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Anticipated BSDEs Driven by Fractional Brownian Motion with a Time-Delayed Generator

Pei Zhang [1,2], Adriana Irawati Nur Ibrahim [1,*] and Nur Anisah Mohamed [1]

[1] Institute of Mathematical Sciences, Faculty of Science, Universiti Malaya, Kuala Lumpur 50603, Malaysia; s2023288@siswa.um.edu.my (P.Z.); nuranisah_mohamed@um.edu.my (N.A.M.)
[2] School of Mathematics and Statistics, Suzhou University, Suzhou 234000, China
* Correspondence: adrianaibrahim@um.edu.my

Abstract: This article describes a new form of an anticipated backward stochastic differential equation (BSDE) with a time-delayed generator driven by fractional Brownian motion, further known as fractional BSDE, with a Hurst parameter $H \in (1/2, 1)$. This study expands upon the findings of the anticipated BSDE by considering the scenario when the driver is fractional Brownian motion rather instead of standard Brownian motion. Additionally, the generator incorporates not only the present and future but also the past. We will demonstrate the existence and uniqueness of the solutions to these equations by employing the fixed point theorem. Furthermore, an equivalent comparison theorem is derived.

Keywords: anticipated; BSDE; fractional Brownian motion; delayed generator; comparison theorem

MSC: 60H10; 60H20

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1. Introduction

The field of theoretical research on backward stochastic differential equations (BSDEs) has made significant progress since the introduction of Pardoux and Peng's [1] first formulation of non-linear BSDEs in 1990. Our analysis focuses on the scenario where there are two adapted processes, $Y_\cdot$ and $Z_\cdot$, used to solve the given type of BSDE:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}B_s, \quad 0 \le t \le T,$$

where the terminal value ξ is square integrable, f is the Lipschitz generator, and B_t is a standard Brownian motion.

Interest in BSDEs has increased since the work of Pardoux and Peng, primarily because of the correlation between BSDEs and stochastic control and partial differential equations (PDEs). Several publications have extensively explored these topics including research that establishes the existence and uniqueness of BSDEs under weaker conditions (He [2]; Abdelhadiet al. [3]; Zhang et al. [4]). Additionally, there have been studies that establish the connection between BSDEs and quasilinear parabolic PDEs (Pardoux and Răşcanu [5]; Ren and Xia [6]). Furthermore, a few cases with an analytical solution of BSDEs, more cases with a numerical solution only (Zhang [7]; Zhao et al. [8]; Gobet et al. [9]), and a wide range of applications of BSDEs in a variety of fields, such as finance, stochastic optimal control problems, physics, and biology (examples can be found in [10–12]).

As the BSDE theory advances, a growing number of models are being investigated. In 2009, Peng and Yang [13] proposed the following concerning anticipated BSDEs, a fundamental category of BSDEs:

$$\begin{cases} Y_t = \xi_T + \int_t^T f(s, Y_s, Z_s, Y_{s+\rho(s)}, Z_{s+\varrho(s)})\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}B_s, \quad 0 \le t \le T; \\ Y_t = \xi_t, \quad T \le t \le T+N; \\ Z_t = \eta_t, \quad T \le t \le T+N, \end{cases}$$

where $\rho(s)$ and $\varrho(s)$ are two deterministic $\mathbb{R}^+$-valued continuous functions defined on $[0, T]$, respectively, which also satisfy the following conditions: $t \le t + \rho(t) \le T + N$, $t \le t + \varrho(t) \le T + N$; $\int_t^T f(s + \rho(s))\,\mathrm{d}s \le L \int_t^{T+N} f(s)\mathrm{d}s$, $\int_t^T f(s + \varrho(s))\,\mathrm{d}s \le L \int_t^{T+N} f(s)\mathrm{d}s$; additionally, the authors established the existence and uniqueness theorem of the aforementioned equations. Feng [14] investigated the existence of a solution for the anticipated BSDE with both Lipschitz and non-Lipschitz generators, denoted as f. Zhang et al. [15] derived solutions for mean-field anticipated BSDEs in the presence of a time-delayed generator function f. Wang and Cui [16] introduced a new type of differential equation called the anticipated backward doubly stochastic differential equation. They used this equation to solve various stochastic control problems by exploiting the relationship between stochastic differential delay equations and anticipated BSDEs.

BSDEs involving time-delayed generators (Delong and Imkeller [17]) are given as below:

$$Y_t = \xi + \int_t^T f(s, Y_{s-h_1(s)}, Z_{s-h_2(s)})\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}B_s, \quad 0 \le t \le T,$$

where $0 \le h_1(s), h_2(s) \le T$, and the generator f is dependent on the past value of a solution. He et al. [2] and Zhuang [18] later examined certain delay and anticipated BSDEs as the generalization of Peng and Yang [13] and Delong and Imkeller [17].

The fractional Brownian motion (fBm, for short) B_t^H with the Hurst parameter $H \in (1/2, 1)$ was first introduced by Kolmogorove [19] in 1940. It is a centered Gaussian process with good properties such as self-similarity and long-wall correlation, which makes it reasonable and efficient to use fBm as a random noise term in stochastic models in the fields of communication engineering, finance, and economics. Consequently, it is crucial to examine the stability, existence, and uniqueness of solutions to BSDEs driven by fBm; that is, the disturbance source of traditional BSDEs will change from white noise to a more general fBm and fractional BSDEs will be obtained. Firstly, Bender [20] established an explicit solution of a type of linear fractional BSDE with a Hurst parameter H by using the solution of linear parabolic PDEs. Soon afterwards, Hu and Peng [21] solved the nonlinear fractional BSDEs with the Hurst parameter $H > 1/2$ by means of the quasiconditional expectation, which has the general form of:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}B_s^H, \quad 0 \le t \le T.$$

The research on fractional BSDEs is steadily increasing. For example, Borkowska [22] examined generalised BSDEs driven by fBm. Wen and Shi [23] solved anticipated backward stochastic differential equations driven by fractional Brownian motion with $H > 1/2$. Inspired by this work, Yu [24] studied the same model under the non-Lipschitz condition, which is weaker than the ones in Wen and Shi [23], and both of their papers obtained some general results via a rigorous approach with an associated Skorohod integral. Recently, Douissi et al. [25] demonstrated a novel anticipated BSDE of the mean-field type while the driver is fractional Brownian motion, which can be viewed as an improvement of the result in the above research. In particular, they used two different fixed pointed theorem methods to prove the existence and uniqueness of the solution to this kind of BSDE. The two methods correspond to two hypotheses of generator f: one is under the Lipschitz continuity condition, and the other is a stronger condition, which makes the proof more convenient.

