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Stronger Forms of Fuzzy Pre-Separation and Regularity Axioms via Fuzzy Topology

Salem Saleh 1,2, Tareq M. Al-shami 3, A. A. Azzam 4,5,* and M. Hosny 6

1 Department of Computer Science, Cihan University-Erbil, Erbil P.O. Box 44001, Iraq; s_wosabi@hoduniv.net.ye
2 Department of Mathematics, Hodeidah University, Hodeidah P.O. Box 3114, Yemen
3 Department of Mathematics, Sana’a University, Sana’a P.O. Box 1247, Yemen; t.alshami@su.edu.ye or tareqalshami83@gmail.com
4 Department of Mathematics, Faculty of Science and Humanities, Prince Sattam Bin Abdulaziz University, Alkharij 11942, Saudi Arabia
5 Department of Mathematics, Faculty of Science, New Valley University, Elkharga 72511, Egypt
6 Department of Mathematics, College of Science, King Khalid University, Abha 61413, Saudi Arabia; maly@kku.edu.sa
* Correspondence: aa.azzam@psau.edu.sa or azzam0911@yahoo.com

Abstract: It is common knowledge that fuzzy topology contributes to developing techniques to address real-life applications in various areas like information systems and optimal choices. The building blocks of fuzzy topology are fuzzy open sets, but other extended families of fuzzy open sets, like fuzzy pre-open sets, can contribute to the growth of fuzzy topology. In the present work, we create some classifications of fuzzy topologies which enable us to obtain several desirable features and relationships. At first, we introduce and analyze stronger forms of fuzzy pre-separation and regularity properties in fuzzy topology called fuzzy pre-$T_i$, $i=0,1,2,3,4$, fuzzy pre-symmetric, and fuzzy pre-$R_i$, $i=0,1,2,3$ by utilizing the concepts of fuzzy pre-open sets and quasi-coincident relation. We investigate more novel properties of these classes and uncover their unique characteristics. By presenting a wide array of related theorems and interconnections, we structure a comprehensive framework for understanding these classes and interrelationships with other separation axioms in this setting. Moreover, the relations between these classes and those in some induced topological structures are examined. Additionally, we explore the hereditary and harmonic properties of these classes.

Keywords: fuzzy sets; fuzzy pre-open set; fuzzy quasi-coincident; fuzzy topology; fuzzy pre-$T_i$ spaces; fuzzy pre-symmetric; fuzzy pre-$R_i$ spaces

MSC: 54A40; 03E72; 54D99

1. Introduction

The concept of fuzzy sets (or $F$-sets), introduced by Zadeh [1], is considered a suitable approach with flexibility in modeling uncertainty and vagueness. By fuzzy sets, we try to get rid of/decrease vagueness from data analysis and facilitate the production of an accurate procedure by decision-makers, which cannot be achieved by classical mathematics. This promotes intellectuals and researchers to benefit from $F$-sets to generate numerous mathematical methods created for data analysis; see, for example, [2,3].

In 1986, Chang [4] made use of $F$-sets to familiarize the fuzzy topological space (or $FTS$). Then many topological concepts in fuzzy settings have been studied by various authors in different aspects as in [5–9]. As is known that fuzzy topology represents a robust mathematical instrument for processing uncertainty in real-world applications; especially, those related to the selection of the best alternative(s). This provides different
applications of fuzzy topology in practical issues and prompts the fast development of the fuzzy topology in a short measure of time, see, [10,11].

Pre-open sets, as a generalization of open sets, were first introduced by Mashhour et al. [12], and since then, various topological properties have been defined in relation to pre-open sets as in [13–15]. Singal-Prakash [16] extended the notion of pre-openness to fuzzy settings. They popularized fuzzy pre-interiors and pre-closure operators and investigated their properties. Separation properties hold a pivotal place in topology, ranking among its most fundamental principles. In the context of fuzzy settings, this concept has garnered significant attention from numerous scholars, as evidenced by the works of authors as in [17–20]. Singal-Prakash [16] extended separation axioms to fuzzy pre-open sets (or \(FPT_i, i = 0, 1, 2, 3, 4\)) and looked at some characterizations of them. To go along with this line, this work initiates strong types of separation axioms via the frame of \(FTS\) and establishes its fundamental features.

The layout of this article is designed as follows. After this introduction, Section 2 is provided to make the readers aware of the concepts and findings that are required to understand this work. Then, in Section 3, it is applied the concepts of fuzzy pre-open sets and quasi-coincident relation to introduce novel stronger forms of fuzzy pre-separation and regularity axioms in fuzzy topology called \(Fp-T_i, i = 0, \frac{1}{2}, 1, 2, 3\) and explore main properties. To complete this line, it is devoted Section 4 to initiate the concepts of \(Fp-R_i, i = 0, 1, 2, 3\) which are stronger than \(FPT_i, i = 0, 1, 2, 3, 4\) and establish basic characterizations. Also, we define the axioms of \(Fp-T_i, i = 3, 4\) and explain the relationships between them and the previous axioms. In Section 5, we elucidate some interrelations between the given separation axioms as well as demonstrate how they are navigated from the original fuzzy topology to classical topologies induced by it. Ultimately, we summarize the manuscript’s contributions and provide a brief conclusion in Section 6.

2. Preliminaries

Through this document, \(U\) refers to a universe set, \(I^U = \{0, 1\}\) is the set of all \(F\)-sets on \(U\), \((U, \sigma)\) means \(FTS\). In the following, we recall some basic definitions and results which are be used in the sequel. For more information; see, [21–25].

Definition 1 ([1]). A \(F\)-set \(H\) on \(U\) is a map \(H : U \rightarrow I\). It can be given as \(H = \{(u, H(u)) : u \in U, H(u) \in I\}\). The fuzzy point (or \(F\)-point), is denoted by \(u_r\), is a \(F\)-set defined as \(u_r(v) = r > 0\) if \(u = v\) and \(u_r(v) = 0\) otherwise for all \(v \in U\). We write \(u_r \in H\) if \(r \leq H(u)\). The set of all \(F\)-points in \(U\) is symbolized as \(FP(U)\). The constant \(F\)-sets \(0, 1\), and \(r\) are defined by \(0(u) = 0, 1(u) = 1\), and \(r(u) = r\) for all \(u \in U\). The characteristic function \(\chi_B\) for any \(B \subset U\) is a \(F\)-set on \(U\) given by \(\chi_B(u) = 1\) if \(u \in B\) and \(\chi_B(u) = 0\) if \(u \notin B\).

Definition 2 ([1,4,19]). For \(F, G \in I^U\). We have the following properties of \(F\)-sets:

(i) \(F \cup G \in I^U\) given by \((F \cup G)(u) = \max\{F(u), G(u)\}\) for all \(u \in U\).
(ii) \(F \cap G \in I^U\) given by \((F \cap G)(u) = \min\{F(u), G(u)\}\) for all \(u \in U\).
(iii) The complement of \(F\) is denoted by \(F^c\) and given by \(F^c(u) = 1 - F(u)\) for all \(u \in U\).
(iv) The support of \(F\) is denoted by \(S(F)\) and given by \(S(F) = \{u \in U : F(u) > 0\}\).
(v) For a map \(f : U \rightarrow V\) and \(H \in I^U\), \(K \in I^V\), \(f(H)\) is a \(F\)-set on \(V\) given by \(f(H)(v) = \sup\{H(u) : u \in f^{-1}(v)\}\) if \(f^{-1}(v) \neq \emptyset\) and \(f(H)(v) = 0\) if \(f^{-1}(v) = \emptyset\). Also, \(f^{-1}(K)\) is a \(F\)-set on \(U\), given by \(f^{-1}(K)(u) = K(f(u))\) for all \(u \in U\).

Lemma 1 ([19]). For \(H, G \in I^U\) and \(\{H_i : i \in I\}\), we have:

(i) \(S(\emptyset) = 0\) and \(S(1) = 1\).
(ii) \(S(H \cap G) = S(H) \cap S(G)\).
(iii) \(S(\cup_{i \in I} H_i) = \cup_{i \in I} S(H_i)\).
A FTS proposed by Chang [4] is the pair \((U, \sigma)\), where \(\sigma \subseteq I^U\) which is closed under finite intersections, arbitrary union, and \(\emptyset, 1\) in \(\sigma\). A F-set \(H\) is called a F-open set (or a FO-set) if \(H \in \sigma\) and the complement of \(H\) is called a F-closed set (or a FC-set). The class of all FO-sets (resp. FC-sets) is denoted as \(\text{FOS}(U)\) (resp. \(\text{FCS}(U)\)). A FTS \((U, \sigma)\) is named a fully-stratified if each constant F-set \(r\) on \(U\) is an element in \(\sigma\).

