Conditions for the Oscillation of Solutions to Neutral Differential Equations of Higher Order

Maryam Al-Kandari

Department of Mathematics, Faculty of Science, Kuwait University, P.O. Box 5969, Safat, Kuwait City 13060, Kuwait; maryam.alkandari@ku.edu.kw

Abstract: In this research, we applied three techniques—the comparison technique, the Riccati technique, and the integral averages technique to analyze and establish various conditions and properties associated with the oscillatory behavior of even-order neutral differential equations. These findings contribute to a better understanding of the dynamics of such equations. To demonstrate the efficacy of these new conditions and properties, we present illustrative examples. This study offers valuable insights into the behavior of neutral differential equations, advancing our knowledge in this field.

Keywords: oscillation conditions; neutral; even-order; differential equation

MSC: 34C10; 34K11

1. Introduction

In this paper, we obtain some oscillation conditions of even-order NDEs of the form

\[
\left(a(x)w^{(\beta-1)}(x)\right)'+b(x)\varphi(z(x)))=0,
\]

where

\[
w(x) = |\xi(x)|^{p-2}\xi(x) + \zeta(x)\xi(\gamma(x)),
\]

and \(\beta \geq 2\), \(p > 1\), \(a, \xi, \zeta \in C([x_0, \infty), [0, \infty))\), \(b \in C([x_0, \infty), \mathbb{R}^+)\), \(a(x) > 0\), \(a'(x) \geq 0\), \(0 \leq \xi(x) < 1\), \(\gamma \in C([x_0, \infty), (0, \infty))\), \(\gamma(x) \leq x\), \(\lim_{x \to \infty} \gamma(x) = \infty\), \(z \in C([x_0, \infty), \mathbb{R})\), \(\varphi \in C(\mathbb{R}, \mathbb{R})\), \(\varphi(\xi) \geq |\xi|^{p-2}\xi\) for \(\xi \neq 0\), \(z(x) \leq x\), \(z'(x) > 0\), \(\lim_{x \to \infty} z(x) = \infty\), and \(\beta\) and \(p\) are even positive integers. Also

\[
\int_{x_0}^{\infty} \frac{1}{a(s)}ds = \infty.
\]

Differential equations serve as powerful mathematical tools for modeling and understanding dynamic systems in various fields, from physics to biology and engineering. They describe how quantities change in relation to each other, capturing the essence of continuous change in natural phenomena. The significance of differential equations lies in their ability to predict and analyze complex behaviors, providing crucial insights into the evolution of systems over time. Their widespread application facilitates advancements in science, technology, and innovation, making them an indispensable tool for solving real-world problems. In essence, differential equations form the backbone of mathematical modeling, enabling us to unravel the intricacies of dynamic processes and make informed decisions, see [1,2].

Neutral differential equations represent a specialized class of differential equations that involve delays in both the dependent variable and its derivatives. These equations play a crucial role in modeling real-world phenomena where past values and their rates
of change impact the present state. The importance of neutral differential equations lies in their ability to capture dynamic systems with memory effects, such as those found in biology, economics, and engineering. By considering delays in the system’s response, neutral differential equations provide a more accurate representation of various time-dependent processes. The study of these equations is essential for gaining insights into the behavior and stability of systems influenced by past states, contributing significantly to the advancement of mathematical modeling and applications in diverse scientific and technological fields, see [3–7].

In recent times, significant progress has been made in the analysis of delay and neutral differential equations of different orders, with several conditions and properties having been identified. These findings have been documented in notable references such as [8–10].

Agarwal et al. [11] used the Riccati method to obtain oscillation conditions for the equation

$$\left( \left( \xi^{(\beta-1)}(x) \right)^{m} \right)' + b(x)\xi^{m}(\gamma(x)) = 0.$$  

Elabbasy et al. [12] used some methods to obtain a comparison for the oscillation of equation

$$a(x)\left( \xi^{(\beta-1)}(x) \right)^{p-2}\xi^{(\beta-1)}(x) \right)' + b(x)\varphi(\xi(\gamma(x))) = 0, \quad p > 1,$$

under

$$\int_{x_0}^{\infty} \frac{1}{a^{1/(p-1)}(s)} ds = \infty.$$  

In [13], Zhang et al. considered the oscillation of equation

$$\left( a(x)\left( \xi^{(\beta-1)}(x) \right)^{a} \right)' + b(x)\xi^{a}(\gamma(x)) = 0,$$

under

$$\int_{x_0}^{\infty} a^{-1/a}(s) ds < \infty.$$  

Bazighifan et al. in [14] considered the equation

$$\left( a(x)\varphi\left( \xi^{(\beta-1)}(x) \right) \right)' + b(x)\varphi(\xi(\gamma(x))) = 0,$$  

where $\varphi(s) = |s|^{p-2}s$ and the obtained properties for the oscillation of (4).