Recently, Aidara and Sylla [26] proved the existence and uniqueness of fractional BSDEs with a delayed generator as follows:

$$Y_t = \xi + \int_t^T f(s, \eta_s, Y_{s-u}, Z_{s-u}) \,\mathrm{d}s - \int_t^T Z(s) \,\mathrm{d}B_s^H, \quad 0 \le t \le T, \tag{1}$$

where $0 \le u \le T$ and the Hurst parameter of fBM H is greater than $1/2$.

Nevertheless, the study of BSDEs driven by fractional Brownian motion has not yet been explored, in which the generator f takes into account not only both present and future times but also the past time. Due to the stability, self-similarity, and auto-correlation of the process increments of fBm, especially the positive correlation of its increments when the Hurst parameter H is greater than $1/2$, the equation after replacing standard Brownian motion with the fBm driver can have significant applications in stochastic optimal control problems with delay. Before that, it is necessary to solve some properties of the solution of such an equation, such as existence and uniqueness and the comparison theorem. In order to advance the theory of BSDEs, our research will concentrate on analyzing the BSDEs in this case. The results of this work need to recall and define the Malliavin derivative and integral operations related to fBm. The results could then enrich the theory of BSDEs and potentially motivate future research into stochastic optimal control issues.

The main aim of this study is to examine fundamental characteristics of a novel kind of BSDEs, specifically Anticipated BSDEs with Delayed Generators (DABSDEs). These DABSDEs are driven by fBm with the Hurst parameter $H \in (1/2, 1)$:

$$\begin{cases} -\mathrm{d}Y_t = f\Big(t, Y_{t-d_1(t)}, Z_{t-d_2(t)}, Y_t, Z_t, Y_{t+d_3(t)}, Z_{t+d_4(t)}\Big) \,\mathrm{d}t - Z_t \,\mathrm{d}B_t^H, \quad 0 \le t \le T; \\ Y_t = \xi_t, \quad T \le t \le T+N; \\ Z_t = \eta_t, \quad T \le t \le T+N, \end{cases} \tag{2}$$

where h_i are four deterministic $\mathbb{R}^+$-valued continuous functions. In particular, we aim to demonstrate the existence and uniqueness of the solutions to these equations, as well as to construct a corresponding comparison theorem.

The subsequent section of this study's framework is structured in the following manner: Section 2 provides a comprehensive introduction to the fractional DABSDE model, which is our proposed new BSDE model. In Section 3, we prove the existence and uniqueness of the adapted solutions to the DABSDE form using the fixed point theorem. Finally, in Section 4, the comparison theorem for this type of model's solutions is derived.

2. Preliminaries

Commence by providing definitions pertaining to fractional Brownian motion and associated assumptions. Additionally, establish a foundation for the article by referencing specific fundamental results of propositions. To enhance the profundity of the discourse, it is recommended that readers refer to scholarly works including those by Decreusefond and Üstünel [27], Hu [28] and Duncan et al. [29].

2.1. Preliminaries on the Fractional Brownian Motion

Let $B^H = \{B_t^H\}_{t \ge 0}$ be a fractional Brownian motion with its index (Hurst parameter) $H \in (0, 1)$, which is defined on the complete probability space $(\Omega, \mathcal{F}, P)$ with the filtration $\mathcal{F}$ generated by fBm $\{B_t^H\}_{t \ge 0}$. Its covariance kernel is given by

$$R_H(s, t) = E_H\Big[B_t^H B_s^H\Big] = \frac{1}{2}\Big(t^{2H} + s^{2H} - |t - s|^{2H}\Big), \;\; s, t \ge 0.$$

In the particular case of $H = 1/2$, the fBm B_t^H is identical to the standard Brownian motion, which has continuous and almost surely non-differentiable sample paths and independent increments, while the fBm with $H \ne 1/2$ does not have this property. Specifically, when $H \in (0, 1/2)$, fBm displays a negative correlation property and the paths are more irregular than those of the standard Brownian motion; however, it exhibits a positive correlation and long-range dependence properties with a more regular path for $H \in (1/2, 1)$. Let $H \geq \frac{1}{2}$, note that this study will only cover the one-dimensional case in order to make the presentation simpler. Next, consider the following definitions as given by Hu [28]. First, we define a Hilbert scalar product $\langle \xi, \eta \rangle_t$ as

$$\langle \xi, \eta \rangle_t = \int_0^t \int_0^t \phi(u-v) \xi_u \eta_v \, \mathrm{d}u \, \mathrm{d}v, \text{ and } \|\xi\|_t^2 = \langle \xi, \xi \rangle_t,$$

where ξ and η are two continuous functions on $[0, T]$ and $\phi(x) := 2H(2H-1)|x|^{2H-2}$ for all $x \in \mathbb{R}$. Let Θ_t be the completion of the continuous functions under this Hilbert norm.

Let $\xi_1, \xi_2, \dots, \xi_k, \dots$ be continuous functions on $[0, T]$ and f is a polynomial of n variables. Denote $\mathcal{P}_T$ as the set of all polynomials of fBm over $[0, T]$, which contains all elements of the following form

$$F(\omega) = f\left(\int_0^T \xi_1(t) \, \mathrm{d}B_t^H, \dots, \int_0^T \xi_n(t) \, \mathrm{d}B_t^H \right).$$

Define the Malliavin derivative D_s^H of the polynomial function F from $L^2(\Omega, \mathcal{F}, P) \to (\Omega, \mathcal{F}, \Theta_t)$ as follows:

$$D_s^H F = \sum_{k=1}^n \frac{\partial f}{\partial x_k} \left(\int_0^T \xi_1(t) \, \mathrm{d}B_t^H, \dots, \int_0^T \xi_n(t) \, \mathrm{d}B_t^H \right) \xi_k(s), \;\; 0 \leq s \leq T.$$

For $F \in \mathcal{P}_T$, let $\mathbb{D}_{1,2}^H$ be the completion of $\mathcal{P}_T$ with respect to the norm

$$\|F\|_{H,1,2} := E\left[\left(\|F\|_T^2 \right)^{\frac{1}{2}} \right] + E\left[\left(\left\| D_s^H F \right\|_T \right)^{\frac{1}{2}} \right].$$

Additionally, we define another derivative

$$\mathbb{D}_t^H F = \int_0^T \phi(t-s) D_s^H F \, \mathrm{d}s.$$

Proposition 1 (Hu [28], Proposition 6.25)**.** *If $F_s : (\Omega, \mathcal{F}, P) \to \Theta_t$ is a continuous process such that $E\left[\|F\|_T^2 + \int_0^T \int_0^T |\mathbb{D}_s^H F_t|^2 \, \mathrm{d}s \, \mathrm{d}t \right] \leq \infty$, denoted as $F_s \in \mathbb{L}_H^{1,2}$, then the Itô-type stochastic integral $\int_0^T F_s \, \mathrm{d}B_s^H$ exists in $L^2(\Omega, \mathcal{F}, P)$ and*

$$E\left[\int_0^T F_s \, \mathrm{d}B_s^H \right] = 0,$$

$$E\left[\int_0^T F_s \, \mathrm{d}B_s^H \right]^2 = E\left[\|F\|_T^2 + \int_0^T \int_0^T \mathbb{D}_s^H F_t \mathbb{D}_t^H F_s \, \mathrm{d}s \, \mathrm{d}t \right].$$

