Valuation of Commodity-Linked Bond with Stochastic Convenience Yield, Stochastic Volatility, and Credit Risk in an Intensity-Based Model

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Abstract: In this study, we consider an intensity-based model for pricing a commodity-linked bond with credit risk. Recently, the pricing of a commodity-linked bond with credit risk under the structural model has been studied. We extend the result using an intensity-based model, stochastic volatility model, and stochastic convenience yield model. In the intensity-based model, the credit event by the counterparty occurs at the time of first jump in a stochastic Poisson process, in which intensity is modeled as the sum of two CIR processes. We assume that the underlying asset follows the stochastic volatility and convenience yield models. Using the measure change technique, we explicitly derive the commodity-linked bond pricing formula in the proposed model. As a result, we provide the explicit solution for the price of the commodity-linked bond with stochastic convenience yield, stochastic volatility, and credit risk as single integrations. In addition, we present several examples to demonstrate the effects of significant parameters on the value of commodity-linked bond using numerical integration. In particular, examples are provided, focusing on the behavior of prices based on effects of recovery rate.

Keywords: commodity-linked bond; credit risk; stochastic volatility; intensity-based model

MSC: 91G20; 91G30

1. Introduction

The options that incorporate credit risk are generally referred to as “vulnerable options” and have been the subject of extensive research in the field of finance. Johnson and Stulz [1] investigated vulnerable options by employing structural approaches to include counterparty credit risk in the option pricing model. Within the classical Black–Scholes model, Klein [2] developed an improved approach to pricing vulnerable options, particularly those affected by correlated default risk. After that, studies on pricing of vulnerable options under the structural model have investigated various modifications to the classical Black–Scholes model, including stochastic volatility models, as shown in studies such as [3–6], jump-diffusion models, as shown in [7–9], stochastic interest rate models [10,11], early counterparty risk models [12], and Markov-modulated processes, as shown in [13–15]. In addition, the Mellin transforms have been employed to solve the PDE for the pricing of vulnerable options. Many researchers have employed the Mellin transforms for pricing vulnerable options under the structural model. Yoon and Kim [16] were the first to apply Mellin transforms to calculate vulnerable European option prices. Several recent studies have demonstrated that the Mellin transforms can be used to solve the PDE for many types of financial derivatives with credit risk (path-dependent option [17], exchange option [18], and foreign equity option [19]).
In the field of finance, there is another model for describing credit risk. Intensity-based models, also known as reduced form models, are used to investigate vulnerable options. In contrast to structural models, which assume default from a firm’s collapsing financial state, intensity-based models take default as a random process with a specified intensity, equal to the arrival rate of events in a Poisson process. In recent years, the pricing models of vulnerable options under the intensity-based model have been studied. Using the intensity-based model with a Gaussian Ornstein—Uhlenbeck process, Fard [20] investigated vulnerable European option pricing under a generalized jump-diffusion model, and Koo and Kim [21] presented an explicit pricing method for a catastrophe put option with exponential jumps and credit risk. Wang [22] considered an intensity-based model based on valuing vulnerable options in discrete time—a Generalized Autoregressive Conditional Heteroscedasticity (GARCH) process. Furthermore, Wang [23,24] improved the studies for vulnerable option pricing models in intensity-based models by employing a stochastic volatility model and a nonnegative intensity process with two components (systematic risk and idiosyncratic risk). In this study, we investigate the financial derivative with credit risk based on an intensity-based model. In the intensity-based model, we specifically deal with a commodity-linked bond with credit risk.

The efficient modeling and pricing of commodity derivatives have been studied for decades. Schwartz [25] established a complete framework for valuing commodity-linked bonds in a pioneering paper, based on the option pricing approaches proposed by Black and Scholes [26], which were further expanded by Cox and Ross [27]. In addition there have been extensive valuation approaches that include consideration of commodity price risk, default risk, and interest rate risk. Carr [28] developed the pricing formula of a commodity-linked bond by considering three types of risk (commodity price risk, credit risk, and interest rate risk). Yan [29] provided a closed-form solution for commodity-linked bonds under a multi-factor model, including stochastic volatility and simultaneous jumps. More recently, Ma et al. [30] developed the pricing models for valuing the commodity-linked bonds with counterparty credit risk using Mellin transforms. We study the pricing model of commodity-linked bond with credit risk, inspired by the work of [30]. We employ an intensity-based model for capturing counterparty credit risk. To our knowledge, the intensity-based model has not been applied to the valuation of commodity-linked bond with credit risk. Furthermore, the underlying assets are assumed to have stochastic volatility and stochastic convenience yield. That is, the contribution of this paper is to extend the work of [30] by using the stochastic volatility model, the stochastic convenience yield, and the intensity-based model.