Anis and Moaaz [15] investigated the oscillatory behavior of even-order neutral differential equations of the form

$$w^{(\beta)}(x) + b(x)\xi(z(x)) = 0$$

where $n \geq 4$ is an even integer and

$$w(x) = \xi(x) + \zeta(x)\xi(\gamma(x)).$$

Guo et al. [16] studied quasi-linear neutral delay differential equations of the form

$$\left( a(x)w^{(\beta-1)}(x) \right)' + b(x)\xi(z(x)) = 0,$$  

where $n$ is even, and

$$w(x) = \xi(x) + \zeta(x)\xi(\gamma(x)).$$

Moaaz et al. [17] investigated the oscillatory properties of solutions of differential equations of the neutral type with the form (5) in the noncanonical case.

In our current research, we obtain the conditions for the oscillation of solutions to (1).

2. Definitions and Lemmas

Definition 1. Equation (1) is called oscillatory if all its solutions are oscillatory.
Lemma 4. Let

\[ L_0 = \{(x, s) : x > s > x_0 \} \quad \text{and} \quad L = \{(x, s) : x \geq s \geq x_0 \}. \]

A function \( W \in C(L, \mathbb{R}) \) is said to belong to the function class \( \varsigma \), written by \( W \in \varsigma \), if

(i) \( W(x, s) > 0 \) on \( L_0 \) and \( W(x, s) = 0 \) for \( x \geq x_0 \) with \( (x, s) \notin L_0 \);

(ii) \( W(x, s) \) has a continuous and nonpositive partial derivative \( \partial W / \partial s \) on \( L_0 \) and \( \varsigma_i \in C(L_0, \mathbb{R}) \) such that

\[ \frac{\partial W(x, s)}{\partial s} = -g(x, s) \sqrt{W(x, s)}. \]

Lemma 1 ([19]). If \( w \in C^\beta([x_0, \infty), (0, \infty)) \) and \( w^{(\beta-1)}(x)w^{(\beta)}(x) \leq 0 \) for \( x \geq x_0 \), then for every \( \nu \in (0, 1) \) there exists a constant \( j > 0 \) such that

\[ |w(\nu x)| \geq jx^{\beta-1}|w^{(\beta-1)}(x)|, \]

for all large \( t \).

Lemma 2 ([20]). Let \( w \in C^\beta([x_0, \infty), (0, \infty)) \) and \( w^{(\beta-1)}(x)w^{(\beta)}(x) \leq 0 \). If \( \lim_{x \to \infty} w(x) \neq 0 \), then for every \( \mu \in (0, 1) \) there exists an \( x_\mu \geq x_0 \) such that

\[ |w(x)| \geq -\frac{H}{(\beta - 1)!}x^{\beta-1}|w^{(\beta-1)}(x)|, \]

for all \( x \geq x_\mu \).

Lemma 3 ([21]). Let \( w(x) \) be a \( \beta \) times differentiable function on \([x_0, \infty), w^{(\beta)}(x) \neq 0 \) on \([x_0, \infty) \) and \( w(x)w^{(\beta)}(x) \leq 0 \). Then,

(I) \( w^{(i)}(x) \), \( i = 1, 2, ..., \beta - 1 \) on \([x_0, \infty) \) such that there exists an \( x_1 \geq x_0 \),

(II) \( i \in \{1, 3, 5, ..., \beta - 1\} \) when \( \beta \) is even, \( i \in \{0, 2, 4, ..., \beta - 1\} \) when \( \beta \) is odd, such that, for \( x \geq x_1 \),

\[ w(x)w^{(i)}(x) > 0, \]

for all \( i = 0, 1, ..., l \) and

\[ (-1)^{\beta+i+1}w(x)w^{(i)}(x) > 0, \]

for all \( i = l + 1, ..., \beta \).

3. Oscillation Results

Lemma 4. Let \( \xi(x) \) is an eventually positive solution of (1). Then,

\[ w(x) > 0, \quad w(x) > 0, \quad w^{(\beta-1)}(x) \geq 0 \quad \text{and} \quad w^{(\beta)}(x) \leq 0, \]

for \( x \geq x_2 \).

