^2 + s_2 \|\nabla(\ln f)\|^2 - s_2 \Delta(\ln f) + \left\{ \frac{s_2}{2} (s_2 + 2s_1 - 1) - 1 \right\}.$$

The equality of the above equation hold if and only if

$$(iii) A_{e_{n+1}} = \left(\begin{array}{cccc|cccc} \zeta_{11}^{n+1} & \cdots & \cdots & \zeta_{1s_1}^{n+1} & 0_{1s_1+1} & \cdots & \cdots & \cdots & 0_{1n} \\ \vdots & \ddots & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \zeta_{s_1 11}^{n+1} & \cdots & \cdots & \zeta_{s_1 s_1}^{n+1} & 0_{s_1 s_1+1} & \cdots & \cdots & \cdots & 0_{s_1 n} \\ \hline 0_{s_1+1 1} & \cdots & \cdots & 0_{s_1+1 s_1} & \mu_1 & \zeta_{s_1+1 s_1+2}^{n+1} & 0 & \cdots & 0_{s_1+1 n} \\ \vdots & \ddots & \ddots & \vdots & \zeta_{s_1+2 s_1+1}^{n+1} & \mu_2 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & 0 & 0 & \mu & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ 0_{n1} & \cdots & \cdots & 0_{ns_1} & 0_{ns_1+1} & 0 & \cdots & 0 & \mu \end{array} \right),$$

where $\mu = \mu_1 + \mu_2$. If $r \in \{n+2, \dots, 2n+1\}$, thus we have

$$(iv) A_{e_r} = \left(\begin{array}{cccc|cccccc} \zeta_{11}^r & \cdots & \cdots & \zeta_{1s_1}^r & 0_{1s_1+1} & \cdots & \cdots & \cdots & 0_{1n} \\ \vdots & \ddots & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \zeta_{s_1 11}^r & \cdots & \cdots & \zeta_{s_1 s_1}^r & 0_{s_1 s_1+1} & \cdots & \cdots & \cdots & 0_{s_1 n} \\ \hline 0_{s_1+11} & \cdots & \cdots & 0_{s_1+1s_1} & \zeta_{s_1+1s_1+1}^r & \zeta_{s_1+1s_1+2}^{n+1} & 0 & \cdots & 0_{s_1+1n} \\ \vdots & \ddots & \ddots & \vdots & \zeta_{s_1+2s_1+1}^{n+1} & -\zeta_{s_1+1s_1+1}^r & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ 0_{n1} & \cdots & \cdots & 0_{ns_1} & 0_{ns_1+1} & 0 & \cdots & 0 & 0 \end{array} \right),$$

(v) If the equality holds in (1) or (2), then $\mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ is mixed totally geodesic in space form $\mathbb{D}_\epsilon^{2n+1}$. Moreover, $\mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ is both $\mathbb{D}_1^{s_1}$ -minimal and $\mathbb{D}_2^{s_2}$ -minimal. Thus, $\mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ is a minimal warped product submanifold in Sasakian space form $\mathbb{D}_\epsilon^{2n+1}$.

Proof. Now we consider the constant sectional curvature $\epsilon = 1$ and $\mathbb{D}_\epsilon^{2n+1} = \mathbb{S}^{2n+1}$ for the product manifold $\mathbb{S}^{2n+1}$. Then, inserting the proceeding value in (23) and (24), we obtain the required result. $\square$

2.2. An Application for Warped Product Submanifold in $\mathbb{R}^{2n+1}$ with $\epsilon = -3$

Theorem 3. Assume that $\phi : \mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ is an isometric immersion of a warped product Legendrian submanifold $\mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ into a Euclidean spaces $\mathbb{R}^{2n+1}$. Then, for each point $x \in \mathbb{D}^n$ and each plane section $\pi_i \subset T_x \mathbb{D}_i^{n_i}$, for $i = 1, 2$, we obtain the following for

(a) $\pi_1 \subset T_x \mathbb{D}_1^{s_1}$ or $\pi_2 \subset T_x \mathbb{D}_2^{s_2}$

$$\delta_{\mathbb{D}^{s_1}}(x) \leq \frac{n^2}{2} \|\mathbb{H}\|^2 + s_2 \|\nabla(\ln f)\|^2 - s_2 \Delta(\ln f).$$

(b) for $\pi_2 \subset T_x \mathbb{D}_2^{s_2}$

$$\delta_{\mathbb{D}^{s_2}}(x) \leq \frac{n^2}{2} \|\mathbb{H}\|^2 + s_2 \|\nabla(\ln f)\|^2 - s_2 \Delta(\ln f).$$

The equality of the above inequality holds as in Theorem 1.

