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Laguerre–Freud Equations for the Gauss Hypergeometric Discrete Orthogonal Polynomials †

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† Dedicated to the memory of Diego Dominici.

Abstract: The Cholesky factorization of the moment matrix is considered for the Gauss hypergeometric discrete orthogonal polynomials. This family of discrete orthogonal polynomials has a weight with first moment given by $\rho_0={}_2F_1\left[\begin{matrix} a,b\\ c+1\end{matrix};\eta\right]$. For the Gauss hypergeometric discrete orthogonal polynomials, also known as generalized Hahn of type I, Laguerre–Freud equations are found, and the differences with the equations found by Dominici and by Filipuk and Van Assche are provided.

Keywords: discrete orthogonal polynomials; Pearson equations; Cholesky factorization; Laguerre–Freud equations; hypergeometric functions; tau functions; Gauss hypergeometric orthogonal polynomials

MSC: 42C05; 33C45; 33C47

1. Introduction

Discrete orthogonal polynomials constitute a well-established and distinguished branch of orthogonal polynomial theory. Various classical monographs have been dedicated to this category of orthogonal polynomials. For instance, the classical case is thoroughly examined in [1], and the Riemann–Hilbert problem has been employed to investigate asymptotics and applications, as discussed in [2]. Authoritative discussions of the subject can be found in [3–6].

Discrete orthogonal polynomials play a role in designing digital filters for signal processing applications. They help in representing signals in a more compact and efficient manner, facilitating analysis and manipulation. Discrete orthogonal polynomials are employed in approximating functions. They provide a basis for representing functions in terms of a series expansion, making it easier to approximate complex functions. Discrete orthogonal polynomials are used in coding theory for designing error-correcting codes. They help in constructing efficient encoding and decoding algorithms, improving the reliability of data transmission in communication systems. Discrete orthogonal polynomials are often involved in combinatorics, providing tools for solving combinatorial problems and deriving combinatorial identities.

Semiclassical discrete orthogonal polynomials, characterized by the satisfaction of a discrete Pearson equation by their weight functions, have received extensive attention in the literature. A comprehensive overview of this topic, along with extensive references, can be found in [7–10]. Furthermore, for certain specific weight types, such as the generalized Charlier and Meixner weights, the corresponding Freud–Laguerre-type equations governing the coefficients of the three-term recurrence have been investigated. Notable works in this area include [11–15].

In our work presented in [16], we harnessed the Cholesky factorization of the moment matrix to delve into the realm of discrete orthogonal polynomials denoted as $\{P_n(x)\}_{n=0}^{\infty}$ on a homogeneous lattice. We directed our focus on semiclassical discrete orthogonal

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polynomials, defined by weight functions constrained by a discrete Pearson equation. This constraint led to the moments being expressible in terms of generalized hypergeometric functions.

We introduced a banded semi-infinite matrix named the Laguerre–Freud structure matrix Ψ, designed to model shifts by ± 1 in the independent variable of the sequence of orthogonal polynomials $\{P_n(x)\}_{n=0}^{\infty}$. Our study also unveiled that the contiguous relations governing the generalized hypergeometric functions have corresponding symmetries within the moment matrix. Furthermore, we established that the 3D Nijhoff–Capel discrete Toda lattice [17,18] offers a description of the contiguous shifts pertaining to the squared norms of the orthogonal polynomials.

In [19], we provided an interpretation for the contiguous transformations of generalized hypergeometric functions by invoking simple Christoffel and Geronimus transformations. Leveraging Geronimus–Uvarov perturbations, we derived determinantal expressions for the shifted orthogonal polynomials. Additionally, our exploration extended to three hypergeometric families and their associated Laguerre–Freud equations in [20].

In the research presented here, we extend and complete the investigations in [16,19,20], adding a new weight not studied so far with the similar techniques adapted to this situation. We delve into the realm of Gauss hypergeometric discrete orthogonal polynomials, which are also known as generalized Hahn polynomials of type I, as discussed in [9,10]. Our primary focus is an in-depth analysis of the Laguerre–Freud structure matrix Ψ. By examining its unique banded structure and its inherent compatibility with both the Toda equation and the Jacobi matrix, we uncover a set of nonlinear equations that govern the coefficients β_n, γ_n in the three-term recursion relations for the orthogonal polynomial sequence.

These nonlinear recurrences for the recursion coefficients take the following form:

$$\begin{aligned}
\gamma_{n+1} &= F_1(n, \gamma_n, \gamma_{n-1}, \dots, \beta_n, \beta_{n-1}, \dots),\\
\beta_{n+1} &= F_2(n, \gamma_{n+1}, \gamma_n, \dots, \beta_n, \beta_{n-1}, \dots),
\end{aligned}$$

with F_1 and F_2 being specific functions. Notably, these relations, referred to as Laguerre–Freud relations by Magnus [21] in reference to [22,23], have been explored in numerous papers for cases like generalized Charlier, generalized Meixner, and Gauss hypergeometric, as documented in [7,14].

A crucial insight that emerges is the role of the τ-function, defined as the Wronskian of the Gauss hypergeometric functions. This τ-function proves to be a valuable solution for each of these systems of nonlinear Laguerre–Freud equations governing the recursion coefficients.

This paper's structure is outlined as follows: In Section 2, we delve into the world of Gauss hypergeometric discrete orthogonal polynomials, as initially introduced in [7,9]. We present the pentadiagonal Laguerre–Freud structure matrix in Theorem 3. Additionally, we delve into the Laguerre–Freud relations in Theorem 4, providing a comparative analysis with Dominici's findings in [7] and those of Filipuk & Van Assche in [14], as well as the work presented in [24].

To round off this introduction, let us begin by summarizing the foundational concepts of discrete orthogonal polynomials. Afterward, we will provide a brief overview of the significant findings from our previous work in [16].

