

Article

Trigonometric Induced Multivariate Smooth Gauss–Weierstrass Singular Integrals Approximation

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Abstract: In this article, we employ the uniform and L_p , $1 \leq p < \infty$ approximation properties of general smooth multivariate singular integral operators over $\mathbb{R}^N$, $N \geq 1$. It is a trigonometric relief approach with detailed applications to the corresponding smooth multivariate Gauss–Weierstrass singular integral operators. The results are quantitative via Jackson-type inequalities involving the first uniform and L_p moduli of continuity.

Keywords: multivariate singular integral operator; multivariate modulus of continuity; multivariate Gauss–Weierstrass operator; uniform and L_p approximation

MSC: 26A15; 41A17; 41A25; 41A35; 26D15; 41A36

1. Introduction

The degree of approximation by univariate and multivariate singular integral operators has been researched extensively in [1–5]. All these sources motivate our current work. In particular we studied the approximation properties of the smooth singular integral operators in [1,3,4]. These are not in general positive operators. Here, we use the uniform and L_p , $p \geq 1$, results of our multivariate general theory [6,7], to establish approximation properties of the smooth Gauss–Weierstrass singular integral operators. The degrees of approximation are given quantitatively by employing the uniform and L_p first moduli of continuity. The fundamental tool here comes from [8], where a multivariate trigonometric Taylor formula is presented. For recent related work, see [9–13]. Other important articles on the topic are [14–18].

In the history of this topic, we mention the monograph [4] published in 2012, which was the first comprehensive source to exclusively address the classic theory of the approximation of singular integrals to the identity-unit operator. The authors there quantitatively studied the basic approximation properties of the general Picard, Gauss–Weierstrass and Poisson–Cauchy singular integral operators over the real line, which are not positive linear operators. In particular, they studied the rate of convergence of these operators to the unit operator, as well as the related simultaneous approximation. This is given via inequalities and with the use of a higher-order modulus of smoothness of the high-order derivative of the involved function. Some of these inequalities have been proved to be sharp. Also, they studied the global smoothness preservation property of these operators. Furthermore, they gave asymptotic expansions of Voronovskaya type for the error of approximation. They continued with the study of related properties of the general fractional Gauss–Weierstrass and Poisson–Cauchy singular integral operators. These properties were studied with respect to the L_p norm, $1 \leq p \leq \infty$. The case of Lipschitz-type functions approximation was studied separately and in detail. Furthermore, they presented the corresponding general approximation theory of general singular integral operators with many applications to the previously under-explored domain of trigonometric singular integrals.



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2. Background on General Theory

Here, $r \in \mathbb{N}$, $m \in \mathbb{Z}_+$, we define

$$\alpha_j := \alpha_{j,r}^{[m]} := \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 1, 2, \dots, r, \\ 1 - \sum_{j=1}^r (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 0. \end{cases} \quad (1)$$

and

$$\delta_k := \delta_{k,r}^{[m]} := \sum_{j=0}^r \alpha_{j,r}^{[m]} j^k, \quad k = 1, 2, \dots, m \in \mathbb{N}. \quad (2)$$

See that

$$\sum_{j=0}^r \alpha_{j,r}^{[m]} = 1, \quad (3)$$

and

$$-\sum_{j=1}^r (-1)^{r-j} \binom{r}{j} = (-1)^r \binom{r}{0}.$$

Let μ_{ξ_n} be a probability Borel measure on $\mathbb{R}^N$, $N \geq 1$, $\xi_n > 0$, $n \in \mathbb{N}$. We now define the multiple smooth singular integral operators

$$\begin{aligned} \theta_n(f; x_1, \dots, x_N) &:= \theta_{r,n}^{[m]}(f; x_1, \dots, x_N) := \\ &\sum_{j=0}^r \alpha_{j,r}^{[m]} \int_{\mathbb{R}^N} f(x_1 + s_1 j, x_2 + s_2 j, \dots, x_N + s_N j) d\mu_{\xi_n}(s), \end{aligned} \quad (4)$$

where $s := (s_1, \dots, s_N)$, $x := (x_1, \dots, x_N) \in \mathbb{R}^N$; $n, r \in \mathbb{Z}$, $m \in \mathbb{Z}_+$, $f : \mathbb{R}^N \rightarrow \mathbb{R}$ is a Borel measurable function, and also $(\xi_n)_{n \in \mathbb{N}}$ is a bounded sequence of positive real numbers; we take $0 < \xi_n \leq 1$.

