

Article

New Stability Results for Periodic Solutions of Generalized Van der Pol Oscillator via Second Bogolyubov's Theorem

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Abstract: A certain class of nonlinear differential equations representing a generalized Van der Pol oscillator is proposed in which we study the behavior of the existing solution. After using the appropriate variables, the first Levinson's change converts the equations into a system with two equations, and the second converts these systems into a Lipschitzian system. Our main result is obtained by applying the Second Bogolubov's Theorem. We established some integrals, which are used to compute the average function of this system and arrive at a new general condition for the existence of an asymptotically stable unique periodic solution. One of the well-known results regarding asymptotic stability appears, owing to the Second Bogolubov's Theorem, and the advantage of this method is that it can be applied not only in the periodic dynamical systems, but also in non-almost periodic dynamical systems.



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1. Introduction

The mathematical model for the system of the Van der Pol equation is a well-known second-order ordinary differential equation with cubic non-linearity. The Van der Pol oscillator is a classical example of a self-oscillatory system and is now considered as a very useful mathematical model that can be used in much more complicated and modified systems. Examples of systems in which the generalized Van der Pol oscillator have recently been proposed are included in [1,2]. In fact, from the point of view of practical implementation, these systems are simpler than the class of systems based on nonlinear differential equations. From a mathematical point of view, they are more complicated due to the presence of nonlinear means. Accurate mathematical analysis of the nature of attractors in such systems, including a rigorous substantiation of the non-linearity hypothesis, is a difficult problem that requires the development of new approaches. In [3], a system based on a non-autonomous Van der Pol oscillator with a delay is considered, which is alternately in the excitation and decay modes due to the periodic change in the parameter responsible for the bifurcation of the birth of a limit cycle. The excitation of oscillations at each new stage of activity is stimulated by the signal generated at the previous stage of activity, which enters through the delay line and undergoes a non-linear transformation using an

auxiliary external signal at the main operating frequency, so that at each successive stage of activity, the phase variable is multiplied by a factor of 2. As a result, for the phases of successive trains of oscillations, a stretching–mapping of a circle (Bernoulli mapping) with chaotic dynamics is realized. Another scheme was proposed in [4], the advantage of which is that it operates in the absence of an auxiliary signal. The system contains two delayed feedback circuits with different delay times. In these circuits, the signal undergoes a nonlinear transformation, respectively, into the second and third harmonics. When they are mixed, a signal appears on a quadratic nonlinear element, which is used to stimulate the excitation of the oscillator at the next stage of its activity. The disadvantage of this scheme is the need to use a high degree of nonlinear transformations, which can be difficult in practice. We now proceed to discuss on the Second Bogolyubov’s Theorem studied in [5–8] and also on the Van der Pol equations. For example, in [8], both are studied under the form

$$u'' + \epsilon(u^2 - 1)u' + (1 + \beta\epsilon)u = \epsilon\omega \sin(s), \quad (1)$$

$$u'' + \epsilon(|u| - 1)u' + (1 + \beta\epsilon)u = \epsilon\omega \sin(s), \quad (2)$$

where $\epsilon > 0$, $s \geq 0$ and β, ω are real constants. It is proven that the unique periodic solution is asymptotically stable by using the Second Bogolyubov’s Theorem. In addition, the authors showed that the amplitude of (1) is $|K| > \sqrt{2}$ and the amplitude of (2) is $|K| > \frac{\pi}{2}$.

In our paper, we proved some results by using the Second Bogolyubov’s Theorem to show the existence, uniqueness, and asymptotic stability for the periodic solution of the oscillatory Van Der Pol system

$$u'' + \epsilon(c_1 u^{2\nu} + c_2 u^{\nu^2} + cu + c_3)u^{2\xi+1} + (1 + \beta\epsilon)u = \epsilon\omega \sin(s), \quad (c_1, c_2) \neq (0, 0), \quad (3)$$

$$u'' + \epsilon(\alpha|u| + c_3)u^{2\xi+1} + (1 + \beta\epsilon)u = \epsilon\omega \sin(s), \quad (4)$$

where $\beta, \omega, c_1, c_2, c, \alpha, c_3 \in \mathbb{R}$, $\xi, \nu \in \mathbb{N}$ and $0 < \epsilon < 1$.

This article introduces and analyzes new stability results for periodic solutions of the generalized Van der Pol oscillator via the Second Bogolyubov’s Theorem. It consists of three sections. In the first one, we presented the general framework of our study and also introduced our main system. In the second section, we recalled most of the preliminary material and we presented the Second Bogolyubov’s Theorem, which will be the key of our proofs. Finally, in Section 3, we stated and proved our results regarding the existence, uniqueness, and asymptotic stability of nonlinear second-order differential equations in the general form. Our study, of course, concluded in Section 4 by outlining the difficulties we faced, as well as novelties and perspectives. In addition, we stated the history of the Van der Pol equation.

2. Basic Results and Operating Principles

Let

$$x' = \epsilon g(s, x, \epsilon), \quad \epsilon > 0, \quad (5)$$

where the function $g \in C^0(\mathbb{R} \times \mathbb{R}^n \times [0.1], \mathbb{R}^n)$ is T -periodic for t and locally Lipschitz with respect to x .

We prepare to use the Second Bogolyubov’s Theorem for (5) and give the next notation

$$g_0(v) = \int_0^T g(r, v, 0) dr. \quad (6)$$

In the case that g is of class C^1 , we note the periodic case of the Second Bogolyubov’s Theorem, which represents a part of the averaging principle: $\det(g_0)'(v_0) \neq 0$ and assures the existence and uniqueness, for $\epsilon > 0$ small, of a T -periodic solution of system (5) in a neighborhood of v_0 , while the fact that all the eigenvalues of the Jacobian matrix $(g_0)'(v_0)$ have a negative real part and also provides its asymptotic stability. We will search for the periodic solution in a neighborhood of $v_0 \in g_0^{-1}(0)$, with $g_0 \in C^1$. Here, we consider $v_0 =$

$(x_1, x_2) = (K \sin(\psi), K \cos(\psi))$, with $K \in \mathbb{R}$, $\psi \in [-\pi, \pi]$ and $g_0(v_0) = (g_{0_1}(v_0), g_{0_2}(v_0))$. The next theorem, Theorem 1, presents the periodic case of the Second Bogolyubov's Theorem, see [8], which is based on the Jacobian matrix of the average function in the vicinity of the value v_0 , where $\det(g_0)'(v_0) > 0$ and $\text{trace}(g_0)'(v_0) < 0$ confirm the existence and uniqueness of the T -periodic solution of the system (5); $\epsilon > 0$ is small enough, so it is asymptotically stable if $\det(g_0)'(v_0) < 0$. Then, the system (5) contains at least one non T -periodic solution that is asymptotically unstable.

Theorem 1 ([8]). Let $g \in C^0(\mathbb{R} \times \Omega \times [0.1], \mathbb{R}^2)$ with $\Omega \subset \mathbb{R}^2$. Let $v_0 \in \Omega$ be such that $g_0(v_0) = 0$ and g_0 is continuously differentiable in a neighborhood of v_0 .

1. If $\det(g_0)'(v_0) \neq 0$, then there exist $\epsilon_0 > 0$ such that for any $\epsilon \in [0, \epsilon_0]$, the system (5) has at least one T -periodic solution $x_\epsilon(0) \rightarrow v_0$ as $\epsilon \rightarrow 0$.
2. If (i) and (ii) are verified, with
 - (i) For some $L > 0$, we have that $\|g(s, v_1, \epsilon) - g(s, v_2, \epsilon)\| \leq L\|v_1 - v_2\|$ for any $s \in [0, T]$, $v_1, v_2 \in \Omega$, $\epsilon \in [0, 1]$.
 - (ii) Given any $\tilde{\gamma} > 0$, there exist $\tilde{\delta} > 0$ and $M \subset [0, T]$ measurable in the sense of Lebesgue with $\text{mes}(M) < \tilde{\gamma}$ such that for every $v \in B_{\tilde{\delta}}(v_0)$, $s \in [0, T] \setminus \{M\}$ and $\epsilon \in [0, \tilde{\delta}]$, we have that $g(s, \cdot, \epsilon)$ is differentiable at v and $\|g'_v(s, v, \epsilon) - g'_v(s, v, 0)\| \leq \tilde{\gamma}$.
 - (iii) If we suppose that

$$\det(g_0)'(v_0) > 0 \text{ and } (g_{0_1})'_{(1)}(v_0) + (g_{0_2})'_{(2)}(v_0) < 0. \quad (7)$$

Then there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$, the system (5) has exactly one T -periodic solution x_ϵ such that $x_\epsilon(0) \rightarrow v_0$ as $\epsilon \rightarrow 0$. Moreover the solution x_ϵ is asymptotically stable, where

$$\|g(s, v, \epsilon)\| = \left\| \begin{pmatrix} g_1(s, v, \epsilon) \\ g_2(s, v, \epsilon) \end{pmatrix} \right\| = \sqrt{g_1(s, v, \epsilon)^2 + g_2(s, v, \epsilon)^2}.$$

3. If $\det(g_0)'(v_0) < 0$. Then there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$, the system (5) has at least one non-asymptotically stable T -periodic solution $x_\epsilon(0) \rightarrow v_0$ as $\epsilon \rightarrow 0$.

Remark 1. We notice here that the first condition (1) of Theorem 1 ensures the existence of a periodic solution, while the second one (2) shows that the solution is unique. Condition (3) gives the asymptotic stability.

3. Main Results and Proofs

In this section, we prove the main results in the present paper, that both Equations (3) and (4) have a unique periodic and asymptotically stable solution. We present these main results in Theorems 2 and 3.