Proposition 2 (Hu [28], Theorem 10.3)**.** *For $i = 1, 2$, let $f_i(s)$ and $g_i(s)$, for $s \in [0, T]$, be real-valued stochastic processes where $g_i(s) \in \mathbb{D}_{1,2}^H$, satisfying $E\left[\int_0^T (|f_i(s)|^2 + |g_i(s)|^2) \, \mathrm{d}s \right] < \infty$. And then, suppose that $\mathbb{D}_t^H g_i(s)$ are continuously differentiable with respect to $0 \leq s, t \leq T$ for almost all $\omega \in \Omega$, and $E\left[\int_0^T \int_0^T |\mathbb{D}_t^H g_i(s)|^2 \, \mathrm{d}s \, \mathrm{d}t \right] < \infty$. Denote*

$$Y_i(t) = \int_0^t f_i(s) \, \mathrm{d}s + \int_0^t g_i(s) \, \mathrm{d}B_s^H, 0 \leq t \leq T.$$

Then

$$Y_1(t)Y_2(t) = \int_0^t Y_1(s)f_2(s)\,\mathrm{d}s + \int_0^t Y_1(s)g_2(s)\,\mathrm{d}B_s^H + \int_0^t Y_2(s)f_1(s)\,\mathrm{d}s + \int_0^t Y_2(s)g_1(s)\,\mathrm{d}B_s^H + \int_0^t \mathbb{D}_s^H Y_1(s)g_2(s)\,\mathrm{d}s + \int_0^t \mathbb{D}_s^H Y_2(s)g_1(s)\,\mathrm{d}s.$$

Meanwhile, the fixed point theorem, which will be used in this paper, is briefly introduced below.

Proposition 3 (Granas and Dugundji [30], Theorem 1.1)**.** *Let (Y,d) be a complete metric space and $F: Y \to Y$ be contractive. Then, F has a unique fixed point u, and $F^n(y) \to u$ for each $y \in Y$.*

Its main idea is to obtain fixed points by constructing contractive mappings in order to obtain the existence and uniqueness of solutions to BSDEs.

2.2. Assumptions

Assume $(\Omega, \mathcal{F}, P)$ is a complete probability space with natural filtration $\mathcal{F}_t$. Consider the sets below:

$L^2(\mathcal{F}_t;\mathbb{R}) := \Big\{\varphi : \Omega \to \mathbb{R} \big| E[|\varphi|^2] < \infty$, and φ is $\mathcal{F}_t -$ measurable$\Big\}$;

$L^2_{\mathcal{F}}(0,T;\mathbb{R}) := \Big\{\varphi : [0,T]\times\Omega \to \mathbb{R} \big|\ E\Big[\int_0^T |\varphi(t)|^2\,\mathrm{d}t\Big] < \infty$, and φ is progressively measurable process$\Big\}$;

$\mathcal{C}^{k,l}([0,T]\times\mathbb{R}) := \Big\{\varphi : [0,T]\times\mathbb{R} \to \mathbb{R} \big|\ \varphi$ is k times differentiable with respect to $t \in [0,T]$ and l times continuously differentiable with respect to $x \in \mathbb{R}\Big\}$;

$\mathcal{C}^{k,l}_{pol}([0,T]\times\mathbb{R}) := \Big\{\varphi \big|\ \varphi \in \mathcal{C}^{k,l}([0,T]\times\mathbb{R})$, where each derivatives of φ are of polynomial growth$\Big\}$;

$\mathcal{W}_{[0,T]} := \Big\{\varphi \big|\ \varphi \in \mathcal{C}^{1,3}_{pol}([0,T]\times\mathbb{R})$ with $\frac{\partial \varphi}{\partial t} \in \mathcal{C}^{0,1}_{pol}([0,T]\times\mathbb{R})\Big\}$.

Let $\tilde{\mathcal{W}}_{[0,T+N]}$ and $\tilde{\mathcal{W}}^H_{[0,T+N]}$ be the completions of $\mathcal{W}_{[0,T+N]}$ under the following norms, respectively:

$$\|\varphi(\cdot)\|_\beta = \Big\{E\Big[\int_0^{T+N} e^{\beta t}|\varphi_t|^2\,\mathrm{d}t\Big]\Big\}^2,$$

$$\|\varphi(\cdot)\|_\beta = \Big\{E\Big[\int_0^{T+N} e^{\beta t}t^{2H-1}|\varphi_t|^2\,\mathrm{d}t\Big]\Big\}^2,$$

where $\beta \geq 0$ is a constant; according to Lemma 7 of Maticiuc and Nie [31], we obtain $\tilde{\mathcal{W}}^H_{[0,T+N]} \subseteq \tilde{\mathcal{W}}_{[0,T+N]} \subseteq \mathcal{W}_{[0,T+N]} \subseteq \mathbb{L}^{1,2}_{H,[0,T+N]}$. Moreover, we shall introduce assumptions regarding h_i. For $0 \leq t \leq T$, suppose $h_i(t)$ are $\mathbb{R}^+$-valued continuous functions, where $i = 1,2,3,4$. Consider the following assumptions:

(A1) For all $0 \leq t \leq T$, $t - h_i(t) \in [0,t]$, $i = 1,2$; $t + h_i(t) \in [t, T+N]$, $i = 3,4$, where N is a positive constant;

(A2) For all non-negative and integrable $f(\cdot)$, there exists a positive constant L, such that
$\int_t^T f(s-h_i(s))\,\mathrm{d}s \le L\int_t^{T+N} f(s)\,\mathrm{d}s,\ i=1,2,$
$\int_t^T f(s+h_i(s))\,\mathrm{d}s \le L\int_t^{T+N} f(s)\,\mathrm{d}s,\ i=3,4.$

The assumption about function h_i in (A1) and (A2) shows the generator f includes not only the past and the present but also the future solutions intuitively, and we can find that the form of the simplest function h that can satisfy (A1) and (A2) is constant delay and $h_i(t)=t$.

