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On the Study of Starlike Functions Associated with the Generalized Sine Hyperbolic Function

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Abstract: Geometric function theory, a subfield of complex analysis that examines the geometrical characteristics of analytic functions, has seen a sharp increase in research in recent years. In particular, by employing subordination notions, the contributions of different subclasses of analytic functions associated with innovative image domains are of significant interest and are extensively investigated. Since $\Re(1+\sinh(z)) \not> 0$, it implies that the class $\mathcal{S}^*_{\sinh}$ introduced in reference third by Kumar et al. is not a subclass of starlike functions. Now, we have introduced a parameter λ with the restriction $0 \leq \lambda \leq \ln(1+\sqrt{2})$, and by doing that, $\Re(1+\sinh(\lambda z)) > 0$. The present research intends to provide a novel subclass of starlike functions in the open unit disk $\mathcal{U}$, denoted as $\mathcal{S}^*_{\sinh\lambda}$, and investigate its geometric nature. For this newly defined subclass, we obtain sharp upper bounds of the coefficients a_n for $n = 2,3,4,5$. Then, we prove a lemma,in which the largest disk contained in the image domain of $q_0(z) = 1+\sinh(\lambda z)$ and the smallest disk containing $q_0(\mathcal{U})$ are investigated. This lemma has a central role in proving our radius problems. We discuss radius problems of various known classes, including $\mathcal{S}^*(\beta)$ and $\mathcal{K}(\beta)$ of starlike functions of order β and convex functions of order β. Investigating $\mathcal{S}^*_{\sinh\lambda}$ radii for several geometrically known classes and some classes of functions defined as ratios of functions are also part of the present research. The methodology used for finding $\mathcal{S}^*_{\sinh\lambda}$ radii of different subclasses is the calculation of that value of the radius $r < 1$ for which the image domain of any function belonging to a specified class is contained in the largest disk of this lemma. A new representation of functions in this class, but for a more restricted range of λ, is also obtained.

Keywords: starlike functions; Janowski starlike function; sine hyperbolic function; radii problems

MSC: 30C45; 30C50

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1. Introduction and Definitions

Complex analysis is one of the major disciplines nowadays due to its numerous applications not just in mathematical science, but also in other fields of study. Among the other disciplines, geometric function theory is an intriguing area of complex analysis that involves the geometrical characteristics of analytical functions. It has been observed that this area is crucial to applied mathematics, particularly in fields like engineering, electronics, nonlinear integrable system theory, fluid dynamics, modern mathematical physics, partial differential equation theory, etc. The foundation of function theory is the theory of univalent functions, and as a consequence of its wide application, new fields of research have emerged with a variety of fascinating results. Below, in the first section, we briefly discuss the basics of function theory, which will help in understanding the terminology used in our results.

Denote $\mathcal{A}$ as the class of all analytic functions f in $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$, which are normalized and of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

and denote $\mathcal{S}$ as the subfamily of $\mathcal{A}$, which consists of univalent functions in $\mathcal{U}$. Also, denote $\mathcal{A}_n$ as the class of analytic functions f of the form $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots$ defined in the open unit disk $\mathcal{U}$. As such, we have $\mathcal{A} = \mathcal{A}_1$. A domain D in the complex plane $\mathbb{C}$ is starlike with respect to $w_0 \in D$ if any line segment or ray joining w_0 to a point $w \in D$ lies in D. Any function that maps $\mathcal{U}$ onto such a domain D is starlike with respect to w_0. We denote $\mathcal{S}^*$ as the class of functions that are starlike with respect to 0. The class $\mathcal{S}^*$ of functions f is analytically defined as $\mathcal{S}^* = \{f \in \mathcal{A} : \Re(zf'(z)/f(z)) > 0,\ z \in \mathcal{U}\}$. Similarly, a set D in $\mathbb{C}$ is convex if it is starlike with respect to each of its points. Any function that maps $\mathcal{U}$ onto such a domain D is known as a convex function, and a class of all such functions is denoted by $\mathcal{K}$. Analytically, a function $f \in \mathcal{K}$ if and only if $zf'(z) \in \mathcal{S}^*$. Also, recall that the relation of subordination between the analytic functions f and g is symbolically written as $f(z) \prec g(z)$, and it holds if there exists a Schwarz function w with $|w(z)| < |z|$ and $w(0) = 0$ such that $f(z) = g(w(z))$. In addition, if g is univalent, then the relation $f(z) \prec g(z)$ holds if and only if $f(\mathcal{U}) \subset g(\mathcal{U})$.

The general families of univalent functions f in $\mathcal{A}$ for which the quantities $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ are subordinate to a univalent function ϕ with a positive real part were discussed by Ma and Minda [1], who defined $\mathcal{S}^*(\phi) = \{f \in \mathcal{A} : zf'(z)/f(z) \prec \phi(z)\ \ z \in \mathcal{U}\}$ and $\mathcal{K}(\phi) = \{f \in \mathcal{A} : 1 + zf''(z)/f'(z) \prec \phi(z)\ \ z \in \mathcal{U}\}$, where $\Re(\phi(z)) > 0$, $\phi'(0) > 0$, and $\mathcal{U}$ were mapped onto a star-shaped domain with respect to 1, and symmetric about the real axis. Several well-known classes can be obtained by specializing the function ϕ. For $\phi(z) = (1 + Mz)/(1 + Nz), -1 \le N < M \le 1$, the class $\mathcal{S}^*(\phi)$ is denoted by $\mathcal{S}^*[M, N]$ and is known as the class of Janowski starlike functions [2]; more specifically, if $\phi(z) = [1 + (1 - 2\alpha)z]/(1 - z)$ and $0 \le \alpha < 1$, the class $\mathcal{S}^*(\phi)$ reduces to the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α. In a similar way, the class $\mathcal{K}(\phi)$ for $\phi(z) = (1 + Mz)/(1 + Nz)$, $-1 \le N < M \le 1$, and $\phi(z) = [1 + (1 - 2\alpha)z]/(1 - z)$, where $0 \le \alpha < 1$, is denoted by $\mathcal{K}[M, N]$ and $\mathcal{K}(\alpha)$, which are known as Janowski convex functions [2] and convex functions of order α, respectively. For $\alpha = 0$, the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ reduce to the prominent classes $\mathcal{S}^*$ and $\mathcal{K}$ of starlike and convex functions, respectively. In the present study, we discuss $\mathcal{S}^*_{\sinh\lambda}$ radii for some already defined classes $S^*(\phi)$ of starlike functions for different choices of ϕ, which will be mentioned in the text wherever required.

In [3], Kumar et al. introduced a subclass of Ma–Minda type functions by choosing $\phi(z) = 1 + \sinh(z)$ associated with sine hyperbolic functions. Since $\Re(1 + \sinh(z)) \not> 0$ and $z \in \mathcal{U}$, the defined class does not belong to the family $\mathcal{S}^*$. To address this problem, Raza et al. [4] introduced a subclass of $\mathcal{S}^*$ by considering $\phi(z) = 1 + \lambda \sinh(z)$, where $0 < \lambda < 1/\sinh(1)$. In a similar way, a subclass $\mathcal{S}^*_{\sinh\lambda}$ can be defined by taking $\phi(z) = 1 + \sinh(\lambda z)$, where $0 \le \lambda \le \ln(1 + \sqrt{2})$. We define it as follows:

$$\mathcal{S}^*_{\sinh\lambda} = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \sinh(\lambda z) = q_0(z),\ z \in \mathcal{U} \right\}.$$

Remark 1. *Since* $\Re(1 + \sinh(\lambda z)) > 0$ *and* $0 \le \lambda \le \ln(1 + \sqrt{2})$ *in* $\mathcal{U}$, $\mathcal{S}^*_{\sinh\lambda} \subset \mathcal{S}^*$.

Remark 2. *We also see that* $q_0(\mathcal{U}) \not\subseteq 1 + \lambda \sinh(\mathcal{U})$ *and* $1 + \lambda \sinh(\mathcal{U}) \not\subseteq q_0(\mathcal{U})$, *where* $0 \le \lambda \le \ln(1 + \sqrt{2})$; *therefore, there is no inclusion relation between the class* $\mathcal{S}^*_{\sinh\lambda}$ *and the class defined in [4].*

From the definition, we see that a function $f \in \mathcal{S}^*_{\sinh\lambda}$ if and only if there exists an analytic function q satisfying the subordination relation $q(z) \prec q_0(z) = 1 + \sinh(\lambda z)$, with $z \in \mathcal{U}$, such that

$$f(z) = z\,\exp\left(\int_0^z \frac{q(t)-1}{t}dt\right). \tag{2}$$

A few examples of functions of our newly defined class $\mathcal{S}^*_{\sinh\lambda}$ are given below. Let us consider the following functions:

$$q_1(z) = 1 + \frac{\lambda z}{3},\;\; q_2(z) = \frac{4+2\lambda z}{4+\lambda z},\;\; q_3(z) = \frac{7+7\lambda z}{7+\lambda z},\; q_4(z) = 1 + \sin(1)\lambda z.$$

Also, since $q_0(z) = 1 + \sinh(\lambda z)$ is univalent in $\mathcal{U}$, $q_i(0) = 1 = q_0(0)$ and $q_i(\mathcal{U}) \subseteq q_0(\mathcal{U})$ for all $i = 1,2,3,4$; this implies that, for each $i = 1,2,3,4$, the relation $q_i \prec q_0$ holds. Thus, from (2), the functions

$$f_1(z) = ze^{\lambda z/3},\;\; f_2(z) = z + \frac{\lambda z^2}{4},\;\; f_3(z) = z\left(\frac{7+\lambda z}{7}\right)^6,\;\; f_4(z) = ze^{\sin(1)\lambda z},$$

belong to the class $\mathcal{S}^*_{\sinh\lambda}$ corresponding to each of the functions q_i respectively.