Proof. Suppose \( \xi(x) \) is an eventually positive solution of (1). Then, \( w(x) > 0 \) and

\[ (aw^{(\beta-1)})'(x) = -b(x)\varphi(\xi(z(x))) \leq 0. \]

which means that \( a(x)w^{(\beta-1)}(x) \) is decreasing and \( w^{(\beta-1)}(x) \) is eventually of one sign.

We claim that \( w^{(\beta-1)}(x) \geq 0 \). Otherwise, if \( x_2 \geq x_1 \) exists such that \( w^{(\beta-1)}(x) < 0 \) for \( x \geq x_2 \),

\[ (aw^{(\beta-1)})(x) \leq \left( aw^{(\beta-1)} \right)(x_2) = -L. \]
where $L > 0$. Integrating the above inequality from $x_2$ to $x$ we find

$$w^{(β-2)}(x) \leq w^{(β-2)}(x_2) - L \int_{x_2}^{x} \frac{1}{a(s)} \, ds.$$ 

Letting $x \to \infty$, we have $\lim_{x \to \infty} w^{(β-2)}(x) = -\infty$, which contradicts the fact that $w(x) > 0$. Hence, we obtain $w^{(β-1)}(x) \geq 0$ for $x \geq x_1$.

From Equation (1), we obtain

$$\left(aw^{(β)}(x)\right) = -\left(a'w^{(β-1)}\right)(x) - b(x)φ(z(x))) \leq 0,$$

this implies that $w^{(β)}(x) \leq 0$, $x \geq x_1$. From Lemma 3, we find (6) holds. The proof is complete. □

**Theorem 1.** Let

$$ξ'(x) + \tilde{K}(x)\tilde{ξ}(z(x)) = 0,$$

be oscillatory, where

$$\tilde{K}(x) : = \frac{μz^{β-1}(x)}{(β - 1)!a(z(x))} K(x),$$

$$K(x) : = b(x)(1 - ζ(z(x))),$$

then, Equation (1) is oscillatory.

**Proof.** Let (1) have a nonoscillatory solution. From Lemma 4, we find that (6) holds. From

$$w(x) = |ξ(x)|^{p-2}ξ(x) + ζ(x)ξ(γ(x)),$$

we see that

$$ξ^{p-1}(x) = w(x) - ζ(x)ξ(γ(x)) \geq w(x) - ζ(x)w(γ(x)) \geq w(x) - ζ(x)w(x)$$

$$\geq (1 - ζ(x))w(x)$$

and so

$$ξ^{p-1}(z(x)) \geq w(z(x))(1 - ζ(z(x))).$$

From (9), we obtain

$$φ(ξ(z(x))) \geq w(z(x))(1 - ζ(z(x))).$$

From (1) and (10), we see that

$$\left(aw^{(β-1)}(x)\right) = -b(x)w(z(x))(1 - ζ(z(x)))$$

$$\leq -w(z(x))b(x)(1 - ζ(z(x)))$$

$$= -K(x)w(z(x))$$

In view of Lemma 2, we obtain

$$w(x) \geq \frac{μ}{(β - 1)!} x^{β-1}w^{(β-1)}(x),$$

for all $x \geq x_2 \geq \max\{x_1, x_μ\}$. Thus, by using (11), we find

$$\left(a(x)w^{(β-1)}(x)\right) + \frac{μz^{β-1}(x)K(x)}{(β - 1)!a(z(x))}(a(z(x))w^{(β-1)}(z(x))) \leq 0.$$
Therefore, we obtain that \( \xi(x) = a(x)w^{(\beta-1)}(x) \) is a positive solution of the differential inequality
\[
\xi''(x) + \tilde{K}(x)\xi(x) \leq 0.
\]
From [18] (Corollary 1), we see that (8) also has a positive solution, a contradiction. This completes the proof. \( \square \)

Now, we find some results.

**Corollary 1.** If
\[
\liminf_{x \to \infty} \int_{z(x)}^{x} \frac{\zeta^{\beta-1}(s)}{\zeta(a(s))} \, ds > \frac{(\beta - 1)!}{\mu e}, \text{ for } \mu \in (0, 1),
\]
then (1) is oscillatory.

