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An Inverse Sturm–Liouville-Type Problem with Constant Delay and Non-Zero Initial Function

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Abstract: We suggest a new statement of the inverse spectral problem for Sturm–Liouville-type operators with constant delay. This inverse problem consists of recovering the coefficient (often referred to as potential) of the delayed term in the corresponding equation from the spectra of two boundary value problems with one common boundary condition. The previous studies, however, focus mostly on the case of zero initial function, i.e., they exploit the assumption that the potential vanishes on the corresponding subinterval. In the present paper, we waive that assumption in favor of a continuously matching initial function, which leads to the appearance of an additional term with a frozen argument in the equation. For the resulting new inverse problem, we pay special attention to the situation when one of the spectra is given only partially. Sufficient conditions and necessary conditions on the corresponding subspectrum for the unique determination of the potential are obtained, and a constructive procedure for solving the inverse problem is given. Moreover, we obtain the characterization of the spectra for the zero initial function and the Neumann common boundary condition, which is found to include an additional restriction as compared with the case of the Dirichlet common condition.

Keywords: Sturm–Liouville-type operator; functional-differential operator; constant delay; initial function; frozen argument; inverse spectral problem

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1. Introduction and Main Results

In recent years, there appeared a considerable interest in the inverse problem of recovering an integrable or a square-integrable potential $q(x)$ in the functional-differential equation

$$-y''(x) + q(x)y(x-a) = \lambda y(x), \quad 0 < x < \pi, \tag{1}$$

with constant delay $a \in (0, \pi)$ from the spectra of two boundary value problems for (1) with one common boundary condition (see [1–17] and references therein). For $a = 0$, this problem becomes the classical inverse Sturm–Liouville problem due to Borg [18,19], but the nonlocal case $a > 0$ requires other approaches. Moreover, it reveals some essentially different effects in the solution of the inverse problem than in the classical situation $a = 0$. For example, the solution of the inverse problem may be non-unique when $a \in (0, 2\pi/5)$ (see [12–14]).

Various equations with delay have been actively studied from the last century in connection with numerous applications (see, e.g., [20–26]). Such equations can be characterized by the possibility for the argument of the unknown function to go beyond its domain. For example, Equation (1) for $a > 0$ includes values of $y(x)$ for $x < 0$. In order to overcome this issue, one should specify an initial function, i.e., to impose $y(x) = f(x)$ for $x \in (-a, 0)$ with some known $f(x)$. In particular, one can put $q(x) = 0$ on $(0, a)$, which actually corresponds to specifying $f = 0$. We distinguish these two ways because rewriting Equation (1) in the form

$$-y''(x) + q^+(x)y(x-a) = \lambda y(x) - r(x), \quad 0 < x < \pi, \tag{2}$$

where $r(x) = q^-(x)f(x-a)$ and

$$q^-(x) = \begin{cases} q(x), & x \in (0,a), \\ 0, & x \in (a,\pi), \end{cases} \qquad q^+(x) = \begin{cases} 0, & x \in (0,a), \\ q(x), & x \in (a,\pi), \end{cases} \tag{3}$$

shows that $f \neq 0$ leads to a non-homogenous equation, while $f = 0$ deals with the corresponding homogenous one. Thus, for posing an eigenvalue problem, it is natural to choose the latter, i.e., to assume that $q(x) = 0$ on $(0,a)$. For this reason, the previous studies of inverse problems for (1) were focused mostly on this case, i.e., the reconstruction of $q(x)$ was actually carried out only for q^+, while q^- was a priori assumed to be zero.

A non-zero initial function f also may be appropriate for posing an eigenvalue problem, but it should be linearly dependent on the unknown function $y(x)$ on $[0,\pi]$ as, e.g.,

$$f(x) = y(0)g(x), \quad -a < x < 0. \tag{4}$$

This example corresponds to the classical theory [22] and ensures a continuous continuation of $y(x)$ to $[-a,0)$ whenever $g(x) \in C[-a,0]$ and $g(0) = 1$. Such continuation, however, is not always required (see, e.g., [25]). So, one can consider more general forms of an initial function such as $f(x) = Ly(x)$ with a linear operator L acting from $W_2^2[0,\pi]$ to $L_\infty(-a,0)$. Then, for keeping L in the frames of a perturbation, a natural requirement would be its relative compactness [27] with respect to the minimal operator of double differentiation. In particular, one can take $Ly(x) = F(y)g(x)$, where $F(y)$ is a linear functional relatively bounded to y'', e.g., $F(y) = y^{(j)}(b)$ for some $b \in [0,\pi]$ and $j \in \{0,1\}$. We will focus, however, on the special case (4).

An attempt to study the inverse problem for Equation (1) with a non-zero initial function $f(x)$ has been made in [16]. However, no dependence of $f(x)$ on $y(x)$ was assumed at all.

In the present paper, we refuse the usual assumption $q^- = 0$ but in favor of the initial function in the form (4). Then, Equation (1) can be rewritten with the so-called frozen argument:

$$-y''(x) + q^+(x)y(x-a) + p(x)y(0) = \lambda y(x), \quad 0 < x < \pi, \quad p(x) := q^-(x)g(x-a). \tag{5}$$

Since the functions $q^-(x)$ and $g(x-a)$ enter only in their product $p(x)$, they cannot be recovered simultaneously from any spectral information. Moreover, the reconstruction of $q^-(x)$ on any subinterval $(\alpha,\beta) \subset (0,a)$ can be possible only if $g(x) \neq 0$ a.e. on $(\alpha-a,\beta-a)$. For these reasons, we consider without loss of generality the canonical situation when $g(x) \equiv 1$.

For $j = 0,1$, $B_j(q)$ denotes the boundary value problem for Equation (1) with a complex-valued potential $q(x) \in L_2(0,\pi)$ under the boundary conditions

$$y'(0) = y^{(j)}(\pi) = 0$$

and under the initial-function condition

$$y(x) = y(0), \quad -a < x < 0. \tag{6}$$

Let $\{\lambda_{n,j}\}_{n\geq 0}$ be the spectrum of $B_j(q)$. Consider the following inverse problem.

Inverse Problem 1. *Given $\{\lambda_{n,0}\}_{n\geq 0}$ and $\{\lambda_{n,1}\}_{n\geq 0}$, find $q(x)$.*

The main results of the present paper (Theorems 1–3) are restricted to the case $a \geq \pi/2$. In accordance with [13,14], the solution of Inverse Problem 1 may be non-unique for $a \in (0, 2\pi/5)$, while the case $a \in [2\pi/5, \pi/2)$ will require an additional investigation. For future reference, however, we will mark those auxiliary assertions below whose proofs automatically extend to any wider ranges of a than just $a \in [\pi/2, \pi)$.

Everywhere below, one and the same symbol $\{\varkappa_n\}$ will denote *different* sequences in l_2. The following theorem gives basic necessary conditions for the solvability of Inverse Problem 1.