1.1. Basics on Orthogonal Polynomials

Given a linear functional $\rho_z \in \mathbb{C}^*[z]$, the corresponding moment matrix is

$$\begin{aligned}
G &= (G_{n,m}),\\
G_{n,m} &= \rho_{n+m},\\
\rho_n &= \langle \rho_z, z^n \rangle, && n, m \in \mathbb{N}_0 := \{0, 1, 2, \dots\},
\end{aligned}$$

with ρ_n the n-th moment of the linear functional ρ_z. If the moment matrix is such that all its truncations, which are Hankel matrices, $G_{i+1,j} = G_{i,j+1}$,

$$G^{[k]} = \begin{bmatrix} G_{0,0} & \cdots & G_{0,k-1} \\ \vdots & & \vdots \\ G_{k-1,0} & \cdots & G_{k-1,k-1} \end{bmatrix} = \begin{bmatrix} \rho_0 & \rho_1 & \rho_2 & \cdots & \rho_{k-1} \\ \rho_1 & \rho_2 & & \iddots & \rho_k \\ \rho_2 & & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & & \vdots \\ \rho_{k-1} & \rho_k & \cdots & \cdots & \rho_{2k-2} \end{bmatrix}$$

are nonsingular; i.e., the Hankel determinants $\Delta_k := \det G^{[k]}$ do not cancel, $\Delta_k \neq 0$, $k \in \mathbb{N}_0$. If this is the case, we have monic polynomials.

$$P_n(z) = z^n + p_n^1 z^{n-1} + \cdots + p_n^n, \quad n \in \mathbb{N}_0, \tag{1}$$

with $p_0^1 = 0$, fulfilling the orthogonality relations

$$\left\langle \rho, P_n(z) z^k \right\rangle = 0, \quad k \in \{0, \dots, n-1\}, \quad \left\langle \rho, P_n(z) z^n \right\rangle = H_n \neq 0,$$

and $\{P_n(z)\}_{n \in \mathbb{N}_0}$ is a sequence of orthogonal polynomials, i.e., $\left\langle \rho, P_n(z) P_m(z) \right\rangle = \delta_{n,m} H_n$ for $n, m \in \mathbb{N}_0$. The symmetric bilinear form $\langle F, G \rangle_\rho := \langle \rho, FG \rangle$, is such that the moment matrix is the Gram matrix of this bilinear form and $\langle P_n, P_m \rangle_\rho := \delta_{n,m} H_n$.

Introducing $\chi(z) := \begin{pmatrix} 1 & z & z^2 \cdots \end{pmatrix}^\top$ the moment matrix is $G = \left\langle \rho, \chi \chi^\top \right\rangle$, and χ is an eigenvector of the *shift matrix*, $\Lambda \chi = x \chi$, where

$$\Lambda := \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ & & & \end{bmatrix}.$$

Hence, $\Lambda G = G \Lambda^\top$, and the moment matrix is a Hankel matrix.

As the moment matrix symmetric its Borel–Gauss factorization is a Cholesky factorization

$$G = S^{-1} H S^{-\top},$$

where S is a lower unitriangular matrix that can be written as

$$S = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ S_{1,0} & 1 & 0 & \ddots \\ S_{2,0} & S_{2,1} & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix},$$

and $H = \operatorname{diag}(H_0, H_1, \dots)$ is a diagonal matrix, with $H_k \neq 0$, for $k \in \mathbb{N}_0$. The Cholesky factorization does hold whenever the principal minors of the moment matrix; i.e., the Hankel determinants Δ_k, do not cancel.

The components $P_n(z)$ of

$$P(z) := S\chi(z), \tag{2}$$

are the monic orthogonal polynomials of the functional ρ.

Proposition 1. *We have the determinantal expressions*

$$H_n = \frac{\Delta_{n+1}}{\Delta_n}, \quad p_n^1 = -\frac{\tilde{\Delta}_n}{\Delta_n},$$

with the Hankel determinants given by

$$\Delta_n := \begin{vmatrix} \rho_0 & \cdots & \rho_{n-2} & \rho_{n-1} \\ \vdots & & & \vdots \\ \rho_{n-2} & \rho_{n-1} & \cdots & \rho_{2n-3} \\ \rho_{n-1} & \cdots & \rho_{2n-3} & \rho_{2n-2} \end{vmatrix}, \quad \tilde{\Delta}_n := \begin{vmatrix} \rho_0 & \cdots & \rho_{n-2} & \rho_{n-1} \\ \vdots & & & \vdots \\ \rho_{n-2} & \rho_{n-1} & \cdots & \rho_{2n-3} \\ \rho_{n} & \cdots & \rho_{2n-2} & \rho_{2n-1} \end{vmatrix}.$$

We introduce the lower Hessenberg semi-infinite matrix

$$J = S\Lambda S^{-1} \tag{3}$$

that has the vector $P(z)$ as eigenvector with eigenvalue z, $JP(z) = zP(z)$. The Hankel condition $\Lambda G = G\Lambda^\top$ and the Cholesky factorization gives

$$JH = (JH)^\top = HJ^\top.$$

As the Hessenberg matrix JH is symmetric, the Jacobi matrix J is tridiagonal. The Jacobi matrix J given in (3) reads

$$J = \begin{bmatrix} \beta_0 & 1 & 0 & \cdots \\ \gamma_1 & \beta_1 & 1 & \ddots \\ 0 & \gamma_2 & \beta_2 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

The eigenvalue equation $JP = zP$ is a three term recursion relation $zP_n(z) = P_{n+1}(z) + \beta_n P_n(z) + \gamma_n P_{n-1}(z)$, that with the initial conditions $P_{-1} = 0$ and $P_0 = 1$ completely determines the sequence of orthogonal polynomials $\{P_n(z)\}_{n=0}^\infty$ in terms of the recursion coefficients β_n, γ_n. The recursion coefficients, in terms of the Hankel determinants, are given by

$$\beta_n = p_n^1 - p_{n+1}^1 = -\frac{\tilde{\Delta}_n}{\Delta_n} + \frac{\tilde{\Delta}_{n+1}}{\Delta_{n+1}}, \quad \gamma_{n+1} = \frac{H_{n+1}}{H_n} = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2}, \quad n \in \mathbb{N}_0. \tag{4}$$

For future use, we introduce the following diagonal matrices $\gamma{:=}\operatorname{diag}(\gamma_1, \gamma_2, \dots)$ and $\beta{:=}\operatorname{diag}(\beta_0, \beta_1, \dots)$ and $J_-{:=}\Lambda^\top\gamma$ and $J_+{:=}\beta + \Lambda$, so that we have the splitting $J = \Lambda^\top\gamma + \beta + \Lambda = J_- + J_+$. In general, given any semi-infinite matrix A, we will write $A = A_- + A_+$, where A_- is a strictly lower triangular matrix and A_+ an upper triangular matrix. Moreover, A_0 will denote the diagonal part of A.