**Theorem 2.** If
\[
\int_{x_0}^{\infty} \left( h(u)K(u) - \frac{1}{4\nu} \left( \frac{h'(u)}{h(u)} \right)^2 b(u) \right) \, du = \infty, \text{ for } \nu \in (0, 1), j > 0,
\]
then (1) is oscillatory, where \( h \in C^1([x_0, \infty), \mathbb{R}^+) \) and
\[
b(x) := \frac{a(x)h(x)}{jz^{\beta-2}(x)z'(x)}.
\]

**Proof.** Let (1) have a nonoscillatory solution. As in the proof of Theorem 1, we arrive at (11). From Lemma 1 with \( \xi = w', j > 0 \) and \( z(x) \leq x \) exist such that
\[
w'(vz(x)) \geq jz^{\beta-2}(x)w^{(\beta-1)}(z(x)) \geq jz^{\beta-2}(x)w^{(\beta-1)}(x).
\]
Define
\[
\psi(x) := h(x) \frac{a(x)w^{(\beta-1)}(x)}{w(vz(x))} > 0,
\]
we have
\[
\psi'(x) = h(x) \frac{a(x)w^{(\beta-1)}(x)}{w(vz(x))} + h(x) \frac{a(x)w^{(\beta-1)}(x)}{w(vz(x))} \frac{w'(vz(x))z'(x)}{w'(vz(x))}.
\]
From (11), we obtain
\[
\psi'(x) \leq \frac{h'(x)}{h(x)} \psi(x) - h(x)K(x) - \nu \frac{w'(vz(x))z'(x)}{w'(vz(x))} \psi(x).
\]
By using (13), we have
\[
\psi'(x) \leq \frac{h'(x)}{h(x)} \psi(x) - h(x)K(x) - \nu \frac{jz^{\beta-2}(x)w^{(\beta-1)}(x)z'(x)}{w'(vz(x))} \psi(x)
\]
\[
\leq \frac{h'(x)}{h(x)} \psi(x) - h(x)K(x) - \nu \frac{jz^{\beta-2}(x)z'(x)h(x)a(x)w^{(\beta-1)}(x)}{a(x)h(x)} \psi(x)
\]
\[
\leq \frac{h'(x)}{h(x)} \psi(x) - h(x)K(x) - \nu \frac{v}{b(x)} \psi^2(x).
\]
Using the inequality [22]

\[ \xi w - u w^{\frac{a+1}{a+1}} \leq \frac{a^a}{(a+1)^{a+1}} \xi^{a+1} u^a, \]

with \( \xi = h'/h, u = v j z^{\beta - 2}(x) z'(x)/(a(x) h(x)) \) and \( w = \psi(x) \), we find

\[ \psi'(x) \leq -h(x) K(x) + \frac{1}{4v} \left( \frac{h'(x)}{h(x)} \right)^2 \frac{a(x) h(x)}{j z^{\beta - 2}(x) z'(x)}. \]  \hfill (15)

Integrating (15) from \( x_1 \) to \( x \), we find

\[
\int_{x_1}^{x} \left( h(u) K(u) - \frac{1}{4v} \left( \frac{h'(u)}{h(u)} \right)^2 b(u) \right) du \leq \psi(x_1) - \psi(x)
\]

which contradicts (12). This completes the proof. \( \square \)

**Theorem 3.** If \( h \in C^1([x_0, \infty), \mathbb{R}^+) \) such that

\[
\limsup_{x \to \infty} \frac{1}{W(x, x_0)} \int_{x_0}^{x} W(x, u) \left( h(u) K(u) - \frac{1}{4v} b(u) \zeta^2(x, u) \right) du = \infty,
\]  \hfill (16)

where

\[
\zeta(x, u) = \frac{h'(s)}{h(s)} - \frac{g(x, s)}{W(x, s)}
\]

then Equation (1) is oscillatory.

**Proof.** Multiplying (14) by \( W(x, s) \) and integrating both sides from \( x_2 \) to \( x \), we obtain

\[
\int_{x_2}^{x} W(x, u) h(u) K(u) du \leq - \int_{x_2}^{x} W(x, u) \psi'(u) du - \int_{x_2}^{x} W(x, u) \frac{v}{b(u)} \psi^2(u) du
\]

\[
+ \int_{x_2}^{x} W(x, u) \frac{h'(u)}{h(u)} \psi(u) du
\]

\[
\leq W(x, x_2) \psi(x_2) - \int_{x_2}^{x} W(x, u) \frac{v}{b(u)} \psi^2(u) du
\]

\[
+ \int_{x_2}^{x} W(x, u) \psi(u) \zeta(x, u) du,
\]

which implies that

\[
\int_{x_2}^{x} W(x, u) h(u) K(u) du \leq W(x, x_2) \psi(x_2)
\]

\[
- \int_{x_2}^{x} W(x, u) \frac{v}{b(u)} \left( \psi^2(u) - \frac{b(u)}{v} \xi(x, u) \psi(u) \right) du.
\]