Theorem 2. Let β, c_1, c_2, c_3, c and $\omega \in \mathbb{R}$, $\xi, \nu \in \mathbb{N}$ and $v_0 \in \mathbb{R}^2$, with $(c_1, c_2) \neq (0, 0)$ and $0 < \epsilon < 1$. If we have the following statements

1. The first claim

$$\begin{aligned} & K^2(\beta^2 + (-c_1 \frac{(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1}{(2\xi+2\nu+2)(2\xi+2\nu) \times \dots \times (2\xi+4)} \\ & \times \frac{(2\xi+1)(2\xi-1) \times \dots \times 3 \times 1}{(2\xi+2) \times \dots \times 4} \\ & + c_2 \frac{(2\xi+2\nu+1)(2\xi+2\nu-1) \times \dots \times 3 \times 1}{(2\xi+2\nu+2)(2\xi+2\nu) \times \dots \times 4}) K^{2\xi+2\nu} \\ & - 2c_3 \frac{(2\xi+1)(2\xi-1) \times \dots \times 3 \times 1}{(2\xi+2)(2\xi) \times \dots \times 4 \times 2} K^{2\xi})^2 = \omega^2. \end{aligned}$$

2. The second claim

$$\begin{aligned}
& -2(c_1 \frac{(2\nu-1)(2\nu-3) \times \dots \times 5 \times 3 \times 1}{(2\xi+2\nu)(2\xi+2\nu-2) \times \dots \times (2\xi+4)} \\
& \times \frac{(2\xi+1)(2\xi-1) \times \dots \times 3 \times 1}{(2\xi+2) \times \dots \times 4 \times 2} \\
& + c_2 \frac{(2\xi+2\nu+1)(2\xi+2\nu-1) \times \dots \times 3 \times 1}{(2\xi+2\nu)(2\xi+2\nu-12) \times \dots \times 4 \times 2}) K^{2\xi+2\nu} \pi \\
& - 2c_3(2\xi+2) \frac{(2\xi+1)(2\xi-1) \times \dots \times 3 \times 1}{(2\xi+2)(2\xi) \times \dots \times 4 \times 2} K^{2\xi} \pi < 0,
\end{aligned}$$

3. The third claim

$$\begin{aligned}
& \pi^2(\beta^2 + (2\xi+2\nu+1)(c_1 \frac{(2\nu-1)(2\nu-3) \times \dots \times 5 \times 3 \times 1}{(2\xi+2\nu+2)(2\xi+2\nu) \times \dots \times (2\xi+4)} \times \\
& \times \frac{(2\xi+1)(2\xi-1) \times \dots \times 3 \times 1}{(2\xi+2) \times \dots \times 4} \\
& + c_2 \frac{(2\xi+2\nu+1)(2\xi+2\nu-1) \times \dots \times 3 \times 1}{(2\xi+2\nu+2)(2\xi+2\nu) \times \dots \times 4})^2 K^{4\xi+4\nu} \\
& + (16\xi+8\nu+8)(c_1 \frac{(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1}{(2\xi+2\nu+2)(2\xi+2\nu) \times \dots \times (2\xi+4)} \\
& \times \frac{(2\xi+1)(2\xi-1) \times \dots \times 3 \times 1}{(2\xi+2) \times \dots \times 4 \times 2} \\
& + c_2 \frac{(2\xi+2\nu+1)(2\xi+2\nu-1) \times \dots \times 3 \times 1}{(2\xi+2\nu+2)(2\xi+2\nu) \times \dots \times 4 \times 2}) \times \\
& \frac{(2\xi+1)(2\xi-1) \times \dots \times 3 \times 1}{(2\xi+2)(2\xi) \times \dots \times 4 \times 2} c_3 K^{4\xi+4\nu-2} \\
& + 4c_3^2(2\xi+1) \frac{(2\xi+1)^2(2\xi-1)^2 \times \dots \times 25 \times 9 \times 1}{(2\xi+2)^2(2\xi)^2 \times \dots \times 16 \times 4} K^{4\xi}) > 0.
\end{aligned}$$

then Equation (3) admits a unique asymptotically stable periodic solution x_ϵ , such that $x_\epsilon(0) \rightarrow v_0$, as $\epsilon \rightarrow 0$.

We are going to prove Theorem 2, both for this end and to be ready to apply to the Second Bogolubov's Theorem; as in [9], we utilize Levinson's changes in order to rewrite (3) as a coupled system.

Let $(z_1, z_2) = (u, u')$, so Equation (3) becomes

$$\begin{cases} z'_1 = z_2, \\ z'_2 = -\epsilon(c_1 z_1^{2\nu} + c_2 z_2^{2\nu} + c z_1 + c_3) z_2^{2\xi+1} - (1 + \beta\epsilon) z_1 + \epsilon\omega \sin(s). \end{cases} \quad (8)$$

Then

$$\begin{pmatrix} z_1(s) \\ z_2(s) \end{pmatrix} = \begin{pmatrix} \cos(s) & \sin(s) \\ -\sin(s) & \cos(s) \end{pmatrix} \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix}.$$

By using the Lipschitz property on x , system (8) will be transformed as

$$\begin{aligned}
x'_1 &= \epsilon(\sin(-s)(-\beta(\cos(s)x_1 + \sin(s)x_2) - (c_1(\cos(s)x_1 + \sin(s)x_2)^{2\nu} \\
& + c_2(-\sin(s)x_1 + \cos(s)x_2)^{2\nu} + c(\cos(s)x_1 + \sin(s)x_2) + c_3)(-\sin(s)x_1 \\
& + \cos(s)x_2)^{2\xi+1} + \omega \sin(s))), \\
x'_2 &= \epsilon(\cos(-s)(-\beta(\cos(s)x_1 + \sin(s)x_2) - (c_1(\cos(s)x_1 + \sin(s)x_2)^{2\nu} \\
& + c_2(-\sin(s)x_1 + \cos(s)x_2)^{2\nu} + c(\cos(s)x_1 + \sin(s)x_2) + c_3)(-\sin(s)x_1 \\
& + \cos(s)x_2)^{2\xi+1} + \omega \sin(s))), \end{aligned} \quad (9)$$

where

$$\begin{aligned}
 g_1(s, x_1, x_2) &= \beta(\cos(s)x_1 + \sin(s)x_2) \sin(s) + (c_1(\cos(s)x_1 + \sin(s)x_2)^{2\nu} \\
 &\quad + c_2(-\sin(s)x_1 + \cos(s)x_2)^{2\nu} + c(\cos(s)x_1 + \sin(s)x_2) + c_3) \\
 &\quad \times (-\sin(s)x_1 + \cos(s)x_2)^{2\xi+1} \sin(s) \\
 &\quad - \omega \sin^2(s), \\
 g_2(s, x_1, x_2) &= -\beta(\cos(s)x_1 + \sin(s)x_2) \cos(s) - (c_1(\cos(s)x_1 + \sin(s)x_2)^{2\nu} \\
 &\quad + c_2(-\sin(s)x_1 + \cos(s)x_2)^{2\nu} + c(\cos(s)x_1 + \sin(s)x_2) + c_3) \\
 &\quad \times (-\sin(s)x_1 + \cos(s)x_2)^{2\xi+1} \cos(s) \\
 &\quad + \omega \sin(s) \cos(s),
 \end{aligned} \tag{10}$$

with $x_1 = K \sin(\psi)$, $x_2 = K \cos(\psi)$. Then

$$\begin{aligned}
 g_1(s, K \sin(\psi), K \cos(\psi)) &= \beta K \sin(s) \sin(\psi + s) \\
 &\quad + c_1 K^{2\xi+2\nu+1} \sin(s) \sin^{2\nu}(\psi + s) \cos^{2\xi+1}(\psi + s) \\
 &\quad + c_2 K^{2\xi+2\nu+1} \sin(s) \cos^{2\xi+2\nu+1}(\psi + s) \\
 &\quad + c K^{2\xi+2} \sin(s) \sin(\psi + s) \cos^{2\xi+1}(\psi + s) \\
 &\quad + c_3 K^{2\xi+1} \sin(s) \cos^{2\xi+1}(\psi + s) - \omega \sin^2(s),
 \end{aligned} \tag{11}$$

$$\begin{aligned}
 g_2(s, K \sin(\psi), K \cos(\psi)) &= -\beta K \cos(s) \sin(\psi + s) \\
 &\quad - c_1 K^{2\xi+2\nu+1} \cos(s) \sin^{2\nu}(\psi + s) \cos^{2\xi+1}(\psi + s) \\
 &\quad - c_2 K^{2\xi+2\nu+1} \cos(s) \cos^{2\xi+2\nu+1}(\psi + s) \\
 &\quad - c K^{2\xi+2} \cos(s) \sin(\psi + s) \cos^{2\xi+1}(\psi + s) \\
 &\quad - c_3 K^{2\xi+1} \cos(s) \cos^{2\xi+1}(\psi + s) + \omega \sin(s) \cos(s).
 \end{aligned} \tag{12}$$

By (6), we give the corresponding average function g_0 as

$$\begin{aligned}
 g_0(v_0) &= \int_0^{2\pi} g(s, v_0) ds, \\
 &= \begin{cases} g_{01}(K \sin(\psi), K \cos(\psi)) = g_{01}(v_0) = \int_0^{2\pi} g_1(s, v_0) ds, \\ g_{02}(K \sin(\psi), K \cos(\psi)) = g_{02}(v_0) = \int_0^{2\pi} g_2(s, v_0) ds. \end{cases}
 \end{aligned} \tag{13}$$

Such that

$$\begin{aligned}
 g_{01}(K \sin(\psi), K \cos(\psi)) &= \int_0^{2\pi} g_1(s, K \sin(\psi), K \cos(\psi)) ds \\
 &= \int_0^{2\pi} \beta K \sin(s) \sin(\psi + s) ds \\
 &\quad + c_1 K^{2\xi+2\nu+1} \int_0^{2\pi} \sin(s) \sin^{2\nu}(\psi + s) \cos^{2\xi+1}(\psi + s) ds \\
 &\quad + c K^{2\xi+2} \int_0^{2\pi} \sin(s) \sin(\psi + s) \cos^{2\xi+1}(\psi + s) ds \\
 &\quad + c_2 K^{2\xi+2\nu+1} \int_0^{2\pi} \sin(s) \cos^{2\xi+2\nu+1}(\psi + s) ds \\
 &\quad + c_3 K^{2\xi+1} \int_0^{2\pi} \sin(s) \cos^{2\xi+1}(\psi + s) ds - \int_0^{2\pi} \omega \sin^2(s) ds,
 \end{aligned}$$

and

$$\begin{aligned}
 g_{02}(K \sin(\psi), K \cos(\psi)) &= \int_0^{2\pi} g_2(s, K \sin(\psi), K \cos(\psi)) ds \\
 &= - \int_0^{2\pi} \beta K \cos(s) \sin(\psi + s) ds \\
 &\quad - c_1 K^{2\xi+2\nu+1} \int_0^{2\pi} \cos(s) \sin^{2\nu}(\psi + s) \cos^{2\xi+1}(\psi + s) ds \\
 &\quad - c_2 K^{2\xi+2\nu+1} \int_0^{2\pi} \cos(s) \cos^{2\xi+2\nu+1}(\psi + s) ds \\
 &\quad - c K^{2\xi+2} \int_0^{2\pi} \cos(s) \sin(\psi + s) \cos^{2\xi+1}(\psi + s) ds \\
 &\quad - c_3 K^{2\xi+1} \int_0^{2\pi} \cos(s) \cos^{2\xi+1}(\psi + s) ds \\
 &\quad + \int_0^{2\pi} \omega \cos(s) \sin(s) ds,
 \end{aligned}$$

where

$$\begin{aligned}
 \int_0^{2\pi} \beta K \sin(s) \sin(\psi + s) ds &= \beta K \cos(\psi) \pi, \quad \int_0^{2\pi} \omega \sin^2(s) ds = \omega \pi, \\
 \int_0^{2\pi} \beta K \cos(s) \sin(\psi + s) ds &= \beta K \sin(\psi) \pi, \quad \int_0^{2\pi} \omega \cos(s) \sin(s) ds = 0.
 \end{aligned}$$

The next proposition will help us to find the average function.