Now, we recall some known basics classes, which will be used in the upcoming results. In this regard, we first define an important class of analytic functions p for $z \in \mathcal{U}$ whose real part is positive. It is denoted by $\mathcal{P}$ and has the series representation of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k,\;\; z \in \mathcal{U}. \tag{3}$$

Let

$$\mathcal{P}_n[M,N] = \left\{p_n(z) = 1 + \sum_{k=n}^{\infty} p_n z^n : p_n(z) \prec \frac{1+Mz}{1+Nz}, -1 \le N < M \le 1\right\}.$$

Then, clearly $\mathcal{P}_n(\alpha) = \mathcal{P}_n[1-2\alpha,-1]$ and $\mathcal{P}_n = \mathcal{P}_n(0)$. The class of all functions $f \in \mathcal{A}_n$, for which $zf'(z)/f(z) \prec (1+Mz)/(1+Nz)$, is denoted by $\mathcal{S}^*_n[M,N]$, and that for $zf'(z)/f(z) \prec [1+(1-2\alpha)z]/(1-z)$ is denoted by $\mathcal{S}^*_n(\alpha)$. The class $\mathcal{M}(\beta)$ consists of functions $f \in \mathcal{A}$ satisfying the relation $\Re[zf'(z)/f(z)] < \beta;\ \beta > 1$ was introduced by Uralegaddi et al. [5]. Also, let

$$\mathcal{S}^*_{\sinh\lambda,n} = \mathcal{A}_n \cap \mathcal{S}^*_{\sinh\lambda},\;\; 0 \le \lambda \le \ln(1+\sqrt{2}),$$

$$\mathcal{S}^*_n(\alpha) = \mathcal{A}_n \cap \mathcal{S}^*(\alpha),\;\; \mathcal{S}^*_{L,n} = \mathcal{A}_n \cap \mathcal{S}^*_L,$$

and $\mathcal{M}_n(\beta) = \mathcal{A}_n \cap \mathcal{M}(\beta)$.

In this paper, we work on finding the radii of starlikeness and convexity, as well as $\mathcal{S}^*_{\sinh\lambda}$ radii for certain subclasses of starlike functions, mentioned above, which mostly have simple geometric interpretations. Besides these subclasses, we also discuss the $\mathcal{S}^*_{\sinh\lambda}$ radii for some families of $\mathcal{A}$ whose functions have been expressed as a ratio between two functions. We denote these families by $\mathcal{F}_i$ $(i = 1,2,3,4)$. In the literature, the very early studies in this direction were due to Kaplan [6] and Read [7], who introduced the class of close-to-convex functions and close-to-starlike functions, respectively. Advanced studies in this direction can be seen, for example, in [8–14].

2. Preliminaries

This section is devoted to some results regarding the coefficient bounds for class $\mathcal{P}$. These are useful in determining the bounds on coefficients of the Taylor series of our newly defined class.

Lemma 1 ([1]). *If $p \in \mathcal{P}$ and is of the form (3), then for any complex number μ,*

$$|p_n| \leq 2, \quad for\ n \geq 1, \tag{4}$$

and

$$\left|p_2 - \mu p_1^2\right| \leq 2max\{1; |2\mu - 1|\}. \tag{5}$$

Lemma 2 ([15]). *Let $p \in \mathcal{P}$ be given by (3), with $0 \leq B \leq 1$ and $B(2B-1) \leq D \leq B$. Then,*

$$\left|p_3 - 2B\, p_1 p_2 + D p_1^3\right| \leq 2.$$

Lemma 3 ([16]). *Let a, b, c, and d be such that $0 < c < 1$, $0 < d < 1$, and*

$$8d(1-d)\left[(cb-2a)^2 + \{c(d+c)-b\}^2\right] + c(1-c)(b-2dc)^2 \leq 4c^2(1-c)^2 d(1-d).$$

If p is in $\mathcal{P}$ and is of the form (3), then

$$\left|a p_1^4 + d p_2^2 + 2c p_1 p_3 - \frac{3}{2} b p_1^2 p_2 - p_4\right| \leq 2.$$

Lemma 4 ([17]). *If $p \in \mathcal{P}_n(\alpha)$, then for $|z| = r$,*

$$\left|\frac{zp'(z)}{p(z)}\right| \leq \frac{2(1-\alpha)nr^n}{(1-r^n)(1+(1-2\alpha)r^n)}.$$

Lemma 5 ([18]). *If $p \in \mathcal{P}_n[M, N]$, then for $|z| = r$,*

$$\left|p(z) - \frac{1 - MNr^{2n}}{1 - N^2 r^{2n}}\right| \leq \frac{(M-N)r^n}{1 - N^2 r^{2n}}.$$

In particular, if $p \in \mathcal{P}_n(\alpha)$, then for $|z| = r$,

$$\left|p(z) - \frac{1 + (1-2\alpha)r^{2n}}{1 - r^{2n}}\right| \leq \frac{2(1-\alpha)r^n}{1 - r^{2n}}.$$

3. Main Results

This section has two subsections. In the first subsection, we derive bounds on the coefficients of the Taylor series for the functions in the class $S^*_{\sinh \lambda}$. We give extremal functions for all the results for which the equalities hold. The main tool for this discussion involves some inequalities that have already been proven for the coefficients of the functions in the class $\mathcal{P}$. We then prove a lemma in which we find the disk of the largest radius with its center on the real axis and contained in the domain $\Omega_{\sinh \lambda} = q_0(\mathcal{U})$, where $q_0(z) = 1 + \sinh(\lambda z)$. In the same lemma, we also obtain the disk of the smallest radius with its center on the real axis such that the domain $\Omega_{\sinh \lambda}$ is contained in it. At the end of the first subsection, we give a new representation for functions in the class $S^*_{\sinh \lambda}$ but for a more restricted range of λ.

3.1. Coefficient Bounds and Inclusion Lemma

Theorem 1. *Let $f \in \mathcal{S}^*_{\sinh \lambda}$ and be of the form (1). Then, for $0 \leq \lambda \leq \ln(1+\sqrt{2})$,*

$$|a_n| \leq \frac{\lambda}{n-1}, \ for\ n = 2, 3, 4, 5.$$

These results are sharp for

$$f(z) = z\exp\left(\int_0^z \frac{\sinh(\lambda t^{n-1})}{t}dt\right) = z + \frac{\lambda}{n-1}z^n + \frac{\lambda^2}{2(n-1)^2}z^{2n-1} + \cdots.$$

Proof. If $f \in \mathcal{S}^*_{\sinh\lambda}$, then, by the definition of class $\mathcal{S}^*_{\sinh\lambda}$,we have

$$\frac{zf'(z)}{f(z)} = 1 + \sinh(\lambda w(z)), \ \ z \in \mathcal{U}, \tag{6}$$

where $w(0) = 0$, and $|w(z)| < 1$. Now, using the definition of class $\mathcal{P}$, we have the relation

$$w(z) = \frac{p(z)-1}{p(z)+1}.$$

Therefore,

$$\begin{aligned} 1+\sinh(\lambda w(z)) &= 1+\sinh\left(\frac{\lambda(p(z)-1)}{p(z)+1}\right) \\ &= 1+\frac{\lambda}{2}p_1 z + \left(\frac{\lambda}{2}p_2 - \frac{\lambda}{4}p_1^2\right)z^2 + \left(\frac{\lambda}{8}p_1^3 - \frac{\lambda}{2}p_1p_2 + \frac{\lambda}{2}p_3 + \frac{\lambda^3}{48}p_1^3\right)z^3 \\ &\quad + \left(\frac{3}{8}\lambda p_1^2 p_2 - \frac{\lambda}{2}p_1p_3 - \frac{\lambda}{16}p_1^4 - \frac{\lambda}{4}p_2^2 - \frac{\lambda^3}{32}p_1^4 + \frac{\lambda}{2}p_4 + \frac{1}{16}\lambda^3 p_1^2 p_2\right)z^4 + \cdots. \end{aligned}$$

Also,

$$\begin{aligned} \frac{zf'(z)}{f(z)} &= 1 + a_2 z + (2a_3 - a_2^2)z^2 + (3a_4 - 3a_2a_3 + a_2^3)z^3 \\ &\quad + (4a_5 - 2a_3^2 + 4a_2^2a_3 - 4a_2a_4 - a_2^4)z^4 + \cdots. \end{aligned}$$

By substituting these values in (6) and comparing the coefficients, we have

$$a_2 = \frac{\lambda}{2}p_1, \tag{7}$$

$$a_3 = \frac{\lambda}{4}\left(p_2 - \frac{(1-\lambda)}{2}p_1^2\right), \tag{8}$$

$$a_4 = \frac{\lambda}{6}\left[p_3 - \left(\frac{4-3\lambda}{4}\right)p_1p_2 + \left(\frac{4\lambda^2 - 9\lambda + 6}{24}\right)p_1^3\right], \tag{9}$$

$$\begin{aligned} a_5 &= -\frac{\lambda}{8}\left[-\frac{1}{144}\left(7\lambda^3 - 27\lambda^2 + 33\lambda - 18\right)p_1^4 + \frac{1}{3}(3-2\lambda)p_1p_3 \right. \\ &\quad \left. + \frac{1}{4}(2-\lambda)p_2^2 - \frac{1}{24}\left(9\lambda^2 - 22\lambda + 18\right)p_1^2p_2 - p_4\right]. \end{aligned} \tag{10}$$