It follows that

\[
\frac{1}{W(x, x_2)} \int_{x_2}^{x} W(x, u) \left( h(u) K(u) - \frac{1}{4v} b(u) \zeta^2(x, u) \right) du
\]

\[
\leq \psi(x_2) - \frac{1}{W(x, x_2)} \int_{x_2}^{x} W(x, u) \left( \psi(u) - \frac{1}{2v} b(u) \zeta(x, u) \right)^2 du,
\]
which implies
\[
\limsup_{x \to \infty} \frac{1}{W(x, x_2)} \int_{x_2}^{x} W(x, u) \left( h(u)K(u) - \frac{1}{4\nu} b(u)\xi^2(x, u) \right) du \leq \psi(x_2).
\]
From (16), we have a contradiction. This completes the proof. \(\square\)

**Corollary 2.** Let
\[
0 < \inf_{s \geq x} \left( \liminf_{x \to \infty} \frac{W(x, s)}{W(x, x_2)} \right) \leq \infty
\]
and
\[
\limsup_{x \to \infty} \frac{1}{W(x, x_2)} \int_{x_2}^{x} W(x, u)b(u)\xi^2(x, u) du < \infty.
\]
If
\[
\limsup_{x \to \infty} \int_{x_2}^{x} \frac{\mu^2(s)}{b(s)} ds = \infty
\]
for \(\mu \in C([x_0, \infty), \mathbb{R})\) and \(\mu(x) = \max\{\mu(x), 0\}\); also,
\[
\limsup_{x \to \infty} \frac{1}{W(x, x_2)} \int_{x_2}^{x} W(x, u) \left( h(u)K(u) - \frac{1}{4\nu} b(u)\xi^2(x, u) \right) du \geq \sup_{x \geq x_0} \mu(x),
\]
then (1) is oscillatory.

**Example 1.** Consider the second-order equation:
\[
\left[ x \left( \xi(x) + \frac{1}{2}\xi \left( \frac{x}{3} \right) \right) \right]'' + \frac{b_0}{x} \xi^2 \left( \frac{x}{2} \right) = 0,
\]
where \(b_0 > 0\) is a constant. Let \(\beta = p = 2, \gamma(x) = 1/2, \gamma(x) = x/3, b(x) = b_0/x, z(x) = x/2, \varphi(\xi) = \xi^2 + \xi\).

Now, we see that
\[
K(x) = b(x)(1 - \xi(z(x))) = \frac{b_0}{2x}
\]
and
\[
b(x) = \frac{a(x)h(x)}{jz^2(x)z'(x)} = \frac{2x^2}{j}.
\]
If we set \(h = x\) then any for constants \(j > 0, 0 < \nu < 1\)
\[
\int_{x_0}^{x} \left( h(u)K(u) - \frac{1}{4\nu} \left( \frac{h'(u)}{h(u)} \right)^2 b(u) \right) du,
\]
\[
= \int_{x_0}^{\infty} \left( \frac{b_0}{2} - \frac{1}{2\nu j} \right) du
\]
\[
= \infty \quad \text{if} \quad b_0 > 1.
\]
From Theorem 2, every solution of Equation (17) is oscillatory if \(b_0 > 1\).

**Example 2.** Let the equation:
\[
\left[ xw(\xi(x) + 1/3\xi(x/2))''' + b_0 \xi(x/3) = 0, \right.
\]
where \( b > 0 \) is a constant and \( \beta = 4, x \geq 1, p = 2, a(x) = x, \zeta(x) = 1/3, \gamma(x) = x/2, b(x) = b_0/x, z(x) = x/3, \varphi(\xi) = \xi. \)

Thus, we see that

\[
\int_{-\infty}^{\infty} a^{-1}(x) dx = \infty.
\]

By Theorem 3, every solution of Equation (18) is oscillatory.

4. Conclusions

In our work, we established new conditions for the oscillatory behavior of a studied equation of even order; we obtained these conditions by applying three methods, which are the comparison method, the Riccati method, and the integral averages method.

In our continuous research regarding this point, we will complete our current work in the near future by studying the same equation but at a different condition to give us different oscillatory theorems; this condition is

\[
\int_{-\infty}^{\infty} a^{-1}(x) dx < \infty.
\]

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