Proposition 1 ([10]). *For any even $a, b \in \mathbb{N}$, we have*

$$\int_0^{2\pi} \sin^a(s) \cos^b(s) ds = \frac{(a-1)(a-3) \times \dots \times 5 \times 3 \times 1}{(a+b)(a+b-2) \times \dots \times (b+2)} \frac{(b-1)(b-3) \times \dots \times 3 \times 1}{b(b-2) \times \dots \times 4 \times 2} 2\pi.$$

Remark 2. *If $a, b \in \mathbb{Z}$ are even integers, then we have*

$$\int_0^{2\pi} \sin^a(s) \cos^b(s) ds = \frac{a-1}{a+b} \int_0^{2\pi} \sin^{a-2}(s) \cos^b(s) ds,$$

and

$$\int_0^{2\pi} \sin^a(s) \cos^b(s) ds = \frac{b-1}{a+b} \int_0^{2\pi} \sin^a(s) \cos^{b-2}(s) ds.$$

If $a \in \mathbb{Z}$ or $b \in \mathbb{Z}$ is an odd integer, we have

$$\int_0^{2\pi} \sin^a(s) \cos^b(s) ds = 0.$$

Lemma 1. *For each $\xi, \nu \in \mathbb{N}$ and $\psi \in [-\pi, \pi]$, we obtain*

1. $\int_0^{2\pi} \sin(s) \sin^{2\nu}(\psi + s) \cos^{2\xi+1}(\psi + s) ds = -\sin(\psi) \frac{(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1}{(2\xi+2\nu+2)(2\xi+2\nu) \times \dots \times (2\xi+4)} \times \frac{(2\xi+1)(2\xi-1) \times \dots \times 5 \times 3 \times 1}{(2\xi+2) \times \dots \times 4} \pi.$
2. $\int_0^{2\pi} \sin(s) \cos^{2\xi+1}(\psi + s) ds = -\sin(\psi) \frac{(2\xi+1)(2\xi-1) \times \dots \times 3 \times 1}{(2\xi+2)(2\xi) \times \dots \times 4 \times 2} 2\pi.$
3. $\int_0^{2\pi} \sin(s) \sin(\psi + s) \cos^{2\xi+1}(\psi + s) ds = 0.$
4. $\int_0^{2\pi} \cos(s) \sin^{2\nu}(\psi + s) \cos^{2\xi+1}(\psi + s) ds = \cos(\psi) \frac{(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1}{(2\xi+2\nu+2)(2\xi+2\nu) \times \dots \times (2\xi+4)} \times \frac{(2\xi+1)(2\xi-1) \times \dots \times 5 \times 3 \times 1}{(2\xi+2) \times \dots \times 4} \pi.$
5. $\int_0^{2\pi} \cos(s) \cos^{2\xi+1}(\psi + s) ds = \cos(\psi) \frac{(2\xi+1)(2\xi-1) \times \dots \times 3 \times 1}{(2\xi+2)(2\xi) \times \dots \times 4 \times 2} 2\pi.$
6. $\int_0^{2\pi} \cos(s) \sin(\psi + s) \cos^{2\xi+1}(\psi + s) ds = 0.$

Proof. We have

$$I = \int_0^{2\pi} \sin(s) \sin^{2\nu}(\psi + s) \cos^{2\xi+1}(\psi + s) ds.$$

Let $\tau = \psi + s$. Then

$$\begin{aligned} & \int_0^{2\pi} \sin(s) \sin^{2\nu}(\psi + s) \cos^{2\xi+1}(\psi + s) ds \\ &= \int_0^{2\pi} \sin(\tau - \psi) \sin^{2\nu}(\tau) \cos^{2\xi+1}(\tau) d\tau \\ &= \int_0^{2\pi} (\sin(\tau) \cos(\psi) - \cos(\tau) \sin(\psi)) \sin^{2\nu}(\tau) \cos^{2\xi+1}(\tau) d\tau \\ &= \cos(\psi) \int_0^{2\pi} \sin^{2\nu+1}(\tau) \cos^{2\xi+1}(\tau) d\tau \\ &\quad - \sin(\psi) \int_0^{2\pi} \sin^{2\nu}(\tau) \cos^{2\xi+2}(\tau) d\tau. \end{aligned}$$

Let

$$I_1 = \int_0^{2\pi} \sin^{2\nu+1}(\tau) \cos^{2\xi+1}(\tau) d\tau,$$

$$I_2 = \int_0^{2\pi} \sin^{2\nu}(\tau) \cos^{2\xi+2}(\tau) d\tau.$$

Using Proposition 1, we have $\forall \xi, \nu \in \mathbb{N}$

$$I_1 = 0.$$

$$\begin{aligned} I_2 &= \frac{(2\nu - 1)(2\nu - 3) \times \dots \times 3 \times 1}{(2\xi + 2\nu + 2)(2\xi + 2\nu) \times \dots \times (2\xi + 4)} \\ &\quad \times \frac{(2\xi + 1)(2\xi - 1) \times \dots \times 3 \times 1}{(2\xi + 2) \times \dots \times 4} \pi. \end{aligned}$$

Then

$$\begin{aligned} I &= -\sin(\psi) \frac{(2\nu - 1)(2\nu - 3) \times \dots \times 3 \times 1}{(2\xi + 2\nu + 2)(2\xi + 2\nu) \times \dots \times (2\xi + 4)} \\ &\quad \times \frac{(2\xi + 1)(2\xi - 1) \times \dots \times 3 \times 1}{(2\xi + 2) \times \dots \times 4} \pi. \end{aligned}$$

It is not difficult to prove the other integrals with the same method. $\square$

Remark 3. For each $\xi, \nu \in \mathbb{N}$ and $\psi \in [-\pi, \pi]$, we obtain

1.

$$\begin{aligned} & \int_0^{2\pi} \sin(s) \cos^{2\xi+2\nu+1}(\psi + s) ds \\ &= -\sin(\psi) \frac{(2\xi + 2\nu + 1)(2\xi + 2\nu - 1) \times \dots \times 3 \times 1}{(2\xi + 2\nu + 2)(2\xi + 2\nu) \times \dots \times 4} \pi. \end{aligned}$$

2.

$$\begin{aligned} & \int_0^{2\pi} \cos(s) \cos^{2\xi+2\nu+1}(\psi + s) ds \\ &= \cos(\psi) \frac{(2\xi + 2\nu + 1)(2\xi + 2\nu - 1) \times \dots \times 3 \times 1}{(2\xi + 2\nu + 2)(2\xi + 2\nu) \times \dots \times 4} \pi. \end{aligned}$$

By substituting these results in (13), we obtain the next Corollary.

Corollary 1. $\forall \xi, \nu \in \mathbb{N}$, $\beta, \omega, c_1, c_2, c_3$ and $K \in \mathbb{R}$ and $\psi \in [-\pi, \pi]$, the average function of functions (11) and (12) are given by

$$\begin{cases} g_{0_1}(K \sin(\psi), K \cos(\psi)) = \beta K \cos(\psi) \pi \\ -K^{2\xi+2\nu+1} \sin(\psi) \pi \left(c_1 \frac{(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1}{(2\xi+2\nu+2)(2\xi+2\nu) \times \dots \times (2\xi+4)} \right. \\ \times \frac{(2\xi+1)(2\xi-1) \times \dots \times 3 \times 1}{(2\xi+2) \times \dots \times 4} + c_2 \frac{(2\xi+2\nu+1)(2\xi+2\nu-1) \times \dots \times 3 \times 1}{(2\xi+2\nu+2)(2\xi+2\nu) \times \dots \times 4} \Big) \\ -c_3 K^{2\xi+1} \sin(\psi) 2 \frac{(2\xi+1)(2\xi-1) \times \dots \times 5 \times 3 \times 1}{(2\xi+2)(2\xi) \times \dots \times 4 \times 2} \pi - \omega \pi, \\ \\ g_{0_2}(K \sin(\psi), K \cos(\psi)) = -\beta K \sin(\psi) \pi \\ -K^{2\xi+2\nu+1} \cos(\psi) \pi \left(c_1 \frac{(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1}{(2\xi+2\nu+2)(2\xi+2\nu) \times \dots \times (2\xi+4)} \right. \\ \times \frac{(2\xi+1)(2\xi-1) \times \dots \times 3 \times 1}{(2\xi+2) \times \dots \times 4} + c_2 \frac{(2\xi+2\nu+1)(2\xi+2\nu-1) \times \dots \times 3 \times 1}{(2\xi+2\nu+2)(2\xi+2\nu) \times \dots \times 4} \Big) \\ -2c_3 K^{2\xi+1} \cos(\psi) \frac{(2\xi+1)(2\xi-1) \times \dots \times 5 \times 3 \times 1}{(2\xi+2)(2\xi) \times \dots \times 4 \times 2} \pi, \end{cases}$$

where $x_1 = K \sin(\psi)$, $x_2 = K \cos(\psi)$. Then

$$\begin{cases} g_{0_1}(x_1, x_2) = \beta x_2 \pi \\ - \left(c_1 \frac{(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1}{(2\xi+2\nu+2)(2\xi+2\nu) \times \dots \times (2\xi+4)} \frac{(2\xi+1)(2\xi-1) \times \dots \times 5 \times 3 \times 1}{(2\xi+2) \times \dots \times 4} \right. \\ + c_2 \frac{(2\xi+2\nu+1)(2\xi+2\nu-1) \times \dots \times 3 \times 1}{(2\xi+2\nu+2)(2\xi+2\nu) \times \dots \times 4} \Big) (x_1^2 + x_2^2)^{\frac{2\xi+2\nu}{2}} x_1 \pi \\ - 2c_3 \frac{(2\xi+1)(2\xi-1) \times \dots \times 3 \times 1}{(2\xi+2)(2\xi) \times \dots \times 4 \times 2} (x_1^2 + x_2^2)^{\frac{2\xi}{2}} x_1 \pi - \omega \pi, \\ \\ g_{0_2}(x_1, x_2) = -\beta x_1 \pi \\ - \left(c_1 \frac{(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1}{(2\xi+2\nu+2)(2\xi+2\nu) \times \dots \times (2\xi+4)} \frac{(2\xi+1)(2\xi-1) \times \dots \times 5 \times 3 \times 1}{(2\xi+2) \times \dots \times 4} \right. \\ + c_2 \frac{(2\xi+2\nu+1)(2\xi+2\nu-1) \times \dots \times 3 \times 1}{(2\xi+2\nu+2)(2\xi+2\nu) \times \dots \times 4} \Big) (x_1^2 + x_2^2)^{\frac{2\xi+2\nu}{2}} x_2 \pi \\ - 2c_3 \frac{(2\xi+1)(2\xi-1) \times \dots \times 3 \times 1}{(2\xi+2)(2\xi) \times \dots \times 4 \times 2} (x_1^2 + x_2^2)^{\frac{2\xi}{2}} x_2 \pi. \end{cases}$$

are continuously differentiable in $\mathbb{R}^2 \setminus \{(0, 0)\}$.

Remark 4. Note that by claim (1) of Theorem 1, if $g_0(x_1, x_2) = 0$ and $\det(g_0)'(x_1, x_2) \neq 0$, $(x_1, x_2) \in \mathbb{R}^2$, the solution of the unperturbed system

$$\begin{cases} u_1(s) = x_1 \cos(s) + x_2 \sin(s), \\ u_2(s) = -x_1 \sin(s) + x_2 \cos(s). \end{cases} \quad (14)$$

is 2π -periodic solution of (8).