By using (7) and (4), we have $|a_2| \leq \lambda$. For the bound on $|a_3|$, in (8), we use (5) for $\mu = \frac{1-\lambda}{2}$ to obtain $|a_3| \leq \frac{\lambda}{2}$. Now, from (9), we have

$$|a_4| = \frac{\lambda}{6}\left|p_3 - 2Bp_1p_2 + Dp_1^3\right|, \tag{11}$$

where

$$B = \left(\frac{4-3\lambda}{8}\right),\ D = \left(\frac{4\lambda^2 - 9\lambda + 6}{24}\right).$$

We also see that

$$B - D = \left(\frac{1}{4} - \frac{1}{6}\lambda^2\right) > 0,\ 0 \leq \lambda \leq \ln\left(1+\sqrt{2}\right),$$

and

$$B(2B-1)-D=\left(\frac{11}{96}\lambda^2-\frac{1}{4}\right)<0,\ 0\leq\lambda\leq\ln\left(1+\sqrt{2}\right).$$

Therefore, by employing Lemma 2 to Equation (11), we have the required result.

Finally, let

$$\begin{aligned} a &= -\frac{1}{144}\left(7\lambda^3-27\lambda^2+33\lambda-18\right),\ b=\frac{1}{36}\left(9\lambda^2-22\lambda+18\right), \\ c &= \frac{(3-2\lambda)}{6},\ d=\frac{(2-\lambda)}{4}. \end{aligned}$$

Thus, (10) takes the form

$$a_5=-\frac{\lambda}{8}\left(ap_1^4++dp_2^2+2cp_1p_3-\frac{3}{2}bp_1^2p_2-p_4\right). \tag{12}$$

Clearly, $0<c<1$ and $0<d<1$. Also, the following inequality holds for all $\lambda\in[0,\ln(1+\sqrt{2})]$.

$$8d(1-d)\left[(cb-2a)^2+\{c(d+c)-b\}^2\right]+c(1-c)(b-2dc)^2<4c^2(1-c)^2d(1-d).$$

Therefore, all the conditions of Lemma 3 are satisfied; thus, from (12), we have

$$|a_5|\leq\frac{\lambda}{4}.$$

□

Lemma 6. *Let* $\Omega_{\sinh\lambda}=\{w\in\mathbb{C}:w(z)=1+\sinh(\lambda z),z\in\mathcal{U}\}$. *Then,*

$$\{w\in\mathbb{C}:|w-1|<\sin(\lambda)\}\subset\Omega_{\sinh\lambda}\subset\{w\in\mathbb{C}:|w-1|<\sinh(\lambda)\},$$

where $0\leq\lambda\leq\ln(1+\sqrt{2})$.

Proof. Consider the distance of $(1,0)$ from the boundary of $\Omega_{\lambda\sinh}$ as

$$h(x)=[\sinh(\cos(x))\cos(\lambda\sin(x))]^2+[\cosh(\lambda\cos(x))\sin(\lambda\sin(x))]^2.$$

Since $h(x)=h(-x)$, we therefore consider $0\leq\theta\leq\pi$. Now, we see that $h'(x)=0$ has 0, $\frac{\pi}{2}$ and π roots in $[0,\pi]$. Also, we see that $h'(x)<0$ for $0<\theta<\frac{\pi}{2}$ and $h'(x)>0$ for $\frac{\pi}{2}<\theta<\pi$. This implies that

$$\max h(x)=h(0)=h(\pi)=[\sinh(\lambda)]^2$$

and

$$\min h(x)=h\left(\frac{\pi}{2}\right)=[\sin(\lambda)]^2.$$

Hence, we have the required result. □

Lemma 7. *Let* $q_0(z)=1+\sinh(\lambda z)$, *where* $0\leq\lambda\leq\ln(1+\sqrt{2})$. *Then, for* $r\in(0,1)$,

$$\min_{|z|=r}\Re q_0(z)=q_0(-r),\qquad \max_{|z|=r}\Re q_0(z)=q_0(r).$$

Proof. For $z=re^{ix}$, where $r\in(0,1)$, we have

$$\Re q_0(z)=1+\sinh(\lambda r\cos(x))\cos(\lambda r\sin(x))=h(x).$$

Now, $h'(x) = 0$ has 0 and π roots in $[0, \pi]$. We also see that $h''(0) < 0$, whereas $h''(\pi) > 0$. Hence, we conclude that

$$\max_{0 \le x \le \pi} h(x) = h(0) = 1 + \sinh(\lambda r), \quad \min_{0 \le x \le \pi} h(x) = h(\pi) = 1 - \sinh(\lambda r).$$

□

Remark 3. *From the above result, it is evident that, for $f \in \mathcal{S}^*_{\sinh\lambda}$,*

$$1 - \sinh(\lambda) < \Re \frac{zf'(z)}{f(z)} < 1 + \sinh(\lambda).$$

Keeping in view the above inequalities, we define an analytic function $G_\lambda(z)$ and a vertical strip Ω_λ, with $0.783 \le \lambda \le \ln(1+\sqrt{2})$ and $7\pi/20 \le \beta < 52\pi/105$, as follows:

$$G_\lambda(z) = \frac{\sinh(\lambda) + \sin(\lambda)}{\pi i} \log\left(\frac{1 + e^{i\beta} z}{1 + e^{-i\beta} z}\right),$$

and

$$\Omega_\lambda = \{w : \sin(\lambda) < \Re(w) < \sinh(\lambda)\}.$$

Thus, $G_\lambda(z)$ is univalent and convex in the open unit disk $\mathcal{U}$, and it maps $\mathcal{U}$ onto Ω_λ.

Theorem 2. *Let $f \in \mathcal{A}$ and $0.783 \le \lambda \le \ln(1+\sqrt{2})$. Then, $f \in \mathcal{S}^*_{\sinh\lambda}$ if and only if*

$$\frac{zf'(z)}{f(z)} - 1 \prec G_\lambda(z) = \frac{\sinh(\lambda) + \sin(\lambda)}{\pi i} \log\left(\frac{1 + e^{i\beta} z}{1 + e^{-i\beta} z}\right),$$

where $7\pi/20 \le \beta < 52\pi/105$.

Proof. Let $f \in \mathcal{S}^*_{\sinh\lambda}$. Then, from Remark 3, $zf'(z)/f(z) - 1$ lies in the vertical strip Ω_λ. Furthermore, $G_\lambda(\mathcal{U}) = \Omega_\lambda$. Now, as G_λ is univalent, hence by subordination principle, the result follows. □

Now, we are in a position to give a new representation to the functions of our class $\mathcal{S}^*_{\sinh\lambda}$.

Theorem 3. *If $0.783 < \lambda < \ln(1+\sqrt{2})$, then $f \in \mathcal{S}^*_{\sinh\lambda}$ for all $z \in \mathcal{U}$ if and only if*

$$f(z) = z \, \exp\left[\frac{\sinh(\lambda) + \sin(\lambda)}{\pi i} \int_0^z \frac{1}{t} \log\left(\frac{1 + e^{i\beta} w(t)}{1 + e^{-i\beta} w(t)}\right) dt\right],$$

where $7\pi/20 \le \beta < 52\pi/105$, and w is an analytic function, with $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathcal{U}$.

Proof. According to Theorem 2 and the definition of subordination $f \in \mathcal{S}^*_{\sinh\lambda}$, there exists a Schwarz function w that is an analytic such that $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathcal{U}$, and

$$\frac{zf'(z)}{f(z)} - 1 = \frac{\sinh(\lambda) + \sin(\lambda)}{\pi i} \ln\left(\frac{1 + e^{i\beta} w(z)}{1 + e^{-i\beta} w(z)}\right)$$

or

$$\frac{f'(z)}{f(z)} - \frac{1}{z} = \frac{\sinh(\lambda) + \sin(\lambda)}{\pi i} \frac{1}{z} \ln\left(\frac{1 + e^{i\beta} w(z)}{1 + e^{-i\beta} w(z)}\right).$$

Therefore,

$$\ln[f(z)] = \ln(z) + \frac{\sinh(\lambda) + \sin(\lambda)}{\pi i} \cdot \int_0^z \frac{1}{t} \ln\left(\frac{1 + e^{i\beta}w(t)}{1 + e^{-i\beta}w(t)}\right) dt$$

This implies the required result. □

3.2. Radius Problems

In this second subsection, we discuss $S^*_{\sinh\lambda}$ radii for various known subclasses of starlike functions. We use Lemma 6 for determining the radius of a disk such that functions in different classes of analytic functions are contained in our class. We start with radius problems for $\mathcal{S}^*_{\sinh\lambda}$ from the two subclasses $\mathcal{S}_n$ and $\mathcal{CS}_n(\alpha)$ due to Ali et al. [19], which are given by

$$\mathcal{S}_n = \left\{ f \in \mathcal{A}_n : \frac{f(z)}{z} \in \mathcal{P}_n \right\},$$

$$\mathcal{CS}_n(\alpha) = \left\{ f \in \mathcal{A}_n : \frac{f(z)}{g(z)} \in \mathcal{P}_n, \;\; g \in \mathcal{S}^*_n(\alpha) \right\}.$$

Theorem 4. *The sharp $\mathcal{S}^*_{\sinh\lambda,n}$ radii for the classes $\mathcal{S}_n$ and $\mathcal{CS}_n(\alpha)$ are given by the following:*

(i). $R_{\mathcal{S}^*_{\sinh\lambda,n}}(\mathcal{S}_n) = \left(\dfrac{\sin(\lambda)}{\sqrt{n^2 + \sin^2(\lambda)} + n} \right)^{1/n}$.