Theorem 1. *For $j = 0, 1$, the following asymptotics holds*

$$\lambda_{n,j} = \rho_{n,j}^2, \quad \rho_{n,j} = n + \frac{1-j}{2} + \frac{\omega}{\pi n} \cos\Big(n + \frac{1-j}{2}\Big)a + \frac{\varkappa_n}{n}, \quad \omega \in \mathbb{C}. \tag{7}$$

Here, the constant ω is determined by the formula

$$\omega = \frac{1}{2}\int_a^\pi q^+(x)\,dx. \tag{8}$$

Moreover, if the spectra $\{\lambda_{n,0}\}_{n\ge0}$ and $\{\lambda_{n,1}\}_{n\ge0}$ correspond to one and the same $q^-(x)$, then

$$i\theta_0(-ir) - \theta_1(-ir) = o(e^{(\pi-a)r}), \quad r \to +\infty, \tag{9}$$

where

$$\begin{aligned} \theta_0(\rho) &= \rho(\Delta_0(\rho^2) - \cos\rho\pi) - \omega \sin\rho(\pi - a), \\ \theta_1(\rho) &= \Delta_1(\rho^2) + \rho\sin\rho\pi - \omega\cos\rho(\pi - a), \end{aligned} \tag{10}$$

while the functions $\Delta_0(\lambda)$ and $\Delta_1(\lambda)$ are determined by the formulae

$$\Delta_0(\lambda) = \prod_{n=0}^\infty \frac{\lambda_{n,0} - \lambda}{(n+1/2)^2}, \quad \Delta_1(\lambda) = \pi(\lambda_{0,1} - \lambda)\prod_{n=1}^\infty \frac{\lambda_{n,1} - \lambda}{n^2}. \tag{11}$$

Condition (9) actually means that Inverse Problem 1 remains overdetermined as in the case $q^- = 0$ (see [6,15]). As will be seen below, it is sufficient to specify only one full spectrum and an appropriate part of the other one. For example, we also consider the following problem.

Inverse Problem 2. *Given $\{\lambda_{n_k,0}\}_{k\in\mathbb{N}}$ and $\{\lambda_{n,1}\}_{n\ge0}$, find $q(x)$.*

Here, $\{n_k\}_{k\in\mathbb{N}}$ is an increasing sequence of non-negative integers. The next theorem gives sufficient conditions as well as necessary conditions on $\{n_k\}_{k\in\mathbb{N}}$ for the uniqueness of $q(x)$.

Theorem 2. *(i) If the system $\sigma_0 := \{\sin(n_k + 1/2)x\}_{k\in\mathbb{N}}$ is complete in $\mathcal{H} := L_2(0, \pi - a)$, then the potential $q(x)$ in Inverse Problem 2 is determined uniquely.*

(ii) Conversely, if the specification of $\{\lambda_{n_k,0}\}_{k\in\mathbb{N}}$ and $\{\lambda_{n,1}\}_{n\ge0}$ uniquely determines $q(x)$, then the defect of σ_0 does not exceed 1, i.e., $\dim(\mathcal{H} \ominus \sigma_0) \le 1$.

Since the system $\{\sin(n + 1/2)x\}_{n\ge0}$ is complete in $L_2(0, \pi)$, this theorem, obviously, implies the unique determination of $q(x)$ by both complete spectra as in Inverse Problem 1.

The use of subspectra in the inverse problem with delay began in [6] for the zero initial function, where necessary and sufficient conditions were obtained on parts of both spectra to ensure the uniqueness of $q^+(x)$ in the case of the Dirichlet common condition at the origin.

We note that the gap between the sufficient and the necessary conditions in Theorem 2 is actually caused by imposing the common Neumann boundary condition. By the same reason, the conditions in Theorem 1 do not suffice for the solvability of Inverse Problem 1.

In the case of the Dirichlet common condition, necessary and sufficient conditions for the solvability of the corresponding inverse problem were obtained in [15] when $q^- = 0$. Here, we provide such conditions in the same case $q^- = 0$ but for the Neumann common condition, which brings to them an additional item. Specifically, the following theorem holds.

Theorem 3. *Arbitrary complex sequences $\{\lambda_{n,0}\}_{n\geq 0}$ and $\{\lambda_{n,1}\}_{n\geq 0}$ of the form (7) sharing one and the same $\omega \in \mathbb{C}$ are the spectra of the problems $B_0(q)$ and $B_1(q)$, respectively, with $q(x) = 0$ a.e. on $(0, a)$ if and only if the exponential types of the functions $\theta_0(\rho)$ and $\theta_1(\rho)$ determined by (10) and (11) do not exceed $\pi - a$ and the following relation is fulfilled:*

$$\lambda_{0,1} \prod_{n=1}^{\infty} \frac{\lambda_{n,1}}{n^2} = \frac{2\omega}{\pi}. \tag{12}$$

The latter relation is an additional characterizing condition, which does not exist in the Dirichlet case [15]. We note that the relevant difference between both cases was pointed out in [12] (see Remark 2 therein).

There are also various studies of recovering the operator with purely frozen argument

$$\ell y := -y''(x) + q(x)y(b), \quad y^{(\alpha)}(0) = y^{(\beta)}(\pi) = 0,$$

from its spectrum, where $b \in [0, \pi]$ and $\alpha, \beta \in \{0, 1\}$ (see [28–35] and references therein). In particular, its unique solvability depends on the value of b as well as on α and β. We note that both related to Inverse Problem 1 situations: $b = 0, \alpha = 1, \beta = 0$ and $b = 0, \alpha = \beta = 1$ belong to the so-called non-generate case, when the solution is unique (see, e.g., [28,29,32]).

We note that Theorem 2 also formally holds for $a = \pi$, which follows from Theorem 4.1 in [28] or Theorem 2 in [29]. In this case, $q^- = q$ and $q^+ = 0$. Then, Equation (1) under the initial-function condition (6) becomes an equation with purely frozen argument:

$$-y''(x) + q(x)y(0) = \lambda y(x), \quad 0 < x < \pi,$$

where $q(x)$ is uniquely determined by the single spectrum $\{\lambda_{n,1}\}_{n\geq 0}$.

The paper is organized as follows. In the next section, we construct transformation operators for a fundamental system of solutions of the homogeneous equation in (2), i.e., when $r(x) = 0$. In Section 3, Green's function of the Cauchy problem for the non-homogeneous Equation (2) under the zero initial conditions is constructed. In Section 4, we study the characteristic functions of the problems $B_j(q)$ and prove Theorem 1. Proofs of Theorems 2 and 3 are given in Section 5 along with a constructive procedure for solving the inverse problems. In the last section, we summarize the main innovations of the paper and discuss the results.

Throughout the paper, we agree that ρ is connected with λ by the relation $\rho^2 = \lambda$, while f' and $f^{(j)}$ denote the partial derivatives of a function f with respect to the *first* argument:

$$f'(x_1, \ldots, x_m) := \frac{d}{dx_1} f(x_1, \ldots, x_m), \quad f^{(j)}(x_1, \ldots, x_m) := \frac{d^j}{dx_1^j} f(x_1, \ldots, x_m).$$

2. Transformation Operators

Let $C(x, \lambda)$ and $S(x, \lambda)$ be solutions of the homogeneous equation in (2), i.e., the equation

$$-y''(x) + q^+(x)y(x - a) = \lambda y(x), \quad 0 < x < \pi, \tag{13}$$

under the initial conditions

$$C(0, \lambda) = S'(0, \lambda) = 1, \quad C'(0, \lambda) = S(0, \lambda) = 0.$$

They form a fundamental system of solutions of equation (13) (see, e.g., [14]).

In this section, we obtain representations for the functions $C(x,\lambda)$ and $S(x,\lambda)$ involving the so-called transformation operators, which connect them with the corresponding solutions of the simplest equation with the zero potential. Specifically, the following lemma holds.