The remainder of this paper is structured as follows. In Section 2, we propose the model used in this study. In Section 3, we derive the explicit pricing formula for the commodity-linked bond under the proposed model. In Section 4, we present some numerical experiments. In Section 5, we provide concluding remarks.

2. Model

In this section, we describe the model for pricing the commodity-linked bond with credit risk and stochastic volatility. We construct the model based on the two-factor model in the work of Schwartz [31] and the stochastic volatility model of Heston [32]. Under the risk neutral measure \( P \), the dynamics are presented by

\[
\begin{align*}
    dS(t) &= (r - \delta(t))S(t)dt + \sigma_1 \sqrt{v_1(t)}S(t)dW_1(t) \\
    dv_1(t) &= a_1(b_1 - v_1(t))dt + \sigma_2 \sqrt{v_1(t)}dW_2(t) \\
    d\delta(t) &= \kappa(\hat{\alpha} - \delta(t))dt + \sigma_3 dW_3(t),
\end{align*}
\]

where \( \delta(t) \) is the stochastic convenience yield rate, \( r \) is the constant risk-free rate, \( \sigma_1, \sigma_2, \) and \( \sigma_3 \) are constant volatilities, \( \kappa \) is the speed of adjustment, \( \hat{\alpha} \) is the long-run mean yield, \( \lambda^* \) is the market price of convenience yield risk, and \( \alpha \) is the long-term mean of
the convenience yield. In addition, \(a_1\) and \(b_1\) in the variance dynamic are constants, while \(W_1(t), W_2(t),\) and \(W_3(t)\) are the standard Brownian motions, and their correlations are such that \(dW_1(t)dW_2(t) = \rho_1 dt\) and \(dW_1(t)dW_3(t) = \rho_2 dt.\)

We consider the intensity-based model to model the counterparty credit risk. We assume that \(N(t)\) is a doubly Poisson process with intensity \(\lambda(t)\), in which the first jump time of \(N(t)\) is \(\tau\). Then, we say that the default occurs at \(\tau\). As in Wang [23], \(\tau\) means the default time, and is defined by

\[
P(\tau > T) = E\left[e^{-\int_0^T \lambda(s)ds}\right],
\]

where \(T\) is the maturity. Following the works of Wang [23,24], we assume that the intensity process is given by

\[
\lambda(t) = \beta v_1(t) + v(t),
\]

where \(v_1(t)\) means a systematic risk, \(\beta > 0\), and \(v(t)\) means an idiosyncratic risk captured by a mean-reverting square root process:

\[
dv(t) = a(b - v(t))dt + \sigma \sqrt{v(t)}dW(t),
\]

where \(a, b,\) and \(\sigma\) are constants, and \(W(t)\) is the standard Brownian motion, independent of \(W_1(t), W_2(t),\) and \(W_3(t).\) We also note that a positive value \(\beta\) ensures that the process \(\lambda(t)\) has positive values. In (2), the intensity process consists of two terms. The first term represents systematic risk, whereas the second term describes idiosyncratic risk of bond issuers.

We first consider the financial derivative with credit risk. If the payoff function of the derivative is \(h(S(T))\), as in Fard [20] and Wang [23,24], the value of the derivative with credit risk at time 0 in the intensity-based model is given by

\[
B = E^P\left[\left\{\frac{\text{e}^{-r(T-\tau)} h(S(T))}{\mathcal{F}(\tau)}\right\}\right|_{0<\tau\leq T} + E^P\left[h(S(T))\right]_{\tau > T}
= \left.E^P\left[h(S(T))\right] + (1-w)E^P\left[\text{e}^{-\int_0^T \lambda(s)ds}h(S(T))\right]\right|_{0<\tau\leq T}.
\]

where \(w\) is the recovery rate of the derivative and \(E^P[\cdot]\) denotes the expectation under the measure \(P\). The first expectation in (4) refers to a default event occurring throughout the lifetime of derivative, while the second expectation in (4) is related to no default events occurring before maturity \(T\).