We are now ready to prove Theorem 2.

Proof of Theorem 2. With the functions in (10), we have (i) and (ii) of claim (2) of Theorem 1 verified for $\Omega = \mathbb{R}^2$. We will treat the question of existence of only one limit cycle (one periodic solution). We should now check (7); for this end, the Jacobian matrix $g_0(x_1, x_2)$ is given by

$$B = J_{g_0}(x_1, x_2) = (g_0)'(x_1, x_2) = \begin{pmatrix} (g_{0_1})'_{x_1}(x_1, x_2) & (g_{0_1})'_{x_2}(x_1, x_2) \\ (g_{0_2})'_{x_1}(x_1, x_2) & (g_{0_2})'_{x_2}(x_1, x_2) \end{pmatrix},$$

with

$$\begin{aligned}
 (g_{01})'_{x_1}(x_1, x_2) = & -(c_1 \frac{(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1}{(2\zeta+2\nu+2)(2\zeta+2\nu) \times \dots \times (2\zeta+4)} \\
 & \times \frac{(2\zeta+1)(2\zeta-1) \times \dots \times 3 \times 1}{(2\zeta+2) \times \dots \times 4} \\
 & + c_2 \frac{(2\zeta+2\nu+1)(2\zeta+2\nu-1) \times \dots \times 3 \times 1}{(2\zeta+2\nu+2)(2\zeta+2\nu) \times \dots \times 4})(x_1^2 + x_2^2)^{\frac{2\zeta+2\nu}{2}} \pi \\
 & - (2\zeta+2\nu)(c_1 \frac{(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1}{(2\zeta+2\nu+2)(2\zeta+2\nu) \times \dots \times (2\zeta+4)} \\
 & \times \frac{(2\zeta+1)(2\zeta-1) \times \dots \times 3 \times 1}{(2\zeta+2) \times \dots \times 4} \\
 & + c_2 \frac{(2\zeta+2\nu+1)(2\zeta+2\nu-1) \times \dots \times 3 \times 1}{(2\zeta+2\nu+2)(2\zeta+2\nu) \times \dots \times 4})(x_1^2 + x_2^2)^{\frac{2\zeta+2\nu-2}{2}} x_1^2 \pi \\
 & - 2c_3 \frac{(2\zeta+1)(2\zeta-1) \times \dots \times 3 \times 1}{(2\zeta+2)(2\zeta) \times \dots \times 4 \times 2} (x_1^2 + x_2^2)^{\frac{2\zeta}{2}} \pi \\
 & - 2c_3(2\zeta) \frac{(2\zeta+1)(2\zeta-1) \times \dots \times 3 \times 1}{(2\zeta+2)(2\zeta) \times \dots \times 4 \times 2} (x_1^2 + x_2^2)^{\frac{2\zeta-2}{2}} x_1^2 \pi.
 \end{aligned}$$

$$\begin{aligned}
 (g_{01})'_{x_2}(x_1, x_2) = & \beta\pi - (2\zeta+2\nu)(c_1 \frac{(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1}{(2\zeta+2\nu+2)(2\zeta+2\nu) \times \dots \times (2\zeta+4)} \\
 & \times \frac{(2\zeta+1)(2\zeta-1) \times \dots \times 3 \times 1}{(2\zeta+2) \times \dots \times 4} \\
 & + c_2 \frac{(2\zeta+2\nu+1)(2\zeta+2\nu-1) \times \dots \times 3 \times 1}{(2\zeta+2\nu+2)(2\zeta+2\nu) \times \dots \times 4})((x_1^2 + x_2^2))^{\frac{2\zeta+2\nu-2}{2}} x_1 x_2 \pi \\
 & - 2c_3(2\zeta) \frac{(2\zeta+1)(2\zeta-1) \times \dots \times 3 \times 1}{(2\zeta+2)(2\zeta) \times \dots \times 4 \times 2} (x_1^2 + x_2^2)^{\frac{2\zeta-2}{2}} x_1 x_2 \pi,
 \end{aligned}$$

$$\begin{aligned}
 (g_{02})'_{x_1}(x_1, x_2) = & -\beta\pi - (2\zeta+2\nu)(c_1 \frac{(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1}{(2\zeta+2\nu+2)(2\zeta+2\nu) \times \dots \times (2\zeta+4)} \\
 & \times \frac{(2\zeta+1)(2\zeta-1) \times \dots \times 3 \times 1}{(2\zeta+2) \times \dots \times 4} \\
 & + c_2 \frac{(2\zeta+2\nu+1)(2\zeta+2\nu-1) \times \dots \times 3 \times 1}{(2\zeta+2\nu+2)(2\zeta+2\nu) \times \dots \times 4})(x_1^2 + x_2^2)^{\frac{2\zeta+2\nu-2}{2}} x_1 x_2 \pi, \\
 & - 2c_3(2\zeta) \frac{(2\zeta+1)(2\zeta-1) \times \dots \times 3 \times 1}{(2\zeta+2)(2\zeta) \times \dots \times 4 \times 2} (x_1^2 + x_2^2)^{\frac{2\zeta-2}{2}} x_1 x_2 \pi,
 \end{aligned}$$

and

$$\begin{aligned}
 (g_{0_2})'_{x_2}(x_1, x_2) = & -(c_1 \frac{(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1}{(2\zeta+2\nu+2)(2\zeta+2\nu) \times \dots \times (2\zeta+4)} \\
 & \times \frac{(2\zeta+1)(2\zeta-1) \times \dots \times 3 \times 1}{(2\zeta+2) \times \dots \times 4} \\
 & + c_2 \frac{(2\zeta+2\nu+1)(2\zeta+2\nu-1) \times \dots \times 3 \times 1}{(2\zeta+2\nu+2)(2\zeta+2\nu) \times \dots \times 4})(x_1^2 + x_2^2)^{\frac{2\zeta+2\nu}{2}} \pi \\
 & - (2\zeta+2\nu)(c_1 \frac{(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1}{(2\zeta+2\nu+2)(2\zeta+2\nu) \times \dots \times (2\zeta+4)} \\
 & \times \frac{(2\zeta+1)(2\zeta-1) \times \dots \times 3 \times 1}{(2\zeta+2) \times \dots \times 4} \\
 & + c_2 \frac{(2\zeta+2\nu+1)(2\zeta+2\nu-1) \times \dots \times 3 \times 1}{(2\zeta+2\nu+2)(2\zeta+2\nu) \times \dots \times 4})(x_1^2 + x_2^2)^{\frac{2\zeta+2\nu-2}{2}} x_2^2 \pi \\
 & - 2c_3 \frac{(2\zeta+1)(2\zeta-1) \times \dots \times 3 \times 1}{(2\zeta+2)(2\zeta) \times \dots \times 4 \times 2} (x_1^2 + x_2^2)^{\frac{2\zeta}{2}} \pi \\
 & - 2c_3(2\zeta) \frac{(2\zeta+1)(2\zeta-1) \times \dots \times 3 \times 1}{(2\zeta+2)(2\zeta) \times \dots \times 4 \times 2} (x_1^2 + x_2^2)^{\frac{2\zeta-2}{2}} x_2^2 \pi.
 \end{aligned}$$

Regarding $\det(B)$, $\text{trace}(B)$, let

$$\begin{aligned}
 U = & -(c_1 \frac{(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1}{(2\zeta+2\nu+2)(2\zeta+2\nu) \times \dots \times (2\zeta+4)} \\
 & \times \frac{(2\zeta+1)(2\zeta-1) \times \dots \times 3 \times 1}{(2\zeta+2) \times \dots \times 4} \\
 & + c_2 \frac{(2\zeta+2\nu+1)(2\zeta+2\nu-1) \times \dots \times 3 \times 1}{(2\zeta+2\nu+2)(2\zeta+2\nu) \times \dots \times 4})\pi, \\
 V = & -(2\zeta+2\nu)(c_1 \frac{(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1}{(2\zeta+2\nu+2)(2\zeta+2\nu) \times \dots \times (2\zeta+4)} \\
 & \times \frac{(2\zeta+1)(2\zeta-1) \times \dots \times 3 \times 1}{(2\zeta+2) \times \dots \times 4} \\
 & + c_2 \frac{(2\zeta+2\nu+1)(2\zeta+2\nu-1) \times \dots \times 3 \times 1}{(2\zeta+2\nu+2)(2\zeta+2\nu) \times \dots \times 4})\pi,
 \end{aligned}$$

and

$$\begin{aligned}
 X &= -2c_3 \frac{(2\zeta+1)(2\zeta-1) \times \dots \times 3 \times 1}{(2\zeta+2)(2\zeta) \times \dots \times 4 \times 2} \pi \\
 Y &= -2c_3(2\zeta) \frac{(2\zeta+1)(2\zeta-1) \times \dots \times 3 \times 1}{(2\zeta+2)(2\zeta) \times \dots \times 4 \times 2} \pi.
 \end{aligned}$$

Then

$$\begin{aligned} \text{trace}(B) &= \text{trace}(J_{g_0}(x_1, x_2)) = (g_{0_1})'_{x_1}(x_1, x_2) + (g_{0_2})'_{x_2}(x_1, x_2) \\ &= (2U + V)(x_1^2 + x_2^2)^{\frac{2\xi+2\nu}{2}} + (2X + Y)(x_1^2 + x_2^2)^{\frac{2\xi}{2}}. \\ \text{trace}(B) &= -2(c_1 \frac{(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1}{(2\xi+2\nu)(2\xi+2\nu-2) \times \dots \times (2\xi+4)} \\ &\quad \times \frac{(2\xi+1)(2\xi-1) \times \dots \times 3 \times 1}{(2\xi+2) \times \dots \times 4 \times 2} \\ &\quad + c_2 \frac{(2\xi+2\nu+1)(2\xi+2\nu-1) \times \dots \times 3 \times 1}{(2\xi+2\nu)(2\xi+2\nu-12) \times \dots \times 4 \times 2})(x_1^2 + x_2^2)^{\frac{2\xi+2\nu}{2}}\pi \\ &\quad - 2c_3(2\xi+2) \frac{(2\xi+1)(2\xi-1) \times \dots \times 3 \times 1}{(2\xi+2)(2\xi) \times \dots \times 4 \times 2}(x_1^2 + x_2^2)^{\frac{2\xi}{2}}\pi. \end{aligned}$$