(ii). $R_{\mathcal{S}^*_{\sinh\lambda,n}}(\mathcal{CS}_n(\alpha)) = \left(\dfrac{\sin(\lambda)}{\sqrt{(1+n-\alpha)^2 + [\sin^2(\lambda) + 2(1-\alpha)\sin(\lambda)]} + (n+1-\alpha)} \right)^{1/n}$.

Proof. (i). Let $f \in \mathcal{S}_n$. Then, $p(z) = f(z)/z \in \mathcal{P}_n$ and

$$\frac{zf'(z)}{f(z)} - 1 = \frac{zp'(z)}{p(z)}.$$

By applying Lemma 4, for $\alpha = 0$, we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{zp'(z)}{p(z)} \right| \le \frac{2nr^n}{1 - r^{2n}}, \text{ for } |z| \le r.$$

From Lemma 6, it implies that the disk $|zf'(z)/f(z) - 1| \le 2nr^n/(1 - r^{2n})$ will contain $\Omega_{\sinh\lambda}$ if

$$\frac{2nr^n}{1 - r^{2n}} \le \sin(\lambda)$$

holds. That is, $f \in \mathcal{S}^*_{\sinh\lambda,n}$ if and only if $(\sin(\lambda))r^{2n} + 2nr^n - sin\lambda \le 0$. Thus, the $\mathcal{S}^*_{\sinh\lambda,n}$ radius of $\mathcal{S}_n$ is the smallest positive root of the equation

$$(\sin(\lambda))r^{2n} + 2nr^n - \sin(\lambda) = 0,$$

is in the interval $(0,1)$. That is,

$$r \le \left(\frac{\sin(\lambda)}{n + \sqrt{n^2 + \sin^2(\lambda)}} \right)^{1/n} = R^*_{\mathcal{S}_{\sinh\lambda,n}}(\mathcal{S}_n).$$

Furthermore, to show the sharpness of $R^*_{\mathcal{S}_{\sinh\lambda,n}}(\mathcal{S}_n)$, we define the function $f_0(z) = z(1 + z^n)/(1 - z^n)$, which, upon differentiation, gives the following:

$$\frac{zf'(z)}{f(z)} - 1 = \frac{2nz^n}{1 - z^{2n}},$$

and at $z = R^{*}_{\mathcal{S}_{\sinh\lambda,n}}(\mathcal{S}_n)$, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = \sin(\lambda).$$

Hence, the proof of part (i) is completed.

(ii). Let $f \in \mathcal{CS}_n(\alpha)$. Then, define functions $h(z) = f(z)/g(z) \in \mathcal{P}_n$ and $g \in \mathcal{S}^*_n(\alpha)$ such that

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \frac{zh'(z)}{h(z)}.$$

Since $h \in \mathcal{P}_n$ and $zg'/g \in \mathcal{P}_n(\alpha)$, then according to Lemma 4 and Lemma 5, respectively, and $|z| = r$, we have

$$\left|\frac{zh'(z)}{h(z)}\right| \leq \frac{2nr^n}{1-r^{2n}} \quad \text{and} \quad \left|\frac{zg'(z)}{g(z)} - \frac{1+(1-2\alpha)r^{2n}}{1-r^{2n}}\right| \leq \frac{2(1-\alpha)r^n}{1-r^{2n}},$$

which by adding gives us the following:

$$\left|\frac{zf'(z)}{f(z)} - \frac{1+(1-2\alpha)r^{2n}}{1-r^{2n}}\right| \leq \frac{2(1+n-\alpha)r^n}{1-r^{2n}}. \tag{13}$$

Also, since $1 - r^{2n} \leq 1 + (1-2\alpha)r^{2n} < 1 + r^{2n}$, it implies that $[1+(1-2\alpha)r^{2n}]/(1-r^{2n}) \geq 1$. Thus, according to Lemma 6, the disk given in (13) lies in $\Omega^*_{\sinh\lambda}$ if

$$\frac{2(1+n-\alpha)r^n}{1-r^{2n}} \leq 1 + \sin(\lambda) - \frac{1+(1-2\alpha)r^{2n}}{1-r^{2n}},$$

and

$$\frac{1+(1-2\alpha)r^{2n}}{1-r^{2n}} \leq 1 + \sin(\lambda)$$

hold. These two inequalities imply, respectively, that

$$r \leq \left(\frac{\sin(\lambda)}{\sqrt{(n+1-\alpha)^2 + [\sin^2\lambda + 2(1-\alpha)\sin(\lambda)]} + (n+1-\alpha)}\right)^{1/n} = r_1$$

and

$$r \leq \left(\sqrt{\frac{\sin(\lambda)}{2(1-\alpha)+\sin(\lambda)}}\right)^{1/n} = r_2.$$

Thus,

$$R_{\mathcal{S}^*_{\sinh\lambda,n}}(\mathcal{CS}_n(\alpha)) = \min\{r_1, r_2\} = r_1.$$

Hence,

$$R_{\mathcal{S}^*_{\sinh\lambda,n}}(\mathcal{CS}_n(\alpha)) = \left(\frac{\sin(\lambda)}{\sqrt{(1+n-\alpha)^2 + [\sin^2\lambda + 2(1-\alpha)\sin(\lambda)]} + (n+1-\alpha)}\right)^{1/n}.$$

For the sharpness of $R_{\mathcal{S}^*_{\sinh\lambda,n}}(\mathcal{CS}_n(\alpha))$, consider the functions below

$$f_0(z) = \frac{z(1+z^n)}{(1-z^n)^{(n+2-2\alpha)/n}} \text{ and } g_0(z) = \frac{z}{(1-z^n)^{2(1-\alpha)/n}}.$$

Then, we have

$$\frac{f_0(z)}{g_0(z)} = \frac{1+z^n}{1-z^n} \text{ and } \frac{zg_0'(z)}{g_0(z)} = \frac{1+(1-2\alpha)z^n}{1-z^n}.$$

Clearly, $\Re(f_0(z)/g_0(z)) > 0$, while $zg_0'/g_0 \in \mathcal{P}_n[1-2\alpha, -1]$, thereby implying that $g_0 \in \mathcal{S}_n^*(\alpha)$; hence, $f_0 \in \mathcal{CS}(\alpha)$. Now, logarithmic differentiation of f_0 gives

$$\frac{zf_0'(z)}{f_0(z)} - 1 = \frac{2(n+1-\alpha)z^n + 2(1-\alpha)z^{2n}}{1-z^{2n}}.$$

At $z = R_{\mathcal{S}^*_{\sinh\lambda,n}}(\mathcal{CS}(\alpha))$, we have

$$\left|\frac{zf_0'(z)}{f_0(z)} - 1\right| = \sin(\lambda).$$

The proof is thus completed. □

Remark 4. *For α with $0 \leq \alpha < 1$, observe that $R_{\mathcal{S}^*_{\sinh\lambda,n}}(\mathcal{S}_n) = R_{\mathcal{S}^*_{\sinh\lambda,n}}(\mathcal{CS}_n(\alpha))$ as α approaches* 1.

Next, we discuss $\mathcal{S}^*_{\sinh\lambda}$ radii for some known subclasses of starlike functions $\mathcal{S}^*_L$, $\mathcal{S}^*_{RL}$, $\mathcal{S}^*_C$, $\mathcal{S}^*_{\mathfrak{C}}$, $\mathcal{S}^*_B$, and $\mathcal{S}^*_{SG}$, in which $zf'(z)/f(z)$ is subordinate to a Ma-and-Minda-type function involving no parameter. Here, we give a brief introduction to these classes.
The first of these is $\mathcal{S}^*_L$, which was introduced by Sokół and Stankiewicz [20] for $\phi(z) = \sqrt{1+z}$. The class $\mathcal{S}^*_L$ can be geometrically interpreted as the set of all those functions $f \in \mathcal{A}$, for which the image of transformation $zf'(z)/f(z)$ lies in the right-half of the lemniscate of Bernoulli $|w^2-1| < 1$. The second subclass $\mathcal{S}^*_{RL}$ of starlike functions associated with the left-half of the shifted lemniscate of Bernoulli has been derived by Mendiratta et al. [21]. In this subclass, they take $\phi(z) = \left(\sqrt{2} - (\sqrt{2}-1)\sqrt{(1-z)/(1+2(\sqrt{2}-1)z)}\right)$. Inspired by the work of Mendiratta et al. [21] in 2016, it was Sharma et al. [8] who introduced the class $\mathcal{S}^*_C$ as a special case of the class $\mathcal{S}^*(\phi)$ by taking $\phi(z) = 1 + 4z/3 + 2z^2/3$, and they studied some properties of the functions in the class $\mathcal{S}^*_C$. The class $\mathcal{S}^*_{\mathfrak{C}} = \mathcal{S}^*(z + \sqrt{1+z^2})$ was considered by Raina and Sokól [22]; they proved that $f \in \mathcal{S}^*_{\mathfrak{C}}$ if and only if $zf'(z)/f(z)$ lies in the region $\{w \in \mathbb{C} : |w^2-1| < 2|w|\}$. Ghandhi and Ravichandran [23] obtained several sufficient conditions for $f \in \mathcal{S}^*_{\mathfrak{C}}$. The fifth class of the present discussion is the class of starlike functions associated with the Bell numbers, which was introduced by Kumar et al. [24]. They denoted this class by $\mathcal{S}^*_B$ and defined it as the class of all functions $f \in \mathcal{A}$, which satisfy the subordination relation $zf'(z/f(z)) \prec e^{e^z-1}$. The last of these is the class $\mathcal{S}^*_{SG}$ of Sigmoid starlike functions defined in [25] by taking $\phi(z) = 2/(1+e^{-z})$. In [25], the authors presented some basic geometric properties of this function, proved some inclusion relationships, investigated coefficient bounds, and discussed first-order differential subordination results.