3. Pricing of Commodity-Linked Bond

In this section, we derive the pricing formula of commodity-linked bond in the proposed model. We follow Carr [28] and Ma et al. [30] in claiming that the firm makes no payments to bond holders prior to the bond’s maturity date, meaning that no dividends or coupons are paid. As a result, the guaranteed payment for commodity-linked bonds is equal to the guaranteed face value of a bond \(F\) plus a call option. The holder of this option has the option to acquire the relevant commodity bundle at a predetermined exercise price \(K\). This payoff mechanism is a significant feature of commodity-linked bonds, and the payoff is defined as

\[
h(S(T)) = F + \max\{S(T) - K, 0\}.
\]

Then, from Equation (4), we define the value of the commodity-linked bond at time 0 under the measure \(Q\) in the proposed framework. In order to start valuing the proposed bond, we state the following well-known proposition (for more details, see Cox et al. [33]).

**Proposition 1.** Assume that the stochastic process \(v(t)\) is given by \(dv(t) = a(b - r(t))dt + \sigma \sqrt{v(t)}dB(t).\) Then, the joint characteristic function of \(\left(\int_0^T v(s)ds, v(T)\right)\) is
where \( u_1 \) and \( u_2 \) are complex variables with nonnegative real parts, and

\[
A(u_1, u_2; t, T) = \frac{2ab}{\sigma^2} \ln \left( \frac{2m(u_1)e^{(m(u_1)+a)(T-t)/2}}{(m(u_1) + a - \sigma^2 u_2)(\epsilon^{m(u_1)(T-t)} - 1) + 2m(u_1)} \right),
\]

\[
m(u_1) = \sqrt{a^2 - 2\sigma^2 u_1},
\]

\[
B(u_1, u_2; t, T) = \frac{2u_1(1 - \epsilon^{m(u_1)(T-t)}) - u_2 \left( 2m(u_1) + (m(u_1) - a) (\epsilon^{m(u_1)(T-t)} - 1) \right)}{(m(u_1) + a - \sigma^2 u_2)(\epsilon^{m(u_1)(T-t)} - 1) + 2m(u_1)}.
\]

To derive the pricing formula of the bond, we should find the characteristic function

\[
f(\phi_1, \phi_2) := E^P \left[ e^{\phi_1 X_T + \phi_2 \int_0^T \lambda(s) ds} \right],
\]

where \( \phi_1 \) and \( \phi_2 \) are complex variables and \( X(T) = \ln S(T) \). The explicit expression of \( f(\phi_1, \phi_2) \) is presented in the following proposition.

**Proposition 2.** In the proposed model, the characteristic function \( f(\phi_1, \phi_2) \) is expressed as

\[
f(\phi_1, \phi_2) = \exp \left[ A_1(\phi_1, \phi_2; T) + A_2(\phi_2; T) - B_1(\phi_1, \phi_2; T)\chi(0) - B_2(\phi_2; T)\sigma(0) \right] \times \exp \left[ g_1(\phi_1) + g_2(\phi_1) + g_3(\phi_1) + g_4(\phi_1) \right]
\]

where

\[
A_1(\phi_1, \phi_2; T) = \frac{2a_1}{\sigma^2} \ln \left( \frac{2m_1(\phi_1, \phi_2)e^{(m_1(\phi_1, \phi_2)+a_1)(T)}}{(m_1(\phi_1, \phi_2) + a_1 - \sigma^2 d_1(\phi_1))(\epsilon^{m_1(\phi_1, \phi_2)(T-t)} - 1) + 2m_1(\phi_1, \phi_2)} \right),
\]

\[
A_2(\phi_2; T) = \frac{2b}{\sigma^2} \ln \left( \frac{2m_2(\phi_2)e^{(m_2(\phi_2)+a_2)(T)}}{(m_2(\phi_2) + a_2 - \sigma^2 d_2(\phi_2))(\epsilon^{m_2(\phi_2)(T-t)} - 1) + 2m_2(\phi_2)} \right),
\]

\[
B_1(\phi_1, \phi_2; T) = \frac{2d_1(\phi_1, \phi_2)(1 - \epsilon^{m_1(\phi_1, \phi_2)(T-t)}) - d_2(\phi_1) \left( 2m_1(\phi_1, \phi_2) + (m_1(\phi_1, \phi_2) - a_1) (\epsilon^{m_1(\phi_1, \phi_2)(T-t)} - 1) \right)}{(m_1(\phi_1, \phi_2) + a_1 - \sigma^2 d_1(\phi_1))(\epsilon^{m_1(\phi_1, \phi_2)(T-t)} - 1) + 2m_1(\phi_1, \phi_2)},
\]