$$\begin{aligned} \det(B) &= \det(g'_0((x_1, x_2))) \\ &= (g_{0_1})'_{x_1}(x_1, x_2)(g_{0_2})'_{x_2}(x_1, x_2) - (g_{0_2})'_{x_1}(x_1, x_2)(g_{0_1})'_{x_2}(x_1, x_2) \\ &= (U^2 + UV)(x_1^2 + x_2^2)^{2\xi+2\nu} + (2UX + UY + VX)(x_1^2 + x_2^2)^{2\xi+2\nu-1} \\ &\quad + (X^2 + XY)(x_1^2 + x_2^2)^{2\xi+2\nu-2} + \beta^2\pi^2. \\ \det(B) &= (2\xi+2\nu+1)(c_1 \frac{(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1}{(2\xi+2\nu+2)(2\xi+2\nu) \times \dots \times (2\xi+4)} \\ &\quad \times \frac{(2\xi+1)(2\xi-1) \times \dots \times 3 \times 1}{(2\xi+2) \times \dots \times 4} \\ &\quad + c_2 \frac{(2\xi+2\nu+1)(2\xi+2\nu-1) \times \dots \times 3 \times 1}{(2\xi+2\nu+2)(2\xi+2\nu) \times \dots \times 4})^2\pi^2(x_1^2 + x_2^2)^{2\xi+2\nu} \\ &\quad + (16\xi+8\nu+8)(c_1 \frac{(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1}{(2\xi+2\nu+2)(2\xi+2\nu) \times \dots \times (2\xi+4)} \\ &\quad \times \frac{(2\xi+1)(2\xi-1) \times \dots \times 3 \times 1}{(2\xi+2) \times \dots \times 4 \times 2} \\ &\quad + c_2 \frac{(2\xi+2\nu+1)(2\xi+2\nu-1) \times \dots \times 3 \times 1}{(2\xi+2\nu+2)(2\xi+2\nu) \times \dots \times 4 \times 2}) \times \\ &\quad \frac{(2\xi+1)(2\xi-1) \times \dots \times 3 \times 1}{(2\xi+2)(2\xi) \times \dots \times 4 \times 2}d(x_1^2 + x_2^2)^{2\xi+2\nu-1} \\ &\quad + 4c_3^2(2\xi+1) \frac{(2\xi+1)^2(2\xi-1)^2 \times \dots \times 25 \times 9 \times 1}{(2\xi+2)^2(2\xi)^2 \times \dots \times 16 \times 4}\pi^2(x_1^2 + x_2^2)^{2\xi} + \beta^2\pi^2, \end{aligned}$$

with $x_1 = K \sin(\psi)$, $x_2 = K \cos(\psi)$. Then

$$\begin{aligned} \text{trace}(B) &= -2(c_1 \frac{(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1}{(2\xi+2\nu)(2\xi+2\nu-2) \times \dots \times (2\xi+4)} \\ &\quad \times \frac{(2\xi+1)(2\xi-1) \times \dots \times 3 \times 1}{(2\xi+2) \times \dots \times 4 \times 2} \\ &\quad + c_2 \frac{(2\xi+2\nu+1)(2\xi+2\nu-1) \times \dots \times 3 \times 1}{(2\xi+2\nu)(2\xi+2\nu-12) \times \dots \times 4 \times 2})K^{2\xi+2\nu}\pi \\ &\quad - 2c_3(2\xi+2) \frac{(2\xi+1)(2\xi-1) \times \dots \times 3 \times 1}{(2\xi+2)(2\xi) \times \dots \times 4 \times 2}K^{2\xi}\pi < 0, \end{aligned}$$

and

$$|K| > \left(\frac{-d(2\xi+2)(2\xi+2\nu)(2\xi+2\nu-2) \times \dots \times (2\xi+4)}{c_1(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1 + c_2(2\xi+2\nu+1)(2\xi+2\nu-1) \times \dots \times (2\xi+3)} \right)^{1/(2\nu)},$$

is defined if

$$\begin{cases} c_3 < 0 \\ c_1(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1 + c_2(2\xi+2\nu+1)(2\xi+2\nu-1) \times \dots \times (2\xi+3) > 0. \end{cases}$$

We discuss the different cases

$$\begin{cases} \text{If } c_3 < 0, c_1 > 0 \text{ and } c_2 > 0, \\ \text{If } c_3 < 0, c_1 = 0 \text{ and } c_2 > 0, \\ \text{If } c_3 < 0, c_2 = 0 \text{ and } c_1 > 0, \\ \text{If } c_3 < 0, c_1 > 0, c_2 < 0 \text{ and } \frac{c_2}{c_1} > \frac{-(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1}{(2\xi+2\nu+1)(2\xi+2\nu-1) \times \dots \times (2\xi+3)}, \\ \text{If } c_3 < 0, c_1 < 0, c_2 > 0 \text{ and } \frac{c_2}{c_1} < \frac{-(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1}{(2\xi+2\nu+1)(2\xi+2\nu-1) \times \dots \times (2\xi+3)}, \end{cases}$$

or

$$\begin{cases} c_3 > 0 \\ c_1(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1 + c_2(2\xi+2\nu+1)(2\xi+2\nu-1) \times \dots \times (2\xi+3) < 0. \end{cases}$$

Then

$$\begin{cases} \text{If } c_3 > 0, c_1 < 0 \text{ and } c_2 < 0, \\ \text{If } c_3 > 0, c_1 = 0 \text{ and } c_2 < 0, \\ \text{If } c_3 > 0, c_2 = 0 \text{ and } c_1 < 0, \\ \text{If } c_3 > 0, c_1 > 0, c_2 < 0 \text{ and } \frac{c_2}{c_1} < \frac{-(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1}{(2\xi+2\nu+1)(2\xi+2\nu-1) \times \dots \times (2\xi+3)}, \\ \text{If } c_3 > 0, c_1 < 0, c_2 > 0 \text{ and } \frac{c_2}{c_1} > \frac{-(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1}{(2\xi+2\nu+1)(2\xi+2\nu-1) \times \dots \times (2\xi+3)}, \end{cases}$$

and

$$\begin{aligned} \det(B) &= \pi^2(\beta^2 + (2\xi+2\nu+1)(c_1 \frac{(2\nu-1)(2\nu-3) \times \dots \times 5 \times 3 \times 1}{(2\xi+2\nu+2)(2\xi+2\nu) \times \dots \times (2\xi+4)} \\ &\quad \times \frac{(2\xi+1)(2\xi-1) \times \dots \times 3 \times 1}{(2\xi+2) \times \dots \times 4} \\ &\quad + c_2 \frac{(2\xi+2\nu+1)(2\xi+2\nu-1) \times \dots \times 3 \times 1}{(2\xi+2\nu+2)(2\xi+2\nu) \times \dots \times 4})^2 K^{4\xi+4\nu} \\ &\quad + (16\xi+8\nu+8)(c_1 \frac{(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1}{(2\xi+2\nu+2)(2\xi+2\nu) \times \dots \times (2\xi+4)} \\ &\quad \times \frac{(2\xi+1)(2\xi-1) \times \dots \times 3 \times 1}{(2\xi+2) \times \dots \times 4 \times 2} \\ &\quad + c_2 \frac{(2\xi+2\nu+1)(2\xi+2\nu-1) \times \dots \times 3 \times 1}{(2\xi+2\nu+2)(2\xi+2\nu) \times \dots \times 4 \times 2}) \times \\ &\quad \frac{(2\xi+1)(2\xi-1) \times \dots \times 3 \times 1}{(2\xi+2)(2\xi) \times \dots \times 4 \times 2} c_3 K^{4\xi+4\nu-2} \\ &\quad + 4c_3^2(2\xi+1) \frac{(2\xi+1)^2(2\xi-1)^2 \times \dots \times 25 \times 9 \times 1}{(2\xi+2)^2(2\xi)^2 \times \dots \times 16 \times 4} K^{4\xi}) > 0. \end{aligned}$$

Now, we check condition $g_0(v_0) = 0$ in Theorem 1.

$$\begin{cases} g_{01}(K \sin(\psi), K \cos(\psi)) = \beta K \cos(\psi) \pi \\ -K^{2\zeta+2\nu+1} \sin(\psi) \pi \left(c_1 \frac{(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1}{(2\zeta+2\nu+2)(2\zeta+2\nu) \times \dots \times (2\zeta+4)} \right. \\ \times \frac{(2\zeta+1)(2\zeta-1) \times \dots \times 3 \times 1}{(2\zeta+2) \times \dots \times 4} + c_2 \frac{(2\zeta+2\nu+1)(2\zeta+2\nu-1) \times \dots \times 3 \times 1}{(2\zeta+2\nu+2)(2\zeta+2\nu) \times \dots \times 4} \Big) \\ -c_3 K^{2\zeta+1} \sin(\psi) 2 \frac{(2\zeta+1)(2\zeta-1) \times \dots \times 5 \times 3 \times 1}{(2\zeta+2)(2\zeta) \times \dots \times 4 \times 2} \pi - \omega \pi = 0, \\ g_{02}(K \sin(\psi), K \cos(\psi)) = -\beta K \sin(\psi) \pi \\ -K^{2\zeta+2\nu+1} \cos(\psi) \pi \left(c_1 \frac{(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1}{(2\zeta+2\nu+2)(2\zeta+2\nu) \times \dots \times (2\zeta+4)} \right. \\ \times \frac{(2\zeta+1)(2\zeta-1) \times \dots \times 3 \times 1}{(2\zeta+2) \times \dots \times 4} + c_2 \frac{(2\zeta+2\nu+1)(2\zeta+2\nu-1) \times \dots \times 3 \times 1}{(2\zeta+2\nu+2)(2\zeta+2\nu) \times \dots \times 4} \Big) \\ -2c_3 K^{2\zeta+1} \cos(\psi) \frac{(2\zeta+1)(2\zeta-1) \times \dots \times 5 \times 3 \times 1}{(2\zeta+2)(2\zeta) \times \dots \times 4 \times 2} \pi = 0. \end{cases}$$

Then

$$\begin{aligned} \cos(\psi) &= \frac{K\beta}{\omega}, \text{ where } \omega \neq 0, \\ \sin(\psi) &= -\left(c_1 \frac{(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1}{(2\zeta+2\nu+2)(2\zeta+2\nu) \times \dots \times (2\zeta+4)} \right. \\ &\times \frac{(2\zeta+1)(2\zeta-1) \times \dots \times 3 \times 1}{(2\zeta+2) \times \dots \times 4} \\ &+ c_2 \frac{(2\zeta+2\nu+1)(2\zeta+2\nu-1) \times \dots \times 3 \times 1}{(2\zeta+2\nu+2)(2\zeta+2\nu) \times \dots \times 4} \Big) \frac{1}{\omega} K^{2\zeta+2\nu+1} \\ &- 2c_3 \frac{(2\zeta+1)(2\zeta-1) \times \dots \times 3 \times 1}{(2\zeta+2)(2\zeta) \times \dots \times 4 \times 2} \frac{1}{\omega} K^{2\zeta+1}, \end{aligned}$$

and also

$$\begin{aligned} &\left(\frac{K\beta}{\omega} \right)^2 + \left(-\left(c_1 \frac{(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1}{(2\zeta+2\nu+2)(2\zeta+2\nu) \times \dots \times (2\zeta+4)} \right. \right. \\ &\times \frac{(2\zeta+1)(2\zeta-1) \times \dots \times 3 \times 1}{(2\zeta+2) \times \dots \times 4} \\ &+ c_2 \frac{(2\zeta+2\nu+1)(2\zeta+2\nu-1) \times \dots \times 3 \times 1}{(2\zeta+2\nu+2)(2\zeta+2\nu) \times \dots \times 4} \Big) \frac{1}{\omega} K^{2\zeta+2\nu+1} \\ &- 2c_3 \frac{(2\zeta+1)(2\zeta-1) \times \dots \times 3 \times 1}{(2\zeta+2)(2\zeta) \times \dots \times 4 \times 2} \frac{1}{\omega} K^{2\zeta+1} \Big)^2 = 1. \end{aligned}$$