Remark 1. The operators $\theta_{r,n}^{[m]}$ are not generally positive (see [2], p. 2).

We observe that

Lemma 1. It holds

$$\theta_{r,n}^{[m]}(c; x_1, \dots, x_N) = c,$$

where c is a constant.

We need

Definition 1. Let $f \in C(\mathbb{R}^N)$, $N \geq 1$. We define the first uniform modulus of continuity of f as

$$\omega_1(f, \delta) := \sup_{\substack{x, y \in \mathbb{R}^N: \\ \|x - y\|_\infty \leq \delta}} |f(x) - f(y)|, \quad \delta > 0, \quad (5)$$

where $\|\cdot\|_\infty$ is the max norm in $\mathbb{R}^N$. The functional $\omega_1(f, \delta)$ is bounded for f being bounded or uniformly continuous, and $\omega_1(f, \delta) \rightarrow 0$ as $\delta \rightarrow 0$, in the case of f being uniformly continuous.

We mention the main uniform general approximation result regarding the operator θ_n .

Theorem 1 ([6]). Here, $f \in C^2(\mathbb{R}^N)$ and let all $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, N$, $N \geq 1$, $|\alpha| := \sum_{i=1}^N \alpha_i = 2$; $x \in \mathbb{R}^N$, and all the partials f_α of order 2, along with $f \in C_B(\mathbb{R}^N)$ (continuous and bounded

functions); or all f_α of order 2, $f \in C_U(\mathbb{R}^N)$ (uniformly continuous functions). Let μ_{ξ_n} be a Borel probability measure on $\mathbb{R}^N$, for $0 < \xi_n \leq 1$, $n \in \mathbb{N}$.

Suppose that for all $\alpha := (\alpha_1, \dots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, N$, $|\alpha| = \sum_{i=1}^N \alpha_i = 2$, $j = 0, 1, \dots, r$, we have that both

$$I_{1j}(\alpha) := \int_{\mathbb{R}^N} \left(1 + \frac{j\|s\|_1}{3\xi_n}\right) \left(\prod_{i=1}^N |s_i|^{\alpha_i}\right) d\mu_{\xi_n}(s), \quad (6)$$

$$I_{2j}(\alpha) := \int_{\mathbb{R}^N} \left(1 + \frac{j\|s\|_1}{3\xi_n}\right) d\mu_{\xi_n}(s), \quad (7)$$

are uniformly bounded in $\xi_n \in (0, 1]$.

Denote ($n \in \mathbb{N}$)

$$\begin{aligned} \Delta_n(x) := & \theta_n(f, x) - f(x) - \left(\sum_{j=0}^r \alpha_j j\right) \sin(1) \left[\sum_{i=1}^N \frac{\partial f(x)}{\partial x_i} \left(\int_{\mathbb{R}^N} s_i d\mu_{\xi_n}(s)\right)\right] \\ & - 2\left(\sum_{j=0}^r \alpha_j j^2\right) \sin^2\left(\frac{1}{2}\right) \left\{\sum_{i=1}^N \left(\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s)\right) \frac{\partial^2 f(x)}{\partial x_i^2} + \right. \\ & \left. \sum_{\substack{i \neq j^*, \\ i, j^* \in \{1, \dots, N\}}} \left(\int_{\mathbb{R}^N} s_i s_{j^*} d\mu_{\xi_n}(s)\right) \frac{\partial^2 f(x)}{\partial x_i \partial x_{j^*}}\right\}. \end{aligned} \quad (8)$$

Then

(i)

$$\begin{aligned} |\Delta_n(x)| \leq & \|\Delta_n(x)\|_\infty \leq \sum_{j=0}^r |\alpha_j| \\ & \left[\left[j^2 \sum_{\substack{\alpha_i \in \mathbb{Z}^+, \\ \alpha: |\alpha|=2}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!}\right) \omega_1(f_\alpha, \xi_n) \int_{\mathbb{R}^N} \left(1 + \frac{j\|s\|_\infty}{3\xi_n}\right) \left(\prod_{i=1}^N |s_i|^{\alpha_i}\right) d\mu_{\xi_n}(s) \right] + \right. \\ & \left. \frac{1}{2} \omega_1(f, \xi_n) \int_{\mathbb{R}^N} \left(1 + \frac{j\|s\|_\infty}{3\xi_n}\right) d\mu_{\xi_n}(s) \right] =: \varphi_{\xi_n}. \end{aligned} \quad (9)$$

In case of all f_α of order 2 and $f \in C_U(\mathbb{R}^N)$ and $\xi_n \rightarrow 0$, as $n \rightarrow \infty$, then $\Delta_n(x)$, $\|\Delta_n(x)\|_\infty \rightarrow 0$ with rates.