Next, we shall discuss assumptions regarding the generator f. Assume that $f(t,\omega,\theta,\vartheta,y,z,\phi,\psi):[0,T]\times\Omega\times L^2(\mathcal{F}_{s'},\mathbb{R})\times L^2(\mathcal{F}_s,\mathbb{R})\times\mathbb{R}^2\times L^2(\mathcal{F}_{r'},\mathbb{R})\times L^2(\mathcal{F}_r,\mathbb{R})\to L^2(\mathcal{F}_t,\mathbb{R})$ is a $\mathcal{C}^{0,1}_{pol}$-continuous function, where $0\le s',s\le t\le r',r\le T+N, t\in[0,T]$, and that it satisfies the following assumptions:

(A3) For all $t\in[0,T]$, there exists a positive constant C, such that

$$\begin{aligned}&\left|f(t,\theta,\vartheta,y,z,\phi,\psi)-f(t,\theta',\vartheta',y',z',\phi',\psi')\right|\\ &\le C\Big(|\theta-\theta'|+t^{H-\frac{1}{2}}|\vartheta-\vartheta'|+|y-y'|+t^{H-\frac{1}{2}}|z-z'|\\ &\quad+E\Big[|\phi-\phi'|+t^{H-\frac{1}{2}}|\psi-\psi'|\Big|\mathcal{F}_t\Big]\Big),\end{aligned}$$

where $y,y',z,z'\in\mathbb{R};\theta,\theta',\vartheta,\vartheta'\in L^2_{\mathcal{F}}(0,t;\mathbb{R});\phi,\phi',\psi,\psi'\in L^2_{\mathcal{F}}(t,T+N;\mathbb{R});$

(A4) $E\Big[\int_0^T|f(t,0,0,0,0,0,0)|^2\mathrm{d_t}\Big]<\infty$, and $f(t,0,0,0,0,0,0)\in L^2_{\mathcal{F}}(0,T+N;\mathbb{R})$.

The above two assumptions ensure the generator f is stronger than the uniformly Lipschitz condition. Compared with other BSDEs driven by standard Brownian motion, the coefficient of z in our assumption (A3) is $t^{H-1/2}$. That means strengthening the condition of the coefficient f with respect to z will make the proof of our results more convenient.

3. Existence and Uniqueness Results

If there exists a pair of processes $(Y_t,Z_t)\in\tilde{\mathcal{W}}_{[0,T+N]}\times\tilde{\mathcal{W}}^H_{[0,T+N]}$ that satisfy the fractional DABSDEs of model (2), we refer to (Y_t,Z_t) as a solution of Equation (2).

Theorem 1. *Let the assumptions (A3) and (A4) be satisfied and for $i=1,2,3,4$, $h_i(t)$ to satisfy (A1) and (A2). Suppose that $\xi_t\in\tilde{\mathcal{W}}_{[T,T+N]}$ and $\eta_t\in\tilde{\mathcal{W}}^H_{[T,T+N]}$, then, for the fractional DABSDE (2), there exists a unique solution $(Y_t,Z_t)_{t\in[0,T+N]}\in\tilde{\mathcal{W}}_{[0,T+N]}\times\tilde{\mathcal{W}}^H_{[0,T+N]}$.*

Proof. The fractional DABSDE given in Equation (2) can be rephrased as

$$\begin{cases}Y_t=\xi_T+\int_t^T f\Big(t,Y_{t-h_1(t)},Z_{t-h_2(t)},Y_t,Z_t,Y_{t+h_3(t)},Z_{t+h_4(t)}\Big)\mathrm{d}s-\int_t^T Z_t\,\mathrm{d}B^H_s, & 0\le t\le T;\\ Y_t=\xi_t, & T\le t\le T+N;\\ Z_t=\eta_t, & T\le t\le T+N.\end{cases}\tag{3}$$

Then, we define the mapping $\mathbb{I}:\tilde{\mathcal{W}}_{[0,T+N]}\times\tilde{\mathcal{W}}^H_{[0,T+N]}\to\tilde{\mathcal{W}}_{[0,T+N]}\times\tilde{\mathcal{W}}^H_{[0,T+N]}$ such that $(Y_\cdot,Z_\cdot)=\mathbb{I}(y_\cdot,z_\cdot)$. For two arbitrary elements $(y_\cdot,z_\cdot),(y'_\cdot,z'_\cdot)\in\tilde{\mathcal{W}}_{[0,T+N]}\times\tilde{\mathcal{W}}^H_{[0,T+N]}$, set $\mathbb{I}(y_\cdot,z_\cdot)=(Y_\cdot,Z_\cdot)$, and $\mathbb{I}(y'_\cdot,z'_\cdot)=(Y'_\cdot,Z'_\cdot)$, we can then define the differences as follows:

$$\begin{aligned}(\hat{Y}_\cdot,\hat{Z}_\cdot)&:=(Y_\cdot-Y'_\cdot,Z_\cdot-Z'_\cdot),\\ (\hat{y}_\cdot,\hat{z}_\cdot)&:=(y_\cdot-y'_\cdot,z_\cdot-z'_\cdot).\end{aligned}$$

We will now prove certain expectations. Applying Itô's formula for $e^{\beta t}|\hat{Y}_t|^2, t \in [0,T]$, and using Proposition 2, we obtain

$$\begin{aligned}
\mathrm{d}\left(e^{\beta t}|\hat{Y}_t|^2\right) &= \beta e^{\beta t}\left|\hat{Y}_t\right|^2\mathrm{d}t + 2e^{\beta t}\left|\hat{Y}_t\right|\mathrm{d}\left|\hat{Y}_t\right| + e^{\beta t}\mathrm{d}\left|\hat{Y}_t\right|^2 \\
&= -2e^{\beta t}\left|\hat{Y}_t\right|\Big|f\left(t, y_{t-h_1(t)}, z_{t-h_2(t)}, y_t, z_t, y_{t+h_3(t)}, z_{t+h_4(t)}\right) \\
&\qquad - f\left(t, y'_{t-h_1(t)}, z'_{t-h_2(t)}, y'_t, z'_t, y'_{t+h_3(t)}, z'_{t+h_4(t)}\right)\Big|\mathrm{d}t \\
&\quad + 2e^{\beta t}\left|\hat{Y}_t\right|\left|\hat{Z}_t\right|\mathrm{d}B_t^H + 2e^{\beta t}\mathbb{D}_t^H\left|\hat{Y}_t\right|\left|\hat{Z}_t\right|\mathrm{d}t.
\end{aligned}$$

Taking the integral on $[t,T]$ and rearranging the terms above,

$$\begin{aligned}
&e^{\beta t}\left|\hat{Y}_t\right|^2 + \beta\int_t^T e^{\beta s}\left|\hat{Y}_s\right|^2\mathrm{d}s + 2\int_t^T e^{\beta s}\left|\hat{Y}_s\right|\left|\hat{Z}_s\right|\mathrm{d}B_s^H + 2\int_t^T e^{\beta s}\mathbb{D}_s^H\left|\hat{Y}_s\right|\left|\hat{Z}_s\right|\mathrm{d}s \\
&= e^{\beta T}\left|\hat{Y}_T\right|^2 + 2\int_t^T e^{\beta s}\left|\hat{Y}_s\right|\Big|f\left(s, y_{s-h_1(s)}, z_{s-h_2(s)}, y_s, z_s, y_{s+h_3(s)}, z_{s+h_4(s)}\right) \\
&\qquad - f\left(s, y'_{s-h_1(s)}, z'_{s-h_2(s)}, y'_s, z'_s, y'_{s+h_3(s)}, z'_{s+h_4(s)}\right)\Big|\mathrm{d}s.
\end{aligned}$$