The lower Pascal matrix is defined by

$$B = (B_{n,m}), \quad B_{n,m} := \begin{cases} \binom{n}{m}, & n \geq m, \\ 0, & n < m, \end{cases}$$

so that

$$\chi(z+1) = B\chi(z).$$

Moreover,

$$B^{-1} = (\tilde{B}_{n,m}), \quad \tilde{B}_{n,m} := \begin{cases} (-1)^{n+m}\binom{n}{m}, & n \geq m, \\ 0, & n < m, \end{cases}$$

and $\chi(z-1) = B^{-1}\chi(z)$. The lower Pascal matrix and its inverse are explicitly given by

$$B = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 0 & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & 1 & 0 & \cdots & \cdots & \cdots \\ 1 & 3 & 3 & 1 & 0 & \cdots & \cdots \\ 1 & 4 & 6 & 4 & 1 & 0 & \cdots \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

$$B^{-1} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & 1 & 0 & \cdots & \cdots & \cdots & \cdots \\ 1 & -2 & 1 & 0 & \cdots & \cdots & \cdots \\ -1 & 3 & -3 & 1 & 0 & \cdots & \cdots \\ 1 & -4 & 6 & -4 & 1 & 0 & \cdots \\ -1 & 5 & -10 & 10 & -5 & 1 & 0 \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

in terms of which we introduce the dressed Pascal matrices, $\Pi{:=}SBS^{-1}$ and $\Pi^{-1}{:=}SB^{-1}S^{-1}$, which are connection matrices; i.e.,

$$P(z+1) = \Pi P(z), \quad P(z-1) = \Pi^{-1}P(z). \tag{5}$$

The lower Pascal matrix can be expressed in terms of its subdiagonal structure as follows

$$B^{\pm 1} = I \pm \Lambda^\top D + \left(\Lambda^\top\right)^2 D^{[2]} \pm \left(\Lambda^\top\right)^3 D^{[3]} + \cdots,$$

where $D = \operatorname{diag}(1, 2, 3, \dots)$ and $D^{[k]} = \frac{1}{k}\operatorname{diag}\left(k^{(k)}, (k+1)^{(k)}, (k+2)^{(k)} \cdots\right)$, in terms of the falling factorials $x^{(k)} = x(x-1)(x-2)\cdots(x-k+1)$. That is,

$$D^{[k]}_n = \frac{(n+k)\cdots(n+1)}{k}, \quad k \in \mathbb{N}, \quad n \in \mathbb{N}_0.$$

The lower unitriangular factor can be also written in terms of its subdiagonals $S = I + \Lambda^\top S^{[1]} + \left(\Lambda^\top\right)^2 S^{[2]} + \cdots$ with $S^{[k]} = \operatorname{diag}\left(S^{[k]}_0, S^{[k]}_1, \dots\right)$. From (2) is clear the following connection between these subdiagonals entries and the coefficients of the orthogonal polynomials given in (1)

$$S^{[k]}_n = p^k_{n+k}.$$

We will use the *shift operators* $T_\pm$ acting over the diagonal matrices as follows

$$T_- \operatorname{diag}(a_0, a_1, \dots) := \operatorname{diag}(a_1, a_2, \dots), \qquad T_+ \operatorname{diag}(a_0, a_1, \dots) := \operatorname{diag}(0, a_0, a_1, \dots).$$

These shift operators have the following important properties, for any diagonal matrix $A = \operatorname{diag}(A_0, A_1, \dots)$

$$\Lambda A = (T_- A)\Lambda, \qquad A\Lambda = \Lambda(T_+ A), \qquad A\Lambda^\top = \Lambda^\top(T_- A), \qquad \Lambda^\top A = (T_+ A)\Lambda^\top.$$

In terms of these shift operators, we find

$$2D^{[2]} = (T_- D)D, \quad 3D^{[3]} = (T_-^2 D)(T_- D)D = 2(T_- D^{[2]})D = 2D^{[2]}(T_-^2 D).$$

Proposition 2. *The inverse matrix S^{-1} of the matrix S expands as follows*

$$S^{-1} = I + \Lambda^\top S^{[-1]} + (\Lambda^\top)^2 S^{[-2]} + \cdots .$$

The subdiagonals $S^{[-k]}$ are given in terms of the subdiagonals of S. In particular,

$$\begin{aligned}
S^{[-1]} &= -S^{[1]}, \\
S^{[-2]} &= -S^{[2]} + (T_- S^{[1]})S^{[1]}, \\
S^{[-3]} &= -S^{[3]} + (T_- S^{[2]})S^{[1]} + (T_-^2 S^{[1]})S^{[2]} - (T_-^2 S^{[1]})(T_- S^{[1]})S^{[1]}.
\end{aligned}$$

Remark 1. *Corresponding expansions for the dressed Pascal matrices are*

$$\Pi^{\pm 1} = I + \Lambda^\top \pi^{[\pm 1]} + (\Lambda^\top)^2 \pi^{[\pm 2]} + \cdots$$

with $\pi^{[\pm n]} = \operatorname{diag}(\pi_0^{[\pm n]}, \pi_1^{[\pm n]}, \dots)$.

Proposition 3 (The dressed Pascal matrix coefficients)**.** *We have*

$$\begin{aligned}
\pi_n^{[\pm 1]} &= \pm(n+1), \quad \pi_n^{[\pm 2]} = \frac{(n+2)(n+1)}{2} \pm p^1_{n+2}(n+1) \mp (n+2)p^1_{n+1} \\
&\qquad\qquad\qquad\quad\;\; = \frac{(n+2)(n+1)}{2} \mp (n+1)\beta_{n+1} \mp p^1_{n+1}, \\
\pi_n^{[\pm 3]} &= \pm\frac{(n+3)(n+2)(n+1)}{3} + \frac{(n+2)(n+1)}{2}p^1_{n+3} - \frac{(n+3)(n+2)}{2}p^1_{n+1} \\
&\quad \pm (n+1)p^2_{n+3} \mp (n+3)p^2_{n+2} \pm (n+3)p^1_{n+2}p^1_{n+1} \mp (n+2)p^1_{n+3}p^1_{n+1}.
\end{aligned} \tag{6}$$

Moreover, the following relations are fulfill

$$\pi^{[1]} + \pi^{[-1]} = 0, \quad \pi^{[2]} + \pi^{[-2]} = 2D^{[2]}, \quad \pi^{[3]} + \pi^{[-3]} = 2((T_-^2 S^{[1]})D^{[2]} - (T_- D^{[2]})S^{[1]}).$$

1.2. Discrete Orthogonal Polynomials and Pearson Equation

We are interested in measures with support on the homogeneous lattice $\mathbb{N}_0$ as follows $\rho = \sum_{k=0}^{\infty} \delta(z-k)w(k)$, with moments given by

$$\rho_n = \sum_{k=0}^{\infty} k^n w(k), \tag{7}$$

and, in particular, with 0-th moment given by

$$\rho_0 = \sum_{k=0}^{\infty} w(k). \tag{8}$$

The weights we consider in this paper satisfy the following *discrete Pearson equation*

$$\theta(k+1)w(k+1) = \sigma(k)w(k), \qquad k \in \mathbb{N}_0. \tag{9}$$

Theorem 1 (Hypergeometric symmetries)**.** *Let the weight w be subject to a discrete Pearson equation of the type (9), where the functions θ, σ are polynomials, with $\theta(0) = 0$. Then,*

(i) *The moment matrix fulfills*

$$\theta(\Lambda)G = B\sigma(\Lambda)GB^\top.$$

(ii) *The Jacobi matrix satisfies*

$$\Pi^{-1}H\theta(J^\top) = \sigma(J)H\Pi^\top,$$

and the matrices $H\theta(J^\top)$ and $\sigma(J)H$ are symmetric.