(ii) If $\frac{\partial f(x)}{\partial x_i} = 0$, $i = 1, \dots, N$, and $f_\alpha(x) = 0$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, N$, with $|\alpha| = 2$, then

$$|\theta_n(f, x) - f(x)| \leq \varphi_{\xi_n}. \quad (10)$$

And $\theta_n(f, x) \rightarrow f(x)$ in the uniformly continuous case.

(iii) Additionally, assume all partials of order ≤ 2 are bounded. Hence,

$$\begin{aligned} \|\theta_n(f) - f\|_\infty \leq & \left(\sum_{j=0}^r |\alpha_j| j\right) (0.8414) \left[\sum_{i=1}^N \left\|\frac{\partial f}{\partial x_i}\right\|_\infty \left(\int_{\mathbb{R}^N} |s_i| d\mu_{\xi_n}(s)\right)\right] + \\ & \left(\sum_{j=0}^r |\alpha_j| j^2\right) (0.4596) \left\{\sum_{i=1}^N \left(\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s)\right) \left\|\frac{\partial^2 f}{\partial x_i^2}\right\|_\infty + \right. \\ & \left. \sum_{\substack{i \neq j^*, \\ i, j^* \in \{1, \dots, N\}}} \left(\int_{\mathbb{R}^N} |s_i| |s_{j^*}| d\mu_{\xi_n}(s)\right) \left\|\frac{\partial^2 f}{\partial x_i \partial x_{j^*}}\right\|_\infty\right\} + \varphi_{\xi_n}. \end{aligned} \quad (11)$$

If all $\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s)$ and $\int_{\mathbb{R}^N} |s_i| |s_{j^*}| d\mu_{\xi_n}(s)$ converge to zero, as $n \rightarrow \infty$, with $\xi_n \rightarrow 0$, and all f_α of order 2, $f \in C_U(\mathbb{R}^N)$, then

$$\|\theta_n(f) - f\|_\infty \rightarrow 0 \text{ with rates, as } \xi_n \rightarrow 0, n \rightarrow +\infty.$$

Next, we deal with $f \in C^m(\mathbb{R}^N)$, $m \in \mathbb{Z}^+$, with $f_\alpha \in L_p(\mathbb{R}^N)$, $|\alpha| = m \in \mathbb{Z}^+$, $p \geq 1$; where f_α denotes the mixed partial $\frac{\partial^{\tilde{j}} f(\cdot, \dots, \cdot)}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}}$, $\alpha_j \in \mathbb{Z}^+$, $j = 1, \dots, N$: $|\alpha| := \sum_{j=1}^N \alpha_j = \tilde{j}$, $\tilde{j} = 1, \dots, m$.

We need

Definition 2 (see also [2], p. 20). We call

$$\Delta_u^r f(x) := \Delta_{u_1, u_2, \dots, u_N}^r f(x_1, \dots, x_N) := \quad (12)$$

$$\sum_{j=0}^r (-1)^{r-j} \binom{r}{j} f(x_1 + ju_1, x_2 + ju_2, \dots, x_N + ju_N).$$

Let $p \geq 1$, the L_p modulus of smoothness of order r is given by

$$\omega_r(f; h)_p := \sup_{\|u\|_2 \leq h} \|\Delta_u^r(f)\|_p, \quad (13)$$

$h > 0$.

We mention

Theorem 2 ([7]). Let $f \in C^m(\mathbb{R}^N)$, $m \in \mathbb{N}$, $N \geq 1$, with $f_\alpha \in L_p(\mathbb{R}^N)$, $|\alpha| = m$, $x \in \mathbb{R}^N$. Let $p, q > 1$: $\frac{1}{p} + \frac{1}{q} = 1$. Here, μ_{ξ_n} is a Borel probability measure on $\mathbb{R}^N$ for $\xi_n > 0$, $(\xi_n)_{n \in \mathbb{N}}$ bounded sequence. Assume for all $\alpha := (\alpha_1, \dots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, N$, $|\alpha| := \sum_{i=1}^N \alpha_i = m$ that we have

$$\int_{\mathbb{R}^N} \left(\left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right)^p d\mu_{\xi_n}(s) < \infty. \quad (14)$$