Then

$$\begin{aligned} &K^2(\beta^2 + \left(-\left(c_1 \frac{(2\nu-1)(2\nu-3) \times \dots \times 3 \times 1}{(2\zeta+2\nu+2)(2\zeta+2\nu) \times \dots \times (2\zeta+4)} \right. \right. \\ &\times \frac{(2\zeta+1)(2\zeta-1) \times \dots \times 3 \times 1}{(2\zeta+2) \times \dots \times 4} \\ &+ c_2 \frac{(2\zeta+2\nu+1)(2\zeta+2\nu-1) \times \dots \times 3 \times 1}{(2\zeta+2\nu+2)(2\zeta+2\nu) \times \dots \times 4} \Big) K^{2\zeta+2\nu} \\ &- 2c_3 \frac{(2\zeta+1)(2\zeta-1) \times \dots \times 3 \times 1}{(2\zeta+2)(2\zeta) \times \dots \times 4 \times 2} K^{2\zeta} \Big)^2) = \omega^2. \end{aligned}$$

□

Example 1. As in [8], the Van der Pol equation is considered

$$u'' + \epsilon(u^2 - 1)u' + (1 + \beta\epsilon)u = \epsilon\omega \sin(s). \quad (15)$$

The existence and stability of the periodic solution was given. The results in our theorem, Theorem 2, show that (15) has exactly one cycle limit that is asymptotically stable if

$$\begin{cases} K^2 \left(\beta^2 + \left(1 - \frac{K^2}{4} \right)^2 \right) = \omega^2, \\ \det(B) = 1 + \beta^2 - K^2 + \frac{3}{16}K^4 > 0, \\ \text{trace}(B) = 2 - K^2 < 0. \end{cases}$$

Theorem 3. For all $\xi \in \mathbb{N}$, $\forall \alpha, \lambda, \beta, \omega \in \mathbb{R}$, $v_0 \in \mathbb{R}^2$, and $0 < \epsilon < 1$, if

$$\begin{aligned} 1. \quad & K^2 \left(\beta^2 + \left(-\alpha |K| K^{2\xi} \frac{4}{(2\xi+3)\pi} - c_3 K^{2\xi} \frac{2(2\xi+1)(2\xi-1) \times \dots \times 5 \times 3 \times 1}{(2\xi+2)(2\xi) \times \dots \times 4 \times 2} \right)^2 \right) - \omega^2 = 0, \\ 2. \quad & -4\alpha K^{2\xi+1} - \lambda \frac{(2\xi+2)2(2\xi+1)(2\xi-1) \times \dots \times 5 \times 3 \times 1}{(2\xi+2)(2\xi) \times \dots \times 4 \times 2} K^{2\xi} \pi < 0, \\ 3. \quad & \alpha^2 \frac{32\xi+32}{(2\xi+3)^2} K^{4\xi+2} - 8(4\xi+1) \frac{(2\xi+1)(2\xi-1) \times \dots \times 5 \times 3 \times 1}{(2\xi+2)(2\xi) \times \dots \times 4 \times 2} K^{4\xi+1} \alpha \lambda \pi \\ & + c_3^2 \frac{4(2\xi+1)^3(2\xi-1) \times \dots \times 9 \times 1}{(2\xi+2)^2(2\xi)^2 \times \dots \times 16 \times 4} K^{4\xi} \pi^2 > 0, \end{aligned}$$

then (4) has a unique periodic asymptotically stable solution.

To prove Theorem 3, as in [9], we will use Levinson's changes to apply Bogolubov's Theorem. Let $(z_1, z_2) = (u, u')$, so Equation (4) becomes

$$\begin{cases} z_1' = z_2, \\ z_2' = -\epsilon(\alpha |z_1| + c_3) z_2^{2\xi+1} - (1 + \beta\epsilon)z_1 + \epsilon\omega \sin(s). \end{cases} \quad (16)$$

Then

$$\begin{pmatrix} z_1(s) \\ z_2(s) \end{pmatrix} = \begin{pmatrix} \cos(s) & \sin(s) \\ -\sin(s) & \cos(s) \end{pmatrix} \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix},$$

with the help of the Lipschitz property for x in the function g . The system (16) becomes

$$\begin{aligned} x_1' &= \epsilon \left(\sin(-s)(-\beta(\cos(s)x_1 + \sin(s)x_2) \right. \\ &\quad \left. - (\alpha |\cos(s)x_1 + \sin(s)x_2| + c_3)(-\sin(s)x_1 + \cos(s)x_2)^{2\xi+1} \right. \\ &\quad \left. + \omega \sin(s)) \right), \\ x_2' &= \epsilon (\cos(-s)(-\beta(\cos(s)x_1 + \sin(s)x_2) \\ &\quad - (\alpha |\cos(s)x_1 + \sin(s)x_2| + c_3)(-\sin(s)x_1 + \cos(s)x_2)^{2\xi+1} \\ &\quad + \omega \sin(s))), \end{aligned}$$

where

$$\begin{cases} g_1(s, x_1, x_2) = \beta(\cos(s)x_1 + \sin(s)x_2) \sin(s) + (\alpha |\cos(s)x_1 + \sin(s)x_2| + c_3) \\ \quad (-\sin(s)x_1 + \cos(s)x_2)^{2\xi+1} \sin(s) - \omega \sin^2(s), \\ g_2(s, x_1, x_2) = -\beta(\cos(s)x_1 + \sin(s)x_2) \cos(s) - (\alpha |\cos(s)x_1 + \sin(s)x_2| + c_3) \\ \quad (-\sin(s)x_1 + \cos(s)x_2)^{2\xi+1} \cos(s) + \omega \sin(s) \cos(s), \end{cases} \quad (17)$$

with $x_1 = K \sin(\psi)$, $x_2 = K \cos(\psi)$. Then

$$\begin{cases} g_1(s, K \sin(\psi), K \cos(\psi)) \\ = \beta K \sin(s) \sin(\psi + s) + \alpha K^{2\xi+1} |K| \sin(s) |\sin(\psi + s)| \cos^{2\xi+1}(\psi + s) \\ + c_3 K^{2\xi+1} \sin(s) \cos^{2\xi+1}(\psi + s) - \omega \sin^2(s), \\ g_2(s, K \sin(\psi), K \cos(\psi)) \\ = -\beta K \cos(s) \sin(\psi + s) - \alpha K^{2\xi+1} |K| \cos(s) |\sin(\psi + s)| \cos^{2\xi+1}(\psi + s) \\ - c_3 K^{2\xi+1} \cos(s) \cos^{2\xi+1}(\psi + s) + \omega \sin(s) \cos(s). \end{cases} \quad (18)$$

By (6), we find the average function g_0 as

$$\begin{aligned} g_0(v_0) &= \int_0^{2\pi} g(s, v_0) ds \\ &= \begin{cases} g_{01}(K \sin(\psi), K \cos(\psi)) = g_{01}(v_0) = \int_0^{2\pi} g_1(s, v_0) ds, \\ g_{02}(K \sin(\psi), K \cos(\psi)) = g_{02}(v_0) = \int_0^{2\pi} g_2(s, v_0) ds, \end{cases} \end{aligned} \quad (19)$$

with

$$\begin{aligned} g_{01}(K \sin(\psi), K \cos(\psi)) &= \int_0^{2\pi} g_1(s, K \sin(\psi), K \cos(\psi)) ds \\ &= \int_0^{2\pi} \beta K \sin(s) \sin(\psi + s) ds \\ &\quad + \alpha |K| K^{2\xi+1} \int_0^{2\pi} \sin(s) |\sin(\psi + s)| \cos^{2\xi+1}(\psi + s) ds \\ &\quad + \int_0^{2\pi} c_3 K^{2\xi+1} \sin(s) \cos^{2\xi+1}(\psi + s) ds - \int_0^{2\pi} \omega \sin^2(s) ds, \end{aligned}$$

and

$$\begin{aligned} g_{02}(K \sin(\psi), K \cos(\psi)) &= \int_0^{2\pi} g_2(s, K \sin(\psi), K \cos(\psi)) ds \\ &= -\int_0^{2\pi} \beta K \cos(s) \sin(\psi + s) ds \\ &\quad - \alpha K^{2\xi+1} |K| \int_0^{2\pi} \cos(s) |\sin(\psi + s)| \cos^{2\xi+1}(\psi + s) ds \\ &\quad - \int_0^{2\pi} c_3 K^{2\xi+1} \cos(s) \cos^{2\xi+1}(\psi + s) ds + \int_0^{2\pi} \omega \cos(s) \sin(s) ds, \end{aligned}$$

with

$$\begin{aligned} \int_0^{2\pi} \beta K \sin(s) \sin(\psi + s) ds &= \beta K \cos(\psi) \pi, \\ \int_0^{2\pi} c_3 K^{2\xi+1} \sin(s) \cos^{2\xi+1}(\psi + s) ds \\ &= -c_3 K^{2\xi+1} \sin(\psi) \frac{(2\xi+1)(2\xi-1) \times \dots \times 3 \times 1}{(2\xi+2)(2\xi) \times \dots \times 4 \times 2} 2\pi, \\ \int_0^{2\pi} \omega \sin^2(s) ds &= \omega \pi, \\ \int_0^{2\pi} \beta K \cos(s) \sin(\psi + s) ds &= \beta K \sin(\psi) \pi, \\ \int_0^{2\pi} c_3 K^{2\xi+1} \cos(s) \cos^{2\xi+1}(\psi + s) ds \\ &= c_3 K^{2\xi+1} \cos(\psi) \frac{(2\xi+1)(2\xi-1) \times \dots \times 3 \times 1}{(2\xi+2)(2\xi) \times \dots \times 4 \times 2} 2\pi, \\ \int_0^{2\pi} \omega \cos(s) \sin(s) ds &= 0. \end{aligned}$$

The next Lemma will be useful to find the average function.