Lemma 1. *Let $a \geq \pi/2$. The functions $S(x,\lambda)$ and $C(x,\lambda)$ admit the representations*

$$S(x,\lambda) = \frac{\sin\rho x}{\rho} + \int_a^x P(x,t)\frac{\sin\rho(x-t)}{\rho}\,dt, \tag{14}$$

$$C(x,\lambda) = \cos\rho x + \int_a^x K(x,t)\cos\rho(x-t)\,dt, \tag{15}$$

where (in accordance with our standing agreement) $\rho^2 = \lambda$ and

$$P(x,t) = \frac{1}{2}\int_{\frac{a+t}{2}}^{x+\frac{a-t}{2}} q^+(\tau)\,d\tau, \tag{16}$$

$$K(x,t) = \frac{1}{2}\int_a^{\frac{a+t}{2}} q^+(\tau)\,d\tau + \frac{1}{2}\int_a^{x+\frac{a-t}{2}} q^+(\tau)\,d\tau. \tag{17}$$

Proof. The assertion for $S(x,\lambda)$ is a particular case of Lemma 1 in [15]. So we will prove only (15) and (17). It is easy to see that the Cauchy problem for $C(x,\lambda)$ is equivalent to the integral equation

$$C(x,\lambda) = \cos\rho x + \int_a^x \frac{\sin\rho(x-t)}{\rho} q^+(t)C(t-a,\lambda)\,dt.$$

Taking into account that $a \geq \pi/2$, we calculate

$$\begin{aligned}
\int_a^x \frac{\sin\rho(x-t)}{\rho} q^+(t)C(t-a,\lambda)\,dt &= \int_a^x \frac{\sin\rho(x-t)}{\rho} q^+(t)\cos\rho(t-a)\,dt \\
&= \int_a^x q^+(t)\cos\rho(t-a)\,dt \int_0^{x-t}\cos\rho\tau\,d\tau \\
&= \frac{1}{2}\int_a^x q^+(t)\,dt\int_0^{x-t}\Big(\cos\rho(t-a+\tau)+\cos\rho(t-a-\tau)\Big)\,d\tau \\
&= \frac{1}{2}\int_a^x q^+(t)\,dt\int_a^{2(x-t)+a}\cos\rho(x-\tau)\,d\tau = \frac{1}{2}\int_a^{2x-a}\cos\rho(x-t)\,dt\int_a^{x+\frac{a-t}{2}} q^+(\tau)\,d\tau \\
&= \frac{1}{2}\int_a^x\Big(\int_a^{x+\frac{a-t}{2}} q^+(\tau)\,d\tau + \int_a^{\frac{a+t}{2}} q^+(\tau)\,d\tau\Big)\cos\rho(x-t)\,dt,
\end{aligned}$$

which finishes the proof. □

Remark 1. *While the imposed restriction $a \geq \pi/2$ is vital for (16) and (17), representations (14) and (15) also remain valid for all smaller $a \geq 0$ but with more complicated kernels. In particular, Lemma 1 in [15] gives an integral equation for $P(x,t)$ for all $a \in [0,\pi/2)$. Moreover, it extends representation (14) to quadratic pencils with two delays.*

The following corollary can be easily checked by direct calculations.

Corollary 1. *The following representations hold:*

$$C(x,\lambda) = \cos\rho x + \omega(x)\frac{\sin\rho(x-a)}{\rho} + \int_a^x K_0(x,t)\frac{\sin\rho(x-t)}{\rho}\,dt, \tag{18}$$

$$C'(x,\lambda) = -\rho \sin \rho x + \omega(x) \cos \rho(x-a) + \int_a^x K_1(x,t) \cos \rho(x-t)\, dt, \tag{19}$$

where

$$\omega(x) = \frac{1}{2}\int_a^x q^+(t)\, dt, \quad K_j(x,t) = \frac{1}{4}\Big(q^+\Big(\frac{a+t}{2}\Big) - (-1)^j q^+\Big(x + \frac{a-t}{2}\Big)\Big), \;\; j = 0, 1. \tag{20}$$

3. Green's Function of the Cauchy Operator

Here, we obtain the solution $z(x,\lambda) = z(x,\lambda;r)$ of the Cauchy problem for the non-homogeneous Equation (2) with an arbitrary free term $r(x)$ under the zero initial conditions

$$z(0,\lambda) = z'(0,\lambda) = 0. \tag{21}$$

In the next section, we will need representations for $z(\pi,\lambda;q^-)$ and $z'(\pi,\lambda;q^-)$.

As in the local case $a = 0$, the function $z(x,\lambda)$ is expected to have the form

$$z(x,\lambda) = \int_0^x G(x,t,\lambda) r(t)\, dt, \tag{22}$$

where $G(x,t,\lambda)$ is called Green's function. Let us obtain an explicit formula for it.

Lemma 2. *Let $a \in [0,\pi]$. Then,*

$$G(x,t,\lambda) = y_t(x-t), \quad t \le x \le \pi, \tag{23}$$

where the function $y_t(x)$ for each fixed $t \in [0,\pi)$ solves the Cauchy problem

$$-y_t''(x) + q_t(x) y_t(x-a) = \lambda y_t(x), \quad 0 < x < \pi - t, \quad y_t(0) = 0, \quad y_t'(0) = 1, \tag{24}$$

with

$$q_t(x) := \begin{cases} 0, & 0 < x < \min\{a, \pi - t\}, \\ q^+(x+t), & a < x < \pi - t. \end{cases} \tag{25}$$

Proof. Since the function $G(x,t,\lambda)$ is uniquely determined by the representation (22), one has the right to impose any restrictions on it that will finally lead to (22). In particular, it is natural to assume that $G(x,t,\lambda)$ is sufficiently smooth and obeys the conditions

$$G(x,x,\lambda) = 0, \quad G'(x,x,\lambda) = 1. \tag{26}$$

Then, substituting (22) into (2) and taking the arbitrariness of $r(x)$ into account, we obtain the relations

$$-G''(x,t,\lambda) = \lambda G(x,t,\lambda), \quad 0 < t < x < a,$$

$$-G''(x,t,\lambda) + q^+(x) G(x-a,t,\lambda) = \lambda G(x,t,\lambda), \quad 0 < t < x - a < \pi - a,$$

$$-G''(x,t,\lambda) = \lambda G(x,t,\lambda), \quad 0 < x - a < t < x < \pi,$$

which, in turn, along with (26) guarantee that (22) is a solution of the problem (2) and (21).

Substituting $x + t$ into the above three relations instead of x, we obtain

$$-G''(x+t,t,\lambda) = \lambda G(x+t,t,\lambda), \quad 0 < x < a - t < a, \tag{27}$$

$$-G''(x+t,t,\lambda) + q^+(x+t) G(x+t-a,t,\lambda) = \lambda G(x+t,t,\lambda), \quad a < x < \pi - t < \pi, \tag{28}$$

$$-G''(x+t,t,\lambda) = \lambda G(x+t,t,\lambda), \quad \max\{0, a-t\} < x < \min\{a, \pi - t\}. \tag{29}$$

Denote $y_t(x) := G(x+t,t,\lambda)$. Then, combining (27) and (29), one can rewrite:

$$-y_t''(x) = \lambda y_t(x), \quad 0 < x < \min\{a, \pi - t\},$$

while (28) takes the form

$$-y_t''(x) + q^+(x+t)y_t(x-a) = \lambda y_t(x), \quad a < x < \pi - t < \pi.$$

Using the designation (25) along with initial conditions (26), we arrive at (24).