Proof. Now we assume that $\mathbb{D}_\epsilon^{2n+1} = \mathbb{R}^{2n+1}$ and constant sectional curvature $\epsilon = -3$ for the Euclidean spaces $\mathbb{R}^{2n+1}$. Then, using these values in (23) and (24), we obtain the required result. $\square$

Remark 3. It should be noticed that Theorem 2 coincides with Theorem 4.1 in [20]. If $f = 1$, then Theorem 2 is generalized the result in [4]. Therefore, our result is a generalization of [4,20].

2.3. Some Applications to Obtain Dirichlet Eigenvalue Inequalities

Now, if the first eigenvalue of the Dirichlet boundary condition is denoted by $v_1(\Sigma) > 0$ on a complete noncompact Riemannian manifold $\mathbb{D}^n$ with the compact domain Σ in $\mathbb{D}^n$, then we have

$$\Delta \sigma + v \sigma = 0, \text{ on } \Sigma \text{ and } \sigma = 0 \text{ on } \partial \Sigma, \quad (51)$$

where Δ is the Laplacian on $\mathbb{D}^n$, and σ is a non-zero function defined on $\mathbb{D}^n$. Then, $v_1(\mathbb{D}^n)$ is expressed as $\inf_{\Sigma} v_1(\Sigma)$.

From the above motivation, assume that f is the non-constant warping function on compact warped product submanifold $\mathbb{D}^n$. Then, the minimum principle on v_1 leads to (see, for instance, [1,9])

$$\int_{\mathbb{D}^n} \|\nabla \sigma\|^2 dV \geq v_1 \int_{\mathbb{D}^n} (\sigma)^2 dV \quad (52)$$

and the equality is satisfied if and only if

$$\Delta \sigma = v_1 \sigma. \quad (53)$$

Implementing the integration along the base manifold $\mathbb{D}^{s_1}$ in Equations (23) and (24), we obtain the following result.

Theorem 4. Assume that $\phi : \mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ is a compact warped product Legendrian submanifold $\mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ into a Sasakian space form $\tilde{\mathcal{D}}^{2n+1}(\epsilon)$. If v_1 is an eigenvalue of the eigenfunction $\sigma = \ln f$ satisfies (53), then we have

$$\begin{aligned} \int_{\mathbb{D}_1 \times \{s_2\}} \delta_{\mathbb{D}^{s_1}}(x) dV &\leq \frac{n^2}{2} \int_{\mathbb{D}_1 \times \{s_2\}} \|\mathbb{H}\|^2 dV + s_2 v_1 \int_{\mathbb{D}_1 \times \{s_2\}} (\ln f)^2 dV \\ &+ \int_{\mathbb{D}_1 \times \{s_2\}} \left\{ \left(\frac{s_1}{2} (s_1 + 2s_2 - 1) - 1 \right) \left(\frac{\epsilon + 3}{4} \right) \right\} dV, \end{aligned} \quad (54)$$

for $\pi_1 \subset T\mathbb{D}_1$. Moreover, we have

$$\begin{aligned} \int_{\mathbb{D}_1 \times \{s_2\}} \delta_{\mathbb{D}^{s_2}}(x) dV &\leq \frac{n^2}{2} \int_{\mathbb{D}_1 \times \{s_2\}} \|\mathbb{H}\|^2 dV + s_2 v_1 \int_{\mathbb{D}_1 \times \{s_2\}} (\ln f)^2 dV \\ &+ \int_{\mathbb{D}_1 \times \{s_2\}} \left\{ \left(\frac{s_2}{2} (s_2 + 2s_1 - 1) - 1 \right) \left(\frac{\epsilon + 3}{4} \right) \right\} dV, \end{aligned} \quad (55)$$

for $\pi_2 \subset T\mathbb{D}_2$.

Proof. As we know from the Stokes theorem, $\int \Delta \sigma dV = 0$ for a compact support. Then, we use the proceeding condition in (23) and (24) by replacing $\sigma = \ln f$, and we easily obtain the result. $\square$