Furthermore, according to Equation (9) and Proposition 24 from Maticiuc and Nie [31], for all $t \in [0,T]$, there exists a suitable constant $M > 0$, such that

$$\frac{t^{2H-1}}{M}Z_t \le \mathbb{D}_t^H Y_t = \frac{\hat{\sigma}_t}{\sigma_t}Z_t \le Mt^{2H-1}Z_t.$$

Therefore,

$$\begin{aligned}
&e^{\beta t}\left|\hat{Y}_t\right|^2 + \beta\int_t^T e^{\beta s}\left|\hat{Y}_s\right|^2\mathrm{d}s + 2\int_t^T e^{\beta s}\left|\hat{Y}_s\right|\left|\hat{Z}_s\right|\mathrm{d}B_s^H + \frac{2}{M}\int_t^T e^{\beta s}s^{2H-1}\left|\hat{Z}_s\right|^2\mathrm{d}s \\
&\le e^{\beta T}\left|\hat{Y}_T\right|^2 + 2\int_t^T e^{\beta s}\left|\hat{Y}_s\right|\Big|f\left(s, y_{s-h_1(s)}, z_{s-h_2(s)}, y_s, z_s, y_{s+h_3(s)}, z_{s+h_4(s)}\right) \\
&\qquad - f\left(s, y'_{s-h_1(s)}, z'_{s-h_2(s)}, y'_s, z'_s, y'_{s+h_3(s)}, z'_{s+h_4(s)}\right)\Big|\mathrm{d}s.
\end{aligned}$$

Taking expectations on both sides and applying the fact that $2AB \le A^2 + B^2$, one has

$$\begin{aligned}
&E\left[e^{\beta t}\left|\hat{Y}_t\right|^2 + \beta\int_t^T e^{\beta s}\left|\hat{Y}_s\right|^2\mathrm{d}s + \frac{2}{M}\int_t^T e^{\beta s}s^{2H-1}\left|\hat{Z}_s\right|^2\mathrm{d}s\right] \\
&\le E\Big[e^{\beta T}\left|\hat{Y}_T\right|^2 + 2\int_t^T e^{\beta s}\left|\hat{Y}_s\right|\Big|f\left(s, y_{s-h_1(s)}, z_{s-h_2(s)}, y_s, z_s, y_{s+h_3(s)}, z_{s+h_4(s)}\right) \\
&\qquad - f\left(s, y'_{s-h_1(s)}, z'_{s-h_2(s)}, y'_s, z'_s, y'_{s+h_3(s)}, z'_{s+h_4(s)}\right)\Big|\mathrm{d}s\Big] \\
&\le E\Big[e^{\beta T}\left|\hat{Y}_T\right|^2 + \frac{\beta}{2}\int_t^T e^{\beta s}\left|\hat{Y}_s\right|^2\mathrm{d}s + \frac{2}{\beta}\int_t^T e^{\beta s}\Big|f\left(s, y_{s-h_1(s)}, z_{s-h_2(s)}, y_s, z_s, y_{s+h_3(s)}, z_{s+h_4(s)}\right) \\
&\qquad - f\left(s, y'_{s-h_1(s)}, z'_{s-h_2(s)}, y'_s, z'_s, y'_{s+h_3(s)}, z'_{s+h_4(s)}\right)\Big|^2\mathrm{d}s\Big].
\end{aligned}$$

By rearranging the terms again, we obtain

$$\begin{aligned}
&E\left[e^{\beta t}\left|\hat{Y}_t\right|^2 - e^{\beta T}\left|\hat{Y}_T\right|^2 + \frac{\beta}{2}\int_t^T e^{\beta s}\left|\hat{Y}_s\right|^2\mathrm{d}s + \frac{2}{M}\int_t^T e^{\beta s}s^{2H-1}\left|\hat{Z}_s\right|^2\mathrm{d}s\right] \\
&\le \frac{2}{\beta}E\Big[\int_t^T e^{\beta s}\Big|f\left(s, y_{s-h_1(s)}, z_{s-h_2(s)}, y_s, z_s, y_{s+h_3(s)}, z_{s+h_4(s)}\right) \\
&\qquad - f\left(s, y'_{s-h_1(s)}, z'_{s-h_2(s)}, y'_s, z'_s, y'_{s+h_3(s)}, z'_{s+h_4(s)}\right)\Big|^2\mathrm{d}s\Big]
\end{aligned} \tag{4}$$

Taking the assumption (A3) and Jensen's inequality into account, we will now determine the expectation for the right side of the preceding equation.

$$
\begin{aligned}
&E\bigg[\int_t^T e^{\beta s}\Big|f\Big(s,y_{s-h_1(s)},z_{s-h_2(s)},y_s,z_s,y_{s+h_3(s)},z_{s+h_4(s)}\Big)\\
&\qquad -f\Big(s,y'_{s-h_1(s)},z'_{s-h_2(s)},y'_s,z'_s,y'_{s+h_3(s)},z'_{s+h_4(s)}\Big)\Big|^2\mathrm{d}s\bigg]\\
&\le C^2E\bigg[\int_t^T e^{\beta s}\bigg(\Big|\hat{y}_{s-h_1(s)}\Big|+(s-h_2(s))^{H-\frac{1}{2}}\Big|\hat{z}_{s-h_2(s)}\Big|+|\hat{y}_s|+s^{H-\frac{1}{2}}|\hat{z}_s|\\
&\qquad +E\bigg[\Big|\hat{y}_{s+h_3(s)}\Big|+(s+h_4(s))^{H-\frac{1}{2}}\Big|\hat{z}_{s+h_4(s)}\Big|\bigg|\mathcal{F}_t\bigg]\bigg)^2\mathrm{d}s\bigg]\\
&\le 12C^2E\bigg[\int_t^T e^{\beta s}\Big(\Big|\hat{y}_{s-h_1(s)}\Big|^2+(s-h_2(s))^{2H-1}\Big|\hat{z}_{s-h_2(s)}\Big|^2+|\hat{y}_s|^2+s^{2H-1}|\hat{z}_s|^2\\
&\qquad +\Big|\hat{y}_{s+h_3(s)}\Big|^2+(s+h_4(s))^{2H-1}\Big|\hat{z}_{s+h_4(s)}\Big|^2\Big)\,\mathrm{d}s\bigg].
\end{aligned}
$$