Finally, note that after solving the Cauchy problem (24) by the standard approach (see, e.g., [14]), it is easy to see that $G(x,t,\lambda)$ is a continuous function with respect to all arguments. Hence, the integral in (22) exists and gives a solution to the Cauchy problem (2) and (21). □

Lemma 3. *Let $a \geq \pi/2$. Then, the following representations hold:*

$$G(x,t,\lambda) = \frac{\sin\rho(x-t)}{\rho}, \quad \max\{0, x-a\} \leq t \leq x \leq \pi, \tag{30}$$

and

$$G(x,t,\lambda) = \frac{\sin\rho(x-t)}{\rho} + \frac{1}{2}\int_{a+t}^{x} \frac{\sin\rho(x-\tau)}{\rho}\,d\tau \int_{\frac{a+t+\tau}{2}}^{x+\frac{a+t-\tau}{2}} q^+(\eta)\,d\eta \tag{31}$$

whenever $0 \leq t \leq x - a \leq \pi - a$.

Proof. By virtue of (24) and Lemma 1, we have the representation

$$y_t(x) = \frac{\sin\rho x}{\rho} + \frac{1}{2}\int_a^x \frac{\sin\rho(x-\tau)}{\rho}\,d\tau \int_{\frac{a+\tau}{2}}^{x+\frac{a-\tau}{2}} q_t(\eta)\,d\eta, \quad 0 \leq x \leq \pi - t,$$

which, in accordance with (23) and (25), leads to (30) and (31). □

By substituting (30) and (31) into (22) and changing the order of integration, we obtain

$$z(x,\lambda) = \int_0^x \Big(r(t) + \frac{1}{2}\int_0^{t-a} r(\tau)\,d\tau \int_{\frac{a+t+\tau}{2}}^{x+\frac{a+\tau-t}{2}} q^+(\eta)\,d\eta \Big) \frac{\sin\rho(x-t)}{\rho}\,dt, \quad 0 \leq x \leq \pi, \tag{32}$$

where $r(x) = 0$ for $x < 0$.

Further, differentiating (30) and (31) with respect to x, we arrive at the formulae

$$G'(x,t,\lambda) = \cos\rho(x-t), \quad \max\{0, x-a\} \leq t \leq x \leq \pi,$$

and

$$G'(x,t,\lambda) = \cos\rho(x-t) + \frac{1}{2}\int_{a+t}^{x} \Big(\int_{\frac{a+t+\tau}{2}}^{x} q^+(\eta)\,d\eta + \int_{x+\frac{a+t-\tau}{2}}^{x} q^+(\eta)\,d\eta \Big) \cos\rho(x-\tau)\,d\tau$$

as soon as $0 \leq t \leq x - a \leq \pi - a$. Substituting them into

$$z'(x,\lambda) = \int_0^x G'(x,t,\lambda) r(t)\,dt,$$

we analogously obtain the representation

$$z'(x,\lambda) = \int_0^x \Big(r(t) + \frac{1}{2}\int_0^{t-a} \Big(\int_{\frac{a+t+\tau}{2}}^{x} q^+(\eta)\,d\eta + \int_{x+\frac{a+\tau-t}{2}}^{x} q^+(\eta)\,d\eta \Big) r(\tau)\,d\tau \Big) \cos\rho(x-t)\,dt. \tag{33}$$

4. Characteristic Functions

Consider the entire functions

$$\Delta_j(\lambda) := C^{(j)}(\pi,\lambda) + z^{(j)}(\pi,\lambda;q^-), \quad j = 0,1. \tag{34}$$

The next lemma holds for any $a \in [0,\pi]$.

Lemma 4. *For $j = 0,1$, eigenvalues of the problem $B_j(q)$ coincide with zeros of $\Delta_j(\lambda)$.*

Proof. Since the sum $C(x,\lambda) + z(x,\lambda;q^-)$ cannot be identically zero, any zero of $\Delta_j(\lambda)$ is an eigenvalue of the problem $B_j(q)$, which, in turn, under our settings has the form

$$-y''(x) + q^+(x)y(x-a) + q^-(x)y(0) = \lambda y(x), \quad y'(0) = y^{(j)}(\pi) = 0. \tag{35}$$

Conversely, let λ be an eigenvalue of $B_j(q)$, and let $y(x)$ be the corresponding eigenfunction, i.e., a nontrivial solution of (35). Then, $y(0) \neq 0$ since, obviously, $y(x) \equiv 0$ otherwise. Without loss of generality, one can assume that $y(0) = 1$, which will imply $y(x) = C(x,\lambda) + z(x,\lambda;q^-)$ due to the uniqueness of solution of the Cauchy problem. Hence, $\Delta_j(\lambda) = y^{(j)}(\pi) = 0$. □

As usual, we call $\Delta_j(\lambda)$ the characteristic function of the problem $B_j(q)$. The following lemma based on the two preceding sections gives representations for both characteristic functions.

Lemma 5. *The characteristic functions admit the representations*

$$\Delta_0(\lambda) = \cos\rho\pi + \omega\frac{\sin\rho(\pi-a)}{\rho} + \int_0^\pi w_0(x)\frac{\sin\rho x}{\rho}\,dx, \quad w_0(x) \in L_2(0,\pi), \tag{36}$$

$$\Delta_1(\lambda) = -\rho\sin\rho\pi + \omega\cos\rho(\pi-a) + \int_0^\pi w_1(x)\cos\rho x\,dx, \quad w_1(x) \in L_2(0,\pi). \tag{37}$$

Moreover, the constant ω is determined by (8), and

$$w_0(\pi-x) = w_1(\pi-x) = q^-(x), \quad 0 < x < a, \tag{38}$$

while for $a < x < \pi$:

$$w_0(\pi-x) = \frac{1}{4}\Big(q^+\Big(\frac{a+x}{2}\Big) - q^+\Big(\pi + \frac{a-x}{2}\Big)\Big) + \frac{1}{2}\int_0^{x-a} q^-(t)\,dt\int_{\frac{a+x+t}{2}}^{\pi+\frac{a+t-x}{2}} q^+(\tau)\,d\tau, \tag{39}$$

$$w_1(\pi-x) = \frac{1}{4}\Big(q^+\Big(\frac{a+x}{2}\Big) + q^+\Big(\pi + \frac{a-x}{2}\Big)\Big)$$
$$+\frac{1}{2}\int_0^{x-a}\Big(\int_{\frac{a+x+t}{2}}^{\pi} q^+(\tau)\,d\tau + \int_{\pi+\frac{a+t-x}{2}}^{\pi} q^+(\tau)\,d\tau\Big)q^-(t)\,dt. \tag{40}$$

Proof. Substituting $x = \pi$ into (18) and (19) and using (8) and (20), we obtain

$$C(\pi,\lambda) = \cos\rho\pi + \omega\frac{\sin\rho(\pi-a)}{\rho} + \int_0^{\pi-a} u_0(x)\frac{\sin\rho x}{\rho}\,dx, \tag{41}$$

$$C'(\pi,\lambda) = -\rho\sin\rho\pi + \omega\cos\rho(\pi-a) + \int_0^{\pi-a} u_1(x)\cos\rho x\,dx, \tag{42}$$

where

$$u_j(\pi-x) = K_j(\pi,x) = \frac{1}{4}\Big(q^+\Big(\frac{a+x}{2}\Big) - (-1)^j q^+\Big(\pi + \frac{a-x}{2}\Big)\Big), \ a < x < \pi, \ j = 0,1. \tag{43}$$