If $N+1 :=\deg\theta(z)$ and $M:=\deg\sigma(z)$, and zeros of these polynomials are $\{-b_i+1\}_{i=1}^N$ and $\{-a_i\}_{i=1}^M$ we write $\theta(z) = z(z+b_1-1)\cdots(z+b_N-1)$ and $\sigma(z) = \eta(z+a_1)\cdots(z+a_M)$. According to (8) the 0-th moment

$$\rho_0 = \sum_{k=0}^\infty w(k) = \sum_{k=0}^\infty \frac{(a_1)_k\cdots(a_M)_k}{(b_1+1)_k\cdots(b_N+1)_k}\frac{\eta^k}{k!} = {}_MF_N\left[\begin{matrix}a_1\cdots\cdots a_M\\ b_1\cdots\cdots b_N\end{matrix};\eta\right].$$

is the generalized hypergeometric function, where we are using the standard notationamc+amc+, see [25]. Then, according to (7), for $n\in\mathbb{N}$, the corresponding higher moments $\rho_n = \sum_{k=0}^\infty k^n w(k)$, are

$$\rho_n = \vartheta_\eta^n\rho_0 = \vartheta_\eta^n\left({}_MF_N\left[\begin{matrix}a_1\cdots\cdots a_M\\ b_1\cdots\cdots b_N\end{matrix};\eta\right]\right), \quad \vartheta_\eta := \eta\frac{\partial}{\partial\eta}.$$

Given a function $f(\eta)$, we consider the Wronskian of the covector

$$\delta(f) := \begin{bmatrix} f & \vartheta f \cdots\cdots \vartheta^n f\end{bmatrix}$$

given by

$$\mathscr{W}_n(f) := \begin{vmatrix} f & \vartheta_\eta f & \vartheta_\eta^2 f & \cdots & \vartheta_\eta^n f\\ \vartheta_\eta f & \vartheta_\eta^2 f & & \iddots & \vartheta_\eta^{n+1} f\\ \vartheta_\eta^2 f & & \iddots & & \vdots\\ \vdots & \iddots & & & \\ \vartheta_\eta^n f & \vartheta_\eta^{n+1} f & \cdots & \cdots & \vartheta_\eta^{2n} f\end{vmatrix}.$$

We refer to this Wronskian as the δ-Wronskian of f. Then, we have that the Hankel determinants $\Delta_n = \det G^{[n]}$ determined by the truncations of the corresponding moment matrix are δ-Wronskians of generalized hypergeometric functions,

$$\Delta_n = \tau_n, \quad \tilde{\Delta}_n = \vartheta_\eta\tau_n, \tag{10}$$

where

$$\tau_n := \mathscr{W}_n\left({}_MF_N\left[\begin{matrix}a_1\cdots\cdots a_M\\ b_1\cdots\cdots b_N\end{matrix};\eta\right]\right).$$

Moreover, using Proposition 1 we get

$$H_n = \frac{\tau_{n+1}}{\tau_n}, \quad p_n^1 = -\vartheta_\eta\log\tau_n, \quad n\in\mathbb{N}_0. \tag{11}$$

The functions τ_k are the well known tau functions [17]. In terms of these τ-functions we have

$$\beta_n = \vartheta_\eta\log\frac{\tau_{n+1}}{\tau_n}, \quad \gamma_{n+1} = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}, \quad n\in\mathbb{N}_0, \tag{12}$$

were we have used (4), (10) and (11).

Theorem 2 (Laguerre–Freud structure matrix)**.** *Let us assume that the weight w solves the discrete Pearson Equation (9) with θ, σ polynomials such that $\theta(0) = 0$, $\deg \theta(z) = N+1$, $\deg \sigma(z) = M$. Then, the Laguerre–Freud structure matrix*

$$\begin{aligned}\Psi &:= \Pi^{-1} H \theta(J^\top) = \sigma(J) H \Pi^\top = \Pi^{-1}\theta(J) H = H\sigma(J^\top)\Pi^\top \\ &= \theta(J+I)\Pi^{-1} H = H\Pi^\top \sigma(J^\top - I),\end{aligned} \tag{13}$$

has only $N+M+2$ possibly nonzero diagonals ($N+1$ superdiagonals and M subdiagonals)

$$\Psi = (\Lambda^\top)^M \psi^{(-M)} + \cdots + \Lambda^\top \psi^{(-1)} + \psi^{(0)} + \psi^{(1)}\Lambda + \cdots + \psi^{(N+1)}\Lambda^{N+1},$$

for some diagonal matrices $\psi^{(k)}$. In particular, the lowest subdiagonal and highest superdiagonal are given by

$$\begin{cases} (\Lambda^\top)^M \psi^{(-M)} = \eta (J_-)^M H, & \psi^{(-M)} = \eta H \prod_{k=0}^{M-1} T_-^k \gamma = \eta \operatorname{diag}\left(H_0 \prod_{k=1}^{M} \gamma_k, H_1 \prod_{k=2}^{M+1} \gamma_k, \dots\right), \\ \psi^{(N+1)}\Lambda^{N+1} = H(J_-^\top)^{N+1}, & \psi^{(N+1)} = H \prod_{k=0}^{N} T_-^k \gamma = \operatorname{diag}\left(H_0 \prod_{k=1}^{N+1} \gamma_k, H_1 \prod_{k=2}^{N+2} \gamma_k, \dots\right). \end{cases}$$

The vector $P(z)$ of orthogonal polynomials fulfill the following structure equations

$$\theta(z)P(z-1) = \Psi H^{-1} P(z), \qquad \sigma(z) P(z+1) = \Psi^\top H^{-1} P(z). \tag{14}$$

The compatibility of the recursion relation, i.e., Eigenfunctions of the Jacobi matrix, and the recursion matrix leads to some interesting equations:

Proposition 4. *The following compatibility conditions for the Laguerre–Freud and Jacobi matrices hold*

$$[\Psi H^{-1}, J] = \Psi H^{-1}, \quad [J, \Psi^\top H^{-1}] \quad = \Psi^\top H^{-1}.$$

1.3. The Toda Flows

Let us define the strictly lower triangular matrix $\Phi := (\vartheta_\eta S) S^{-1}$.