Lemma 2. For each $\xi \in \mathbb{N}$ and $\psi \in [-\pi, \pi]$

1. $\int_0^{2\pi} \sin(s) |\sin(\psi + s)| \cos^{2\xi+1}(\psi + s) ds = \frac{-4}{2\xi+3} \sin(\psi).$
2. $\int_0^{2\pi} \cos(s) |\sin(\psi + s)| \cos^{2\xi+1}(\psi + s) ds = \frac{4}{2\xi+3} \cos(\psi).$

Proof. 1. $\forall \psi \in [0, \pi], \xi \in \mathbb{N}$, we have

$$|\sin(\psi + s)| = \begin{cases} \sin(\psi + s), & \text{if } \psi + s \in [0, \pi] \\ -\sin(\psi + s), & \text{if } \psi + s \in [-\pi, 0], \end{cases}$$

which implies that

$$|\sin(\psi + s)| = \begin{cases} \sin(\psi + s), & \text{if } s \in [-\psi, \pi - \psi] \\ -\sin(\psi + s), & \text{if } s \in [-\pi - \psi, -\psi]. \end{cases}$$

Then

$$\begin{aligned} & \int_0^{2\pi} \sin(s) |\sin(\psi + s)| \cos^{2\xi+1}(\psi + s) ds \\ &= \int_0^{\pi-\psi} \sin(s) \sin(\psi + s) \cos^{2\xi+1}(\psi + s) ds \\ & \quad - \int_{\pi-\psi}^{-\psi} \sin(s) \sin(\psi + s) \cos^{2\xi+1}(\psi + s) ds \\ & \quad + \int_{-\psi}^{2\pi} \sin(s) \sin(\psi + s) \cos^{2\xi+1}(\psi + s) ds \\ &= \frac{-1}{2\xi+3} \sin(\psi) - \frac{2}{2\xi+3} \sin(\psi) + \frac{-1}{2\xi+3} \sin(\psi) = \frac{-4}{2\xi+3} \sin(\psi). \end{aligned}$$

If $\psi \in [-\pi, 0]$, we have

$$\begin{aligned} & \int_0^{2\pi} \sin(s) |\sin(\psi + s)| \cos^{2\xi+1}(\psi + s) ds \\ &= - \int_0^{-\psi} \sin(s) \sin(\psi + s) \cos^{2\xi+1}(\psi + s) ds \\ & \quad + \int_{-\psi}^{\pi-\psi} \sin(s) \sin(\psi + s) \cos^{2\xi+1}(\psi + s) ds \\ & \quad - \int_{\pi-\psi}^{2\pi} \sin(s) \sin(\psi + s) \cos^{2\xi+1}(\psi + s) ds \\ &= -\frac{1}{2\xi+3} \sin(\psi) + \frac{-2}{2\xi+3} \sin(\psi) - \frac{1}{2\xi+3} \sin(\psi) \\ &= \frac{-4}{2\xi+3} \sin(\psi). \end{aligned}$$

2. The other integral can be proven in the same way. $\square$

These last results, together with (19), give the next Corollary.

Corollary 2. $\forall \xi \in \mathbb{N}, \beta, \omega, \alpha, \lambda$ and $K \in \mathbb{R}$ and $\psi \in [-\pi, \pi]$. The average function of (18) is given by

$$\begin{aligned}
g_{0_1}(K \sin(\psi), K \cos(\psi)) &= \beta K \cos(\psi) \pi - \alpha |K| K^{2\zeta+1} \frac{4}{2\zeta+3} \sin(\psi) \\
&\quad - c_3 K^{2\zeta+1} \sin(\psi) \frac{(2\zeta+1)(2\zeta-1) \times \dots \times 3 \times 1}{(2\zeta+2)(2\zeta) \times \dots \times 4 \times 2} 2\pi - \omega \pi, \\
g_{0_2}(K \sin(\psi), K \cos(\psi)) &= -\beta K \sin(\psi) \pi - \alpha |K| K^{2\zeta+1} \frac{4}{2\zeta+3} \cos(\psi) \\
&\quad - c_3 K^{2\zeta+1} \cos(\psi) \frac{(2\zeta+1)(2\zeta-1) \times \dots \times 3 \times 1}{(2\zeta+2)(2\zeta) \times \dots \times 4 \times 2} 2\pi.
\end{aligned}$$

By noting $x_1 = K \sin(\psi)$, $x_2 = K \cos(\psi)$, we get

$$\begin{aligned}
g_{0_1}(x_1, x_2) &= \beta x_2 \pi - \frac{4}{2\zeta+3} \alpha (x_1^2 + x_2^2)^{\frac{2\zeta+1}{2}} x_1 \\
&\quad - \lambda \frac{(2\zeta+1)(2\zeta-1) \times \dots \times 3 \times 1}{(2\zeta+2)(2\zeta) \times \dots \times 4 \times 2} 2(x_1^2 + x_2^2)^{\frac{2\zeta}{2}} x_1 \pi - \omega \pi, \\
g_{0_2}(x_1, x_2) &= -\beta x_1 \pi - \frac{4}{2\zeta+3} \alpha (x_1^2 + x_2^2)^{\frac{2\zeta+1}{2}} x_2 \\
&\quad - \lambda \frac{(2\zeta+1)(2\zeta-1) \times \dots \times 3 \times 1}{(2\zeta+2)(2\zeta) \times \dots \times 4 \times 2} 2(x_1^2 + x_2^2)^{\frac{2\zeta}{2}} x_2 \pi,
\end{aligned}$$

and it is continuously differentiable in $\mathbb{R}^2 \setminus \{(0,0)\}$.

Remark 5. By claim (1) of Theorem 1, if $g_0(x_1, x_2) = 0$ and $\det(g_0)'(x_1, x_2) \neq 0$, $(x_1, x_2) \in \mathbb{R}^2$, the solution of the unperturbed system

$$\begin{cases} u_1(s) = x_1 \cos(s) + x_2 \sin(s), \\ u_2(s) = -x_1 \sin(s) + x_2 \cos(s), \end{cases} \quad (20)$$

is a 2π -periodic solution of (16).

Lemma 3. Let $v_0 \in \mathbb{R}^2$, $v_0 \neq 0$. The function (17) satisfies (ii) in Theorem 1 for any $\alpha, \lambda, \beta, \omega \in \mathbb{R}$ and $\zeta \in \mathbb{N}$.

Proof. In order to prove that the function (17) satisfies condition (ii), we use the same as in [8], which is enough to show that

$$g(s, v) = |[v]_1 \cos(s) + [v]_2 \sin(s)|,$$

satisfies this condition (ii), where $[v]_i$ is the i -th component of the vector $v \in \mathbb{R}^2$ and $g \in C^0([0, 2\pi] \times \mathbb{R}^2, \mathbb{R})$.

If $[v_0]_2 \neq 0$, let

$$\eta(v) = \arctan\left(-\frac{[v]_1}{[v]_2}\right),$$

while when $[v_0]_2 = 0$, let

$$\eta(v) = \arctan\left(-\frac{[v]_1}{[v]_2}\right),$$

for

$$[v_0]_1 [v]_2 < 0, \quad \eta(v) = \frac{\pi}{2},$$

and, respectively,

$$\eta(v) = \arctan\left(-\frac{[v]_1}{[v]_2}\right) + \pi,$$

for $[v_0]_1 [v]_2 > 0$.

Note that the function $v \rightarrow \eta(v)$ is continuous in every sufficiently small neighborhood of v_0 . Fix $\tilde{\gamma} > 0$. Let M be the union of two intervals centered in $\eta(v_0)$ (when $\eta(v_0) < 0$,

take $\eta(v_0) + 2\pi$ instead) and, respectively, $\eta(v_0) + \pi$, each of length $\frac{\tilde{\gamma}}{2}$, denoting M_1 and M_2 . Take $\tilde{\delta} > 0$ such that $\eta(v) \in M_1, \forall v \in B_{\tilde{\delta}}(v_0)$ and $\eta(v) + \pi \in M_2, \forall \|v - v_0\| \leq \tilde{\delta}$. This implies that for fixed $s \in [0, 2\pi] \setminus M$. We have

$$[v]_1 \cos(s) + [v]_2 \sin(s),$$

has a constant sign for all $v \in B_{\tilde{\delta}}(v_0)$, which affirms that $g(s, \cdot)$ is differentiable and

$$g'_v(s, v) = g'_v(s, v_0), \forall v \in B_{\tilde{\delta}}(v_0).$$

Hence, (ii) is fulfilled. $\square$

We are now in position to prove Theorem 3.

Proof of Theorem 3. Condition (i) for claim (2) of Theorem 1. In the case $\Omega = \mathbb{R}^2$ is verified, since the function in (17) is an absolute value function with respect to (x_1, x_2) . We will prove the existence of only one limit cycle (one periodic solution). We check conditions (7) of Theorem 1 and then demonstrate that $g_0(v_0) = 0$ by using Corollary 2.

Let the Jacobian matrix $g_0(x_1, x_2)$ given as

$$\begin{aligned} B &= J_{g_0}(x_1, x_2) \\ &= (g_0)'(x_1, x_2) \\ &= \begin{pmatrix} (g_{01})'_{x_1}(x_1, x_2) & (g_{01})'_{x_2}(x_1, x_2) \\ (g_{02})'_{x_1}(x_1, x_2) & (g_{02})'_{x_2}(x_1, x_2) \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} (g_{01})'_{x_1}(x_1, x_2) &= -\frac{4}{2\tilde{\zeta}+3} \alpha(x_1^2 + x_2^2)^{\frac{2\tilde{\zeta}+1}{2}} - \frac{8\tilde{\zeta}+4}{2\tilde{\zeta}+3} \alpha(x_1^2 + x_2^2)^{\frac{2\tilde{\zeta}-1}{2}} x_1^2 \\ &\quad - \lambda \frac{2(2\tilde{\zeta}+1)(2\tilde{\zeta}-1) \times \dots \times 3 \times 1}{(2\tilde{\zeta}+2)(2\tilde{\zeta}) \times \dots \times 4 \times 2} (x_1^2 + x_2^2)^{\frac{2\tilde{\zeta}}{2}} \pi \\ &\quad - \lambda \frac{(2\tilde{\zeta})2(2\tilde{\zeta}+1)(2\tilde{\zeta}-1) \times \dots \times 3 \times 1}{(2\tilde{\zeta}+2)(2\tilde{\zeta}) \times \dots \times 4 \times 2} (x_1^2 + x_2^2)^{\frac{2\tilde{\zeta}-2}{2}} x_1^2 \pi, \\ (g_{01})'_{x_2}(x_1, x_2) &= \beta\pi - \frac{8\tilde{\zeta}+4}{2\tilde{\zeta}+3} \alpha(x_1^2 + x_2^2)^{\frac{2\tilde{\zeta}-1}{2}} x_1 x_2 \\ &\quad - \lambda \frac{(2\tilde{\zeta})2(2\tilde{\zeta}+1)(2\tilde{\zeta}-1) \times \dots \times 3 \times 1}{(2\tilde{\zeta}+2)(2\tilde{\zeta}) \times \dots \times 4 \times 2} (x_1^2 + x_2^2)^{\frac{2\tilde{\zeta}-2}{2}} x_1 x_2 \pi, \end{aligned}$$

and

$$\begin{aligned} (g_{02})'_{x_1}(x_1, x_2) &= -\beta\pi - \frac{8\tilde{\zeta}+4}{2\tilde{\zeta}+3} \alpha(x_1^2 + x_2^2)^{\frac{2\tilde{\zeta}-1}{2}} x_1 x_2 \\ &\quad - \lambda \frac{(2\tilde{\zeta})2(2\tilde{\zeta}+1)(2\tilde{\zeta}-1) \times \dots \times 3 \times 1}{(2\tilde{\zeta}+2)(2\tilde{\zeta}) \times \dots \times 4 \times 2} (x_1^2 + x_2^2)^{\frac{2\tilde{\zeta}-2}{2}} x_1 x_2 \pi, \\ (g_{02})'_{x_2}(x_1, x_2) &= -\frac{4}{2\tilde{\zeta}+3} \alpha(x_1^2 + x_2^2)^{\frac{2\tilde{\zeta}+1}{2}} - \frac{8\tilde{\zeta}+4}{2\tilde{\zeta}+3} \alpha(x_1^2 + x_2^2)^{\frac{2\tilde{\zeta}-1}{2}} x_2^2 \\ &\quad - \lambda \frac{2(2\tilde{\zeta}+1)(2\tilde{\zeta}-1) \times \dots \times 3 \times 1}{(2\tilde{\zeta}+2)(2\tilde{\zeta}) \times \dots \times 4 \times 2} (x_1^2 + x_2^2)^{\frac{2\tilde{\zeta}}{2}} \pi \\ &\quad - \lambda \frac{(2\tilde{\zeta})2(2\tilde{\zeta}+1)(2\tilde{\zeta}-1) \times \dots \times 3 \times 1}{(2\tilde{\zeta}+2)(2\tilde{\zeta}) \times \dots \times 4 \times 2} (x_1^2 + x_2^2)^{\frac{2\tilde{\zeta}-2}{2}} x_2^2 \pi. \end{aligned}$$