For $\tilde{j} = 1, \dots, m$, and $\alpha := (\alpha_1, \dots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, N$, $|\alpha| := \sum_{i=1}^N \alpha_i = \tilde{j}$, call

$$c_{\alpha, n, \tilde{j}} := \int_{\mathbb{R}^N} \prod_{i=1}^N s_i^{\alpha_i} d\mu_{\xi_n}(s). \quad (15)$$

Then

$$\begin{aligned} \|E_{r,n}^{[m]}\|_p &:= \left\| \theta_{r,n}^{[m]}(f; x) - f(x) - \sum_{\tilde{j}=1}^m \delta_{\tilde{j},r}^{[m]} \left(\sum_{|\alpha|=\tilde{j}} \frac{c_{\alpha, n, \tilde{j}} f_\alpha(x)}{\left(\prod_{i=1}^N \alpha_i! \right)} \right) \right\|_{p,x} \leq \\ &\left[\left(\frac{N^m}{(m-1)!} \right)^{\frac{p}{q}} \left(\frac{m}{(q(m-1)+1)^{\frac{p}{q}}} \right) \left(\sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^N \alpha_i!} \right. \right. \\ &\left. \left. \left[\int_{\mathbb{R}^N} \left[\left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right]^p d\mu_{\xi_n}(s) \right] \omega_r(f_\alpha, \xi_n)_p^p \right]^{\frac{1}{p}} \right] \end{aligned} \quad (16)$$

As $n \rightarrow \infty$ and $\xi_n \rightarrow 0$, by (16), we obtain that $\|E_{r,n}^{[m]}\|_p \rightarrow 0$ with rates.

One also gets by (16) that

$$\begin{aligned} & \left\| \theta_{r,n}^{[m]}(f; x) - f(x) \right\|_{p,x} \leq \\ & \sum_{\tilde{j}=1}^m \left| \delta_{\tilde{j},r}^{[m]} \right| \left(\sum_{|\alpha|=\tilde{j}} \frac{|c_{\alpha,n,\tilde{j}}|}{\prod_{i=1}^N \alpha_i!} \|f_\alpha\|_p \right) + \text{R.H.S. (16)}, \end{aligned} \quad (17)$$

given that $\|f_\alpha\|_p < \infty$, $|\alpha| = \tilde{j}$, $\tilde{j} = 1, \dots, m$.

Assuming that $c_{\alpha,n,\tilde{j}} \rightarrow 0$, $\xi_n \rightarrow 0$, as $n \rightarrow \infty$, we get $\left\| \theta_{r,n}^{[m]}(f) - f \right\|_p \rightarrow 0$, that is $\theta_{r,n}^{[m]} \rightarrow I$ the unit operator, in L_p norm, with rates.

We make

Remark 2. Notice that ($p > 1$)

$$\begin{aligned} & \int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) \leq \\ & \left[\int_{\mathbb{R}^N} \left(\left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right)^p d\mu_{\xi_n}(s) \right]^{\frac{1}{p}} < \infty, \end{aligned} \quad (18)$$

as assumed in Theorem 2.

By (18), we get that

$$\int_{\mathbb{R}^N} \prod_{i=1}^N |s_i|^{\alpha_i} d\mu_{\xi_n}(s) < \infty. \quad (19)$$

Hence, $c_{\alpha,n,\tilde{j}} \in \mathbb{R}$.

We mention also the following trigonometric induced alternative L_p approximation result for θ_n operators.

Theorem 3 ([7]). Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $0 < \xi_n \leq 1$, $n \in \mathbb{N}$. Here, we deal with $f \in C^2(\mathbb{R}^N)$, $N \geq 1$, with $f, f_\alpha \in L_p(\mathbb{R}^N)$, $|\alpha| = 2$, where $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, N$, and $|\alpha| = \sum_{i=1}^N \alpha_i$; $x \in \mathbb{R}^N$. Let μ_{ξ_n} be a Borel probability measure on $\mathbb{R}^N$. Suppose that for all $\alpha : |\alpha| = 2$, $j = 0, 1, \dots, r$, we have that both

$$I_{1j}(\alpha) := \int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{p\alpha_i} \right) \left(1 + \frac{j^p \|s\|_2^p}{\xi_n^p} \right) d\mu_{\xi_n}(s), \quad (20)$$

$$I_2 := \int_{\mathbb{R}^N} \left(\frac{\|s\|_2}{\xi_n} \right)^p d\mu_{\xi_n}(s), \quad (21)$$

are uniformly bounded in $\xi_n \in (0, 1]$.