Proposition 5.

(i) *The semi-infinite vector P fulfills*

$$\vartheta_\eta P = \Phi P. \tag{15}$$

(ii) *The Sato–Wilson equations holds*

$$-\Phi H + \vartheta_\eta H - H\Phi^\top = JH.$$

Consequently, $\Phi = -J_-$ and $n \in \mathbb{N}_0$ we have $\vartheta_\eta \log H_n = J_{n,n}$.

Moreover,

Proposition 6 (Toda)**.** *The following equations hold*

$$\Phi = (\vartheta_\eta S) S^{-1} = -\Lambda^\top \gamma, \quad (\vartheta_\eta H) H^{-1} = \beta.$$

In particular, for $n, k-1 \in \mathbb{N}$, we have

$$\vartheta_\eta p^1_n = -\gamma_n, \quad \vartheta_\eta p^k_{n+k} = -\gamma_{n+k} p^{k-1}_{n+k-1}, \quad \vartheta_\eta \log H_n = \beta_n.$$

The functions $q_n := \log H_n$, $n \in \mathbb{N}$, *satisfy the Toda equations*

$$\vartheta^2_\eta q_n = \mathrm{e}^{q_{n+1}-q_n} - \mathrm{e}^{q_n - q_{n-1}}.$$

For $n \in \mathbb{N}$, *we also have* $\vartheta_\eta P_n(z) = -\gamma_n P_{n-1}(z)$.

Proposition 7. *The following Lax equation holds* $\vartheta_\eta J = [J_+, J]$. *The recursion coefficients satisfy the following Toda system*

$$\vartheta_\eta \beta_n = \gamma_{n+1} - \gamma_n, \tag{16a}$$
$$\vartheta_\eta \log \gamma_n = \beta_n - \beta_{n-1}, \tag{16b}$$

for $n \in \mathbb{N}_0$ *and* $\beta_{-1} = 0$. *Consequently, we get*

$$\vartheta^2_\eta \log \gamma_n = \gamma_{n+1} - 2\gamma_n + \gamma_{n-1}.$$

For the compatibility of (5) and (15) we obtain $\vartheta_\eta(\Pi) = [\Phi, \Pi]$. In the general case the dressed Pascal matrix Π is a lower unitriangular semi-infinite matrix that possibly has an infinite number of subdiagonals. However, for the case when the weight $w(z) = v(z)\eta^z$ satisfies the Pearson Equation (9), with v independent of η, that is $\theta(k+1)v(k+1)\eta = \sigma(k)v(k)$, the situation improves as we have the banded semi-infinite matrix Ψ that models the shift in the z variable as in (14). From the previous discrete Pearson equation, we see that $\sigma(z) = \eta\kappa(z)$ with κ, θ η-independent polynomials in z

$$\theta(k+1)v(k+1) = \eta\kappa(k)v(k).$$

Proposition 8. *Let us assume a weight* w *satisfying the Pearson equation* (9). *Then, the Laguerre–Freud structure matrix* Ψ *given in* (13) *satisfies*

$$\vartheta_\eta(\eta^{-1}\Psi^\top H^{-1}) = [\Phi, \eta^{-1}\Psi^\top H^{-1}], \tag{17a}$$
$$\vartheta_\eta(\Psi H^{-1}) = [\Phi, \Psi H^{-1}]. \tag{17b}$$

Relations (17a) *and* (17b) *are* gauge *equivalent.*

2. Gauss Hypergeometric Weights

The choice of $\sigma(z) = \eta(z+a)(z+b)$ and $\theta(z) = z(z+c)$ results in the following Pearson equation:

$$(k+1)(k+1+c)w(k+1) = \eta(k+a)(k+b)w(k)$$

with solutions proportional to: $w(z) = \frac{(a)_z(b)_z}{(c+1)_z}\frac{\eta^z}{z!}$. As documented in [7,9], this weight function falls under the category of generalized Hahn weight of type I. The first moment is given by $\rho_0 = {}_2F_1\left[\begin{matrix} a, b \\ c+1 \end{matrix}; \eta\right]$, which is expressed as the Gauss hypergeometric function. For the specific case when $\eta = 1$, Hahn introduced these discrete orthogonal polynomials in [26]. Notably, the standard "Hahn polynomials" commonly found in the literature have parameters set as $a = \alpha + 1$, $b = -N$, and $c = -N - 1 - \beta$, where N is a positive integer.

The Gauss hypergeometric series is convergent for $|\eta| < 1$ and divergent for $|\eta| > 1$. For $|\eta| = 1$, $\eta \neq 1$, the series is absolutely convergent whenever $\operatorname{Re}(c+1-a-b) > 0$ and convergent and not absolutely convergent for $-1 < \operatorname{Re}(c+1-a-b) \leq 0$. Finally, for $\operatorname{Re}(c+1-a-b) = -1$, the series is convergent for $\operatorname{Re}(a+b) > \operatorname{Re} ab$ and divergent for $\operatorname{Re}(a+b) \leq \operatorname{Re} ab$. See [27].

Remark 2. *By referring to previous comments and utilizing (12), we establish that in terms of the following δ-Wronskian of the Gauss hypergeometric function*

$$\tau_n := \mathscr{W}_n\left({}_2F_1\left[\begin{matrix} a, b \\ c+1 \end{matrix} ; \eta\right]\right),$$

we can derive explicit expressions for the recursion coefficients:

$$\beta_n = \vartheta_\eta \log \frac{\tau_{n+1}}{\tau_n}, \quad \gamma_{n+1} = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2},$$

where n belongs to the set of non-negative integers, denoted as $\mathbb{N}_0$.