For a F-set \(H\) in \((U, \sigma)\), the F-closure of \(H \in I^U\) in \((U, \sigma)\) is the intersection of all FC-sets containing \(H\) and it is denoted by \(\text{cl}(H)\). The F-interior of \(H\), denoted by \(\text{int}(H)\), is the union all FO-sets contained in \(H\). It is clear that \(u_t \in \text{int}(H)\) if and only if there is \(G \in \sigma\) such that \(u_t \in G \subseteq H\). We observe that \(H\) is FO-set (resp. FC-set) if and only if \(H = \text{int}(H)\) (resp. \(H = \text{cl}(H)\)).

\[\text{Definition 3 (}[4]\). For a FTS \((U, \sigma)\) and \(Y \subseteq U\), the collection \(\sigma_Y = \{\chi_Y \cap H : H \in \sigma\}\) is a FT on \(Y\). The pair \((Y, \sigma_Y)\) is called a FT-subspace of \((U, \sigma)\).\]

\[\text{Definition 4 ([26])}. A F-point \(u_t\) is said to be quasi-coincident with a F-set \(H\) in \(U\), denoted by \(u_t \sim q H\), if there is \(u \in U\) such that \(r + H(u) > 1\). In general, \(H \sim q G\) if \(H(u) + G(u) > 1\) for some \(u \in U\). If \(H\) is not quasi-coincident with \(G\), then we write \(H \not\sim q G\).\]

\[\text{Proposition 1 ([9,18,26])}. For two F-sets \(F, G\) in \((U, \sigma)\) and \(u_t \in FP(U)\), we have:

\[(i)\] \(u_t \sim q F \iff u_t \in F^c\), in general \(F \sim q G \iff F \subseteq G^c\).

\[(ii)\] \(F \cap G = \emptyset \implies F \sim q G\).

\[(iii)\] \(F \sim q G, H \subseteq G \implies F \sim q H\).

\[(iv)\] \(F \subseteq G \iff (u_t \sim q F \iff u_t \sim q G), u_t \in FP(U)\).

\[(v)\] \(u_t \cap \bigcap_{i \in I} K_i \implies u_t \sim q K_i\), for all \(i \in I\).

\[(vi)\] \(u_t \not\sim q v \iff u_t \not\sim q v_t\).

\[(vii)\] \(u_t \sim q v_t \iff u_t \sim q v\) or \((u_t = v\) and \(r + t > 1\)).

\[(viii)\] For a map \(f : U \rightarrow V\), \(H \in I^U\), \(G \in I^V\), and \(u_t \in FP(U)\), we have:

\[\bullet\] \(f(u_t) \sim q G \implies u_t \sim q f^{-1}(G),\) and \(u_t \sim q H \implies f(u_t) \sim q f(H)\).

\[\bullet\] \(u_t \sim q f^{-1}(G)\) if \(f(u_t) \in G\), and \(f(u_t) \in f(H)\) if \(u_t \in H\).

\[\text{Definition 5 ([16])}. A F-set \(H\) in FTS \((U, \sigma)\) is called a fuzzy pre-open set (briefly, FpO-set) if \(H \subseteq \text{int}(cl(H))\). The class of all FpO-sets in \((U, \sigma)\) is denoted by \(\text{FpO}(U)\). The complement of FpO-set is named a fuzzy pre-closed set (or FpC-set) and \(\text{FpC}(U)\) refers to the class of all FpC-sets in \((U, \sigma)\).

\[\text{Definition 6 ([16])}. In a FTS \((U, \sigma)\), the fuzzy pre-closure \(cl_p(G)\) and pre-interior \(\text{int}_p(G)\) of any \(G \in I^U\) are given as \(cl_p(G) = \cap\{H : H \in \text{FpC}(U), G \subseteq H\}\) and \(\text{int}_p(G) = \cup\{H : H \in \text{FpO}(U), H \subseteq G\}\), respectively.

\[\text{Notation}.\] For a FTS \((U, \sigma)\) and \(u_t \in FP(U), \sigma_u\) refers to a FpO-set containing \(u_t\) and it is called a Fp-open neighborhood (or FpO-nbd) of \(u_t\). In general, \(\sigma u\) refers to a FpO-set containing \(G\).

\[\text{Theorem 1 ([16])}. For any \(G, H \in I^U\) in \((U, \sigma)\), the next properties hold:

\[(i)\] \(cl_p(\emptyset) = \emptyset\).

\[(ii)\] \(G \in \text{FpC}(U, \sigma) \iff G = cl_p(G)\).

\[(iii)\] If \(G \subseteq H\), then \(cl_p(G) \subseteq cl_p(H)\).

\[(iv)\] \(cl_p(cl_p(G)) = cl_p(G)\).

\[(v)\] \(cl_p(G \cup H) \supseteq cl_p(G) \cup cl_p(H)\).

\[(vi)\] \(cl_p(G \cap H) \subseteq cl_p(G) \cap cl_p(H)\).\]
Definition 7 \([\text{[27]}]\). A F-set \(K\) in a FTS \((\mathcal{U}, \sigma)\) is named:

(i) A fuzzy regular pre-closed set (briefly, FpC-set) if \(K = \text{cl}_p(\text{int}_p(K))\).

(ii) A fuzzy locally pre-closed set (briefly, FLpC-set) if there is \(H \in \text{FO}(\mathcal{U})\) and \(G \in \text{FC}(\mathcal{U})\) such that \(K = H \cap G\).

Definition 8 \([\text{[28]}]\). A F-set \(K\) in a FTS \((\mathcal{U}, \sigma)\) is named a fuzzy pre-generalized closed set (briefly, FpgC-set) if \(\text{cl}_p(K) \subseteq H\) whenever \(K \subseteq H\) and \(H \in \text{FP}(\mathcal{U})\). The collection of all FpgC-sets in \((\mathcal{U}, \sigma)\) is denoted by \(\text{FpgC}(\mathcal{U})\). The complement of FpgC-set is a fuzzy pre generalized open set (or FpgO-set). The class of all FpgO-sets is symbolized by \(\text{FpgO}(\mathcal{U})\).

Remark 1 \([\text{[27,28]}]\). In a FTS \((\mathcal{U}, \sigma)\), we have:

(i) Every FpC-set (FpO-set) is a FpgC-set (FpgO-set), but not conversely.

(ii) Every FpC-set is a FLpC-set, but not conversely.

Proposition 2 \([\text{[27,28]}]\). A FpgC-set in a FTS \((\mathcal{U}, \sigma)\) is a FpC-set \(\iff\) it is FLpC-set.

Definition 9 \([\text{[18]}]\). A FTS \((\mathcal{U}, \sigma)\) is named:

(i) \(FT_0\) iff for any \(u, v_1 \in \text{FP}(\mathcal{U})\) with \(u \sqsubseteq v_1\) implies \(u \sqsupseteq \text{cl}(u)\) or \(\text{cl}(u) \sqsupseteq v_1\).

(ii) \(FT_1\) iff \(u \in \text{FC}(\mathcal{U})\) for any F-point \(u \in \text{FP}(\mathcal{U})\).

(iii) \(FT_2\) iff for any \(u, v_1 \in \text{FP}(\mathcal{U})\) with \(u \sqsubseteq v_1\), there are \(O_u\) and \(O_{v_1} \in \sigma\) such that \(O_u \sqsubseteq O_{v_1}\).

(iv) \(FR_0\) iff for any \(u, v_1, v_2 \in \text{FP}(\mathcal{U})\) with \(u \sqsubseteq v_1\) implies \(\text{cl}(u) \sqsupseteq v_1\).

(v) \(FR_1\) iff for any \(u, v_1 \in \text{FP}(\mathcal{U})\) with \(u \sqsubseteq v_1\), there are \(O_u, O_{v_1} \in \sigma\) such that \(O_u \sqsupseteq O_{v_1}\).

(vi) \(FR_2\) (or F-regular) iff for any \(u \in \text{FP}(\mathcal{U})\) and any FC-set \(G\) with \(u \sqsubseteq G\), there are \(O_u, O_{G} \in \sigma\) such that \(O_u \sqsupseteq O_{G}\).

Note. Evidently, \(FT_2 \implies FT_1 \implies FT_0\). Moreover, \(FR_2 \implies FR_1 \implies FR_0\). Singal-Prakash \([\text{[16]}]\) defined the fuzzy pre-separation axioms as follows.

Definition 10. A FTS \((\mathcal{U}, \sigma)\) is called:

(i) Fuzzy pre-\(T_0\) (or \(FPT_0\)) iff for any \(u, v_1 \in \text{FP}(\mathcal{U})\) \((u \neq v)\) there is \(FpO\)-set \(K\) such that \(u \in K \subseteq (v_1)^\circ\) or \(v_1 \in K \subseteq (u)^\circ\).