2.4. An Applications for Brochler Formulas

Theorem 5. Assume that $\phi : \mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ is a compact warped product Legendrian submanifold $\mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ into a Sasakian space form $\tilde{\mathcal{D}}^{2n+1}(\epsilon)$. If v_1 is an eigenvalue of the eigenfunction $\sigma = \ln f$ satisfies (53), then we have

$$\begin{aligned} \int_{\mathbb{D}_1 \times \{s_2\}} Ric(\nabla \ln f, \nabla \ln f) dV &\geq \frac{v_1}{s_2} \int_{\mathbb{D}_1 \times \{s_2\}} \delta_{\mathbb{D}^{s_1}}(x) dV - \frac{n^2 v_1}{2s_2} \int_{\mathbb{D}_1 \times \{s_2\}} \|\mathbb{H}\|^2 dV \\ &+ \frac{v_1}{s_2} \int_{\mathbb{D}_1 \times \{s_2\}} \left\{ 1 - \left(\frac{s_1}{2} (s_1 + 2s_2 - 1) \right) \right\} \left(\frac{\epsilon + 3}{4} \right) dV \\ &- \int_{\mathbb{D}_1 \times \{s_2\}} \|\nabla^2 \ln f\|^2 dV, \end{aligned} \quad (56)$$

for $\pi_1 \subset T\mathbb{D}_1$. Moreover, we have

$$\int_{\mathbb{D}_1 \times \{s_2\}} Ric(\nabla \ln f, \nabla \ln f) dV \geq \frac{v_1}{s_2} \int_{\mathbb{D}_1 \times \{s_2\}} \delta_{\mathbb{D}^{s_2}}(x) dV - \frac{n^2 v_1}{2s_2} \int_{\mathbb{D}_1 \times \{s_2\}} \|\mathbb{H}\|^2 dV$$

$$\begin{aligned}
& + \frac{v_1}{s_2} \int_{\mathbb{D}_1 \times \{s_2\}} \left\{ 1 - \left(\frac{s_2}{2} (s_2 + 2s_1 - 1) \right) \right\} \left(\frac{\epsilon + 3}{4} \right) dV \\
& - \int_{\mathbb{D}_1 \times \{s_2\}} \|\nabla^2 \ln f\|^2 dV,
\end{aligned} \tag{57}$$

for $\pi_2 \subset T\mathbb{D}_2$.

Proof. If σ is the first eigenfunction of the Laplacian $\Delta\sigma = \operatorname{div}(\nabla\sigma)$ for $\mathbb{D}^n$ connected to the first non-zero eigenvalue v_1 , such that, $\Delta\sigma = -v_1\sigma$, then recalling the Bochner formula (see [40]) that gives the following relation of the differentiable function σ denoted at the Riemannian manifold as:

$$\frac{1}{2} \Delta \|\nabla\sigma\|^2 = \|\nabla^2\sigma\|^2 + \operatorname{Ric}(\nabla\sigma, \nabla\sigma) + g(\nabla\sigma, \nabla(\Delta\sigma)).$$

By the integration of the previous equation, using the Stokes theorem, we have

$$\int_{\mathbb{D}_1 \times \{s_2\}} \|\nabla^2\sigma\|^2 dV + \int_{\mathbb{D}_1 \times \{s_2\}} \operatorname{Ric}(\nabla\sigma, \nabla\sigma) dV + \int_{\mathbb{D}_1 \times \{s_2\}} g(\nabla\sigma, \nabla(\Delta\sigma)) dV = 0. \tag{58}$$

Now, using $\Delta\sigma = v_1\sigma$ and making some rearrangement in Equation (58), we derive

$$\int_{\mathbb{D}_1 \times \{s_2\}} \|\nabla\sigma\|^2 dV = \frac{1}{v_1} \left(\int_{\mathbb{D}_1 \times \{s_2\}} \|\nabla^2\sigma\|^2 dV + \int_{\mathbb{D}_1 \times \{s_2\}} \operatorname{Ric}(\nabla\sigma, \nabla\sigma) dV \right). \tag{59}$$

Taking the integration in (23) and (24) and inserting the above equation, we obtain the desired results. $\square$

3. Chern's Problem: Finding the Conditions under Which Warped Products Must Be Minimal

In this section, we provide the partial answer to the Chern problem [41], that is, the necessary condition for a warped product Legendrian submanifold to be a minimal in Sasakian space form $\tilde{\mathcal{D}}^{2n+1}(\epsilon)$.

Corollary 1. Let $\phi : \mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ be an isometric immersion of a warped product Legendrian submanifold $\mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ into a Sasakian space form $\tilde{\mathcal{D}}^{2n+1}(\epsilon)$. Then, for each point $x \in \mathbb{D}^n$ and each $\pi_1 \subset T_x \mathbb{D}_1^{n_1}$, we have

$$\delta_{\tilde{\mathbb{D}}^{s_1}}(x) + s_2 \Delta(\ln f) \leq \left\{ \frac{s_1}{2} (s_1 + 2s_2 - 1) - 1 \right\} \left(\frac{\epsilon + 3}{4} \right) + s_2 \|\nabla(\ln f)\|^2. \tag{60}$$

and if the equality satisfies, then ϕ is minimal.