\[
B_2(\phi_2; T) = \frac{2d_2(\phi_2)(1 - \epsilon^{m_2(\phi_2)(T-t)})}{(m_2(\phi_2) + a_2)(\epsilon^{m_2(\phi_2)(T-t)} - 1) + 2m_2(\phi_2)},
\]

\[
m_1(\phi_1, \phi_2) = \sqrt{\sigma^2 - 2\sigma^2 d_1(\phi_1, \phi_2)},
\]

\[
m_2(\phi_2) = \sqrt{\sigma^2 - 2\sigma^2 d_2(\phi_2)},
\]

\[
d_1(\phi_1, \phi_2) = \frac{\phi_1 c_1 \rho_1 a_1}{c_2} - \frac{1}{2} \left( \phi_1 c_1^2 - \phi_2^2 c_1^2 \left( 1 - \rho_1^2 \right) - 2\phi_2 \beta \right),
\]

\[
d_2(\phi_2) = \frac{\phi_2 c_2}{c_2},
\]

\[
g_1(\phi_1) = \phi_1 \ln \left( \frac{\sigma T}{c_2} \right) - \frac{c_1 \phi_1 a_1}{c_2} \chi(0),
\]

\[
g_2(\phi_1) = -\phi_1 \left( \sigma T + \frac{\chi(0) - \bar{\chi}}{\kappa} \left( 1 - e^{-\kappa T} \right) \right),
\]

\[
g_3(\phi_1) = \frac{\phi_2^2 c_2^2}{T} \int_0^T \frac{2}{1 - \rho_2^2} M^2(u) du,
\]

\[
g_4(\phi_1) = \frac{\phi_2^2 c_2^2}{T} \int_0^T \frac{2}{1 - \rho_2^2} M^2(u) du,
\]
**Proof.** Recall the dynamics in (1). Then, we can rewrite the dynamics as

\[
\frac{dS(t)}{S(t)} = (r - \delta(t))dt + \sigma_1 \sqrt{v_1(t)} \left( \frac{1}{\sqrt{1 - \rho_1^2}} dW_1^s(t) + \rho_1 dW_2(t) \right),
\]

\[
dv_1(t) = a_1(b_1 - v_1(t))dt + \sigma_2 \sqrt{v_1(t)} dW_2(t),
\]

\[
d\delta(t) = \kappa (\hat{\alpha} - \delta(t))dt + \sigma_3 \left( \frac{\rho_2}{\sqrt{1 - \rho_1^2}} dW_1^s(t) + \frac{1 - \rho_1^2 - \rho_2^2}{1 - \rho_1^2} dW_3^s(t) \right),
\]

where \(W_1^s(t), W_2(t),\) and \(W_3^s(t)\) are the independent Brownian motions. From the above form of \(S(T),\) we have that

\[
x(T) = \ln S(T) = \ln S(0) + rT - \frac{\sigma_1 \rho_1}{\sigma_2} (v_1(0) + a_1 b_1 T) - \int_0^T \delta(u) du
\]

\[
- \frac{1}{2} \left( \sigma_1^2 - \frac{2a_1 \sigma_1 \rho_1}{\sigma_2} v_1(u) + \frac{\sigma_1 \rho_1}{\sigma_2} v_1(T) \right)
\]

\[
+ \sigma_1 \sqrt{1 - \rho_1^2} \int_0^T \sqrt{v_1(u)} dW_1^s(u).
\]

Using the dynamics in (2) and (8), we have the following explicit expression of \(f(\phi_1, \phi_2):\)

\[
f(\phi_1, \phi_2) = \mathbb{E}^P \left[ e^{\phi_1 x(T) + \phi_2 \int_0^T \lambda(s) ds} \right]
\]

\[
= \mathbb{E}^P \left[ \exp \left( \frac{\phi_1 \rho_1}{\sigma_2} \int_0^T v_1(u) du + \frac{\phi_1 c_1}{\sigma_2} v_1(T) \right) \right.
\]

\[
\times \exp \left( \phi_1 c_1 \sqrt{1 - \rho_1^2} \int_0^T \sqrt{v_1(u)} dW_1^s(u) \right) \left. \right] \times \mathbb{E}^P \left[ \exp \left( \phi_2 \int_0^T v(u) du \right) \right] \times \exp(g_1(\phi_1))
\]