(ii) Fuzzy pre-\(T_1\) (or \(FPT_1\)) iff each F-point is a FpC-set.

(iii) Fuzzy pre-\(T_2\) (or \(FPT_2\)) iff for any \(u, v_1 \in \text{FP}(\mathcal{U})\) \((u \neq v)\) there are \(FpO\)-sets \(K, G\) such that \(u \in K \subseteq (v_1)^\circ\), \(v_1 \in G \subseteq (u)^\circ\) and \(K \subseteq G\).

(iv) Fuzzy pre-regular (or \(FPR\)) iff for any \(u \in \text{FP}(\mathcal{U})\) and \(F \in \text{FC}(\mathcal{U})\), there are \(H, G \in \text{FC}(\mathcal{U})\) such that \(u \in H \subseteq F \subseteq G \subseteq G^\circ\).

(v) Fuzzy pre-normal (or \(FPN\)) iff for any two \(FC\)-sets \(F, G\) with \(F \subseteq G^\circ\), there are \(H, K \in \text{FC}(\mathcal{U})\) such that \(F \subseteq H, G \subseteq K\) and \(H \subseteq K^\circ\).

(vi) \(FPT_3\) (resp. \(FPT_4\)) iff it is \(FPR\) (resp. \(FPN\)) and \(FPT_1\).

3. Stronger Forms of Fuzzy Pre-\(T_i\) Spaces, \(i = 0, 1, 2\)

Here, we introduce new stronger classes of fuzzy pre-separation axioms called, \(FpT_i\), \(i = 0, 1, 2, 3, 4\) spaces via \(FpO\)-sets and discuss some of their properties. We build some elucidative example to elaborate the obtained findings and relationships.

Definition 11. A FTS \((\mathcal{U}, \sigma)\) is said to be:

(i) Fuzzy pre-\(T_0\) (briefly, \(FpT_0\)) iff for each \(u, v_1 \in \text{FP}(\mathcal{U})\) with \(u \sqsubseteq v_1\) there is \(O_{v_1} \in \text{FP}(\mathcal{U})\) such that \(v_1 \sqsupseteq O_{v_1}\) or there is \(O_{v_1} \in \text{FP}(\mathcal{U})\) such that \(u \sqsupseteq O_{v_1}\).

(ii) Fuzzy pre-\(T_1\) (briefly, \(FpT_1\)) iff for each \(u, v_1 \in \text{FP}(\mathcal{U})\) with \(u \sqsubseteq v_1\) there are \(O_u, O_{v_1} \in \text{FP}(\mathcal{U})\) such that \(v_1 \sqsupseteq O_{v_1}\) and \(u \sqsupseteq O_{v_1}\).

(iii) Fuzzy pre-\(T_2\) (briefly, \(FpT_2\)) iff for each \(u, v_1 \in \text{FP}(\mathcal{U})\) with \(u \sqsubseteq v_1\) there are \(O_u, O_{v_1} \in \text{FP}(\mathcal{U})\) such that \(O_u \sqsubseteq O_{v_1}\).
Lemma 2. For a FTS \((U, \sigma)\), \(u_r \in FP(U)\), and \(H \subseteq I^U\), we have:

(i) \(u_r \in int_p(H) \iff \exists O_{u_r} \subseteq FpO(U)\) such that \(O_{u_r} \subseteq H\),
(ii) \(u_r \in \text{cl}_p(H) \iff O_{u_r} \subseteq \text{cl}_p(H)\) for any \(O_{u_r} \subseteq FpO(U)\),
(iii) \(G \subseteq \text{cl}_p(H) \iff G \subseteq \text{cl}_p(H)\) for any \(G \subseteq FpO(U)\).

Proof. (i): Obvious.
(ii): \(u_r \in \text{cl}_p(H) \iff u_r \subseteq \text{cl}_p(H)\) for any \(O_{u_r} \subseteq FpO(U)\).
(iii): \(G \subseteq \text{cl}_p(H) \iff G \subseteq \text{cl}_p(H)\) for any \(G \subseteq FpO(U)\).

\(\square\)

In the next, we give some characterizations of \(FpT_0\) space, \(i = 0, 1, 2\).

Theorem 2. A FTS \((U, \sigma)\) is \(FpT_0\) \(\iff\) for any \(u_r, v_1 \in FP(U)\) with \(u_r \ni v_1\) implies \(u_r \ni \text{cl}_p(v_1)\) or \(\text{cl}_p(u_r) \ni v_1\).

Proof. Necessity, if \((U, \sigma)\) is \(FpT_0\) and \(u_r \ni v_1\) for any \(u_r, v_1 \in FP(U)\), there is \(O_{u_r} \subseteq FpO(U)\) such that \(v_1 \ni O_{u_r}\) or there is \(O_{v_1} \subseteq FpO(U)\) such that \(u_r \ni O_{v_1}\). From Lemma 2(ii), we obtain \(u_r \ni \text{cl}_p(v_1)\) or \(\text{cl}_p(u_r) \ni v_1\).

Conversely, let \(u_r \ni v_1\). By the given assumption \(u_r \ni \text{cl}_p(v_1)\) or \(\text{cl}_p(u_r) \ni v_1\), we obtain from Lemma 2(ii) that there is \(O_{u_r} \subseteq FpO(U)\) such that \(v_1 \ni O_{u_r}\) or there is \(O_{v_1} \subseteq FpO(U)\) with \(u_r \ni O_{v_1}\). Hence \((U, \sigma)\) is \(FpT_0\).

\(\square\)

Remark 2. Evidently, each \(FpT_0\) is \(FpT_0\), but the converse does not hold. This is demonstrated in the next example.

Example 1. Let \(U = \{u, v\}\). Consider two F-sets on \(U\) as follows:
\(F_1 = \{(u_{0.75}, v_1)\}\) and \(F_2 = \{(u_1, v_0)\}\). Clearly, the class \(\sigma = \{0, 1, F_1, F_2\}\) is a FT on \(U\). One can check that \(F_2\) is a FpO-set because \(cl(F_2) = 0.1\) and \(int(cl(F_2)) = 0.1\). So, \(F_2 \subseteq \text{int}(cl(F_2)) = 0.1\). On the other hand, \(F_2 \not\subseteq \sigma\), so it is not FO-set in \((U, \sigma)\). Evidently, \((U, \sigma)\) is \(FpT_0\) but not \(FT_0\). Indeed, for \(u_{0.8} \ni v_{0.9}\) there is no FO-sets \(O_{u_{0.8}}\) and \(O_{v_{0.9}}\) such that \(u_{0.8} \ni O_{v_{0.9}}\).

Theorem 3. For a FTS \((U, \sigma)\), the next items are equivalent:

(i) \((U, \sigma)\) is \(FpT_0\)
(ii) The pre-closures for any two different crisp F-points are distinct.

Proof. (i) \(\Rightarrow\) (ii). Assume that \((U, \sigma)\) is \(FpT_0\) and \(u_1, v_1\) are two different crisp F-points in \(U\). Since \((U, \sigma)\) is \(FpT_0\), there is \(O_{u_1} \subseteq FpO(U)\) such that \(v_1 \ni O_{u_1}\) or there is \(O_{v_1} \subseteq FpO(U)\) such that \(u_1 \ni O_{v_1}\). Consider \(v_1 \ni O_{u_1}\) implies \(v_1 \subseteq (O_{u_1})^c\) which is \(FP\)-set in \((U, \sigma)\).

So, we have, \(v_1 \subseteq \text{cl}_p(v_1) \subseteq (O_{u_1})^c\). Clearly, \(u_1 \notin (O_{u_1})^c\), \(\text{cl}_p(v_1) \subseteq (O_{u_1})^c\), so we obtain \(u_1 \notin \text{cl}_p(v_1)\). But, \(u_1 \subseteq \text{cl}_p(v_1)\). Therefore, \(\text{cl}_p(u_1) \neq \text{cl}_p(v_1)\).

(ii) \(\Rightarrow\) (i). Suppose that \(u_r \ni v_1\) for any \(u_r, v_1 \in FP(U)\). In particular, \(u_1 \ni v_1\) where \(u_1, v_1\) are two different crisp F-points in \(U\) such that \(\text{cl}_p(u_1) \neq \text{cl}_p(v_1)\). Clearly, \((\text{cl}_p(u_1))^c \subseteq (u_1)^c\).

Since \(u_1 \subseteq u_1\), \((u_1)^c \subseteq (u_1)^c\). Therefore, \((\text{cl}_p(u_1))^c \subseteq (u_1)^c\). By the given \(v_1 \notin \text{cl}_p(u_1)\), it follows that \(v_1 \in (\text{cl}_p(u_1))^c\). Thus \(v_1 \subseteq (\text{cl}_p(u_1))^c = \text{cl}_p(v_1)\) which is \(FpO\)-set contains \(v_1\) with \(O_{v_1} = (\text{cl}_p(u_1))^c \subseteq (u_1)^c\). This implies that \(O_{v_1} \ni u_1\). Hence, \((U, \sigma)\) is \(FpT_0\).