The second result is:

Corollary 2. Let $\phi : \mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ be an isometric immersion of a warped product Legendrian submanifold $\mathbb{D}^n = \mathbb{D}_1^{s_1} \times_f \mathbb{D}_2^{s_2}$ into a Sasakian space form $\tilde{\mathcal{D}}^{2n+1}(\epsilon)$. Then, for each point $x \in \mathbb{D}^n$ and each $\pi_2 \subset T_x \mathbb{D}_2^{n_2}$, we have

$$\delta_{\tilde{\mathbb{D}}^{s_2}}(x) + s_2 \Delta(\ln f) \leq \left\{ \frac{s_2}{2} (s_2 + 2s_1 - 1) - 1 \right\} \left(\frac{\epsilon + 3}{4} \right) + s_2 \|\nabla(\ln f)\|^2. \tag{61}$$

and if the equality satisfies, then ϕ is minimal.

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References

1. Bishop, R.L.; O’Neil, B. Manifolds of negative curvature. *Trans. Am. Math. Soc.* **1969**, *145*, 1–9.
2. Hawking, S.W.; Ellis, G.F.R. *The Large Scale Structure of Space Time*; Cambridge University Press: Cambridge, UK, 1973.
3. O’Neil, B. *Semi-Riemannian Geometry with Applications to Relativity*; Academic Press: New York, NY, USA, 1983.
4. Chen, B. Y. δ -invariants, inequalities of submanifolds and their applications. In *Topics in Differential Geometry*; Editura Academiei Romane: Bucharest, Romania, 2008; pp. 29–55.
5. Besse, A. *Einstein Manifolds*; Springer: New York, NY, USA; Berlin, Germany, 1987.
6. Chen, B.Y. A tour through δ -invariants: From Nash embedding theorem to ideal immersions, best ways of living and beyond. *Publ. Inst. Math.* **2013**, *94*, 108.
7. Chen, B.Y.; Dillen, F.; Verstraelen, L.; Vrancken, L. Characterization of Riemannian space forms, Einstein spaces, and conformally flat spaces. *Proc. Am. Math. Soc.* **1999**, *128*, 589–598.
8. Chen B.-Y. *Pseudo-Riemannian Geometry, δ -Invariants, and Applications*; World Scientific: Hackensack, NJ, USA, 2011.
9. Chen, B.Y. On isometric minimal immersions from warped products into real space forms. In *Proceedings of the Edinburgh Mathematical Society*; Cambridge University Press: Cambridge, UK, 2002; Volume 45, pp. 579–587.
10. Chen, B.-Y.; Blaga, A.M.; Vilcu, G.E. Differential geometry of submanifolds in Complex space forms involving δ -invariants. *Mathematics* **2022**, *10*, 591.
11. Chen, B.Y.; Dillen, F.; Vrancken, L. Lagrangian submanifolds in complex space forms attaining equality in a basic inequality. *J. Math. Anal. Appl.* **2012**, *387*, 139–152.
12. Chen, B.Y. Geometry of warped product submanifolds: A survey. *J. Adv. Math. Stud.* **2013**, *6*, 1–43.
13. Chen, B.Y.; Dillen, F. Optimal general inequalities for Lagrangian submanifolds in complex space forms. *J. Math. Anal. Appl.* **2011**, *379*, 229–239.
14. Chen, B.-Y.; Dillen, F.; Van der Veken, J.; Vrancken, L. Curvature inequalities for Lagrangian submanifolds: The final solution. *Differ. Geom. Appl.* **2013**, *31*, 808–819.
15. Chen, B.-Y.; Dillen, F. δ -invariants for Lagrangian submanifolds of complex space forms, in Riemannian Geometry and Applications. In *Proceedings of the RIGA 2011*, Riga, Latvia, 16–18 February 2011; pp. 75–94.
16. Chen, B.-Y.; Prieto-Martín, A. Classification of Lagrangian submanifolds in complex space forms satisfying basic equality involving $\delta(2, 2)$. *Differ. Geom. Appl.* **2012**, *30*, 107–123.
17. Tripathi, M.M. Improved Chen–Ricci inequality for curvature-like tensors and its applications. *Differ. Geom. Appl.* **2011**, *29*, 685–698.
18. Vilcu, G.-E. On Chen invariants and inequalities in quaternionic geometry. *J. Inequal. Appl.* **2013**, *2013*, 66.
19. Chen, B.-Y. An optimal inequality for CR-warped products in complex space forms involving CR δ -invariant. *Int. J. Math.* **2012**, *23*, 3.
20. Mustafa, A.; Ozel, C.; Pigazzini, A.; Kaur, R.; Shanker, G. First Chen inequality for general warped product submanifolds of a Riemannian space form and applications. *arXiv* **2021**, arXiv:2109.08911.
21. Hasegawa, I.; Mihai, I. Contacts CR-warped product submanifolds in Sasakian manifolds. *Geom. Dedicata* **2003**, *102*, 143–150.
22. Ali, A.; Laurian-Ioan, P. Geometric classification of warped products isometrically immersed in Sasakian space forms. *Math. Nachr.* **2018**, *292*, 234–251.
23. Ali, A.; Uddin, S.; Othman, W.A.M. Geometry of warped product pointwise semi-slant submanifolds of Kaehler manifolds. *Filomat* **2017**, *32*, 3771–3788.
24. Ali, A.; Luarian, P. Geometry of warped product immersions of Kenmotsu space forms and its applications to slant immersions. *J. Geom. Phys.* **2017**, *114*, 276–290.
25. Ali, A.; Ozel, C. Geometry of warped product pointwise semi-slant submanifolds of cosymplectic manifolds and its applications. *Int. J. Geom. Methods Mod. Phys.* **2017**, *14*, 175002.