The first expectation in (8) can be obtained by applying Proposition 1 and the law of iterated expectations, as follows:

\[
= \mathbb{E}^P \left[ \exp \left( \frac{\phi_1 \rho_1}{\sigma_2} \int_0^T v_1(u) du + \frac{\phi_1 c_1}{\sigma_2} v_1(T) \right) \right.
\]

\[
\times \exp \left( \phi_1 c_1 \sqrt{1 - \rho_1^2} \int_0^T \sqrt{v_1(u)} dW_1^s(u) \right) \left. \right] \times \exp \left( \phi_2 \int_0^T v(u) du \right)
\]

\[
= \mathbb{E}^P \left[ \exp \left( \frac{\phi_1 \rho_1}{\sigma_2} \int_0^T v_1(u) du + \frac{\phi_1 c_1}{\sigma_2} v_1(T) \right) \right.
\]

\[
\times \exp \left( \phi_1 c_1 \sqrt{1 - \rho_1^2} \int_0^T \sqrt{v_1(u)} dW_1^s(u) \right) \left. \right] \sigma(W_2(t))_{0 \leq t \leq T}
\]

\[
= \exp \left( d_1(\phi_1, \phi_2) \int_0^T v_1(u) du + d_2(\phi_1) v_1(T) \right)
\]

\[
= \exp(A_1(\phi_1, \phi_2; T) - B_1(\phi_1, \phi_2; T) v_1(0)),
\]

where \(\sigma(W_2(t))_{0 \leq t \leq T}\) is the \(\sigma\)-field generated by \(W_2(t).\) Similarly, \(\mathbb{E}^P \left[ \exp \left( \phi_2 \int_0^T v(u) du \right) \right] \)

is calculated.

In addition, since

\[
\int_0^T \delta(u) du = \hat{\alpha} T + \frac{\delta(0) - \hat{\alpha}}{\kappa} (1 - e^{\hat{\alpha} T}) + \frac{\sigma_3 c_1}{\kappa} \int_0^T M(u) dW_1^s(u) + \frac{\sigma_3 c_2}{\kappa} \int_0^T M(u) dW_3^s(u),
\]
where \( c_1 = \frac{\rho_2}{\sqrt{1 - \rho_1^2}} \), \( c_2 = \sqrt{\frac{1 - \rho_1^2}{1 - \rho_1^2}} \), then
\[
E^P \left[ \exp \left( -\phi_1 \int_0^T \delta(u) du \right) \right] = \exp \left[ g_2(\phi_1) + g_3(\phi_1) + g_4(\phi_1) \right].
\]
This completes the proof. \( \Box \)

We can obtain the characteristic function of the logarithm of the underlying asset in Proposition 2. We derive the commodity-linked bond pricing formula explicitly using the characteristic function. To obtain the bond price, the Fourier inversion technique and the measure change technique are applied.

**Proposition 3.** The value at time 0 of the commodity-linked bond with credit risk, under the proposed model, is given by
\[
B = (1 - w)e^{-rT} (E_1 - E_2 + F \cdot f(0, -1)) + we^{-rT} (E_3 - E_4) + w \cdot F \cdot e^{-rT}
\]
where
\[
E_1 = \frac{1}{2} f(1, -1) + \frac{1}{\pi} \int_0^\infty \text{Re} \left[ \frac{e^{-i\phi_1 \ln K f(1 + i\phi_1, -1)}}{i\phi_1} \right] d\phi_1
\]
\[
E_2 = \frac{1}{2} f(0, -1) + \frac{1}{\pi} \int_0^\infty \text{Re} \left[ \frac{e^{-i\phi_1 \ln K f(i\phi_1, 1, -1)}}{i\phi_1} \right] d\phi_1
\]
\[
E_3 = \frac{1}{2} f(1, 0) + \frac{1}{\pi} \int_0^\infty \text{Re} \left[ \frac{e^{-i\phi_1 \ln K f(1 + i\phi_1, 0)}}{i\phi_1} \right] d\phi_1
\]
\[
E_4 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left[ \frac{e^{-i\phi_1 \ln K f(i\phi_1, 1, 0)}}{i\phi_1} \right] d\phi_1
\]