\(\square\)

Remark 3. In the theorem mentioned above, the requirement for crispness is necessary.

The following example illustrates this observation.
Example 2. Consider \( U = [0, 1] \) and the \( F \)-sets \( G, H \) on \( U \) which are given as follows: \( G(n) = 1 - 2n \) for \( 0 \leq n < 0.5 \) and \( G(n) = 0.5 \) for \( 0.5 \leq n \leq 1 \), \( H(n) = 0 \) for \( 0 \leq n < 0.5 \) and \( H(n) = 0.5 \) for \( 0.5 \leq n \leq 1 \). Then the class \( \sigma = \{ 0, 1 \} \) is a FT on \( U \). One can check that not every pair of different crisp \( F \)-points have distinct pre-closures and also, we can verify that \((U, \sigma)\) is not \( FpT_0 \).

Theorem 4. A FTS \((U, \sigma)\) is \( FpT_1 \) if and only if for each \( u, v \in FP(U) \) with \( u, v \notin FP(U) \) implies \( u, v \notin cl_p(u) \) and \( cl_p(u \cap v) \).

Proof. By using an analogous approach to the given in Theorem 2. □

Theorem 5. A FTS \((U, \sigma)\) is \( FpT_1 \) if and only if for each \( u, v \in FP(U) \) with \( u, v \notin FP(U) \) implies \( cl_p(u) = u \) for all \( u \in FP(U) \).

Proof. Necessity, assume that \((U, \sigma)\) is \( FpT_1 \) and \( u, v \in FP(U) \) with \( u, v \notin FP(U) \). Then there is \( FP\)-set \( O_U \) such that \( u, v \in O_U \). This implies \( O_U \subseteq \{ u \} \). So, \( \{ u \} \) is \( FP\)-set. Thus, \( u \in FP(U) \). Hence, \( cl_p(u) = u \).

Conversely, for any \( u, v \in FP(U) \) with \( u, v \notin FP(U) \), we have \( u \subseteq \{ u \} \) and \( v \subseteq \{ v \} \). From Theorem 2(iii), we obtain \( cl_p(u) \subseteq \{ u \} \). From Lemma 2(ii), we obtain \( cl_p(u) \subseteq \{ u \} \).

Correspondingly, suppose that \( u, v \in FP(U) \) with \( u \notin FP(U) \) and \( v \notin FP(U) \). By the given, there is \( O_u \in FP(O(U)) \) such that \( cl_p(O_u) \subseteq \{ u \} \). From Lemma 2(ii), there is \( O_v \in FP(O(U)) \) such that \( O_v \subseteq \{ v \} \). This finishes the proof. □

Theorem 6. A FTS \((U, \sigma)\) is \( FpT_2 \) if and only if for each \( u, v \in FP(U) \) with \( u, v \notin FP(U) \) implies \( cl_p(u) \subset cl_p(u \cap v) \).

Proof. Necessity, assume that \((U, \sigma)\) is \( FpT_2 \) and \( u, v \in FP(U) \) with \( u, v \notin FP(U) \). Then there are \( O_u \), \( O_v \in FP(O(U)) \) such that \( O_u \subseteq \{ u \} \). From Lemma 2(ii), we obtain \( cl_p(O_u) \subseteq \{ u \} \).

Conversely, suppose that \( u, v \in FP(U) \) for any \( u, v \in FP(U) \). By the given, there is \( O_u \in FP(O(U)) \) such that \( cl_p(O_u) \subseteq \{ u \} \). From Lemma 2(ii), there is \( O_v \in FP(O(U)) \) such that \( O_v \subseteq \{ v \} \). This finishes the proof. □

Theorem 7. For a \( FpT_2 \) space \((U, \sigma)\), we have \( u \in \{ cl_p(K) : u \in K \} \).

Proof. Assume that \((U, \sigma)\) is \( FpT_2 \) and \( u \in FP(U) \). Then for each \( u \), \( v \in FP(U) \), there are \( FP\)-sets \( O_v \) and \( O_u \) such that \( K \subseteq O_v \). From Lemma 2(ii), we have \( \{ v \} \subseteq \{ u \} \). From (iv) of Proposition 1, we have \( \{ cl_p(K) : u \in K \} \subseteq \{ u \} \). From (iv) of Proposition 1, we have \( \{ cl_p(K) : u \in K \} \subseteq \{ u \} \). This completes the proof. □

Theorem 8. In a \( FpT_1 \) space \((U, \sigma)\). If \( K \in FP(O(U)) \) such that \( K \subseteq \{ cl_p(K) : u \in K \} \subseteq \{ u \} \). Then \((U, \sigma)\) is \( FpT_2 \).

Proof. Assume that \((U, \sigma)\) is \( FpT_1 \) and \( u \in FP(U) \). Then for each \( u \), \( v \in FP(U) \), there are \( FP\)-sets \( O_v \) and \( O_u \) such that \( K \subseteq O_v \). From Lemma 2(ii), we have \( \{ v \} \subseteq \{ u \} \). From (iv) of Proposition 1, we have \( \{ cl_p(K) : u \in K \} \subseteq \{ u \} \). From (iv) of Proposition 1, we have \( \{ cl_p(K) : u \in K \} \subseteq \{ u \} \). This completes the proof. □

Proposition 3. A FTS \((U, \sigma)\) is \( FpT_2 \) if every crisp \( F \)-point is \( FP\)-set.

Proof. It is obvious. □

Theorem 9. Each \( FpT_i \) space \((U, \sigma)\) is \( FpT_{i-1} \), \( i = 1, 2 \).

Proof. It is clear. □

Theorem 10. A \( F \)-subspace \((Y, \sigma_Y)\) of a \( FpT_1 \) space \((U, \sigma)\) is \( FpT_i \), \( i = 0, 1, 2 \).

Proof. It is clear. □
Proof. We will prove the case \( i = 1 \). The proofs for the rest of the cases are analogous.

Assume that \( u_r, v_t \) are two \( F \)-points in \( Y \subset U \) with \( u_r \bar{q} v_t \), which are also \( F \)-points in \( U \) and \( u_t \bar{q} v_t \). Since \((U, \sigma)\) is \( Fp-T_1 \), there is \( O_u, O_v \in FpO(U) \) with \( v_t \bar{q} O_u \) and \( u_t \bar{q} O_v \). Thus \( O_u \cap \chi_Y \) and \( O_v \cap \chi_Y \) are \( FpO \)-sets in \((Y, \sigma_Y)\). Put \( O_u^* = O_u \cap \chi_Y \) and \( O_v^* = O_v \cap \chi_Y \), then \( v_t \bar{q} O_u^* \) and \( u_t \bar{q} O_v^* \). Hence, the result is established. \( \square \)

Definition 12. A FTS \((U, \sigma)\) is named \( Fp-T_{\frac{1}{2}} \) iff every \( Fp_{gC} \)-set in \((U, \sigma)\) is a \( FpC \)-set.

Proposition 4. A FTS \((U, \sigma)\) is \( Fp-T_{\frac{1}{2}} \) \iff Every \( Fp_{gC} \)-set is a \( FLpC \)-set.

Proof. Assume that \((U, \sigma)\) is \( Fp-T_{\frac{1}{2}} \), then any \( Fp_{gC} \)-set is \( FpC \)-set. By Remark 1, every \( FpC \)-set is \( FLpC \)-set. Thus, the result holds.

Conversely, it directly follows from Proposition 2. \( \square \)

Definition 13. A FTS \((U, \sigma)\) is named \( Fp \)-symmetric iff for any \( u_r, v_t \in FP(U) \) with \( u_r \bar{q} cl_p(v_t) \) implies \( v_t \bar{q} cl_p(u_r) \).

Theorem 11. For a FTS \((U, \sigma)\), the following items are equivalent:

(i) \((U, \sigma)\) is \( Fp \)-symmetric
(ii) \( cl_p(u_r) \bar{q} K \) for any \( FpC \)-set \( K \) with \( u_r \bar{q} K \).