26. Li, Y.; Abolarinwa, A.; Alkhaldi, A.; Ali, A. Some Inequalities of Hardy Type Related to Witten-Laplace Operator on Smooth Metric Measure Spaces. *Mathematics* **2022**, *10*, 4580.
27. Li, Y.; Aldossary, M.T.; Abdel-Baky, R.A. Spacelike Circ. Surfaces Minkowski 3-Space. *Symmetry* **2023**, *15*, 173.
28. Li, Y.; Chen, Z.; Nazra, S.H.; Abdel-Baky, R.A. Singul. Timelike Dev. Surfaces Minkowski 3-Space. *Symmetry* **2023**, *15*, 277.
29. Li, Y.; Alkhaldi, A.; Ali, A.; Abdel-Baky, R.A.; Saad, M.K. Investigation of ruled surfaces and their singularities according to Blaschke frame in Euclidean 3-Space. *AIMS Math.* **2023**, *8*, 13875–13888. <https://doi.org/10.3934/math.2023709>.
30. Li, Y.; Eren, K.; Ayvaci, K.H.; Ersoy, S. The developable surfaces with pointwise 1-type Gauss map of Frenet type framed base curves in Euclidean 3-space. *AIMS Math.* **2023**, *8*, 2226–2239.
31. Li, Y.; Abdel-Salam, A.A.; Saad, M.K. Primitivoids Curves Minkowski Plane. *AIMS Math.* **2023**, *8*, 2386–2406.
32. Mihai, I. Contact $\mathcal{CR}$ -warped product submanifolds in Sasakian space forms. *Geom. Dedicata* **2004**, *109*, 165–173.
33. Srivastava, S.K.; Mofarreh, F.; Kumar, A.; Ali, A. Characterizations of $\mathcal{PR}$ -Pseudo-Slant Warped Product Submanifold of Para-Kenmotsu Manifold with Slant Base. *Symmetry* **2022**, *14*, 1001.
34. Chen, B.Y. *Differential Geometry of Warped Product Manifolds and Submanifolds*; World Scientific: Singapore, 2017.
35. Sasahara, T. A class of bi-minimal Legendrian submanifolds in Sasakian space forms. *Math. Nachr.* **2014**, *287*, 79–90.
36. Yano, K.; Kon, M. *CR-Submanifolds of Kaehlerian and Sasakian Manifolds*; Birkhauser: Boston, MA, USA, 1983.
37. Yano, Y.; Kon, M. *Structures on Manifolds*; World Scientific: Singapore, 1984.
38. Mihai, I.; Presură, I. An improved first Chen inequality for Legendrian submanifolds in Sasakian space forms. *Period. Math. Hung.* **2017**, *74*, 220–226.
39. Atceken, M. Contact CR-warped product submanifolds in Sasakian space forms. *Hacet. J. Math. Stat.* **2015**, *44*, 23–32.
40. Berger, M.; Gauduchon, P.; Mazet, E. *Le Spectre d'une Variété Riemannienne, Lectures Notes in Mathematics*; Springer: Berlin, Germany, 1971; Volume 194.
41. Chern, S.S. *Minimal Submanifold in a Riemannian Manifold*; University of Kansas: Lawrence, KS, USA, 1968.

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