**Proof.** In the intensity-based model, the price \( B \) of vulnerable commodity-linked bond at time 0 under the risk neutral measure \( P \) can be represented as
\[
B = e^{-rT} E^P \left[ \{(S(T) - K)^+ + F\} \mathbf{1}_{\{\tau > T\}} \right] + we^{-rT} E^P \left[ \{(S(T) - K)^+ + F\} \mathbf{1}_{\{0 < \tau \leq T\}} \right].
\]
By using \( \mathbf{1}_{\{0 < \tau \leq T\}} = 1 - \mathbf{1}_{\{\tau > T\}} \), we have
\[
B = (1 - w)e^{-rT} E^P \left[ e^{-\int_0^T \lambda(s) ds} \{(S(T) - K)^+ + F\} \right] + we^{-rT} E^P \left[ \{(S(T) - K)^+ + F\} \right]
\]
\[
:= (1 - w)e^{-rT} (E_1 - E_2 + F \cdot E^P \left[ e^{-\int_0^T \lambda(s) ds} \right]) + we^{-rT} (E_3 - E_4) + w \cdot F \cdot e^{-rT}
\]
where
\[
E_1 = E^P \left[ e^{x(T) - \int_0^T \lambda(s) ds} \cdot \mathbf{1}_{\{S(T) > K\}} \right],
\]
\[
E_2 = E^P \left[ e^{-\int_0^T \lambda(s) ds} \cdot \mathbf{1}_{\{S(T) > K\}} \right],
\]
\[
E_3 = E^P \left[ e^{x(T)} \cdot \mathbf{1}_{\{S(T) > K\}} \right],
\]
\[
E_4 = E^P \left[ \mathbf{1}_{\{S(T) > K\}} \right].
\]
To calculate \( E_1 \), we introduce a new measure \( Q_1 \) as
\[
\frac{dQ_1}{dP} := \frac{e^{x(T) - \int_0^T \lambda(u) du}}{E \left[ e^{x(T) - \int_0^T \lambda(u) du} \right]}.
\]
Then, we obtain that parameters in the base case are used: party is modeled through an intensity-based model. Based on the studies in [23,30], the

\[ x \]

also employ the quadrature method to numerically calculate the integrals in Proposition 3.

4. Numerical Example

Under the measure \( Q_1 \), the characteristic function of \( x(T) \) is given by

\[
E^{Q_1} \left[ e^{i\phi_1 x(T)} \right] = E^P \frac{e^{x(T) - \int_0^T \lambda(s)ds} e^{i\phi_1 x(T)}}{E^P \left[ e^{x(T) - \int_0^T \lambda(s)ds} \right]} = f(1, -1) / f(1, -1).
\]

In addition, the marginal distribution of \( x(T) \) under the measure \( Q_1 \) is given by

\[
F(x(T); x) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \text{Re} \left[ \frac{e^{-i\phi_1 x} E^{Q_1} \left[ e^{i\phi_1 x(T)} \right]}{i\phi_1} \right] d\phi_1.
\]

Then, we have

\[
E_1 = E^P \left[ e^{x(T) - \int_0^T \lambda(s)ds} \cdot 1_{\{x(T)>\ln K\}} \right]
= E^P \left[ e^{x(T) - \int_0^T \lambda(s)ds} \right] E^{Q_1} \left\{ \frac{dQ_1}{dP} \cdot 1_{\{x(T)>\ln K\}} \right\}
= \frac{1}{2} f(1, -1) + \frac{1}{\pi} \int_0^\infty \text{Re} \left[ \frac{e^{-i\phi_1 \ln K f(1, -1)} / f(1, -1)}{i\phi_1} \right] d\phi_1.
\]

For \( E_2 \), we introduce another measure \( Q_2 \) as

\[
\frac{dQ_2}{dP} := \frac{e^{-\int_0^T \lambda(u)du}}{E \left[ e^{-\int_0^T \lambda(u)du} \right]}.
\]

Then, we obtain that

\[
E_2 = E^P \left[ e^{-\int_0^T \lambda(s)ds} \cdot 1_{\{x(T)>\ln K\}} \right]
= E^P \left[ e^{-\int_0^T \lambda(s)ds} \right] E^{Q_2} \left\{ \frac{dQ_2}{dP} \cdot 1_{\{x(T)>\ln K\}} \right\}
= \frac{1}{2} f(0, -1) + \frac{1}{\pi} \int_0^\infty \text{Re} \left[ \frac{e^{-i\phi_1 \ln K f(1, -1)} / f(0, -1)}{i\phi_1} \right] d\phi_1.
\]