Proof. (i) \( \Rightarrow \) (ii). Assume that \( K \) is a \( FpC \)-set on \( U \) with \( u_r \bar{q} K \). Then we have \( cl_p(v_t) \subseteq K \) for any \( v_t \in K \). Thus, from Proposition 1(iii), we have \( u_r \bar{q} cl_p(v_t) \). Since \((U, \sigma)\) is \( Fp \)-symmetric, we obtain \( v_t \bar{q} cl_p(u_r) \) for any \( v_t \in K \) and so by Lemma 2(ii) we have for any \( v_t \in K \), there is \( O_v \in FpO(U) \) with \( u_t \bar{q} O_v \). Take \( H = \cup \{ O_v : v_t \in K \text{ and } u_t \bar{q} O_v \} \), then \( H = O_K \) and \( u_t \bar{q} H \) implies \( u_r \in H^c \) and so, \( cl_p(u_r) \subseteq H^c \); that is, \( cl_p(u_r) \bar{q} H \). Therefore, \( cl_p(u_r) \bar{q} K \).

(ii) \( \Rightarrow \) (i). It directly follows from the hypothesis. \( \square \)

Corollary 1. A FTS \((U, \sigma)\) is \( Fp \)-symmetric \iff for any \( u_r \in FP(U) \), \( u_r \) is a \( FpgC \)-set.

Proof. It follows from Theorem 11 and Lemma 2(ii). \( \square \)

Remark 4. Evidently, each \( Fp-T_{\frac{1}{2}} \) space \((U, \sigma)\) is \( Fp \)-symmetric, but not conversely.

Example 3. Let \( U = \{ a, b, c, d \} \) and \( \sigma = \{ 0, 1, a_1 \lor b_1, c_1 \lor d_1 \} \). Then \( \sigma \) is a FT on \( U \). One can verify that \( \sigma \) is \( Fp \)-symmetric. But not \( Fp-T_{\frac{1}{2}} \). Indeed, \( cl_p(a_1) \neq a_1 \). Further, \( \sigma \) is not \( Fp-T_{\frac{1}{2}} \).

Proposition 5. A FTS \((U, \sigma)\) is \( Fp-T_{\frac{1}{2}} \) \iff it is \( Fp \)-symmetric and \( Fp-T_0 \).

Proof. Clearly, if \((U, \sigma)\) is \( Fp-T_1 \), then it is \( Fp \)-symmetric and \( Fp-T_0 \). Conversely, assume \((U, \sigma)\) is \( Fp \)-symmetric and \( Fp-T_0 \). If \( u_r \bar{q} v_t \), we have either \( u_r \bar{q} cl_p(v_t) \) or \( v_t \bar{q} cl_p(u_r) \). By \( Fp \)-symmetric, we obtain \( u_r \bar{q} cl_p(v_t) \) and \( v_t \bar{q} cl_p(u_r) \) for any \( u_r, v_t \in FP(U) \). This completes the proof. \( \square \)

Corollary 2. For a \( Fp \)-symmetric space \((U, \sigma)\). The next properties are equivalent:

(i) \((U, \sigma)\) is \( Fp-T_0 \)
(ii) \((U, \sigma)\) is \( Fp-T_{\frac{1}{2}} \)
(iii) \((U, \sigma)\) is \( Fp-T_{\frac{1}{2}} \).

Proof. It is obvious. \( \square \)
4. Stronger Forms of Fuzzy Pre-$R_i$ Spaces, $i = 0, 1, 2, 3$

Here, we apply the concept of $FpO$-sets to introduce new stronger classes of fuzzy pre-separation axioms called, $Fp-R_i$, $i = 0, 1, 2, 3$ spaces and examine some of its properties.

**Definition 14.** An FTS $(U, \sigma)$ is named:

(i) Fuzzy pre $R_0$ (briefly, $Fp-R_0$) iff $cl_p(u_r) \subseteq O_{u_r}$ for each $FpO$-set $O_{u_r}$ and $u_r \in O_{u_r}$.

(ii) Fuzzy pre $R_1$ (briefly, $Fp-R_1$) iff for each $u_r, v_1 \in FP(U)$ with $u_r qcl_p(v_1)$, there are $FpO$-sets $O_{u_r}, O_{v_1}$ containing $u_r, v_1$, respectively, such that $O_{u_r} \sigma O_{v_1}$.

**Proposition 6.** A FTS $(U, \sigma)$ is $Fp-R_0$ iff $u_r qcl_p(v_1)$ implies $v_1 qcl_p(u_r)$ for all $u_r, v_1 \in FP(U)$.

**Proof.** Necessity. It follows from Definition 14 and Lemma 2(ii).

Conversely. Assume that for any $u_r, v_1 \in FP(U)$ with $v_1 qcl_p(u_r)$ implies $u_r qcl_p(v_1)$. By Lemma 2(ii), we obtain $v_1 \sigma O_{u_r}$ and from Proposition 1(iv), we have $cl_p(u_r) \subseteq O_{u_r}$ for all $O_{u_r} \in FpO(U)$. Therefore, $(U, \sigma)$ is $Fp-R_0$.

**Corollary 3.** A FTS $(U, \sigma)$ is $Fp-R_0$ iff $cl_p(u_r) \subseteq \{ O_{u_r} : O_{u_r} \in FpO(U) \}$ for all $u_r \in FP(U)$.

**Theorem 12.** For a FTS $(U, \sigma)$ and $K \in FpC(U)$, the next items are equivalent:

(i) $(U, \sigma)$ is $Fp-R_0$.

(ii) $u_r qK$ implies there is a $FpO$-set $O_K$ such that $u_r \sigma O_K$.

(iii) $u_r qK$ implies $cl_p(u_r) \subseteq O_K$.

(iv) $u_r qcl_p(v_1)$ implies $cl_p(u_r) \subseteq O_{v_1}$.

**Proof.** (i) $\implies$ (ii). Assume that $K \in FpC(U)$ with $u_r qK$, we have $u_r \subseteq K^c = O_{u_r}$, that is, $O_{u_r}$ is a $FpO$-set containing $u_r$. Since $(U, \sigma)$ is $Fp-R_0$, we obtain $cl_p(u_r) \subseteq K^c = O_{u_r}$ implies that $K \subseteq (cl_p(u_r))^c = O_K$. Clearly, $u_r \subseteq cl_p(u_r)$ and so, $(cl_p(u_r))^c \subseteq (u_r)^c$. This means that $u_r q(cl_p(u_r))^c = O_K$.

(ii) $\implies$ (iii). Let $K \in FpC(U)$ with $u_r qK$. By given there is $O_K \in FpO(U)$ such that $u_r \sigma O_K$. From Lemma 2(ii), we obtain $cl_p(u_r) \subseteq O_K$.

(iii) $\implies$ (iv). It is obvious.

(iv) $\implies$ (i). Assume $u_r, v_1 \in FP(U)$ with $u_r qcl_p(v_1)$. By (iv) we have, $cl_p(u_r) \subset cl_p(v_1)$. Clearly, $v_1 \subseteq cl_p(v_1)$ and so, by Proposition 1(iii) we obtain, $v_1 \sigma O_{cl_p(u_r)}$. Therefore, $(U, \sigma)$ is $Fp-R_0$.

**Theorem 13.** A FTS $(U, \sigma)$ is $Fp-R_1$ iff for any $u_r, v_1 \in FP(U)$ with $u_r qcl_p(v_1)$, there are $O_{cl_p(u_r)}, O_{cl_p(v_1)} \in FpO(U)$ such that $O_{cl_p(u_r)} \sigma O_{cl_p(v_1)}$.

**Proof.** It directly follows from Theorem 12 and Definition 14.

**Proposition 7.** Each $Fp-R_i$ space $(U, \sigma)$ is $Fp-R_0$.

**Proof.** Assume that $u_r qcl_p(v_1)$ for any $u_r, v_1 \in FP(U)$. Since $(U, \sigma)$ is $Fp-R_i$, we have from the above theorem that there are $O_{u_r}, O_{v_1} \in FpO(U)$ such that $O_{u_r} \sigma O_{v_1}$. This implies that $u_r qO_{v_1}$. By Proposition 1(iv), $cl_p(v_1) \subseteq O_{v_1}$. Hence $(U, \sigma)$ is $Fp-R_0$.

**Theorem 14.** A $F$-subspace $(Y, \sigma_Y)$ of $Fp-R_i$ space $(U, \sigma)$ is $Fp-R_i$, $i = 0, 1$.

**Proof.** We will demonstrate the proof for the case $i = 1$.

Assume that $u_r qcl_p(v_1)$ for any $u_r, v_1 \in FP(Y)$ which are also, $F$-points in $U$. By given, $(U, \sigma)$ is $Fp-R_1$, there are $O_{u_r}, O_{v_1} \in FpO(U)$ with $O_{u_r} \sigma O_{v_1}$. Take $O^{*}_{u_r} = O_{u_r} \cap \chi_Y$ and $O^{*}_{v_1} = O_{v_1} \cap \chi_Y$, which are $FpO$-set in $(Y, \sigma_Y)$ such that $O^{*}_{u_r} \sigma O^{*}_{v_1}$.