By substituting the preceding result into inequality (4), letting $t=0$ and assuming (A1) and (A2), we are able to derive the following:

$$
\begin{aligned}
&E\bigg[\int_0^T e^{\beta s}\bigg(\frac{\beta}{2}|\hat{Y}_s|^2+\frac{2}{M}s^{2H-1}|\hat{Z}_s|^2\bigg)\mathrm{d}s\bigg]\\
&\le \frac{24C^2}{\beta}E\bigg[\int_0^T e^{\beta s}\Big(\Big|\hat{y}_{s-h_1(s)}\Big|^2+(s-h_2(s))^{2H-1}\Big|\hat{z}_{s-h_2(s)}\Big|^2+|\hat{y}_s|^2+s^{2H-1}|\hat{z}_s|^2\\
&\qquad +\Big|\hat{y}_{s+h_3(s)}\Big|^2+(s+h_4(s))^{2H-1}\Big|\hat{z}_{s+h_4(s)}\Big|^2\Big)\,\mathrm{d}s\bigg]\\
&\le \frac{24C^2(2L+1)}{\beta}E\bigg[\int_0^{T+N} e^{\beta s}\Big(|\hat{y}_s|^2+s^{2H-1}|\hat{z}_s|^2\Big)\mathrm{d}s\bigg].
\end{aligned}
$$

Multiplying $\frac{M}{2}$ by both sides of the preceding inequality yields

$$
\begin{aligned}
&E\bigg[\int_0^T e^{\beta s}\bigg(\frac{M\beta}{4}|\hat{Y}_s|^2+s^{2H-1}|\hat{Z}_s|^2\bigg)\mathrm{d}s\bigg]\\
&\le \frac{12C^2(2L+1)M}{\beta}E\bigg[\int_0^{T+N} e^{\beta s}\Big(|\hat{y}_s|^2+s^{2H-1}|\hat{z}_s|^2\Big)\mathrm{d}s\bigg].
\end{aligned}
$$

Letting $\beta=12C^2(2L+1)M+\frac{4}{M}$, we obtain

$$
\begin{aligned}
&E\bigg[\int_0^T e^{\beta s}\Big(|\hat{Y}_s|^2+s^{2H-1}|\hat{Z}_s|^2\Big)\mathrm{d}s\bigg]\\
&\le \frac{1}{2}E\bigg[\int_0^{T+N} e^{\beta s}\Big(|\hat{y}_s|^2+s^{2H-1}|\hat{z}_s|^2\Big)\mathrm{d}s\bigg].
\end{aligned}
$$

Hence,

$$
\big\|(\hat{Y}_\cdot,\hat{Z}_\cdot)\big\|_\beta\le\frac{1}{\sqrt{2}}\|(\hat{y}_\cdot,\hat{z}_\cdot)\|_\beta.
$$

Thus, the mapping $\mathbb{I}$ we constructed is a strict contraction on $\tilde{\mathcal{W}}_{[0,T+N]}\times\tilde{\mathcal{W}}^H_{[0,T+N]}$; using the Method of Contraction Mapping, this contraction mapping $\mathbb{I}$ has a fixed point that is unique. This indicates that Equation (3) has a unique solution $(Y_t,Z_t)\in\tilde{\mathcal{W}}_{[0,T+N]}\times\tilde{\mathcal{W}}^H_{[0,T+N]}$ such that $\mathbb{I}(y_t,z_t)=(Y_t,Z_t)$. $\square$

4. Comparison Theorem

In this part, we present the following comparison theorem for fractional DABSDEs under a one-dimensional case:

$$\begin{cases} Y_t = \xi_T + \int_t^T f(s, Y_{s-h_1(s)}, Y_s, Z_s, Y_{s+h_3(s)})\, \mathrm{d}s - \int_t^T Z_s\, \mathrm{d}B_s^H, & 0 \le t \le T; \\ Y_t = \xi_t, & T \le t \le T+N; \\ Z_t = \eta_t, & T \le t \le T+N. \end{cases}$$

We will start by presenting the conventional approach of applying the comparison theorem to fractional BSDEs.

Lemma 1 (Hu et al. [32], Theorem 12.3)**.** *Define $\eta_t = \eta_0 + \int_0^t b_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}B_s^H$, where b_s and σ_s are bounded deterministic functions and $\sigma_s > 0$. For $j = 1,2$, assume ξ_T^j are continuously differentiable and are of polynomial growth, and $f_j(t,x,y,z)$ and $\frac{\partial}{\partial y} f_j(t,x,y,z)$ are uniformly Lipschitz continuous with respect to y and z. Assume that the solutions of the following two classical fractional BSDEs are $(y^{(1)}, z^{(1)})$ and $(y^{(2)}, z^{(2)})$, respectively:*

$$\begin{cases} \mathrm{d}y_t^{(j)} = -f_j\left(t, \eta_t, y_t^{(j)}, z_t^{(j)}\right) \mathrm{d}t + z_t^{(j)} \mathrm{d}B_t^H, \\ y_T^{(j)} = \xi_T^{(j)}. \end{cases}$$

If $\xi_T^{(1)} \le \xi_T^{(2)}$, $f_1(t,x,y,z) \le f_2(t,x,y,z)$ for $t \in [0,T]$, then

$$y_t^{(1)} \le y_t^{(2)}, \quad a.s.$$

Suppose $(Y_\cdot^{(j)}, Z_\cdot^{(j)})$, $j = 1,2$, represent the solutions of the following two one-dimensional fractional DABSDEs:

$$\begin{cases} Y_t^{(j)} = \xi_T^{(j)} + \int_t^T f_j\left(s, Y_{s-h_1(s)}^{(j)}, Y_s^{(j)}, Z_s^{(j)}, Y_{s+h_3(s)}^{(j)}\right) \mathrm{d}s - \int_t^T Z_s^{(j)}\, \mathrm{d}B_s^H, & 0 \le t \le T; \\ Y_t^{(j)} = \xi_t^{(j)}, & T \le t \le T+N; \\ Z_t^{(j)} = \eta_t^{(j)}, & T \le t \le T+N, \end{cases} \tag{5}$$

Hence, we can derive the following theorem:

Theorem 2. *Suppose that for $i = 1,2,3,4$, $h_i(t)$ satisfy the assumptions (A1) and (A2), for $j = 1,2$, $f_j(t,\cdot)$ satisfy (A3) and (A4), and $\xi_t^{(j)} \in \tilde{\mathcal{W}}_{[T,T+N]}$. Furthermore, suppose that*

(i) $\xi_t^{(1)} \le \xi_t^{(2)}$;

(ii) *the generator $f_2(t,\theta,y,z,\phi)$ is increasing with respect to θ and ϕ;*

(iii) $f_1\left(t, y_{t-h_1(t)}, y_t, z_t, y_{t+h_3(t)}\right) \le f_2\left(t, y_{t-h_1(t)}, y_t, z_t, y_{t+h_3(t)}\right)$, $y_{t-h_1(t)} \in L_{\mathcal{F}}^2(0,t)$, $y_{t+h_3(t)} \in L_{\mathcal{F}}^2(t, T+N)$.