Theorem 3 (The Gauss hypergeometric Laguerre–Freud structure matrix)**.** *For a Gauss hypergeometric weight; i.e., $\sigma = \eta(z+a)(z+b)$ and $\theta = z(z+c)$, we find*

$$\begin{aligned} p^1_{n+1} &= (n+2)\beta_{n+2} + \beta_{n+1} + \frac{\eta-1}{\eta+1}\Big(\gamma_{n+3} + \gamma_{n+2} + \beta^2_{n+2} + \frac{(n+2)(n+1)}{2}\Big) \\ &\quad + \frac{1}{\eta+1}\big(\eta ab + (\eta(a+b)-c)\beta_{n+2} + (\eta(a+b)+c)(n+2)\big), \\ \pi^{[2]}_n &= \frac{1}{1+\eta}\big((n+2)(n+1) - \eta ab - (\eta(a+b)-c)\beta_{n+2} - (\eta(a+b)+c)(n+2)\big) \\ &\quad + \frac{1-\eta}{1+\eta}(\gamma_{n+3} + \gamma_{n+2} + \beta^2_{n+2}) - (n+2)(\beta_{n+2} + \beta_{n+1}). \end{aligned}$$

The Laguerre–Freud structure matrix is

$$\Psi = \begin{bmatrix} \eta(\gamma_1+(\beta_0+a)(\beta_0+b))H_0 & (\beta_0+\beta_1+c)H_1 & H_2 & 0 & \cdots \\ \eta(\beta_0+\beta_1+a+b)H_1 & \begin{matrix}\eta(\gamma_1+\gamma_2+(\beta_1+a)(\beta_1+b) \\ +\beta_0+\beta_1+a+b)H_1\end{matrix} & (\beta_1+\beta_2+c-1)H_2 & H_3 & \ddots \\ \eta H_2 & \eta(\beta_1+\beta_2+a+b+1)H_2 & \begin{matrix}\eta(\gamma_2+\gamma_3+(\beta_2+a)(\beta_2+b) \\ +2(\beta_1+\beta_2+a+b)+\pi^{[2]}_0)H_2\end{matrix} & (\beta_2+\beta_3+c-2)H_3 & \ddots \\ 0 & \eta H_3 & \eta(\beta_2+\beta_3+a+b+2)H_3 & \begin{matrix}\eta(\gamma_3+\gamma_4+(\beta_3+a)(\beta_3+b) \\ +3(\beta_2+\beta_3+a+b)+\pi^{[2]}_1)H_1\end{matrix} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Proof. In this case, due to the properties of the polynomials σ and θ, the Freud–Laguerre matrix exhibits a specific diagonal structure:

$$\Psi = (\Lambda^\top)^2\psi^{(-2)} + \Lambda^\top\psi^{(-1)} + \psi^{(0)} + \psi^{(1)}\Lambda + \psi^{(2)}\Lambda^2.$$

This structure consists of two subdiagonals, two superdiagonals, and the main diagonal. On one hand, starting with the equation $\Psi = \sigma(J)H\Pi^\top$, we can derive the Laguerre–Freud structure matrix as follows:

$$\begin{aligned} \Psi = \underbrace{\eta\Lambda^\top\gamma\Lambda^\top\gamma H}_{\text{second subdiagonal}} &+ \underbrace{\eta(\Lambda^\top\gamma\Lambda^\top\gamma HD\Lambda + \Lambda^\top\gamma(\beta+b)H + (\beta+a)\Lambda^\top\gamma H)}_{\text{first subdiagonal}} \\ &+ \underbrace{\eta\big(\Lambda^\top\gamma\Lambda H + \Lambda\Lambda^\top\gamma H + (\beta+a)(\beta+b)H + (\Lambda^\top\gamma(\beta+b) + (\beta+a)\Lambda^\top\gamma)HD\Lambda}_{\text{main diagonal}} \\ &+ \underbrace{\Lambda^\top\gamma\Lambda^\top\gamma H\pi^{[2]}\Lambda^2\big)}_{\text{main diagonal}} \end{aligned}$$

$$
\begin{aligned}
&+\eta\Big(\underbrace{\big((\beta+a)(\beta+b)+\Lambda^\top\gamma\Lambda+\Lambda\Lambda^\top\gamma\big)HD\Lambda+(\beta+a)\Lambda H+\Lambda(\beta+b)H}_{\text{first superdiagonal}}\\
&\qquad\underbrace{+\Lambda^\top(\beta+b)H\pi^{[2]}\Lambda^2+(\beta+a)\Lambda^\top H\pi^{[2]}\Lambda^2+\Lambda^\top\gamma\Lambda^\top\gamma H\pi^{[3]}\Lambda^3\Big)}_{\text{first superdiagonal}}\\
&+\eta\Big(\underbrace{\Lambda^\top\gamma\Lambda^\top\gamma H\pi^{[4]}\Lambda^4+\big(\Lambda^\top\gamma(\beta+b)+(\beta+a)\Lambda^\top\gamma\big)H\pi^{[3]}\Lambda^3}_{\text{second superdiagonal}}\\
&\quad\underbrace{+\big(\Lambda^\top\gamma\Lambda+\Lambda\Lambda^\top\gamma+(\beta+a)(\beta+b)\big)H\pi^{[2]}\Lambda^2+\big(\Lambda(\beta+b)+(\beta+a)\Lambda\big)HD\Lambda+\Lambda^2H\Big)}_{\text{second superdiagonal}}.
\end{aligned}
\tag{18}
$$

On the other hand, by using the equation $\Psi=\Pi^{-1}H\theta(J^\top)$, we can deduce:

$$
\begin{aligned}
\Psi=&\underbrace{H(\Lambda^\top)^2+\Lambda^\top DH\Lambda^\top(\beta+T_-\beta+c)+(\Lambda^\top)^2\pi^{[-2]}H(\gamma+T_+\gamma+\beta^2+c\beta)}_{\text{second subdiagonal}}\\
&\underbrace{-(\Lambda^\top)^3\pi^{[-3]}H(\beta+T_-\beta+c)\gamma\Lambda+(\Lambda^\top)^4\pi^{[-4]}H(T_-\gamma)\gamma\Lambda^2}_{\text{second subdiagonal}}\\
&+\underbrace{H\Lambda^\top(\beta+T_-\beta+c)-\Lambda^\top DH(\gamma+T_+\gamma+\beta^2+c\beta)}_{\text{first subdiagonal}}\\
&+\underbrace{(\Lambda^\top)^2\pi^{[-2]}H(\beta+T_-\beta+c)\gamma\Lambda-(\Lambda^\top)^3\pi^{[-3]}H(T_-\gamma)\gamma\Lambda^2}_{\text{first subdiagonal}}\\
&+\underbrace{H(\gamma+T_+\gamma+\beta^2+c\beta)-\Lambda^\top DH(\beta+T_-\beta+c)\gamma\Lambda+(\Lambda^\top)^2\pi^{[-2]}H(T_-\gamma)\gamma\Lambda^2}_{\text{main diagonal}}\\
&+\underbrace{H(\beta+T_-\beta+c)\gamma\Lambda-\Lambda^\top DH(T_-\gamma)\gamma\Lambda^2}_{\text{first superdiagonal}}+\underbrace{H(T_-\gamma)\gamma\Lambda^2}_{\text{second superdiagonal}}
\end{aligned}
\tag{19}
$$

From (18), we can extract the first two subdiagonals of Ψ, which are as follows:

$$
\psi^{(-2)}=\eta H\gamma T_-\gamma,\qquad \psi^{(-1)}=\eta\gamma T_+\gamma T_+H+\gamma(\beta+b)H+(T_-\beta+a)\gamma H,
$$