The proof for the other case can be proved in a similar manner.
Proposition 8. In a FTS $(U, \sigma)$, each $Fp-T_i$ is $Fp-R_{i-1}$ for $i = 1, 2$.

Proof. It is clear. □

Remark 5. The converse of the above proposition may not be true. Consider Example 3, one can verify that $(U, \sigma)$ is $Fp-R_0$ but not $Fp-T_1$.

Proposition 9. A FTS $(U, \sigma)$ is $Fp-T_i$ if and only if it is both $Fp-T_{i-1}$ and $Fp-R_{i-1}$ for $i = 1, 2$.

Proof. We will demonstrate for the case $i = 2$.

Necessity. It follows from Proposition 8 and Theorem 9.

Conversely. Assume that $(U, \sigma)$ is $Fp-T_1, Fp-T_1$, and $u, \tilde{v}_1$ for any $u, v_1 \in FP(U)$. By Theorem 2, we have $u, \tilde{v}_1 \in FP(U)$. By Theorem 2, we have $u, \tilde{v}_1 \in FpC(U)$ with $O_{u, \tilde{v}_1} \in FOS(U)$ with $O_{u, \tilde{v}_1} \in FOS(U)$.

The result holds. The proof for the other case can be proved in a similar manner. □

Definition 15. A FTS $(U, \sigma)$ is said to be:

(i) Fuzzy pre-regular (briefly, $Fp-R_2$) if for any $u, \tilde{v} \in FP(U)$ and any $H \in FpC(U)$ with $u, \tilde{v} \in H$, there are $O_{u, \tilde{v}} \in FOS(U)$ such that $O_{u, \tilde{v}} \in FOS(U)$.

(ii) Fuzzy pre-normal (briefly, $Fp-R_3$) if for any $G, H \in FpC(U)$ with $G \cong H$, there are $O_{G, H} \in FOS(U)$ such that $O_{G, H} \in FOS(U)$.

(iii) $Fp-T_3(Fp-T_4)$ if it is $Fp-R_2(Fp-R_3)$ and $Fp-T_1$.

First, we show that the axioms $Fp-R_i, i = 0, 1, 2, 3$ and $Fp-T_i, i = 1, 2, 3, 4$ are harmonic.

Theorem 15. For a FTS $(U, \sigma)$, we have:

$$Fp - R_3 \land Fp - R_0 \implies Fp - R_2 \implies Fp - R_1 \implies Fp - R_0$$

Proof. (i) Assume that $H$ is any $FpC$-set with $u, \tilde{v} \in FP(U)$. Since $(U, \sigma)$ is $Fp-R_3$, we have $O_{u, \tilde{v}} \in FOS(U)$ such that $O_{u, \tilde{v}} \in FOS(U)$.

Conversely. Assume that $(U, \sigma)$ is $Fp-R_3, Fp-R_3$, and $u, \tilde{v}_1$ for any $u, v_1 \in FP(U)$. By Theorem 2, we have $u, \tilde{v}_1 \in FpC(U)$ with $O_{u, \tilde{v}_1} \in FOS(U)$ with $O_{u, \tilde{v}_1} \in FOS(U)$.

The proof for the other case can be proved in a similar manner. □

Theorem 16. For a FTS $(U, \sigma)$, we have:

$$Fp - T_4 \implies Fp - T_3 \implies Fp - T_2 \implies Fp - T_1 \implies Fp - T_0$$

Proof. Assume that $(U, \sigma)$ is $Fp-T_4$ this means that it is $Fp-R_3$ and $Fp-T_1$. So that $(U, \sigma)$ is $Fp - R_0$ (by Proposition 8). Suppose that $u, \tilde{v} \in FP(U)$ and $H$ is a $FpC$-set with $u, \tilde{v} \in H$, we have from Theorem 12(3), $cl_{Fp}(u, \tilde{v}) \in H$ where $cl_{Fp}(u, \tilde{v})$ and $h_E$ are $FpC$-sets. Since $(U, \sigma)$ is $Fp-R_3$, there are $FpC$-sets $O_{cl_{Fp}(u, \tilde{v})} \in FOS(U)$ such that $O_{cl_{Fp}(u, \tilde{v})} \in FOS(U)$.

Thus $(U, \sigma)$ is $Fp-R_2$. We obtain the required result.

From the above theorems, Definition 15 and Proposition 9, we obtain the next result.

Corollary 4. For a FTS $(U, \sigma)$, the next implications hold:

$$Fp - T_4 \implies Fp - T_3 \implies Fp - T_2 \implies Fp - T_1 \implies Fp - T_0$$

$$Fp - R_3 \land Fp - R_0 \implies Fp - R_2 \implies Fp - R_1 \implies Fp - R_0.$$
Theorem 17. For a FTS $(U, \sigma)$, the next items are equivalent:

(i) $(U, \sigma)$ is $Fp-R_2$

(ii) For any $u_r \in FP(U)$ and any $O_{u_r} \in FpO(U)$, there is $H \in FpO(U)$ containing $u_r$ such that $cl_p(H) \subseteq O_{u_r}$.

**Proof.** (i) $\Rightarrow$ (ii). Assume that $u_r \in FP(U)$ and $O_{u_r} \in FpO(U)$ containing $u_r$. Then $O_{u_r}^c = G$ which is a FPC-set. Evidently, $O_{u_r} \subseteq O_{u_r}^c$ and so, $u_r \in O_{u_r}^c$. Since $(U, \sigma)$ is $Fp-R_2$, there are FPO-sets $H, O_{O_{u_r}}$ containing $u_r$ and $O_{u_r}$, respectively, such that $H \subseteq O_{O_{u_r}}$ and so, $cl_p(H) \subseteq O_{O_{u_r}}$. Clearly, $O_{u_r} \subseteq O_{O_{u_r}} = O_G$ and so, $O_{u_r} \cap O_{O_{u_r}} \subseteq O_{u_r}$. Therefore, $cl_p(H) \subseteq O_{u_r}$.

(ii) $\Rightarrow$ (i). Let $u_r \in FP(U), G \in FpC(U)$ with $u_r \notin G$, then $u_r \notin G$ which is FPO-set containing $u_r$. By hypothesis, there is $H \in FpO(U)$ containing $u_r$ such that $cl_p(H) \subseteq O_{u_r} = G^c$ implies that $G \subseteq [cl_p(H)]^c = O_G$. Clearly, $cl_p(H) \subseteq [cl_p(H)]^c = O_G$ and so, $H \subseteq O_G$. We obtain the required result. \qed

Theorem 18. A FTS $(U, \sigma)$ is $Fp-R_2$ $\iff$ for any $u_r \in FP(U)$ and $H \in FpC(U)$ with $u_r \notin H$, there are $O_{u_r}, O_H \in FpO(U)$ such that $cl_p(O_{u_r}) \subseteq cl_p(H)$. \qed

**Proof.** Assume that $u_r \in FP(U)$ and $H \in FpC(U)$ with $u_r \notin H$. Since $(U, \sigma)$ is $Fp-R_2$, there are $O_{u_r}, O_H$ such that $H \subseteq O_{u_r}$. From Lemma 2(iii), we obtain $cl_p(O_{u_r}) \subseteq cl_p(H)$.

Conversely, It directly follows from the hypothesis. \qed

Theorem 19. For a FTS $(U, \sigma)$, the next items are equivalent:

(i) $(U, \sigma)$ is $Fp-R_3$

(ii) For each $K \in FpC(U)$ and any $O_K \in FpO(U)$, there is $O_K^c \in FpO(U)$ containing $K$ such that $cl_p(O_K^c) \subseteq K$.

**Proof.** Assume that $K \in FpC(U)$ and $O_K \in FpO(U)$ containing $K$, we have $O_K^c \in FpC(U)$. Clearly, $O_K \subseteq O_K^c \subseteq O_K$. Since $(U, \sigma)$ is $Fp-R_3$, there are $O_{K, O_K} \in FpO(U)$ such that $O_{K, O_K} \subseteq O_K$ and so, $cl_p(O_{K, O_K}) \subseteq O_K$. Since $O_{K, O_K} \subseteq O_K = K$, we have $O_{K, O_K} \subseteq O_K$ and so, $cl_p(O_{K, O_K}) \subseteq O_K$. We obtain the required result.

Conversely. Assume that $H, K \in FpC(U)$ with $H \subseteq K$. Then $H \subseteq K^c = O_K$ which is FPO-set containing $H$. By hypothesis, there is $O_H \in FpO(U)$ such that $cl_p(O_H) \subseteq K^c = O_K$, we have $K \subseteq [cl_p(O_H)]^c = O_K$. Since $cl_p(O_H) \subseteq [cl_p(O_H)]^c = O_K$, then $O_K \subseteq O_H$. We obtain the result. \qed

Theorem 20. A FTS $(U, \sigma)$ is $Fp-R_3$ $\iff$ for any $G, K \in FpC(U)$ with $G \subseteq K$, there are $O_G, O_K \in FpO(U)$ such that $cl_p(O_G) \subseteq cl_p(O_K)$. \qed

**Proof.** It bears similarity to that in Theorem 18. \qed

Definition 16. A FTS $(U, \sigma)$ is named fuzzy pre generalized regular (or Fpg-R2) if for any FpgC-set $H$ with $u_r \notin H$, there are $O_{u_r}, O_H \in FOS(U)$ such that $O_{u_r} \subseteq O_H$.