Theorem 5. *The sharp $\mathcal{S}^*_{\sinh\lambda}$ radii for some particular subclasses of starlike functions are*

$$\begin{aligned}
(i). \quad & R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_L) &=& (2-\sin(\lambda))\sin(\lambda),\\
(ii). \quad & R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_{RL}) &=& \tfrac{\sin(\lambda)\left[2+2\sqrt{2}+\left(3+2\sqrt{2}\right)\sin(\lambda)\right]}{-1+2\sqrt{2}+2\sqrt{2}\sin^2(\lambda)+4\sin(\lambda)+2\sin^2(\lambda)},\\
(iii). \quad & R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_C) &=& \tfrac{-2+\sqrt{4+6\sin(\lambda)}}{2},\\
(iv). \quad & R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_{\mathfrak{C}}) &=& \tfrac{(2+\sin(\lambda))\sin(\lambda)}{2+2\sin(\lambda)},\\
(v). \quad & R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_B) &=& \ln(1+\ln(1+\sin(\lambda))),\\
(vi). \quad & R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_{SG}) &=& \ln\left(\tfrac{1+\sin(\lambda)}{1-\sin(\lambda)}\right), \textit{with } 0\le\lambda< arc\sin\left(\tfrac{e-1}{e+1}\right).
\end{aligned}$$

Proof. (i). If $f\in\mathcal{S}^*_L$, then $zf'(z)/f(z)\prec\sqrt{1+z}$ and, hence, for $|z|=r$, we have

$$\left|\frac{zf'(z)}{f(z)}-1\right|\le 1-\sqrt{1-r}.$$

Thus, according to Lemma 6, $f\in\mathcal{S}^*_{\sinh\lambda}$ if the inequality

$$1-\sqrt{1-r}\le\sin(\lambda)$$

holds, and it further simplifies

$$r\le(2-\sin(\lambda))\sin(\lambda)=R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_L).$$

For the sharpness of $R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_L)$, we consider the function below

$$f_0(z)=\frac{4z\exp\left\{2\sqrt{1+z}-1\right\}}{\left(1+\sqrt{1+z}\right)^2}.$$

The logarithmic differentiation of f_0 gives $zf_0'(z)/f_0(z)=\sqrt{1+z}$, which, upon putting $z=-R^*_{\mathcal{S}_{\sinh\lambda}}(\mathcal{S}^*_L)$, yields

$$\left|zf_0'(z)/f_0(z)-1\right|=\sin(\lambda),$$

and this proves the sharpness of $R^*_{\mathcal{S}_{\sinh\lambda}}(\mathcal{S}^*_L)$.

(ii). From the definition of the class $\mathcal{S}^*_{RL}$, we can write $zf'(z)/f(z)\prec\phi(z)$ with

$$\phi(z)=\sqrt{2}-\left(\sqrt{2}-1\right)\sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}}$$

For $z=-r$, where $|z|=r$, we have

$$\left|\frac{zf'(z)}{f(z)}-1\right|\le(1-\sqrt{2})+(\sqrt{2}-1)\sqrt{\frac{1+r}{1-2(\sqrt{2}-1)r}}$$

According to Lemma 6, the above disk will contain $\Omega_{\sinh\lambda}$ and, hence, $f\in\mathcal{S}^*_{\sinh\lambda}$ if

$$(1-\sqrt{2})+(\sqrt{2}-1)\sqrt{\frac{1+r}{1-2(\sqrt{2}-1)r}}\le\sin(\lambda),$$

which, after some simplification, gives

$$r \leq \frac{\sin(\lambda)\left[2+2\sqrt{2})+3\sin(\lambda)+2\sqrt{2}\sin(\lambda)\right]}{-1+2\sqrt{2}+2\sqrt{2}\sin^2(\lambda)+4\sin(\lambda)+2\sin^2(\lambda)} = R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_{RL}).$$

For the sharpness of $R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_{RL})$, we define the following function

$$f_0(z) = z\exp\left(\int_0^z \frac{q(t)-1}{t}dt\right)$$

where

$$q(z) = \sqrt{2} - \left(\sqrt{2}-1\right)\sqrt{\frac{1-z}{1+2\left(\sqrt{2}-1\right)z}} = \frac{zf_0'(z)}{f_0(z)}.$$

At $z = -R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_{RL})$, we have

$$\left|\frac{zf_0'(z)}{f_0(z)} - 1\right| = \sin(\lambda),$$

which proves the sharpness of $R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_{RL})$.

(iii). Let $f \in \mathcal{S}^*_C$. Then, $zf'(z)/f(z) \prec 1 + 4z/3 + 2z^2/3$ and, hence, for $|z| = r$, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{4}{3}r + \frac{2}{3}r^2.$$

From Lemma 6, the last inequality implies that $f \in \mathcal{S}^*_{\sinh\lambda}$ if $\frac{4}{3}r + \frac{2}{3}r^2 \leq \sin(\lambda)$, or $2r^2 + 4r - 3\sin(\lambda) \leq 0$. Thus, $R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_C)$ is the smallest positive root of the equation $2r^2 + 4r - 3\sin(\lambda) = 0$, which is

$$R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_C) = \frac{\sqrt{2(2+3\sin(\lambda))}-2}{2}.$$

For the sharpness of $R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_C)$, we define the following function

$$f_0(z) = z\exp\left(\frac{4z+z^2}{3}\right)$$

such that

$$\frac{zf_0'(z)}{f_0(z)} = 1 + \frac{4}{3}z + \frac{2}{3}z^2,$$

which, at $z = R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_C)$, yields

$$\left|\frac{zf_0'(z)}{f_0(z)} - 1\right| = \sin(\lambda),$$

and this proves the sharpness of $R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_C)$.

(iv). Let $f \in \mathcal{S}^*_{\mathfrak{C}}$. Then, $zf'(z)/f(z) \prec z + \sqrt{1+z^2}$. Hence, for $|z| = r$, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq r + \sqrt{1+r^2} - 1.$$

According to Lemma 6, the above disk will contain $\Omega_{\sinh\lambda}$ and, hence, $f \in \mathcal{S}^*_{\sinh\lambda}$ if $r + \sqrt{1+r^2} - 1 \leq \sin(\lambda)$ or $\sqrt{1+r^2} \leq (1 - r + \sin(\lambda))$, which further implies that

$$r \leq \frac{\sin(\lambda)(2+\sin(\lambda))}{2(1+\sin(\lambda))} = R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_{\mathfrak{C}}).$$

To verify the sharpness of $R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_{\mathfrak{C}})$, we consider the function

$$f_0(z) = z\ \exp\left(\int_0^z \frac{q(t)-1}{t} dt\right)$$

with $q(z) = z + \sqrt{1+z^2} = zf_0'(z)/f_0(z)$. At $z = R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_{\mathfrak{C}})$, we have

$$\left|\frac{zf_0'(z)}{f_0(z)} - 1\right| = \frac{\sin(\lambda)(2+\sin(\lambda))}{2(1+\sin(\lambda))} + \left[\left(\frac{\sin^2\lambda + 2\sin(\lambda) + 2}{2(1+\sin(\lambda))}\right)^2\right]^{1/2} - 1 = \sin(\lambda).$$

This proves the sharpness of $R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_{\mathfrak{C}})$.

(v). Let $f \in \mathcal{S}^*_B$. Then, $zf'(z)/f(z) \prec e^{e^z-1}$. Therefore, for $|z| = r$, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le e^{e^r-1} - 1.$$

The above disk will contain $\Omega_{\sinh\lambda}$ and, hence, $f \in \mathcal{S}^*_{\sinh\lambda}$, if

$$e^{e^r-1} - 1 \le \sin(\lambda).$$

After some simple computation, we have

$$r \le \ln(1 + \ln(1+\sin(\lambda))) = R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_B).$$

For the sharpness of $R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_B)$, we consider the function

$$f_0(z) = z\exp\left(\int_0^z \frac{q(t)-1}{t} dt\right),\ q(z) = e^{e^z-1}.$$

At $z = R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_B)$, we have

$$\left|\frac{zf_0'(z)}{f_0(z)} - 1\right| = \sin(\lambda).$$

This proves the sharpness of $R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_B)$.