From (19), we can determine the first two superdiagonals of Ψ, which are as follows:

$$
\psi^{(1)}=(\beta+T_-\beta+c)\gamma-(T_+D)(T_+H)\gamma T_+\gamma,\qquad \psi^{(2)}=H(T_-\gamma)\gamma.
$$

We can derive an expression for the main diagonal that is independent of $\pi^{[+2]}$ or $\pi^{[-2]}$ by equating the terms associated with the main diagonal in the two previous expressions, namely (18) and (19). This yields:

$$
\begin{aligned}
&(\eta-1)(T_+\gamma+\gamma+\beta^2)+\beta[\eta(a+b)-c]+\eta ab\\
&\qquad+(\eta+1)(T_+D)[T_+\beta+\beta]+(T_+D)[\eta(a+b)+c]=T_+^2(\pi^{[-2]}-\eta\pi^{[2]}).
\end{aligned}
$$

Referring back to (6), which states that $\pi_n^{[\pm2]}=\frac{(n+2)(n+1)}{2}\mp(n+1)\beta_{n+1}\mp p_{n+1}^1$, and substituting this into the previous expression, we can eliminate p_{n+1}^1. We can then replace this result in the expression for $\pi^{[2]}$, ensuring that the main diagonal is no longer dependent on it. Examining each term in the first equation and isolating p_{n+1}^1, we obtain

$$p^1_{n+1} = (n+2)\beta_{n+2} + \beta_{n+1} + \frac{\eta-1}{\eta+1}\Big(\gamma_{n+3}+\gamma_{n+2}+\beta^2_{n+2}+\frac{(n+2)(n+1)}{2}\Big) + \frac{1}{\eta+1}\big(\eta ab + (\eta(a+b)-c)\beta_{n+2} + (\eta(a+b)+c)(n+2)\big),$$

and the diagonal matrix entries of $\pi^{[2]}$ are

$$\pi^{[2]}_n = \frac{1}{1+\eta}\big((n+2)(n+1) - \eta ab - (\eta(a+b)-c)\beta_{n+2} - (\eta(a+b)+c)(n+2)\big) + \frac{1-\eta}{1+\eta}(\gamma_{n+3}+\gamma_{n+2}+\beta^2_{n+2}) - (n+2)(\beta_{n+2}+\beta_{n+1}).$$

Simplifying further, we obtain the Laguerre–Freud matrix as

$$\begin{aligned}\Psi &= \eta(\Lambda^\top)^2 T_-^2 H + \eta\Lambda^\top\big(a+b+T_+D+\beta+T_-\beta\big)T_-H \\ &\quad + \eta\big(T_+\gamma+\gamma+(\beta+a)(\beta+b)+T_+(D)(a+b+T_+\beta+\beta)+T_+^2\pi^{[2]})\big)H \\ &\quad + (c-T_+D+\beta+T_-\beta)T_-H\Lambda + T_-^2 H\Lambda^2.\end{aligned}$$

□

Now, let's investigate the compatibility condition $[\Psi H^{-1}, J] = \Psi H^{-1}$.

Theorem 4 (Laguerre–Freud equations for Gauss hypergeometric)**.** *The Gauss hypergeometric recursion coefficients satisfy the following Laguerre–Freud relations*

$$\begin{aligned}&(\eta^2-1)\big((\beta_{n+1}+\beta_n)\gamma_{n+1} - (\beta_{n-1}+\beta_n)\gamma_n\big) \\ &\quad + \eta\big(\beta_n(2\beta_n+a+b+c) + 2(\gamma_{n+1}+\gamma_n) + n(a+b-c+n-1) + ab\big) \\ &\quad + (\eta+1)\big((\eta(a+b)-c-(\eta+1)n)(\gamma_{n+1}-\gamma_n) + (\eta+1)\gamma_n\big) = 0,\end{aligned} \tag{20a}$$

$$\begin{aligned}&(\eta+1)((n-1)\beta_n + (n+1)\beta_{n+1}) + (\eta-1)(\gamma_{n+2}-\gamma_n+\beta^2_{n+1}-\beta^2_n+n) \\ &\quad + (\eta(a+b)-c)(\beta_{n+1}-\beta_n) + \eta(a+b)+c = 0.\end{aligned} \tag{20b}$$

Proof. We analyze the compatibility $[\Psi H^{-1}, J] = \Psi H^{-1}$ by diagonals. In both sides of the equation, we find matrices whose only non-zero diagonals are the main diagonal, the first and second subdiagonals, and the first and second superdiagonals. Equating the non-zero diagonals of both matrices, two identities for the second superdiagonal and subdiagonal are obtained. From the remaining diagonals, we obtain the two Laguerre–Freud equations (we obtain the same equality from the first subdiagonal and from the first superdiagonal). Firstly, by simplifying, we obtain that

$$\begin{aligned}\Psi H^{-1} &= \eta(\Lambda^\top)^2(T_-\gamma)\gamma + \eta\Lambda^\top\gamma(T_+(D)+(\beta+b)+T_-(\beta+a)) \\ &\quad + \frac{\eta}{\eta+1}\big(2(T_+\gamma+\gamma+\beta^2) + (a+b+c)\beta + ab + T_+D(a+b-c) + T_+DT_+^2D\big) \\ &\quad + (\beta+T_-\beta+c-T_+(D))\Lambda + \Lambda^2.\end{aligned}$$

From the main diagonal, clearing, we obtain

$$\begin{aligned}&(1-\eta)\beta_{n-1}\gamma_n + (\eta-1)\beta_{n+1}\gamma_{n+1} + \beta_n(\frac{\eta}{\eta+1}(2\beta_n+a+b+c) + (\eta-1)(\gamma_{n+1}-\gamma_n)) \\ &\quad + (\frac{\eta}{\eta+1}(2(\gamma_{n+1}+\gamma_n) - nc + n(a+b+n-1) + ab) - (\eta+1)(n(\gamma_n-\gamma_{n+1})-\gamma_n) \\ &\quad + (\eta(a+b)-c)(\gamma_{n+1}-\gamma_n)) = 0\end{aligned}$$

and we get Equation (20a).