Further, substituting $r = q^-$ and $x = \pi$ into (32) and (33), we arrive at

$$z(\pi, \lambda; q^-) = \int_0^\pi v_0(x) \frac{\sin \rho x}{\rho} \, dx, \quad z'(\pi, \lambda; q^-) = \int_0^\pi v_1(x) \cos \rho x \, dx, \tag{44}$$

where

$$v_0(\pi - x) = v_1(\pi - x) = q^-(x), \quad 0 < x < a, \tag{45}$$

$$v_0(\pi - x) = \frac{1}{2} \int_0^{x-a} q^-(t) \, dt \int_{\frac{a+x+t}{2}}^{\pi + \frac{a+t-x}{2}} q^+(\tau) \, d\tau, \quad a < x < \pi, \tag{46}$$

$$v_1(\pi - x) = \frac{1}{2} \int_0^{x-a} \Big(\int_{\frac{a+x+t}{2}}^{\pi} q^+(\tau) \, d\tau + \int_{\pi + \frac{a+t-x}{2}}^{\pi} q^+(\tau) \, d\tau \Big) q^-(t) \, dt, \quad a < x < \pi. \tag{47}$$

According to (34), (41), (42), and (44), we obtain (36) and (37) with

$$w_j(x) = u_j(x) + v_j(x), \quad j = 0, 1, \tag{48}$$

where $u_0(x) = u_1(x) = 0$ on $(\pi - a, \pi)$. Finally, substituting (43) and (45)–(47) into (48), we arrive at (38)–(40). □

In the rest of this section, we provide auxiliary facts about arbitrary functions of the form (36) and (37) and give the proof of Theorem 1.

Lemmas 6–8 below are valid for any fixed $a \in [0, 2\pi]$. By the standard approach (see, e.g., [19,36]) involving Rouché's theorem, one can prove the following assertion.

Lemma 6. *For $j = 0, 1$, any $\Delta_j(\lambda)$ has infinitely many zeros $\{\lambda_{n,j}\}_{n \geq 0}$ of the form (7).*

The next assertion for $a = 0$ can be found in [19], but the proof does not depend on the value of a as soon as it ranges within $[0, 2\pi]$.

Lemma 7. *Any functions of the forms (36) and (37) are determined by their zeros uniquely. Moreover, the representations in (11) hold.*

Now, we are in position to give the proof of Theorem 1.

Proof of Theorem 1. The asymptotics (7) is a direct corollary of Lemmas 5 and 6. It remains to make note that, by virtue of (10), (36), and (37) along with Lemma 7, we have

$$i\theta_0(\rho) - \theta_1(\rho) = i \int_0^\pi w_0(x) \sin \rho x \, dx - \int_0^\pi w_1(x) \cos \rho x \, dx = \frac{\theta_+(\rho) - \theta_-(\rho)}{2},$$

where, according to (38),

$$\theta_+(\rho) = \int_0^{\pi - a} (w_0 - w_1)(x) \exp(i\rho x) \, dx, \quad \theta_-(\rho) = \int_0^\pi (w_0 + w_1)(x) \exp(-i\rho x) \, dx,$$

which implies (9). □

Statements analogous to the next lemma are often used for finding necessary and sufficient conditions for the solvability of inverse problems, i.e., a characterization of the spectral data (see Remark 2 in [36]). For its proof, we will follow a new simple idea suggested in [36].

Lemma 8. *For $j = 0, 1$, let $\{\lambda_{n,j}\}_{n \geq 0}$ be arbitrary complex sequences of the form (7). Then, the function $\Delta_j(\lambda)$ constructed by the corresponding formula in (11) has the form (36) or (37), respectively.*

Proof. Since the assertion of the lemma for $j = 0$ formally follows from Lemma 6 in [15], we focus on the case $j = 1$. Let a sequence $\{\lambda_{n,1}\}_{n\geq 0}$ of the form (7) be given. First, let all values $\lambda_{n,1}$ be distinct and $\lambda_{0,1} = 0$. Denote $\rho_{-n,1} := -\rho_{n,1}$ for $n \geq 1$. By virtue of Lemma 2 in [36], the system $\{\exp(i\rho_{n,1}x)\}_{n\in\mathbb{Z}}$ is a Riesz basis in $L_2(-\pi, \pi)$. Moreover, the asymptotics (7) implies $\{\theta(\rho_{n,1})\}_{n\in\mathbb{Z}} \in l_2$, where $\theta(\rho) := \rho \sin \rho\pi - \omega \cos \rho(\pi - a)$ and ω is as in (7). Hence, there exists a unique function $W_1(x) \in L_2(-\pi, \pi)$ obeying the relations

$$\theta(\rho_{n,1}) = \int_{-\pi}^{\pi} W_1(x) \exp(i\rho_{n,1}x)\, dx, \quad n \in \mathbb{Z}.$$

Obviously, $W_1(x)$ is even. Thus, $\lambda_{n,1} = (\rho_{n,1})^2$, $n \geq 0$, are zeros of the function $\Delta_1(\lambda)$ determined by (37) with $w_1(x) = 2W_1(x)$. By Lemma 6, $\Delta_1(\lambda)$ has no other zeros, while by Lemma 7, it admits the second representation in (11), which finishes the proof for a simple sequence $\{\lambda_{n,1}\}_{n\geq 0}$ containing a zero element.

For the general case, it is sufficient to note that multiplying $\Delta_1(\lambda)$ with any function

$$h(\lambda) := \prod_{n\in A} \frac{\lambda - \tilde{\lambda}_{n,1}}{\lambda - \lambda_{n,1}}, \quad A \subset \mathbb{N} \cup \{0\}, \quad \#A < \infty,$$

preserves the form (37) and changes only $w_1(x)$. Indeed, we have

$$h(\lambda)\Delta_1(\lambda) = -\rho \sin \rho\pi + \omega \cos \rho(\pi - a) + H(\lambda),$$

where

$$H(\lambda) = (1 - h(\lambda))\big(\rho \sin \rho\pi - \omega \cos \rho(\pi - a)\big) + h(\lambda) \int_0^{\pi} w_1(x) \cos \rho x\, dx.$$

The function $H(\lambda)$ is whole as soon as $\lambda_{n,1}$ are zeros of $\Delta_1(\lambda)$. Moreover, in the ρ-plane, we, obviously, have $H(\rho^2) \in L_2(-\infty, +\infty)$ and $H(\rho^2) = o(\exp(|\mathrm{Im}\, \rho|\pi))$ as $\rho \to \infty$. Thus, by virtue of the Paley–Wiener theorem (see, e.g., [37]), it has the form

$$H(\lambda) = \int_0^{\pi} \tilde{w}_1(x) \cos \rho x\, dx, \quad \tilde{w}_1(x) \in L_2(0, \pi),$$

which finishes the proof completely. □

Finally, let us give one more auxiliary assertion, which will be used in the proof of Theorem 2. Let $\{n_k\}_{k\in\mathbb{N}}$ be an increasing sequence of non-negative integers. Without loss of generality, assume that multiple elements in the subspectrum $\{\lambda_{n_k,0}\}_{k\in\mathbb{N}}$ are neighboring, i.e.,

$$\lambda_{n_k,0} = \lambda_{n_{k+1},0} = \ldots = \lambda_{n_{k+m_k-1},0},$$

where m_k is the multiplicity of the value $\lambda_{n_k,0}$ in this subspectrum. Put

$$\mathcal{S} := \{1\} \cup \{k : \lambda_{n_k,0} \neq \lambda_{n_{k-1},0},\ k \geq 2\}$$

and consider the functional system $\sigma := \{s_n(x)\}_{n\in\mathbb{N}}$, where

$$s_{k+\nu}(x) := \Big(n_k + \frac{1}{2}\Big) \frac{d^\nu}{d\lambda^\nu} \frac{\sin \rho x}{\rho}\Big|_{\lambda=\lambda_{n_k,0}}, \quad k \in \mathcal{S}, \quad \nu = \overline{0, m_k - 1}.$$

Lemma 9. *The system σ is a Riesz basis in $\mathcal{H}_b := L_2(0, b)$ if and only if so is the system $\sigma_0 = \{\sin(n_k + 1/2)x\}_{k\in\mathbb{N}}$. Moreover, they have equal defects, i.e.,* $\dim(\mathcal{H}_b \ominus \sigma_0) = \dim(\mathcal{H}_b \ominus \sigma)$.