(vi). Let $f \in S^*_{SG}$. Then, $zf'(z)/f(z) \prec 2/(1+e^{-z})$. Thus, for $|z| = r < 1$, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le 1 - \frac{2}{1+e^r}.$$

Now, according to Lemma 6, the above disk will contain $\Omega_{\sinh\lambda}$ if

$$1 - \frac{2}{1+e^r} \le \sin\lambda,$$

which, upon solving for r, yields

$$r \le \ln(\frac{1+\sin\lambda}{1-\sin\lambda}),\ \ \textit{provided that}\ \ln(\frac{1+\sin\lambda}{1-\sin\lambda}) < 1.$$

That is,

$$r \le \ln(\frac{1+\sin\lambda}{1-\sin\lambda}) = R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_{SG}),\ \ \textit{with}\ \ 0 \le \lambda < arc\sin(\frac{e-1}{e+1}).$$

The sharpness is proven for

$$f_0(z) = z\exp\int_0^z \frac{q(t)-1}{t} dt,\ \ \textit{where}\ q(z) = \frac{2}{1+e^{-z}}$$

That is, $zf_0'(z)/f_0(z) = 2/(1+e^{-z})$, which at $z = R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_{SG})$ yields

$$\frac{zf_0'(z)}{f_0(z)} = 1 + \sin\lambda.$$

$\square$

Remark 5. *Since* $\ln((1+\sin\lambda)/(1-\sin\lambda))$ *is greater than one for arc* $\sin((e-1)/(e+1)) < \lambda \leq \ln(1+\sqrt{2})$, *this shows that for this interval of* λ, $\mathcal{S}^*_{SG} \subset \mathcal{S}^*_{\sinh\lambda}$.

Corollary 1. *For* $\lambda = \ln(1+\sqrt{2})$:

(i). $R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_L) = (2 - \sin\Big(\ln(1+\sqrt{2})\Big))\sin\Big(\ln(1+\sqrt{2})\Big) \approx 0.9478395308.$

(ii). $R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_{RL}) = \frac{\sin(\ln(1+\sqrt{2}))\left[2+2\sqrt{2}+\left(3+2\sqrt{2}\right)\sin(\ln(1+\sqrt{2}))\right]}{-1+2\sqrt{2}+2\sqrt{2}\sin^2(\ln(1+\sqrt{2}))+4\sin(\ln(1+\sqrt{2}))+2\sin^2(\ln(1+\sqrt{2}))}$
$\approx 0.9237689095.$

(iii). $R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_C) = \dfrac{-2+\sqrt{4+6\sin\Big(\ln(1+\sqrt{2})\Big)}}{2} \approx 0.468815850.$

(iv). $R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_{\mathfrak{C}}) = \dfrac{(2+\sin\Big(\ln(1+\sqrt{2})\Big))\sin\Big(\ln(1+\sqrt{2})\Big)}{2+2\sin\Big(\ln(1+\sqrt{2})\Big)} \approx 0.6035780393.$

(v). $R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_B) = \ln\Big(1+\ln(1+\sin\Big(\ln(1+\sqrt{2})\Big)\Big) \approx 0.4522791114.$

Corollary 2. *For* $\lambda = 0.48$:

$$R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_{SG}) = \ln\left(\frac{1+\sin(0.48)}{1-\sin(0.48)}\right) \approx 0.9991406561.$$

Now, we discuss $\mathcal{S}^*_{\sinh\lambda}$ radii for a few such subclasses of starlike functions in each of which $zf'(z)/f(z)$ is subordinate to a Ma-and-Minda-type function involving a single parameter. These subclasses are $\mathcal{BS}^*(\alpha)$, $\mathcal{M}(\beta)$, $\mathcal{SS}^*(\eta)$, $\mathcal{S}^*_T(\nu)$, and $\mathcal{S}^*_R$. The class $\mathcal{BS}^*(\alpha)$, for $0 \leq \alpha < 1$, was introduced by Kargar et al. [26]. They studied coefficient bounds and obtained subordination results for this class. The class $\mathcal{M}(\beta)$ consists of functions $f \in \mathcal{A}$ satisfying the relation $\Re[zf'(z)/f(z)] < \beta$; $\beta > 1$ was introduced by Uralegaddi et al. [5]. The third class that will be considered for a $\mathcal{S}^*_{\sinh\lambda}$ radius is the class $\mathcal{SS}^*(\eta)$ of strongly starlike functions, which was studied in [27,28]. The fourth class of this discourse is the subclass $\mathcal{S}^*_T(\nu)$ of starlike functions, which was introduced by Deniz [29] for $\phi(z) = e^{(z+\nu z^2/2)}$, ($\nu$ is an integer, with $\nu \geq 1$) in connection with the new generalization of telephone numbers. The fifth class for which we investigate the $\mathcal{S}^*_{\sinh\lambda}$ radius is the class $\mathcal{S}^*_R$ associated with a rational function, $\phi(z) = 1 + \left[(zk+z^2)/(k^2-kz)\right]$, which was introduced and studied by Kumar and Ravichandran in [30].

Theorem 6. *The sharp $\mathcal{S}^*_{\sinh\lambda}$ radii for some particular subclasses of starlike functions are the following:*

$$
\begin{array}{llll}
(i). & R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{BS}^*(\alpha)) & = & \begin{cases} \sin(\lambda) & \text{if} \quad \alpha = 0 \\ \frac{\sqrt{1+4\alpha\sin^2\lambda}-1}{2\alpha\sin(\lambda)} & \text{if} \quad 0<\alpha<1 \end{cases} \\
(ii). & R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{M}(\beta)) & = & \frac{\sin(\lambda)}{2(\beta-1)+\sin(\lambda)}, \ \text{for } \beta>1, \\
(iii). & R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{SS}^*(\eta)) & = & \frac{(1+\sin(\lambda))^{1/\eta}-1}{(1+\sin(\lambda))^{1/\eta}+1}, \ (0<\eta\leq 1), \\
(iv). & R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_T(\nu)) & = & \frac{\sqrt{1+2\nu\ln(1+\sin(\lambda))}-1}{\nu}, \ \nu\in\mathbb{N}, \\
(v). & R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_R) & = & \frac{2k\sin(\lambda)}{1+\sin(\lambda)+\sqrt{1+(6+\sin(\lambda))\sin(\lambda)}}.
\end{array}
$$

Proof. (i). Let $f\in\mathcal{BS}^*(\alpha)$. Then, $zf'(z)/f(z)\prec 1+z/(1-\alpha z^2)$, with $0\leq\alpha<1$. Hence, for $|z|=r$, we have

$$\left|\frac{zf'(z)}{f(z)}-1\right|\leq\frac{r}{1-\alpha r^2}.$$

From the use of Lemma 6, we conclude that the above disk will be contained in $\Omega_{\sinh\lambda}$ and, hence, $f\in\mathcal{S}^*_{\sinh\lambda}$ if $r/(1-\alpha r^2)\leq\sin(\lambda)$.

Case 1 : For $\alpha=0$, we have

$$r\leq\sin(\lambda)=R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{BS}^*(\alpha)).$$

The function $g_0(z)=ze^z$ proves the sharpness.

Case 2 : For $0<\alpha<1$, the inequality $r/(1-\alpha r^2)\leq\sin(\lambda)$ implies $\alpha(\sin(\lambda))r^2+r-\sin(\lambda)\leq 0$. Thus, $R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{BS}^*(\alpha))$ is the smallest positive root of the equation $\alpha(\sin(\lambda))r^2+r-\sin(\lambda)=0$, which, upon solving, yields

$$r\leq\frac{\sqrt{1+4\alpha\sin^2\lambda}-1}{2\alpha\sin(\lambda)}=R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{BS}^*(\alpha)).$$

To prove the sharpness of $R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{BS}^*(\alpha))$, we consider the function

$$f_0(z)=z\ \exp\left(\int_0^z\frac{q(t)-1}{t}dt\right),\ \text{with } q(z)=1+\frac{z}{1-\alpha z^2}=\frac{zf_0'(z)}{f_0(z)}.$$

At $z=R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{BS}^*(\alpha))$, we have

$$\left|\frac{zf_0'(z)}{f_0(z)}-1\right|=\frac{2\alpha\sin(\lambda)\left(\sqrt{1+4\alpha\sin^2\lambda}-1\right)}{2\alpha\left(\sqrt{1+4\alpha\sin(\lambda)}-1\right)}=\sin(\lambda).$$

Thus, the sharpness of $R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{BS}^*(\alpha))$ is verified.

(ii). Let $f\in\mathcal{M}(\beta)$. Then, by using Lemma 5 for $n=1$, we have

$$\left|\frac{zf'(z)}{f(z)}-\frac{1+(1-2\beta)r^2}{1-r^2}\right|\leq\frac{2(\beta-1)r}{1-r^2}.$$

Clearly, the inequality $1 + (1-2\beta)r^2/(1-r^2) \leq 1$ holds. Therefore, according to Lemma 6, the above disk contains $\Omega_{\sinh\lambda}$, or equivalently, $f \in \mathcal{S}^*_{\sinh\lambda}$ if

$$1 - \sin(\lambda) \leq \frac{1+(1-2\beta)r^2}{1-r^2} \text{ and } \frac{2(\beta-1)r}{1-r^2} \leq \frac{1+(1-2\beta)r^2}{1-r^2} - 1 + \sin(\lambda).$$

After some computation, we have

$$(2(\beta-1) + \sin(\lambda))r^2 \leq \sin(\lambda),$$

and

$$(2(\beta-1) + \sin(\lambda))r^2 + 2(\beta-1)r - \sin(\lambda) \leq 0,$$

which further imply, respectively, that

$$r \leq \sqrt{\frac{\sin(\lambda)}{2(\beta-1)+\sin(\lambda)}}, \text{ and } r \leq \frac{\sin(\lambda)}{2(\beta-1)+\sin(\lambda)}.$$

Hence, $R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{M}(\beta))$ is the minimum of these two, and it is

$$r \leq \frac{\sin(\lambda)}{2(\beta-1)+\sin(\lambda)} = R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{M}(\beta)).$$

For the sharpness of $R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{M}(\beta))$, we define the following function

$$f_0(z) = z\exp\left(\int_0^z \frac{q(t)-1}{t}dt\right),$$

with $q(z) = 1 + 2(\beta-1)z + [2(\beta-1)+\sin(\lambda)]z^2$. Then, at $z = R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{M}(\beta))$, we have

$$\left|\frac{zf_0'(z)}{f_0(z)} - 1\right| = \sin(\lambda),$$

which proves the sharpness of $R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{M}(\beta))$.