We obtain the following expressions from the first superdiagonal and the first subdiagonal:

$$\begin{aligned}((\eta+1)(\beta_n+\beta_{n+1}+n(\beta_{n+1}-\beta_n))+(\eta-1)(\gamma_{n+2}-\gamma_n+\beta_{n+1}^2-\beta_n^2+n)\\+c(1+\beta_n-\beta_{n+1})+\eta(a+b)(1+\beta_{n+1}-\beta_n)=0\end{aligned}$$

which results in Equation (20b). □

We now proceed with the compatibility condition $[\Psi H^{-1}, J_-] = \vartheta_\eta(\Psi H^{-1})$. Recall that $J_- := \Lambda^\top \gamma$ and $\vartheta_\eta = \eta\frac{\mathrm{d}}{\mathrm{d}\eta}$. As we will see, we do not obtain any further equations beyond those already obtained in Theorem 4.

Proposition 9. *The recursion coefficients for the Gauss hypergeometric discrete orthogonal polynomials satisfy the following Laguerre–Freud relations:*

$$\vartheta_\eta(\beta_n+\beta_{n+1}+c-n)=\gamma_{n+2}-\gamma_n, \tag{21a}$$

$$\begin{aligned}\vartheta_\eta\Big(\frac{\eta}{\eta+1}(2(\gamma_{n+1}+\gamma_n+\beta_n^2)+c(\beta_n-n)+n(n-1)+(a+b)(\beta_n+n)+ab)\Big)=\\ \gamma_{n+1}(\beta_n+\beta_{n+1}+c-n)-\gamma_n(\beta_{n-1}+\beta_n+c-(n-1)),\end{aligned} \tag{21b}$$

$$\begin{aligned}&\vartheta_\eta(\eta\gamma_{n+1}(n+a+b+\beta_n+\beta_{n+1}))=\\ &\frac{\eta}{\eta+1}\gamma_{n+1}(2(\gamma_{n+2}-\gamma_n+\beta_{n+1}^2-\beta_n^2)+(a+b+c)(\beta_{n+1}-\beta_n)+2n+(a+b-c))\end{aligned} \tag{21c}$$

$$\vartheta_\eta(\eta\gamma_{n+1}\gamma_{n+2})=\eta\gamma_{n+1}\gamma_{n+2}(\beta_{n+2}-\beta_n+1). \tag{21d}$$

Proof. From the diagonals of $[\Psi H^{-1}, J_-] = \vartheta_\eta(\Psi H^{-1})$ we get

(i) From the first superdiagonal we obtain (21a)
(ii) From the main diagonal cleaning up we get (21b).
(iii) From the first subdiagonal we get, symplifying (21c).
(iv) Finally, from the second subdiagonal we get (21d).

□

Remark 3. *We observe that* (21a) *follow from the Toda Equations* (16a)*, and* (21d) *follow from Toda Equation* (16b)*. Moreover, the two remaining equations (from the main diagonal and the first subdiagonal) coincide with those presented in Theorem* 4*.*

Remark 4. *Dominici in* [7] *(Theorem 4) found the following Laguerre–Freud equations*

$$\begin{aligned}(1-\eta)\nabla(\gamma_{n+1}+\gamma_n)&=\eta v_n\nabla(\beta_n+n)-u_n\nabla(\beta_n-n),\\ \Delta\nabla(u_n-\eta v_n)\gamma_n&=u_n\nabla(\beta_n-n)+\nabla(\gamma_{n+1}+\gamma_n),\end{aligned}$$

with $u_n := \beta_n+\beta_{n+1}-n+c+1$ *and* $v_n := \beta_n+\beta_{n-1}+n-1+a+b$*. Therefore, the first one is of type* $\gamma_{n+1} = F_1(n,\gamma_n,\gamma_{n-1},\beta_n,\beta_{n-1})$*, of length two, and the second of the form* $\beta_{n+1} = F_2(n,\gamma_{n+1},\gamma_n,\gamma_{n-1},\beta_n,\beta_{n-1},\beta_{n-2})$*, is of length three. .*

Remark 5. *Filipuk and Van Assche in* [14] *(Equations (3.6) and (3.9)) introduce new non local variables* (x_n, y_n)*,*

$$\begin{aligned}\beta_n&=x_n+\frac{n+(n+a+b)\eta-c-1}{1-\eta},\\ \frac{1-\eta}{\eta}\gamma_n&=y_n+\sum_{k=0}^{n-1}x_k+\frac{n(n+a+b-c-2)}{1-\eta}.\end{aligned}$$

Then, in [14] *(Theorem 3.1) Equations (3.13) and (3.14) for* (x_n, y_n) *are found, of length* 0 *and* 1 *respectively, in the new variables. Recall that these new variables are non-local and involve all* *the previous recursion coefficients. In this respect, the meaning of length is not so clear. The nice feature in this case is that [14] (Equations (3.13) and (3.14)) are discrete Painlevé equations that, combined with the Toda equations, lead to a differential system for the new variables x_n and y_n that, after suitable transformation, can be reduced to Painlevé VI σ-equation. Recently, it has been shown in [24] that this system is equivalent to $dP(D_4^{(1)}/D_4^{(1)})$, known as the difference Painlevé V.*

3. Conclusions and Outlook

In their studies of integrable systems and orthogonal polynomials, Adler and van Moerbeke have thoroughly used the Gauss–Borel factorization of the moment matrix (see [28–30]). This strategy has been extended and applied by us in different contexts, such as CMV orthogonal polynomials, matrix orthogonal polynomials, multiple orthogonal polynomials, and multivariate orthogonal polynomials (see [31–33]). For a general overview, see [34].

Recently, we extended those ideas to the discrete world (see [16]). In particular, we applied that approach to the study of the consequences of the Pearson equation on the moment matrix and Jacobi matrices. For that description, a new banded matrix is required, the Laguerre–Freud structure matrix, which encodes the Laguerre–Freud relations for the recurrence coefficients. We have also found that the contiguous relations fulfilled by generalized hypergeometric functions, which determine the moments of the weight, describe a discrete Toda hierarchy known as the Nijhoff–Capel equation (see [18]). In [19], we study the role of Christoffel and Geronimus transformations in the description of the mentioned contiguous relations, as well as the use of Geronimus–Christoffel transformations to characterize the shifts in the spectral independent variable of the orthogonal polynomials.

In this paper, we delve deeper into that program and further explore the discrete semiclassical cases. We find Laguerre–Freud relations for the recursion coefficients of Gauss hypergeometric discrete orthogonal polynomials. We observe that a solution of the Laguerre–Freud-type nonlinear equations for the recursion coefficients is provided by the τ-function, which is defined as a Wronskian of the Gauss hypergeometric functions, respectively. Notice that in [20], we presented a study similar to the one in this paper for the hypergeometric cases ${}_1F_2, {}_2F_2$ and ${}_3F_2$.

For the future, we will extend these techniques to multiple discrete orthogonal polynomials [35] and their relations with the transformations presented in [36], as well as quadrilateral lattices [37,38].

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