We will find $\det(B)$ and $\text{trace}(B)$. Let

$$\begin{aligned} u &= -\alpha \frac{4}{2\zeta + 3}, v = -\alpha \frac{8\zeta + 4}{2\zeta + 3} \\ x &= -\lambda \frac{2(2\zeta + 1)(2\zeta - 1) \times \dots \times 3 \times 1}{(2\zeta + 2)(n - 1) \times \dots \times 4 \times 2} \pi, \\ y &= -\lambda \frac{(2\zeta)2(2\zeta + 1)(2\zeta - 1) \times \dots \times 3 \times 1}{(2\zeta + 2)(2\zeta) \times \dots \times 4 \times 2} \pi. \end{aligned}$$

Then

$$\begin{aligned} \text{trace}(B) &= \text{trace}(J_{g_0}(x_1, x_2)) = (g_{01})'_{x_2}(x_1, x_2) + (g_{02})'_{x_1}(x_1, x_2) \\ &= (2u + v)(x_1^2 + x_2^2)^{\frac{2\zeta+1}{2}} + (2x + y)(x_1^2 + x_2^2)^{\frac{2\zeta}{2}} \\ &= \frac{-8\zeta - 12}{2\zeta + 3} \alpha (x_1^2 + x_2^2)^{\frac{2\zeta+1}{2}} \\ &\quad - \lambda \frac{(2\zeta + 2)2(2\zeta + 1)(2\zeta - 1) \times \dots \times 3 \times 1}{(2\zeta + 2)(2\zeta) \times \dots \times 4 \times 2} (x_1^2 + x_2^2)^{\frac{2\zeta}{2}} \pi, \end{aligned}$$

so

$$\text{trace}(B) = -4\alpha (x_1^2 + x_2^2)^{\frac{2\zeta+1}{2}} - \lambda \frac{(2\zeta + 2)2(2\zeta + 1)(2\zeta - 1) \times \dots \times 3 \times 1}{(2\zeta + 2)(2\zeta) \times \dots \times 4 \times 2} (x_1^2 + x_2^2)^{\frac{2\zeta}{2}} \pi,$$

and

$$\begin{aligned} \det(B) &= \det(g'_0(x_1, x_2)) \\ &= (g_{01})'_{x_1}(x_1, x_2)(g_{02})'_{x_2}(x_1, x_2) - (g_{02})'_{x_1}(x_1, x_2)(g_{01})'_{x_2}(x_1, x_2) \\ &= (u^2 + uv)(x_1^2 + x_2^2)^{2\zeta+1} + (2ux + uy + vx)(x_1^2 + x_2^2)^{\frac{4\zeta+1}{2}} \\ &\quad + (x^2 + xy)(x_1^2 + x_2^2)^{2\zeta}, \end{aligned}$$

so

$$\begin{aligned} \det(B) &= f^2 \frac{32\zeta + 32}{(2\zeta + 3)^2} (x_1^2 + x_2^2)^{2\zeta+1} \\ &\quad - 8(4\zeta + 1) \frac{(2\zeta + 1)(2\zeta - 1) \times \dots \times 5 \times 3 \times 1}{(2\zeta + 2)(2\zeta) \times \dots \times 4 \times 2} (x_1^2 + x_2^2)^{\frac{4\zeta+1}{2}} \alpha \lambda \pi \\ &\quad + c_3^2 \frac{4(2\zeta + 1)^3(2\zeta - 1) \times \dots \times 9 \times 1}{(2\zeta + 2)^2(2\zeta)^2 \times \dots \times 16 \times 4} (x_1^2 + x_2^2)^{2\zeta} \pi^2, \end{aligned}$$

with $x_1 = K \sin(\psi)$, $x_2 = K \cos(\psi)$. Then

$$\text{trace}(B) = -4\alpha K^{2\zeta+1} - \lambda \frac{(2\zeta + 2)2(2\zeta + 1)(2\zeta - 1) \times \dots \times 3 \times 1}{(2\zeta + 2)(2\zeta) \dots 4 \times 2} K^{2\zeta} \pi < 0.$$

Thus

$$|K| > \frac{-\lambda}{2\alpha} \frac{(2\zeta + 2)(2\zeta + 1)(2\zeta - 1) \times \dots \times 3 \times 1}{(2\zeta + 2)(2\zeta) \times \dots \times 4 \times 2} \pi, \alpha \neq 0,$$

is defined if

$$\begin{cases} c_3 < 0, \\ \alpha > 0, \end{cases} \quad \text{or} \quad \begin{cases} c_3 > 0, \\ \alpha < 0. \end{cases}$$

On the other hand, if $\alpha = 0$, we have

$$\text{trace}(B) = -\lambda \frac{(2\xi + 2)2(2\xi + 1)(2\xi - 1) \times \dots \times 3 \times 1}{(2\xi + 2)(2\xi) \dots 4 \times 2} K^{2\xi} \pi < 0,$$

then $c_3 > 0$ and

$$\begin{aligned} \det(B) &= \alpha^2 \frac{32\xi + 32}{(2\xi + 3)^2} K^{4\xi+2} - 8(4\xi + 1) \frac{(2\xi + 1)(2\xi - 1) \times \dots \times 3 \times 1}{(2\xi + 2)(2\xi) \times \dots \times 4 \times 2} K^{4\xi+1} \alpha \lambda \pi \\ &+ c_3^2 \frac{4(2\xi + 1)^3(2\xi - 1) \times \dots \times 9 \times 1}{(2\xi + 2)^2(2\xi)^2 \times \dots \times 16 \times 4} K^{4\xi} \pi^2 > 0. \end{aligned}$$

Now, we should check that $g_0(v_0) = 0$ from Theorem 1.

$$\begin{cases} g_{01}(K \sin(\psi), K \cos(\psi)) \\ = \beta K \cos(\psi) \pi - \alpha |K| K^{2\xi+1} \frac{4}{2\xi+3} \sin(\psi) - c_3 K^{2\xi+1} \sin(\psi) \frac{(2\xi+1)(2\xi-1) \times \dots \times 3 \times 1}{(2\xi+2)(2\xi) \times \dots \times 4 \times 2} 2\pi - \omega \pi = 0, \\ g_{02}(K \sin(\psi), K \cos(\psi)) \\ = -\beta K \sin(\psi) \pi - \alpha |K| K^{2\xi+1} \frac{4}{2\xi+3} \cos(\psi) - c_3 K^{2\xi+1} \cos(\psi) \frac{(2\xi+1)(2\xi-1) \times \dots \times 3 \times 1}{(2\xi+2)(2\xi) \times \dots \times 4 \times 2} 2\pi = 0. \end{cases}$$

So, we have

$$\begin{aligned} \cos(\psi) &= \frac{K\beta}{\omega}, \text{ where } \omega \neq 0, \\ \sin(\psi) &= -\alpha |K| K^{2\xi+1} \frac{4}{(2\xi+3)\omega\pi} - c_3 K^{2\xi+1} \frac{2(2\xi+1)(2\xi-1) \times \dots \times 3 \times 1}{\omega(2\xi+2)(2\xi) \times \dots \times 4 \times 2}, \end{aligned}$$

then

$$\begin{aligned} \left(\frac{K\beta}{\omega}\right)^2 + \left(-\alpha |K| K^{2\xi+1} \frac{4}{(2\xi+3)\omega\pi} - c_3 K^{2\xi+1} \frac{2(2\xi+1)(2\xi-1) \times \dots \times 3 \times 1}{\omega(2\xi+2)(2\xi) \times \dots \times 4 \times 2}\right)^2 &= 1, \\ K^2 \left(\beta^2 + \left(-\alpha |K| K^{2\xi} \frac{4}{(2\xi+3)\pi} - c_3 K^{2\xi} \frac{2(2\xi+1)(2\xi-1) \times \dots \times 5 \times 3 \times 1}{(2\xi+2)(2\xi) \times \dots \times 4 \times 2}\right)^2\right) &= \omega^2. \end{aligned}$$

□

Example 2. As in [8], the Van der Pol equation is considered

$$u'' + \epsilon(u^2 - 1)u' + (1 + \beta\epsilon)u = \epsilon\omega \sin(s), \quad (21)$$

and asymptotic stability was shown. The results in our theorem, Theorem 3, show that (21) has exactly one cycle limit that is asymptotically stable if

$$\begin{cases} K^2 \left(\beta^2 + \left(1 - \frac{4}{3\pi} |K|\right)^2\right) = \omega^2, \\ \det(B) = \pi^2(1 + \beta^2) + \frac{32}{9} K^2 - 2\pi |K| > 0, \\ \text{trace}(B) = 2\pi - 4|K| < 0. \end{cases}$$

4. Conclusions

In this study, sufficient conditions of existence, uniqueness, and asymptotic stability of the solution for a class of almost periodic dynamical systems with an infinite set of nonlinear differential equations were shown. Our novelties are located mainly in the impact of non-linearity, where:

1. We proved that both Equations (3) and (4) have a unique periodic solution which is asymptotically stable.
2. The use of the Second Bogolyubov's Theorem in a simple and effective way, accompanied by Levinson's changes.

3. We have overcome the difficulty of having a polynomial function and treated it with the Jacobian matrix, Lipschitz property, and some useful integrals.
4. We improved the early results in the literature.

This type of equation dates back to the famous Dutch scientist Balthazar Van der Pol, which was one of his best works with a major contribution to the development of some branches of modern mathematics and physics, as well as radio engineering to be more precise. His work was mainly related with the equation that bears his name and it has a surprisingly wide range of applications in daily life and the natural sciences. The equation discusses modeling processes in the human body in a good and basic way that includes most studies related to the internal organs of the human body and the nervous system, in addition to the vocal cords. The well-known result regarding the asymptotic stability of such models arises due to the Second Bogolobov's Theorem, and the advantage of this method is that it can be applied not only in periodic dynamical systems, but also in almost nonperiodic dynamical systems, see [11–16].

Open problem. It is interesting to determine where the methods in this paper can be extended and also be applicable for other finite time stability analysis like “Finite-Time Synchronization of Quantized Markovian-Jump Time-Varying Delayed Neural Networks via an Event-Triggered Control Scheme under Actuator Saturation”. This open problem will be our next project.

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