Proof. Let there exist d linearly independent entire functions $h_\nu(\lambda)$, $\nu = \overline{1, d}$, of the form

$$h_\nu(\lambda) = \int_0^b f_\nu(x) \frac{\sin \rho x}{\rho}\, dx, \quad f_\nu(x) \in L_2(0, b), \tag{49}$$

whose zeros have the common part $\{(n_k + 1/2)^2\}_{k\in\mathbb{N}}$. In other words, the space $\mathcal{H}_b \ominus \sigma_0$ contains at least d linearly independent functions $f_\nu(x)$. Consider the meromorphic function

$$F(\lambda) := \prod_{k=1}^{\infty} \frac{\lambda_{n_k,0} - \lambda}{(n_k + 1/2)^2 - \lambda}.$$

Then, the entire (after removing singularities) function $\tilde{h}_\nu(\lambda) := F(\lambda)h_\nu(\lambda)$ has the form

$$\tilde{h}_\nu(\lambda) = \int_0^b \tilde{f}_\nu(x)\frac{\sin\rho x}{\rho}\,dx, \quad \tilde{f}_\nu(x) \in L_2(0,b). \tag{50}$$

Indeed, as in the proof of Lemma 2 in [36], one can show that $|F(\rho^2)| < C_\delta$ whenever

$$|\rho \pm (n_k + 1/2)| \geq \delta, \quad k \in \mathbb{N},$$

for each fixed $\delta > 0$. Hence, we have $|\rho\tilde{h}_\nu(\rho^2)| \leq C_\delta|\rho h_\nu(\rho^2)|$ for such ρ. Thus, according to (49), the function $\rho\tilde{h}_\nu(\rho^2)$ is square-integrable on the line $\rho = i\delta$, while the maximum modulus principle for analytic functions gives $\rho\tilde{h}_\nu(\rho^2) = o(\exp(|\mathrm{Im}\,\rho|b))$ as $\rho \to \infty$ in the entire plane. Using the Paley–Wiener theorem [37] and taking the oddness of $\rho\tilde{h}_\nu(\rho^2)$ into account, we obtain (50). Obviously, the functions $\tilde{h}_\nu(\lambda)$, $\nu = \overline{1,d}$, are linearly independent, and their zeros have the common part $\{\lambda_{n_k,0}\}_{k\in\mathbb{N}}$ with account of multiplicity. Therefore, $\dim(\mathcal{H} \ominus \sigma_0) \leq \dim(\mathcal{H} \ominus \sigma)$. Analogously, one can prove the inequality $\dim(\mathcal{H} \ominus \sigma_0) \geq \dim(\mathcal{H} \ominus \sigma)$.

We have proved the second assertion of the lemma, which means, in particular, that the systems σ and σ_0 can be complete in $\mathcal{H}_b$ only simultaneously. Hence, by virtue of Proposition 1.8.5 in [19], the simultaneous Riesz-basisness follows from their quadratical closeness

$$\sum_{k=1}^{\infty} \|s_k - s_k^0\|^2_{L_2(0,b)} < \infty, \quad s_k^0(x) := \sin\gamma_k x, \quad \gamma_k := n_k + \frac{1}{2}.$$

The last inequality, in turn, is ensured by the estimate

$$s_k(x) - s_k^0(x) = \sin\rho_{n_k,0}x - \sin\gamma_k x + O\Big(\frac{1}{k^2}\Big)$$
$$= 2\cos\frac{(\rho_{n_k,0} + \gamma_k)x}{2}\sin\frac{(\rho_{n_k,0} - \gamma_k)x}{2} + O\Big(\frac{1}{k^2}\Big) = O\Big(\frac{1}{k}\Big), \quad k \to \infty,$$

which holds uniformly in $x \in [0,b]$. □

5. Solution of the Inverse Problems

When the functions $w_0(x)$ and $w_1(x)$ are specified, relations (38)–(40) can be considered as a nonlinear integral equation with respect to $q(x) = q^-(x) + q^+(x)$. The following lemma actually implies its unique solvability.

Lemma 10. *For any functions $w_0(x), w_1(x), q^-(x) \in L_2(0, \pi - a)$, the linear system consisting of (39) and (40) has a unique solution $q^+(x) \in L_2(a, \pi)$.*

Proof. Summing up equations (39) and (40) and then subtracting one from the other, we obtain

$$\left.\begin{aligned} 2(w_1 + w_0)(\pi - x) &= q^+\Big(\frac{a+x}{2}\Big) + 2\int_0^{x-a} q^-(t)\,dt\int_{\frac{a+x+t}{2}}^{\pi} q^+(\tau)\,d\tau, \\ 2(w_1 - w_0)(\pi - x) &= q^+\Big(\pi + \frac{a-x}{2}\Big) + 2\int_0^{x-a} q^-(t)\,dt\int_{\pi+\frac{a+t-x}{2}}^{\pi} q^+(\tau)\,d\tau, \end{aligned}\right\} \quad a < x < \pi.$$

Changing the variable, we arrive at the relations

$$2(w_1 + w_0)(\pi + a - 2x) = q^+(x) + 2\int_0^{2(x-a)} q^-(t)\,dt\int_{x+\frac{t}{2}}^{\pi} q^+(\tau)\,d\tau, \quad a < x < \frac{a+\pi}{2},$$

$$2(w_1 - w_0)(2x - \pi - a) = q^+(x) + 2\int_0^{2(\pi - x)} q^-(t)\,dt \int_{x+\frac{t}{2}}^{\pi} q^+(\tau)\,d\tau, \quad \frac{a+\pi}{2} < x < \pi.$$

Then, changing the order of integration in the last two formulae, we obtain the system

$$2(w_1 + w_0)(\pi + a - 2x) = q^+(x) + 2\int_x^{2x-a} q^+(t)\,dt \int_0^{2(t-x)} q^-(\tau)\,d\tau$$
$$+2\int_{2x-a}^{\pi} q^+(t)\,dt \int_0^{2(x-a)} q^-(\tau)\,d\tau, \quad a < x < \frac{a+\pi}{2},$$
$$2(w_1 - w_0)(2x - \pi - a) = q^+(x) + 2\int_x^{\pi} q^+(t)\,dt \int_0^{2(t-x)} q^-(\tau)\,d\tau, \quad \frac{a+\pi}{2} < x < \pi.$$

Using the designations

$$W(x) := \begin{cases} 2(w_1 + w_0)(\pi + a - 2x), & a < x < \dfrac{a+\pi}{2}, \\ 2(w_1 - w_0)(2x - \pi - a), & \dfrac{a+\pi}{2} < x < \pi, \end{cases} \tag{51}$$

$$Q(x,t) := \begin{cases} 2\displaystyle\int_0^{2(x-a)} q^-(\tau)\,d\tau, & a < 2x - a < t < \pi, \\ 2\displaystyle\int_0^{2(t-x)} q^-(\tau)\,d\tau, & a < x < t < \min\{2x - a, \pi\}, \end{cases} \tag{52}$$

one can rewrite the latter system as a Volterra integral equation of the second kind:

$$W(x) = q^+(x) + \int_x^{\pi} Q(x,t) q^+(t)\,dt, \quad a < x < \pi, \tag{53}$$

which possesses a unique solution $q^+(x) \in L_2(a,\pi)$ (see, e.g., [38]). □

Proof of Theorem 2. First of all, note that due to (7), the value ω is always determined by specifying $\{\lambda_{n,1}\}_{n\geq 0}$ via the formula

$$\omega = \pi \lim_{k\to\infty} \tilde{n}_k \frac{\rho_{\tilde{n}_k,1} - \tilde{n}_k}{\cos \tilde{n}_k a}, \tag{54}$$

where the natural sequence $\{\tilde{n}_k\}$ is chosen so that $|\cos \tilde{n}_k a| \geq c > 0$. Alternatively, in accordance with (37), one can use the relation

$$\omega = \lim_{n\to\infty} \Big(\Delta_1(\xi_n^2) + \xi_n \sin \xi_n \pi\Big), \quad \xi_n = \frac{2\pi n}{\pi - a}, \tag{55}$$

where $\Delta_1(\lambda)$ is constructed by the second formula in (11).