(iii). If $f \in \mathcal{SS}^*(\eta)$, then $zf'(z)/f(z) \prec ((1+z)/(1-z))^\eta$, with $0 < \eta \leq 1$. Therefore, for $|z| = r$, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \left(\frac{1+r}{1-r}\right)^\eta - 1.$$

The above disk is contained in $\Omega_{\sinh\lambda}$ as given in Lemma 6 if

$$\left(\frac{1+r}{1-r}\right)^\eta - 1 \leq \sin(\lambda).$$

After some simplification, we have

$$r \leq \frac{(1+\sin(\lambda))^{1/\eta} - 1}{(1+\sin(\lambda))^{1/\eta} + 1} = R^*_{\sinh\lambda}(\mathcal{SS}^*(\eta)).$$

To prove the sharpness of $R^*_{\sinh\lambda}(\mathcal{SS}^*(\eta))$, we consider the function

$$f_0(z) = z\exp\left(\int_0^z \frac{q(t)-1}{t}\right)$$

and $q(z) = \left(\frac{1+z}{1-z}\right)^{\eta} = \frac{zf_0'(z)}{f_0(z)}$. At $z = R^*_{\sinh\lambda}(\mathcal{SS}^*(\eta))$, we easily obtain

$$\left|\frac{zf_0'(z)}{f_0(z)} - 1\right| = \sin(\lambda).$$

(iv). Let $f \in \mathcal{S}^*_T(\nu)$ $(\nu \in \mathbb{N})$. Then, $zf'/f(z) \prec e^{(z+\nu z^2/2)}$. Therefore, for $|z| = r$, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq e^{r+\nu\frac{r^2}{2}} - 1.$$

Using Lemma 6, the last given disk is contained in $\Omega_{\sinh\lambda}$ and, hence, $f \in \mathcal{S}^*_{\sinh\lambda}$ if $e^{r+\nu\frac{r^2}{2}} - 1 \leq \sin(\lambda)$. Furthermore, it implies that

$$r + \nu r^2/2 \leq \ln(1 + \sin(\lambda)).$$

Thus, $R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_T)$ is the smallest positive root of the equation $\nu r^2/2 + r - \ln(1+\sin(\lambda)) = 0$. By simple computation, we obtain

$$R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_T) = \frac{\sqrt{1 + 2\nu\ln(1 + \sin(\lambda))} - 1}{\nu}.$$

For the sharpness of $R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_T(\nu))$, we consider the function

$$f_0(z) = z\exp\left(\int_0^z \frac{q(t)-1}{t}dt\right),\ q(z) = e^{z+\nu\frac{z^2}{2}} = \frac{zf_0'(z)}{f_0(z)}.$$

At $z = R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_T(\nu))$, we have

$$\left|\frac{zf_0'(z)}{f_0(z)} - 1\right| = \sin(\lambda).$$

(v). Let $f \in \mathcal{S}^*_R$. Then, $zf'(z)/f(z) \prec 1 + \left[(zk + z^2)/(k^2 - kz)\right]$. For $|z| = r$, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{kr + r^2}{k^2 - kr}.$$

According to Lemma 6, the above disk will be contained in $\Omega_{\sinh\lambda}$, and, hence, $f \in \mathcal{S}^*_{\sinh\lambda}$ if $(rk + r^2)/(k^2 - kr) \leq \sin(\lambda)$. Furthermore, computation yields $r^2 + k(1 + \sin(\lambda))r - k^2\sin(\lambda) \leq 0$. Thus, $R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_R)$ is the smallest root of the equation $r^2 + k(1+\sin(\lambda))r - k^2\sin(\lambda) = 0$. For the sharpness of $R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{S}^*_R)$, we consider the function

$$f_0(z) = z\ \exp\left(\int_0^z \frac{q(t)-1}{t}dt\right),$$

and with $q(z) = 1 + \frac{zk+z^2}{k^2-kz} = \frac{zf_0'(z)}{f_0(z)}$, we have

$$\left|\frac{zf_0'(z)}{f_0(z)} - 1\right| = \sin(\lambda).$$

□

Remark 6. *For $\beta = 2$ and $\eta = 1$, in the above parts (ii) and (iii), respectively, we have*

$$R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{M}(\beta)) = R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{SS}^*(\eta)).$$

4. Functions Defined in Terms of Ratio of Functions

Now, we discuss the radius problem of classes denoted by $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$ and defined as

$$\begin{aligned}
\mathcal{F}_1 &= \left\{ f \in \mathcal{A}_n : \Re\left(\frac{f(z)}{g(z)}\right) > 0 \text{ and } \Re\left(\frac{g(z)}{z}\right) > 0,\ g \in \mathcal{A}_n \right\}, \\
\mathcal{F}_2 &= \left\{ f \in \mathcal{A}_n : \Re\left(\frac{f(z)}{g(z)}\right) > 0 \text{ and } \Re\left(\frac{g(z)}{z}\right) > \frac{1}{2},\ g \in \mathcal{A}_n \right\}, \\
\mathcal{F}_3 &= \left\{ f \in \mathcal{A}_n : \left|\frac{f(z)}{g(z)} - 1\right| < 1 \text{ and } \Re\left(\frac{g(z)}{z}\right) > 0,\ g \in \mathcal{A}_n \right\}, \\
\mathcal{F}_4 &= \left\{ f \in \mathcal{A}_n : \left|\frac{f(z)}{g(z)} - 1\right| < 1, \text{for convex function } g \in \mathcal{A}_n \right\}.
\end{aligned}$$

Theorem 7. *The sharp $\mathcal{S}^*_{\sinh\lambda,n}$ radii for functions in the classes $\mathcal{F}_1$, $\mathcal{F}_2$ $\mathcal{F}_3$, and $\mathcal{F}_4$, respectively, are:*

$$\begin{aligned}
(i).\ R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{F}_1) &= \left(\sqrt{1+4n^2\csc^2\lambda} - 2n\ \csc\lambda\right)^{1/n}, \\
(ii).\ R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{F}_2) &= \left(\frac{\sqrt{9n^2+4n\sin\lambda+4\sin^2\lambda} - 3n}{2(n+\sin\lambda)}\right)^{1/n}, \\
(iii).\ R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{F}_3) &= \left(\frac{\sqrt{9n^2+4n\sin\lambda+4\sin^2\lambda} - 3n}{2(n+\sin\lambda)}\right)^{1/n}, \\
(iv).\ R_{\mathcal{S}^*_{\sinh\lambda}}(\mathcal{F}_4) &= \left(\frac{\sqrt{(n+1)^2+4(n-1+\sin\lambda)\sin\lambda} - (n+1)}{2(n-1+\sin\lambda)}\right)^{1/n}.
\end{aligned}$$

Proof. (i). Let $f \in \mathcal{F}_1$. Then, there is $g \in \mathcal{A}_n$ such that

$$\Re\left(\frac{f(z)}{g(z)}\right) > 0, \quad \text{and} \quad \Re\left(\frac{g(z)}{z}\right) > 0.$$

Let us choose $p(z) = f(z)/g(z)$ and $h(z) = g(z)/z$. Then, clearly $p, h \in \mathcal{P}_n$, and so we easily obtain

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \left|\frac{zp'(z)}{p(z)}\right| + \left|\frac{zh'(z)}{h(z)}\right|.$$

By applying Lemma 4 for p and h in the above inequality, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{4nr^n}{1-r^{2n}}.$$

Now, by using Lemma 6, the function $f \in \mathcal{S}^*_{\sinh\lambda,n}$ if

$$\frac{4nr^n}{1-r^{2n}} \leq \sin(\lambda),$$

or, equivalently,

$$\sin\lambda r^{2n} + 4nr^n - \sin\lambda \leq 0.$$

Thus, $r = \left(\sqrt{1+4n^2\csc^2\lambda} - 2n\ \csc\lambda\right)^{1/n} = R_{\mathcal{S}^*_{\sinh\lambda,n}}(\mathcal{F}_1)$ is the smallest positive root of the equation

$$\sin(\lambda)r^{2n} + 4nr^n - \sin(\lambda) = 0.$$

For the sharpness of $R_{\mathcal{S}^*_{\sinh\lambda,n}}(\mathcal{F}_1)$, we consider the following two functions

$$f_0(z) = z\left(\frac{1+z^n}{1-z^n}\right)^2, \quad \text{and} \quad g_0(z) = z\left(\frac{1+z^n}{1-z^n}\right).$$

Then, it is obvious for $f_0(z)/g_0(z) = (1+z^n)/(1-z^n) = g_0(z)/z$ that $\Re(f_0(z)/g_0(z)) > 0$ and $\Re(g_0(z)/z) > 0$. Its further implies that $f_0 \in \mathcal{F}_1$ and also that

$$\frac{zf_0'(z)}{f_0(z)} - 1 = \frac{4nz^n}{1-z^{2n}} = \sin(\lambda)$$

at $z = R_{\mathcal{S}^*_{\sinh\lambda,n}}(\mathcal{F}_1)$. This proves the sharpness of $R_{\mathcal{S}^*_{\sinh\lambda,n}}(\mathcal{F}_1)$.