Remark 6. Clearly, each Fpg-R2 space $(U, \sigma)$ is $Fp-R_2$, but not conversely.

Example 4. Consider $U = \{u, v\}$ and $\sigma = \{\emptyset, \{u\}, G = \{(0.5, v_0.3), K = \{(0.5, v_0.7)\}\}$, then $\sigma$ is a FST on $U$. One can verify that $(U, \sigma)$ is $Fp-R_2$ but not Fpg-R2.
Theorem 21. A FTS \((U, \sigma)\) is \(F_{pg-R_2} \iff \) it is both \(F_{p-R_2}\) and \(F_{p-T_1}^{\perp}\).

Proof. If \((U, \sigma)\) is \(F_{pg-R_2}\), then by Remark 6 it is \(F_{p-R_2}\). For any \(F_{pgC}\)-set \(H\) and \(F\)-point \(u\) with \(u \notin \partial H\), there are \(O_u, O_H \in \text{FOS}(U)\) such that \(O_u \nsubseteq O_H\) implies \(O_u \nsubseteq \partial H\). By Lemma 2(ii), \(u \notin \partial \{\text{cl}_p(H)\}^c\) and so, \(\text{cl}_p(H) \subseteq H\). Hence, \(\text{cl}_p(H) = H\). This mean that each \(F_{pgC}\)-set is \(FC\)-set. Hence \((U, \sigma)\) is \(F_{p-T_1}^{\perp}\).

The converse part is obvious. \(\square\)

5. More Characterizations and Relations

Here, we will discuss more characterizations and investigate the relations between these classes and the induced topologies. The relationships for \(F_{p-T_i}, i = 0, 1, 2\) and other separation axioms are presented. First, let us present the next interesting results.

By Lemma 1 and utilizing the properties of the characteristic function, one can prove the following propositions.

Proposition 10. For a TS \((U, \tau)\), the next structures form FTS on \(U\) generated by \(\tau\):

(i) \(\sigma_\tau = \{K \in I^1 : S(K) \in \tau\}\)

(ii) \(\sigma_\tau = \{\chi_k : K \in \tau\}\).

Proposition 11. In a FTS \((U, \sigma)\), the next structures form crisp topologies on \(U\) induced by \(\sigma\):

(i) \(\tau_\sigma = \{S(K) : K \in \sigma\}\)

(ii) \(\tau_\sigma = \{K \subseteq U : \chi_K \in \sigma\}\).

Now, it is easy to prove the next result.

Proposition 12. For a TS \((U, \tau)\) and a FTS \((U, \sigma)\), we have:

(i) \(\chi \in \sigma_\tau\) for any \(\chi \in I = [0, 1]\).

(ii) For any \(K \in \tau\), \(\chi_K \in \sigma_\tau\). Moreover \(\sigma_\tau \subseteq \sigma_\tau\).

(iii) \(\tau = \tau_\sigma\) and \(\sigma \subseteq \sigma_\sigma\).

Theorem 22. \((U, \sigma_\tau)\) is \(F_{p-T_i} \iff (U, \tau)\) is \(F_{p-T_i}, i = 0, 1\)

Proof. For the case \(i = 1\). Necessity. Assume that \((U, \sigma_\tau)\) is \(F_{p-T_1}\) and \(u \neq v\), we have \(u \notin \partial v\). In particular, \(u \notin \partial v\) implies that there are \(O_u, O_v \in \text{FpO}(U)\) such that \(v \nsubseteq O_u, O_v\). Take \(O_u = S(O_u) \in \tau\) and \(O_v = S(O_v) \in \tau\) such where \(O_u, O_v\) are pre-open sets with \(v \nsubseteq O_u\) and \(u \notin O_v\). Hence \((U, \tau)\) is \(F_{p-T_1}\).

Conversely. Assume that \((U, \tau)\) is \(F_{p-T_1}\) and for any \(u, v \in \text{FpO}(U)\) with \(u \notin \partial v\), we have either \(u \neq v\) or \((u = v, r + t < 1)\). If \(u \neq v\), there are pre-open sets \(O_u\) and \(O_v\) with \(u \nsubseteq O_v\) and \(v \nsubseteq O_u\). Take \(O_u = \chi_{O_u} \in \sigma_\tau\) and \(O_v = \chi_{O_v} \in \sigma_\tau\) where \(O_u, O_v \in \text{FpO}(U)\), then \(v \nsubseteq \partial O_u\) and \(u \notin \partial O_v\). The result holds.

If \((u = v, r + t < 1)\). Take \(O_u = \tau \in \sigma_\tau\) and \(O_v = \tau \in \sigma_\tau\) to be the required \(\text{FpO}\)-sets. Hence \((U, \sigma_\tau)\) is \(F_{p-T_1}\). The proof for the other case is similar. \(\square\)

Theorem 23. If \((U, \sigma)\) is \(F_{p-T_i}\), then \((U, \tau_\sigma)\) is \(F_{p-T_i}, i = 0, 1\)

Proof. As a sample we prove the case \(i = 1\).

Assume that \((U, \sigma)\) is \(F_{p-T_1}\) and \(u \in U\). Then \(\text{cl}_p(u_1) = u_1\) and so \(u_1 \in \text{FpO}(U)\) implies that \(S(u_1) = U - \{u_1\} \in \tau_\sigma\). It follows that \(\{u_1\}\) is pre-closed for all \(u_1 \in U\). The proof is complete.

Similarly, the proof for the other case can be conducted. \(\square\)

Note. Is the converse being true, we leave this point as open problem.
Theorem 24. For a fully stratified FTS $(U, \sigma)$ we have, $(U, \sigma)$ is $\text{Fp-T}_i \iff (U, \tau_i)$ is $\text{pre-T}_i$, $i = 0, 1$.

Proof. For the case $i = 1$. The necessary part follows from the above theorem.

Conversely. Assume that, $(U, \tau_i)$ is $\text{pre-T}_1$ and $u_i \not\in O_1$ for any $u_i, v_i \in FP(U)$, we have either $u \neq v$ or $(u = v, r + t < 1)$. If $u \neq v$, there are $O_u = S(\chi_{O_u}) \in \tau_v$ and $O_v = S(\chi_{O_v}) \in \tau_u$ which are pre-open sets with $u \notin O_v$ and $v \notin O_u$. Now, put $O_u = \chi_{O_u} \in \sigma$ and $O_v = \chi_{O_v} \in \sigma$, where $O_u, O_v \in FP(O(U))$, then $v \not\in O_u$ and $u \not\in O_v$. The result holds.

If $(u = v, r + t < 1)$. Take $O_u = \tau \in \sigma$ and $O_v = \sigma \in \sigma$ to be the required $\text{FpO}$-sets.

Hence $(U, \sigma)$ is $\text{Fp-T}_1$. The proof for the other case is similar. $\square$

Theorem 25. For a fully stratified FTS $(U, \sigma)$. If $(U, \tau_i)$ is $\text{pre-T}_i$, then $(U, \sigma)$ is $\text{Fp-T}_i$, $i = 0, 1, 2$.

Proof. For the case $i = 0$. Assume that $(U, \tau_i)$ is $\text{pre-T}_i$ and for any $u_i, v_i \in FP(U)$ with $u_i \not\in O_i$ we have, either $u \neq v$ or $(u = v, r + t < 1)$. If $u \neq v$, there is $O_u = \tau_u \in \tau_v$ which is a pre-open set such that $v \notin O_u$ or there is $O_v \in \tau_u$ which is a pre-open set such that $u \notin O_v$. Now put $O_u = \chi_{O_u} \in \sigma$ which is a $\text{FpO}$-set with $\tau \not\in O_u$ or take $O_v = \chi_{O_v} \in \sigma$ which is a $\text{FpO}$-set with $\tau \not\in O_v$. The result holds. If $(u = v, r + t < 1)$. Take $O_u = \tau \in \sigma$ and $O_v = \sigma \in \sigma$ to be the required $\text{FpO}$-sets.

Hence $(U, \sigma)$ is $\text{Fp-T}_1$. The proof for the other cases is similar. $\square$

Theorem 26. If $(U, \tau)$ is $\text{pre-T}_2$, then $(U, \sigma)$ is $\text{FpT}_2$.