(i) Let the system σ_0 be complete in $\mathcal{H}$. Since, according to Lemma 7, the characteristic function $\Delta_1(\lambda)$ is uniquely determined by its zeros, so is also $w_1(x)$ in (37). By virtue of (38), the function $w_0(x)$ coincides with $w_1(x)$ a.e. on $(\pi - a, \pi)$, i.e., it becomes known too.

By differentiating (36) $\nu = \overline{0, m_k - 1}$ times and substituting $\lambda = \lambda_{n_k,0}$ for $k \in \mathcal{S}$, we arrive at the relations

$$\beta_n = \int_0^{\pi - a} w_0(x) s_n(x)\,dx, \quad n \in \mathbb{N}, \tag{56}$$

where m_k, $\mathcal{S}$ and $s_n(x)$ were defined before Lemma 9 and

$$\beta_{k+\nu} = -(n_k + 1)\,\frac{d^\nu}{d\lambda^\nu}\Big(\cos\rho\pi + \omega\frac{\sin\rho(\pi - a)}{\rho} + \gamma(\lambda)\Big)\Big|_{\lambda=\lambda_{n_k,0}}, \quad k \in \mathcal{S}, \quad \nu = \overline{0, m_k - 1}, \tag{57}$$

$$\gamma(\lambda) = \int_{\pi - a}^{\pi} w_1(x)\frac{\sin\rho x}{\rho}\,dx. \tag{58}$$

Hence, by virtue of Lemma 9, the function $w_0(x)$ is determined uniquely also on $(0, \pi - a)$. Thus, it remains to recall representations (3) and (38), as well as to apply Lemma 10.

(ii) Assume that $q(x)$ is uniquely determined by $\{\lambda_{n_k,0}\}_{k\in\mathbb{N}}$ and $\{\lambda_{n,1}\}_{n\geq 0}$ and, to the contrary, that $\dim(\mathcal{H}\ominus\sigma_0) > 1$. Then, according to Lemma 9, we have $\dim(\mathcal{H}\ominus\sigma) > 1$, i.e., there exist at least two linearly independent functions $f_1(x), f_2(x) \in L_2(0, \pi - a)$ such that

$$\int_0^{\pi-a} f_\nu(x) s_n(x)\, dx = 0, \quad n \in \mathbb{N}, \quad \nu = 1, 2. \tag{59}$$

Let $\tilde{q}^+(x)$ be a solution of the integral equation

$$W(x) + \alpha_1 F_1(x) + \alpha_2 F_2(x) = \tilde{q}^+(x) + \int_x^\pi Q(x,t)\tilde{q}^+(t)\, dt, \quad a < x < \pi, \tag{60}$$

where $W(x)$ is defined in (51), while

$$F_\nu(x) := \begin{cases} 2f_\nu(\pi + a - 2x), & a < x < \dfrac{a+\pi}{2}, \\ -2f_\nu(2x - \pi - a), & \dfrac{a+\pi}{2} < x < \pi, \end{cases} \quad \nu = 1, 2. \tag{61}$$

According to (53), we have $\tilde{q}^+(x) = q^+(x) + \alpha_1 g_1(x) + \alpha_2 g_2(x)$, where

$$g_\nu(x) = F_\nu(x) + \int_x^\pi Q_1(x,t) F_\nu(t)\, dt, \quad \nu = 1, 2,$$

while $Q_1(x,t)$ is the resolvent kernel for the kernel $Q(x,t)$. Choose α_1 and α_2 so that they do not vanish simultaneously and

$$\frac{1}{2}\int_a^\pi \tilde{q}^+(x)\, dx = \omega. \tag{62}$$

Since the functions $F_1(x)$ and $F_2(x)$ are linearly independent, so are $g_1(x)$ and $g_2(x)$. Hence, $\tilde{q}^+ \neq q^+$. Continue $\tilde{q}^+(x)$ to $(0, a)$ as zero and consider the function $\tilde{q}(x) = q^-(x) + \tilde{q}^+(x)$. By virtue of (62) and Lemma 5, the characteristic functions $\tilde{\Delta}_0(\lambda)$ and $\tilde{\Delta}_1(\lambda)$ of the problems $B_0(\tilde{q})$ and $B_1(\tilde{q})$, respectively, have the forms

$$\tilde{\Delta}_0(\lambda) = \cos\rho\pi + \omega\frac{\sin\rho(\pi - a)}{\rho} + \int_0^\pi \tilde{w}_0(x)\frac{\sin\rho x}{\rho}\, dx, \quad \tilde{w}_0(x) \in L_2(0, \pi),$$

$$\tilde{\Delta}_1(\lambda) = -\rho\sin\rho\pi + \omega\cos\rho(\pi - a) + \int_0^\pi \tilde{w}_1(x)\cos\rho x\, dx, \quad \tilde{w}_1(x) \in L_2(0, \pi),$$

and $\tilde{w}_j(x) = w_j(x)$ a.e. on $(\pi - a, \pi)$ for $j = 0, 1$. Moreover, analogously to (53), we have

$$\tilde{q}^+(x) + \int_x^\pi Q(x,t)\tilde{q}^+(t)\, dt = \begin{cases} 2(\tilde{w}_1 + \tilde{w}_0)(\pi + a - 2x), & a < x < \dfrac{a+\pi}{2}, \\ 2(\tilde{w}_1 - \tilde{w}_0)(2x - \pi - a), & \dfrac{a+\pi}{2} < x < \pi. \end{cases}$$

Comparing this with (51), (60), and (61), we obtain $\tilde{w}_1(x) = w_1(x)$ a.e. on $(0, \pi)$ and

$$\tilde{w}_0(x) = w_0(x) + \alpha_1 f_1(x) + \alpha_2 f_2(x) \ \text{ a.e. on } \ (0, \pi - a). \tag{63}$$

Hence, the spectra of $B_1(q)$ and $B_1(\tilde{q})$ coincide. Moreover, according to (56)–(59) and (63), the sequence $\{\lambda_{n_k,0}\}_{k\in\mathbb{N}}$ is a subsequence of zeros of $\tilde{\Delta}_0(\lambda)$. Hence, $\{\lambda_{n_k,0}\}_{k\in\mathbb{N}}$ is a subspectrum also of the problem $B_0(\tilde{q})$. Thus, we obtained another potential $\tilde{q} \neq q$ with the same spectral data $\{\lambda_{n_k,0}\}_{k\in\mathbb{N}}$ and $\{\lambda_{n,1}\}_{n\geq 0}$ as q has. This contradiction finishes the proof. □

Now, we are in a position to give a constructive procedure for solving Inverse Problem 1 (Algorithm 1).