Therefore, $Y_t^{(1)} \le Y_t^{(2)}$ almost surely.

Proof. Consider the BSDEs below:

$$\begin{cases} Y_t^{(1)} = \xi_T^{(1)} + \int_t^T f_1\left(s, Y_{s-h_1(s)}^{(1)}, Y_s^{(1)}, Z_s^{(1)}, Y_{s+h_3(s)}^{(1)}\right) \mathrm{d}s - \int_t^T Z_s^{(1)}\, \mathrm{d}B_s^H, & 0 \le t \le T; \\ Y_t^{(1)} = \xi_t^{(1)}, & T \le t \le T+N. \end{cases} \tag{6}$$

As previously stated, $\left(Y_\cdot^{(1)}, Z_\cdot^{(1)}\right)$ is the solution of the fractional DABSDEs with only one dimension given in Equation (5). Then, we also consider the fractional DABSDEs as follows:

$$\begin{cases} Y_t^{(3)} = \xi_T^{(2)} + \int_t^T f_2(s, Y_{s-h_1(s)}^{(1)}, Y_s^{(3)}, Z_s^{(3)}, Y_{s+h_3(s)}^{(1)})\, ds - \int_t^T Z_s^{(3)}\, dB_s^H, & 0 \le t \le T; \\ Y_t^{(3)} = \xi_t^{(2)}, & T \le t \le T+N. \end{cases} \tag{7}$$

We observe that the above equation has a unique solution $\left(Y_t^{(3)}, Z_t^{(3)}\right) \in \tilde{\mathcal{W}}_{[0,T+N]} \times \tilde{\mathcal{W}}_{[0,T]}^H$. Since $\xi_t^{(1)} \le \xi_t^{(2)}$, $f_1\left(s, Y_{s-h_1(s)}^{(1)}, y, z, Y_{s+h_3(s)}^{(1)}\right) \le f_2\left(s, Y_{s-h_1(s)}^{(1)}, y, z, Y_{s+h_3(s)}^{(1)}\right)$, considering Equations (6) and (7), by Lemma 1, we obtain

$$Y_t^{(1)} \le Y_t^{(3)} \quad a.s.$$

Let

$$\begin{cases} Y_t^{(4)} = \xi_T^{(2)} + \int_t^T f_2\left(s, Y_{s-h_1(s)}^{(3)}, Y_s^{(4)}, Z_s^{(4)}, Y_{s+h_3(s)}^{(3)}\right) ds - \int_t^T Z_s^{(4)}\, dB_s^H, & 0 \le t \le T; \\ Y_t^{(4)} = \xi_t^{(2)}, & T \le t \le T+N. \end{cases} \tag{8}$$

Consider Equations (7) and (8) and note that since $f_2(t, \theta, y, z, \phi)$ is increasing in θ and ϕ and that $Y_t^{(1)} \le Y_t^{(3)}$, one could obtain $f_2\left(s, Y_{s-h_1(s)}^{(1)}, y, z, Y_{s+h_3(s)}^{(1)}\right) \le f_2\left(s, Y_{s-h_1(s)}^{(3)}, y, z, Y_{s+h_3(s)}^{(3)}\right)$. Similarly to the above case, we obtain

$$Y_t^{(3)} \le Y_t^{(4)} \quad a.s.$$

For $n = 5, 6, 7, \ldots$, take into consideration the BSDEs as follows:

$$\begin{cases} Y_t^{(n)} = \xi_T^{(2)} + \int_t^T f_2\left(s, Y_{s-h_1(s)}^{(n-1)}, Y_s^{(n)}, Z_s^{(n)}, Y_{s+h_3(s)}^{(n-1)}\right) ds - \int_t^T Z_s^{(n)}\, dB_s^H, & 0 \le t \le T; \\ Y_t^{(n)} = \xi_t^{(2)}, & T \le t \le T+N. \end{cases}$$

Similarly, we obtain

$$Y_t^{(4)} \le Y_t^{(5)} \le \cdots \le Y_t^{(n)} \le \ldots, \quad a.s.$$

We shall now show that $\left\{Y_t^{(n)}, Z_t^{(n)}\right\}_{n \ge 4}$ are Cauchy sequences. Denote $\hat{Y}_t^{(n)} := Y_t^{(n)} - Y_t^{(n-1)}$, $\hat{Z}_t^{(n)} := Z_t^{(n)} - Z_t^{(n-1)}$, $n \ge 4$, then from inequality (4), we obtain

$$\begin{aligned} & E\left[e^{\beta t}\left|\hat{Y}_t^{(n)}\right|^2 - e^{\beta T}\left|\hat{Y}_T^{(n)}\right|^2 + \frac{\beta}{2}\int_t^T e^{\beta s}\left|\hat{Y}_s^{(n)}\right|^2 ds + \frac{2}{M}\int_t^T e^{\beta s} s^{2H-1}\left|\hat{Z}_s^{(n)}\right|^2 ds\right] \\ & \le \frac{2}{\beta} E\left[\int_t^T e^{\beta s}\left| f_2\left(s, Y_{s-h_1(s)}^{(n-1)}, Y_s^{(n)}, Z_s^{(n)}, Y_{s+h_3(s)}^{(n-1)}\right)\right.\right. \\ & \qquad \left.\left. - f_2\left(s, Y_{s-h_1(s)}^{(n-2)}, Y_s^{(n-1)}, Z_s^{(n-1)}, Y_{s+h_3(s)}^{(n-2)}\right)\right|^2 ds\right] \end{aligned}$$

Taking into consideration the assumptions (A1)–(A3) and Jensen's inequality, along with the fact that $(a+b+c+d)^2 \le 4(a^2+b^2+c^2+d^2)$ and setting $t = 0$, we obtain

$$\begin{aligned} & E\left[\int_0^T e^{\beta s}\left(\frac{\beta}{2}\left|\hat{Y}_s^{(n)}\right|^2 + \frac{2}{M} s^{2H-1}\left|\hat{Z}_s^{(n)}\right|^2\right) ds\right] \\ & \le \frac{2C^2}{\beta} E\left[\int_0^T e^{\beta s}\left(\left|\hat{Y}_s^{(n-1)}\right| + \left|\hat{Y}_s^{(n)}\right| + s^{H-\frac{1}{2}}\left|\hat{Z}_s^{(n)}\right| + E\left[\left|\hat{Y}_t^{(n-1)}\right| \middle| \mathcal{F}_t\right]\right)^2 ds\right] \\ & \le \frac{8C^2(2L+1)}{\beta} E\left[\int_0^T e^{\beta s}\left(\left|\hat{Y}_s^{(n-1)}\right|^2 + \left|\hat{Y}_s^{(n)}\right|^2 + s^{2H-1}\left|\hat{Z}_s^{(n)}\right|^2 + \left|\hat{Y}_s^{(n-1)}\right|^2\right) ds\right]. \end{aligned}$$