Proof. Assume that $u_i \not\in O_i$ for any $u_i, v_i \in FP(U)$, we have either $u \neq v$ or $(u = v, r + t < 1)$.

If $u \neq v$. Since $(U, \tau)$ is $\text{pre-T}_2$, there are pre-open sets $O_u, O_v$, such that $O_u \cap O_v = \emptyset$.

Now put $O_u = \chi_{O_u} \in \sigma$ and $O_v = \chi_{O_v} \in \sigma$. Where $O_u, O_v \in FP(O(U))$ with $O_u \not\in O_v$. The result holds.

If $(u = v, r + t < 1)$. Take $O_u = \tau \in \sigma$ and $O_v = \sigma \in \sigma$ to be the required $\text{FpO}$-sets.

Hence $(U, \sigma)$ is $\text{Fp-T}_2$. $\square$

Lemma 3. For a TS $(U, \tau)$ and $\tau \in FP(U)$, we have:

(i) $u \in \sigma_r$ for all $0 \neq r < 1$.

(ii) $\text{cl}_{\sigma_r}(\chi_{\{x\}}) = \chi_{\text{cl}_{\sigma_r}(\chi_{\{x\}})}$

Proof. It is obvious. $\square$

Theorem 27. $(U, \tau)$ is $\text{pre-T}_0 \iff (U, \sigma_r)$ is $\text{Fp-T}_0$.

Proof. “$\Rightarrow$”. Assume that $u_i \in FP(U)$. From Lemma 3, we have $u_i \in \sigma_r$, that is $\text{cl}_{\sigma_r}(u_i) = u_i \subseteq O_u$, for each $O_u \in FP(O(U))$. When $r = 1$. Since $(U, \tau)$ is $\text{pre-T}_0$, we have $\text{cl}_{\tau}(u_1) = O_{u_1}$. Hence $(U, \sigma_r)$ is $\text{Fp-T}_0$.

“$\Leftarrow$”. Assume that $(U, \sigma_r)$ is $\text{Fp-T}_0$ and $u \in \text{cl}_{\tau}(v)$, then $u_i \not\in \text{cl}_{\sigma_r}(v_i)$ implies $u_i \not\in O_v$. By Lemma 2(ii), we obtain $\tau \not\in \text{cl}_{\sigma_r}(u_i)$ implies $v \in \text{cl}_{\tau}(u)$. Hence $(U, \tau)$ is $\text{pre-T}_0$. $\square$

Theorem 28. If $(U, \tau)$ is $\text{pre-T}_1$, then $(U, \sigma_r)$ is $\text{Fp-T}_1$.

Proof. Assume that $(U, \tau)$ is $\text{pre-T}_1$ and $u_i, v_i \in FP(U)$ with $u_i \not\in O_i$, then either $u \neq v$ or $(u = v, r + t < 1)$. If $u \neq v$. Now we have two cases:

(a) If $u \neq v \implies \text{cl}_{\tau}(u) \neq \text{cl}_{\tau}(v)$ or $\text{cl}_{\tau}(u) = \text{cl}_{\tau}(v)$

• If $\text{cl}_{\tau}(u) \neq \text{cl}_{\tau}(v)$, there are pre-open sets $O_u, O_v$ such that $O_u \cap O_v = \emptyset$. Put $O_u = \chi_{O_u} \in \sigma_r$ and $O_v = \chi_{O_v} \in \sigma_r$ which are $\text{FpO}$-sets with $O_u \not\in O_v$. The result holds.
• $\text{cl}_\tau(u) = \text{cl}_\tau(v)$, this case is impossible because $(U, \tau)$ is pre-$R_1$.

(b) If $(u = v, r + t < 1)$. Take $O_{u_t} = r \in \sigma_\tau$ and $O_{v_t} = t \in \sigma_\tau$ to be the required $F_pO$-sets. Hence $(U, \sigma_\tau)$ is $F_pR_1$.

\[ \square \]

Note. Is the converse being true, we leave this point as an open problem.

Theorem 29. $(U, \sigma_\tau)$ is $F_pR_2$ $\iff (U, \tau)$ is $p$-regular.

Proof. “$\Rightarrow$”. Assume that $G$ is pre-closed set in $(U, \tau)$ with $u \notin G$, we have $H = \chi_G$ is a FC-set which is $F_pC$-set with $\chi_G H$. Since $(U, \sigma_\tau)$ is $F_pR_2$, there is $O_{u_t}, O_H \in \text{FOS}(U)$ such that $O_{u_t} \subseteq O_H$. Hence, there are $O_{u_t} \subseteq O_G \subseteq T$ such that $O_{u_t} = \chi_{O_{u_t}}O_H = \chi_{O_G}$ and $O_{u} \cap O_G = \emptyset$. Therefore, $(U, \tau)$ is $p$-regular.

“$\Leftarrow$”. Assume that $H \in F_pC(U)$ with $u \notin H$, then there is pre-closed set $B$ in $(U, \tau)$ such that $H = \chi_{O_{u_t}}$ with $u \notin B$. Since $(U, \tau)$ is $p$-regular, there are $O_{u_t}, O_B \subseteq \tau$ such that $O_{u_t} \cap O_B = \emptyset$. Put $O_{u_t} = \chi_{O_B}$ and $O_H = \chi_{O_{u_t}}$, then there are $O_{u_t}$ and $O_H \subseteq \sigma_\tau$ with $O_{u_t} \subseteq O_H$. Hence $(U, \sigma_\tau)$ is $F_pR_2$. \[ \square \]

Remark 7. Clearly, each $FT_i$ space $(U, \sigma)$ is $F_pT_i$ space, for $i = 0, 1, 2$.

Remark 8. Clearly, each $Fp-T_i$ space $(U, \sigma)$ is $FPT_i$ space, for $i = 0, 1, 2$.

Corollary 5. From the Definitions 9 and 10, Theorems 15 and 16, and Remarks 7 and 8, we can summarize the interrelationships between $FT_i$, $Fp-T_i$, and $FPT_i$ separation axioms as follows:

\[
\begin{align*}
FT_2 & \Rightarrow FT_1 \Rightarrow FT_0 \\
Fp - T_2 & \Rightarrow Fp - T_1 \Rightarrow Fp - T_0 \\
FPT_2 & \Rightarrow FPT_1 \Rightarrow FPT_0
\end{align*}
\]

6. Conclusions

Recently, it has been demonstrated that topology and its generalizations, like supra topology and infra topology, are robust frameworks for analyzing data and modeling real-world problems; see, [29–31]. Topological structure is a beneficial tool to prove many mathematical results as well. For instance, connectedness is a key point to prove the theorem of intermediate value, and compactness is a necessary condition to guarantee obtaining the maximum and minimum values for the continuous functions. Also, it is well known that separation axioms provide some categories for topological spaces [32,33] and help to prove some interesting properties of compactness and connectedness. With the goal of shedding light on the properties of separability in the framework of fuzzy topologies, we have written this article.

In this study, we have applied the concept of fuzz pre-open sets to introduce and analyze some stronger classes of fuzzy pre-separation axioms in fuzzy topology called $Fp-T_i, i = 0, 1, 2, 3, 4$, $Fp$-symmetric, and $Fp-R_i, i = 0, 1, 2, 3$. The paper thoroughly investigated the fundamental properties and unique characteristics of these classes. A comprehensive framework has been established through the presentation of related theorems and relations, demonstrating their interrelationships with other separation axioms in this context. The implications in Corollary 4 hold have been shown, but we cannot obtain examples to show that the converse in these implications may not be true, in general, expect in the case $Fp-T_0 \not\Rightarrow FT_0$. Additionally, the relations between these separation axioms and that in induced fuzzy topology have been studied, as well as the hereditary and harmonic properties of these classes have been explored.

As a future work, we intend to discuss the current fuzzy separation axioms using the other generalizations of fuzzy open sets such as fuzzy semi-open and fuzzy $b$-open
sets. Moreover, we are going to popularize the characterizations of these classes in the environments of fuzzy soft and fuzzy multi settings and look at the possible applications of them.

**Author Contributions:** Conceptualization, S.S. and T.M.A.-s.; Methodology, S.S., T.M.A.-s., A.A.A. and M.H.; Formal Analysis, S.S., T.M.A.-s., A.A.A. and M.H.; Writing—Original Draft Preparation, S.S. and T.M.A.-s.; Writing—Review & Editing, S.S., T.M.A.-s., A.A.A. and M.H.; Funding Acquisition, A.A.A. and M.H. All authors have read and agreed to the published version of the manuscript.

**Funding:** The fourth author extends her appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through a research groups program under grant RGPF2/310/44.

**Data Availability Statement:** Data are contained within the article.

**Conflicts of Interest:** The authors declare no conflict of interest.

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