Algorithm 1 Constructive procedure for solving Inverse Problem 1

Let the spectra $\{\lambda_{n,0}\}_{n\geq 0}$ and $\{\lambda_{n,1}\}_{n\geq 0}$ be given. Then:
(i) Construct the functions $\Delta_0(\lambda)$ and $\Delta_1(\lambda)$ by the formulae in (11);
(ii) Find the value ω by (54) or (55);
(iii) Calculate the functions $w_0(x)$ and $w_1(x)$ in (36) and (37) by inverting the corresponding Fourier transforms:

$$w_0(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} a_n \sin nx, \quad w_1(x) = \frac{2}{\pi}\sum_{n=0}^{\infty} b_n \cos nx,$$

where

$$a_n = n(\Delta_0(n^2) - (-1)^n) + \omega(-1)^n \sin na, \ \ n \geq 1, \quad b_n = \Delta_1(n^2) - \omega(-1)^n \cos na, \ \ n \geq 0;$$

(iv) Find $q^-(x) \in L_2(0,a)$ by any relation in (38) and put $q^-(x) = 0$ for $x \in (a,\pi)$;
(v) Construct the functions $W(x)$ and $Q(x,t)$ by the formulae (51) and (52), respectively, and find $q^+(x) \in L_2(a,\pi)$ by solving the Volterra integral Equation (53);
(vi) Finally, construct $q(x) = q^-(x) + q^+(x)$, where $q^+(x) = 0$ on $(0,a)$.

This algorithm can be easily extended to Inverse Problem 2 if $\{\sin(n_k + 1/2)x\}_{k\in\mathbb{N}}$ is a Riesz basis in $L_2(0,\pi - a)$. Then, by virtue of Lemma 9, so is the system $\{s_n(x)\}_{n\in\mathbb{N}}$. Therefore, on step (iii), the function $\omega_0(x)$ can be constructed in accordance with (56) by the formula

$$w_0(x) = \sum_{n=1}^{\infty} \beta_n s_n^*(x), \quad 0 < x < \pi - a,$$

where the coefficients β_n are determined by relations (57) and (58), while $\{s_n^*(x)\}_{n\in\mathbb{N}}$ is the biorthogonal basis to the basis $\{\overline{s_n(x)}\}_{n\in\mathbb{N}}$. It remains to note that, according to (38), the knowledge of $w_0(x)$ on $(\pi - a, \pi)$ is excessive since $w_1(x)$ has been found completely.

Proof of Theorem 3. Let us begin with the necessity part. According to (10), (36), and (37), we have

$$\theta_0(\rho) = \int_0^{\pi} w_0(x) \sin \rho x \, dx, \quad \theta_1(\rho) = \int_0^{\pi} w_1(x) \cos \rho x \, dx.$$

Hence, by virtue of (3) and (38), the exponential types of $\theta_0(\rho)$ and $\theta_1(\rho)$ do not exceed $\pi - a$. Finally, the relation (12) follows from Lemmas 5 and 7 after substituting $\lambda = 0$ into (37) and the second formula in (11). Indeed, according to (38) and (40), the assumption $q^- = 0$ implies

$$\int_0^{\pi} w_1(x)\, dx = \frac{1}{4}\int_a^{\pi} \left(q^+\left(\frac{a+x}{2}\right) + q^+\left(\pi + \frac{a-x}{2}\right)\right) dx = \frac{1}{2}\int_a^{\pi} q^+(x)\, dx = \omega. \quad (64)$$

For the sufficiency, we construct the functions $\Delta_0(\lambda)$ and $\Delta_1(\lambda)$ by the formulae in (11) using the given sequences $\{\lambda_{n,0}\}_{n\geq 0}$ and $\{\lambda_{n,1}\}_{n\geq 0}$. By virtue of Lemma 8, these functions have the forms (36) and (37), respectively, with some $w_0(x), w_1(x) \in L_2(0,\pi)$, which, in turn, vanish a.e. on $(\pi - a, \pi)$ by the first condition along with the Paley–Wiener theorem [37].

By virtue of Lemma 10, there exists a unique solution $q^+(x) \in L_2(a,\pi)$ of the system (39) and (40) with $q^-(x) = 0$. As in (64), we calculate

$$\tilde{\omega} := \int_0^{\pi - a} w_1(x)\, dx = \frac{1}{2}\int_a^{\pi} q^+(x)\, dx$$

and, hence,

$$\Delta_1(0) = \omega + \tilde{\omega}. \quad (65)$$

On the other hand, the second formula in (11) and condition (12) imply $\Delta_1(0) = 2\omega$, which, along with (65), gives $\tilde{\omega} = \omega$. Consider the problems $B_0(q)$ and $B_1(q)$ with the potential

$$q(x) = \begin{cases} 0, & x \in (0,a), \\ q^+(x), & x \in (a,\pi). \end{cases}$$

According to Lemma 5, $\Delta_0(\lambda)$ and $\Delta_1(\lambda)$ are their characteristic functions, respectively. Hence, $\{\lambda_{n,j}\}_{n\geq 0}$ is the spectrum of $B_j(q)$ for $j = 0, 1$. □

6. Conclusions and Discussing the Results

The paper thus connects two different directions in the inverse spectral theory, namely: for operators with constant delay [1–17] and for operators with a frozen argument [28–35], which have been developed independently before the present study. Such a fusion is naturally caused by replacing the standard assumption of the vanishing of the potential $q(x)$ on $(0, a)$ in equation (1) by imposing a continuously matching initial function (4). This leads to the appearance of a new term with a frozen argument at zero in Equation (5). Alternative forms of an initial function may give rise to considering also other equations with frozen argument

$$-y''(x) + q^+(x)y(x-a) + q^-(x)g(x-a)y^{(j)}(b) = \lambda y(x), \quad 0 < x < \pi, \tag{66}$$

where $g(x) \in L_\infty(0, a)$ and $b \in [0, \pi]$, while $j \in \{0, 1\}$, or more general equations

$$-y''(x) + q^+(x)y(x-a) + q^-(x)Ly(x) = \lambda y(x), \quad 0 < x < \pi, \tag{67}$$

with some known linear operator $L : W_2^2[0, \pi] \to L_\infty(-a, 0)$ under the reasonable assumption of the relative compactness with respect to the operator of double differentiation.

The usual restriction $q^-(x) = 0$ means that the two spectra must carry excessive information about the potential. For this reason, the reconstruction of $q(x)$ given only parts of the spectra was initiated in [6]. In particular, necessary and sufficient conditions for arbitrary subspectra guaranteeing the uniqueness of the potential were established. Later in [15], necessary and sufficient conditions for the solvability of the inverse problem from the complete spectra were obtained. Due to the overdetermination, these conditions besides the asymptotics also included some restrictions on the growth of certain entire functions constructed by the spectra.

Refusing the assumption $q^-(x) = 0$ would obviously lead to an increase of the required information for the unique recovery of $q(x)$. However, Theorem 2 shows that one of the spectra can still be specified partially. This effect is caused by the unique determination of the corresponding operator with the purely frozen argument, when $q^+(x) = 0$, from only one spectrum.

The proof of Theorem 2 gave Algorithm 1 for solving the inverse problem, which can be implemented numerically. We note that, in spite of the growing interest in recovering operators with constant delay, still no numerical results in this direction are known. For implementing Algorithm 1, one can adapt the numerical method suggested in [39] for integro-differential operators and involving approximations by entire functions of exponential type.

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