Multiplying $\frac{M}{2}$ by both sides of the preceding inequality yields

$$E\left[\int_0^T e^{\beta s}\left(\frac{M\beta}{4}\left|\hat{Y}_s^{(n)}\right|^2 + s^{2H-1}\left|\hat{Z}_s^{(n)}\right|^2\right)\mathrm{d}s\right] \leq \frac{4C^2(2L+1)M}{\beta}E\left[\int_0^T e^{\beta s}\left(\left|\hat{Y}_s^{(n-1)}\right|^2 + \left|\hat{Y}_s^{(n)}\right|^2 + s^{2H-1}\left|\hat{Z}_s^{(n)}\right|^2 + \left|\hat{Y}_s^{(n-1)}\right|^2\right)\mathrm{d}s\right].$$

Letting $\beta = 16MC^2(2L+1) + \frac{4}{M}, M > 2$, we obtain

$$E\left[\int_0^T e^{\beta s}\left(\left|\hat{Y}_s^{(n)}\right|^2 + s^{2H-1}\left|\hat{Z}_s^{(n)}\right|^2\right)\mathrm{d}s\right] \leq \frac{1}{4}E\left[\int_0^T e^{\beta s}\left(\left|\hat{Y}_s^{(n)}\right|^2 + s^{2H-1}\left|\hat{Z}_s^{(n)}\right|^2\right)\mathrm{d}s\right] + \frac{1}{2}E\left[\int_0^T e^{\beta s}\left|\hat{Y}_s^{(n-1)}\right|^2\mathrm{d}s\right].$$

Furthermore,

$$\begin{aligned} &E\left[\int_0^T e^{\beta s}\left(\left|\hat{Y}_s^{(n)}\right|^2 + s^{2H-1}\left|\hat{Z}_s^{(n)}\right|^2\right)\mathrm{d}s\right] \\ &\leq \frac{2}{3}E\left[\int_0^T e^{\beta s}\left|\hat{Y}_s^{(n-1)}\right|^2\mathrm{d}s\right] \\ &\leq \frac{2}{3}E\left[\int_0^T e^{\beta s}\left(\left|\hat{Y}_s^{(n-1)}\right|^2 + s^{2H-1}\left|\hat{Z}_s^{(n-1)}\right|^2\right)\mathrm{d}s\right]. \end{aligned}$$

Therefore,

$$E\left[\int_0^T e^{\beta s}\left(\left|\hat{Y}_s^{(n)}\right|^2 + s^{2H-1}\left|\hat{Z}_s^{(n)}\right|^2\right)\mathrm{d}s\right] \leq \left(\frac{2}{3}\right)^{n-4}E\left[\int_0^T e^{\beta s}\left(\left|\hat{Y}_s^{(4)}\right|^2 + s^{2H-1}\left|\hat{Z}_s^{(4)}\right|^2\right)\mathrm{d}s\right].$$

It follows that $\left(\hat{Y}_t^{(n)}, \hat{Z}_t^{(n)}\right)_{n\geq 4}$ are Cauchy sequences in $\tilde{\mathcal{W}}_{[0,T+N]} \times \tilde{\mathcal{W}}_{[0,T]}^H$. For all $0 \leq t \leq T$, let $(Y_\cdot, Z_\cdot)$ be the limit of $\left(\hat{Y}_t^{(n)}, \hat{Z}_t^{(n)}\right)$; therefore, when $n \to \infty$,

$$\begin{aligned} &E\left[\int_t^T e^{\beta s}\left|f_2\left(s, Y_{s-h_1(s)}^{(n-1)}, Y_s^{(n)}, Z_s^{(n)}, Y_{s+h_3(s)}^{(n-1)}\right) - f_2\left(s, Y_{s-h_1(s)}, Y_s, Z_s, Y_{s+h_3(s)}\right)\right|^2\mathrm{d}s\right] \\ &\leq 4C^2E\left[\int_t^T e^{\beta s}\left(\left|Y_s^{(n)} - Y_s\right|^2 + s^{2H-1}\left|Z_s^{(n)} - Z_s\right|^2 + 2L\left|Y_s^{(n-1)} - Y_s\right|^2\right)\mathrm{d}s\right] \longrightarrow 0. \end{aligned}$$

Consequently, (Y_t, Z_t) is the solution of the fractional DABSDE as follows:

$$\begin{cases} Y_t = \xi_T^{(2)} + \int_t^T f_2(s, Y_{s-h_1(s)}, Y_s, Z_s, Y_{s+h_3(s)})\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}B_s^H, & 0 \leq t \leq T; \\ Y_t = \xi_t^{(2)}, & T \leq t \leq T+N. \end{cases}$$

Then, based on Theorem 1 regarding the uniqueness of the solution, we obtain

$$Y_t^{(2)} = Y_t, \quad a.s.$$

Given that

$$Y_t^{(1)} \leq Y_t^{(3)} \leq Y_t^{(4)} \leq Y_t,$$

we achieve the desired outcome, which is $Y_t^{(1)} \leq Y_t^{(2)}$, *a.s.* □

5. Conclusions

This work specifically examines a category of fractional anticipated BSDEs with delayed generators. It explores many features of the solution to these equations, including their existence, uniqueness, and the comparison theorem. The research concepts, conditions, and assumptions for investigating the solvability of the BSDE model in the existing literature vary. Fractional Brownian motion as a driving noise in our models (2) has self-similarity, steady increment, and long memory properties at $H \in (1/2, 1)$. This implies that our suggested fractional DABSDEs have wide applicability in various real-world domains, including finance and physics. This article utilises the fBm driver and a more general generator f, which leads to more intricate methods compared to the original studies conducted by Peng and Yang [13] and Delong and Imkeller [17]. Furthermore, the simpler fixed point theorem in this study makes the assumptions required more stringent, and we then obtain the comparison theorem of solutions in the one-dimensional case, which further broadens the theoretical research of BSDEs. It is important to note that, once the solvability of fractional DABSDEs has been discussed, a new issue arises in establishing the relationship between their solution and the solution of parabolic partial differential equations with some special conditions. Furthermore, akin to the study of anticipated BSDEs driven by fractional Brownian motion under mean field limits in Douissi et al. [25], our model has the potential to be employed in stochastic optimal control issues. This aspect, however, necessitates further investigation in future studies.

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