Since \( E_3 \) and \( E_4 \) can determined in the similar way using new measures and \( E^P \left[ e^{-\int_0^T \lambda(s)ds} \right] = f(0, -1) \), the proof is completed. □

4. Numerical Example

In this section, we present several numerical examples to investigate the impact of several variables on commodity-linked bond pricing when the credit risk of counterparty is modeled through an intensity-based model. Based on the studies in [23,30], the parameters in the base case are used: \( S(0) = 20, K = 20, F = 10, T = 1, r = 0.04, \sigma_1 = 0.4, \sigma_2 = 0.1, \sigma_3 = 0.2, \sigma = 0.1, \rho_1 = -0.26, \rho_2 = 0.3, a_1 = 1.15, b_1 = 0.03, a = 2, b = 0.01, \alpha = 0.08, \lambda = 0.25, \kappa = 0.7, \delta(0) = 0.02, \nu_1(0) = 0.03, \beta = 1.2, w = 0.4 \). We also employ the quadrature method to numerically calculate the integrals in Proposition 3.
Figure 1 shows bond values against strike price \( K \) for four different recovery rates \( w = 0.1, 0.3, 0.5, 0.7 \) in the proposed model. From Figure 1, we observe that bond values decrease for the strike \( K \) from 15 to 25 and have similar values for sufficiently large values of \( K \). Also, as expected, we note that bond values increase as the recovery rate \( w \) becomes higher. In other words, the bond prices are higher when there is little credit risk by the counterparty compared to high credit risk cases. Figure 2 shows the bond values against the time to maturity \( T \) for different values of recovery rates. The bond values increase with an increase in both \( T \) and \( w \). This means that bond prices are an increasing function of maturity. We can also see that the function is exponential, not linear.

Figure 3 shows bond values against \( \beta \) values in the intensity process. We observe that, as \( \beta \) grows larger, bond values decrease, as expected. That is, the bond prices are a decreasing function of \( \beta \) for all recovery rates. This is because the value of the intensity process increases as \( \beta \) increases. As a result, the probability of a default increases, and the prices decrease. Also, because \( \beta \) values affect the intensity process, \( \beta \) has little impact on the value of bonds when \( w \) has a large value (close to 1). Figure 4 shows bond values against initial volatility \( v_1(0) \) of the dynamics \( v_1(t) \). We can see that the bond prices are
decreasing linear functions of $v_1(0)$ for all recovery rates. That is, from Figure 4, we find that bond values fall as $v_1(0)$ increases. This is because a higher level of $v_1(0)$ connects to a larger default probability.

![Figure 3](image1.png)

**Figure 3.** Value of bonds against $\beta$ for recovery rates $w = 0.1, 0.3, 0.5, 0.7$.

![Figure 4](image2.png)

**Figure 4.** Value of bonds against $v_1(0)$ for recovery rates $w = 0.1, 0.3, 0.5, 0.7$.

5. Concluding Remarks

In this study, we incorporate the stochastic convenience model and the stochastic volatility model to characterize the dynamics of the underlying assets. Additionally, we examine credit risk of counterparty using an intensity-based model. In the proposed model, we develop the commodity-linked bond pricing formula by using the measure change technique and the closed-form solution of the characteristic function obtained from the law of iterated expectation. Since the pricing formula is explicitly provided with single integral, the bond values are calculated using the numerical integration methods. Finally, we provide the graphs to illustrate the impacts of significant parameters on bond value.

This study is based on the works of Wang [23] and Ma et al. [30]. Wang [23] developed vulnerable option pricing models using the intensity-based model under a stochastic volatility model, and Ma et al. [30] improved the pricing models of the commodity-linked bonds with credit risk under the structural model. As a consequence, the main contribution of this study is an extension of the work of Ma et al. [30], using the credit risk model of Wang [23]. These results will be applicable to various bond-linked derivatives. However, our results have a limitation. In contrast to Ma et al. [30], who studied the valuation of
commodity-linked bond with credit risk using the structural model, the credit risk model used in this paper is an intensity-based model. That is, the price of commodity-linked bonds with credit risk should be investigated using the structural model. This will be studied further in the future.

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