(ii). If $f \in \mathcal{F}_2$, then there exist a function $g \in \mathcal{A}_n$ such that

$$\Re\left(\frac{f(z)}{g(z)}\right) > 0, \text{ and } \Re\left(\frac{g(z)}{z}\right) > \frac{1}{2}.$$

Let us set $p(z) = f(z)/g(z)$ and $h(z) = g(z)/z$. Then, clearly $p \in \mathcal{P}_n$ and $h \in \mathcal{P}_n(1/2)$. Also, it yields

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \left|\frac{zp'(z)}{p(z)}\right| + \left|\frac{zh'(z)}{h(z)}\right|.$$

Therefore, by using Lemma 4 for p and h, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{nr^n(3+r^n)}{1-r^{2n}}, \text{ for } |z| = r.$$

Now, by virtue of Lemma 6, the function $f \in \mathcal{S}^*_{\sinh\lambda,n}$ if

$$\frac{nr^n(3+r^n)}{1-r^{2n}} \leq \sin(\lambda),$$

or

$$(n+\sin\lambda)r^{2n} + 3nr^n - \sin(\lambda) \leq 0,$$

Thus, $R_{\mathcal{S}^*_{\sinh\lambda,n}}(\mathcal{F}_2)$ is the smallest positive root of the equation

$$(n+\sin\lambda)r^{2n} + 3nr^n - \sin(\lambda) = 0.$$

To verify the sharpness of $R_{\mathcal{S}^*_{\sinh\lambda,n}}(\mathcal{F}_2)$, we consider the functions

$$f_0(z) = \frac{z(1+z^n)}{(1-z^n)^2}, \text{ and } g_0(z) = \frac{z}{1-z^n},$$

and with simple computation, we have

$$\frac{f_0(z)}{g_0(z)} = \frac{1+z^n}{1-z^n}, \text{ and } \frac{g_0(z)}{z} = \frac{z}{1-z^n}.$$

Also, $\Re(f_0(z)/g_0(z)) > 0$, and $\Re(g_0(z)/z) > 1/2$. This proves that $f_0 \in \mathcal{F}_2$ and also we easily obtain

$$\frac{zf_0'(z)}{f_0(z)} - 1 = \frac{3nz^n + nz^{2n}}{1-z^{2n}}.$$

By putting $z = R_{\mathcal{S}^*_{\sinh\lambda,n}}(\mathcal{F}_2)$, it yields

$$\left|\frac{zf_0'(z)}{f_0(z)} - 1\right| = \sin(\lambda),$$

and this confirms the sharpness.

(iii). Let $f \in \mathcal{F}_3$. Then, by definition, there is $g \in \mathcal{A}_n$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1 \text{ and } \Re\left(\frac{g(z)}{z}\right) > 0.$$

If we put $p(z) = g(z)/f(z)$ and $h(z) = g(z)/z$, then the above inequalities becomes

$$\left|\frac{1}{p(z)} - 1\right| < 1 \Leftrightarrow \Re p(z) > \frac{1}{2} \text{ and } \Re h(z) > 0.$$

Hence, $p \in \mathcal{P}_n(\frac{1}{2})$, and $h \in \mathcal{P}_n$. By simple computation, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \left|\frac{zp'(z)}{p(z)}\right| + \left|\frac{zh'(z)}{h(z)}\right|.$$

By applying Lemma 4 for p and h, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{3nr^n + nr^{2n}}{1 - r^{2n}}.$$

Proceeding on the same lines as in above part (**ii**), we obtain

$$R_{\mathcal{S}^*_{\sinh\lambda,n}}(\mathcal{F}_3) = \left(\frac{\sqrt{9n^2 + 4n\sin(\lambda) + 4\sin^2\lambda} - 3n}{2(n + \sin(\lambda))}\right)^{1/n}.$$

For the sharpness of $R_{\mathcal{S}^*_{\sinh\lambda,n}}(\mathcal{F}_3)$, we consider the following functions

$$f_0(z) = \frac{z(1+z^n)^2}{1 - z^n} \text{ and } g_0(z) = \frac{z(1+z^n)}{1 - z^n}.$$

However, since $|f_0(z)/g_0(z) - 1| = |(1 + z^n) - 1| < 1$, it implies that $\Re(g_0(z)/z) > 0$, and, hence, $f_0 \in \mathcal{F}_3$. Also

$$\frac{zf_0'(z)}{f_0(z)} - 1 = \frac{(3nz^n - nz^{2n})}{1 - z^{2n}} = \sin(\lambda),$$

at $z = R_{\mathcal{S}^*_{\sinh\lambda,n}}(\mathcal{F}_3)$ and this proves the sharpness of $R_{\mathcal{S}^*_{\sinh\lambda,n}}(\mathcal{F}_3)$.

(iv). If $f \in \mathcal{F}_4$, then a convex function $g \in \mathcal{A}_n$ exists such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1.$$

Or, equivalently, we write it as $\Re(g(z)/f(z)) > 1/2$. Also, since every convex function is starlike of order $1/2$, it therefore follows from Lemma 5 that

$$\left|\frac{zg'(z)}{g(z)} - \frac{1}{1 - r^{2n}}\right| \leq \frac{r^n}{1 - r^{2n}}. \tag{14}$$

If we put $h(z) = g(z)/f(z)$, then $h \in \mathcal{P}_n(1/2)$. Hence, from $f(z) = g(z)/h(z))$, we have

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} - \frac{zh'(z)}{h(z)}.$$

Therefore, from the inequality (14) and Lemma 4, we easily have

$$\left|\frac{zf'(z)}{f(z)} - \frac{1}{1 - r^{2n}}\right| \leq \frac{(n+1)r^n + nr^{2n}}{1 - r^{2n}}.$$

Since $1/(1 - r^{2n}) \leq 1$, therefore, accodring to Lemma 6, the above disk will be contained in $\Omega_{\sinh\lambda}$, and, hence, $f \in \mathcal{S}^*_{\sinh\lambda,n}$ if

$$\frac{(n+1)r^n + nr^{2n}}{1 - r^{2n}} \leq \frac{1}{1 - r^{2n}} - (1 - \sin(\lambda)),$$

or, equivalently, we have

$$(n + \sin(\lambda) - 1)r^{2n} + (n+1)r^n - \sin(\lambda) \leq 0.$$

Thus, $R_{\mathcal{S}^*_{\sinh\lambda,n}}(\mathcal{F}_4)$ is the smallest positive root of the equation

$$(n + \sin(\lambda) - 1)r^{2n} + (n+1)r^n - \sin(\lambda) = 0.$$

For the sharpness of $R_{\mathcal{S}^*_{\sinh\lambda,n}}(\mathcal{F}_4)$, consider the functions

$$f_0(z) = \frac{z(1+z^n)}{(1-z^n)^{1/n}}, \text{ and } g_0(z) = \frac{z}{(1-z^n)^{1/n}}.$$

Also, $|f_0(z)/g_0(z) - 1| = |(1+z^n) - 1| < 1$, and

$$\Re\left(1 + \frac{zg_0''(z)}{g_0'(z)}\right) = \Re\left(\frac{1+nz^n}{1-z^n}\right) > 0,$$

thus showing that $g_0(z)$ is convex. Thus, $f_0 \in \mathcal{F}_4$. Also,

$$\left|\frac{zf_0'(z)}{f_0(z)} - 1\right| = \left|\frac{(n+1)z^n - (n-1)z^{2n}}{1-z^{2n}}\right|,$$

which, at $z = R_{\mathcal{S}^*_{\sinh\lambda,n}}(\mathcal{F}_4)$, yields

$$\left|\frac{zf_0'(z)}{f_0(z)} - 1\right| = \sin(\lambda),$$

thus proving the sharpness of $R_{\mathcal{S}^*_{\sinh\lambda,n}}(\mathcal{F}_4)$. $\square$

5. Conclusions

As some portion of the image of the function $1 + \sinh(z)$ is not in the right-half plan, we introduced λ with $0 \leq \lambda \leq \ln(1+\sqrt{2})$, as factor of z and obtained the Ma-Minda-type function $1 + \sinh(\lambda z)$, whose image is entirely in the right-half plane for all values of λ in the above specified interval. Thus, we defined a new subclass of starlike functions $S^*_{\sinh\lambda}$. We obtained first four sharp coefficient bounds and $S^*_{\sinh\lambda}$ radii of some well recognized subclasses of analytic functions. Still, there are so many directions, for example, Hankel determinants, for both the functions of this class and for its inverse functions, Fekete-Szegö type inequality, logarithmic coefficients, partial sums problems, sufficiency criteria, convolution preserving property and many more in which researchers can show the essence of their abilities. In addition, with the association of $q(z) = 1 + \sinh(\lambda z)$, one can defined the class of convex functions, close-to-convex functions, bounded turnings and etc.

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