Denote ($n \in \mathbb{N}$)

$$\begin{aligned} \Delta_n(x) &:= \theta_n(f, x) - f(x) - \left(\sum_{j=0}^r \alpha_j j \right) \sin(1) \left[\sum_{i=1}^N \frac{\partial f(x)}{\partial x_i} \left(\int_{\mathbb{R}^N} s_i d\mu_{\xi_n}(s) \right) \right] \\ &\quad - 2 \left(\sum_{j=0}^r \alpha_j j^2 \right) \sin^2\left(\frac{1}{2}\right) \left\{ \sum_{i=1}^N \left(\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s) \right) \frac{\partial^2 f(x)}{\partial x_i^2} + \right. \end{aligned} \quad (22)$$

$$\sum_{\substack{i \neq j^*, \\ i, j^* \in \{1, \dots, N\}}} \left(\int_{\mathbb{R}^N} s_i s_{j^*} d\mu_{\xi_n}(s) \right) \frac{\partial^2 f(x)}{\partial x_i \partial x_{j^*}} \Bigg\}.$$

Then, it holds

$$\begin{aligned} \|\Delta_n(x)\|_p &\leq \left(\sum_{j=0}^r |\alpha_j|^q \right)^{\frac{1}{q}} \left\{ \frac{2}{(q+1)^{\frac{1}{q}}} \left(\frac{N(N+1)+2}{2} \right)^{\frac{1}{q}} \right\} \\ &\left\{ \sum_{j=0}^r \left[j^{2p} \sum_{|\alpha|=2} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right)^p \omega_1(f_\alpha, \xi_n)_p^p \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{p\alpha_i} \right) \left(1 + \frac{j^p \|s\|_2^p}{\xi_n^p} \right) d\mu_{\xi_n}(s) \right) \right. \right. \\ &\quad \left. \left. + \omega_1(f, \xi_n)_p^p \left(\int_{\mathbb{R}^N} \left(1 + \frac{j^p \|s\|_2^p}{\xi_n^p} \right) d\mu_{\xi_n}(s) \right) \right] \right\}^{\frac{1}{p}}. \end{aligned} \quad (23)$$

As $n \rightarrow \infty$ and $\xi_n \rightarrow 0$, by (23), we obtain that $\|\Delta_n\|_p \rightarrow 0$ with rates. One also gets by (23) that

$$\begin{aligned} \|\theta_n(f, x) - f(x)\|_{p,x} &\leq \\ &\left(\sum_{j=0}^r |\alpha_j| j \right) \sin(1) \left[\sum_{i=1}^N \left\| \frac{\partial f}{\partial x_i} \right\|_p \left(\int_{\mathbb{R}^N} |s_i| d\mu_{\xi_n}(s) \right) \right] \\ &+ 2 \left(\sum_{j=0}^r |\alpha_j| j^2 \right) \sin^2\left(\frac{1}{2}\right) \left\{ \sum_{i=1}^N \left(\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s) \right) \left\| \frac{\partial^2 f}{\partial x_i^2} \right\|_p + \right. \\ &\quad \left. \sum_{\substack{i \neq j^*, \\ i, j^* \in \{1, \dots, N\}}} \left(\int_{\mathbb{R}^N} |s_i s_{j^*}| d\mu_{\xi_n}(s) \right) \left\| \frac{\partial^2 f}{\partial x_i \partial x_{j^*}} \right\|_p \right\} + R.H.S. (23), \end{aligned} \quad (24)$$

given that $\|f_\alpha\|_p < \infty$, $|\alpha| = \tilde{j}$, $\tilde{j} = 1, 2$. Assuming that $\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s)$, $\int_{\mathbb{R}^N} |s_i s_{j^*}| d\mu_{\xi_n}(s)$, $i, j^* \in \{1, \dots, N\}$, $i \neq j^*$, converge to zero as $\xi_n \rightarrow 0$, we get $\|\theta_n(f, x) - f(x)\|_p \rightarrow 0$, that is $\theta_n \rightarrow I$ the unit operator, in L_p norm, with rates.

Furthermore, we mention the following trigonometric-based alternative L_1 approximation result for θ_n operators.

Theorem 4 ([7]). Let $0 < \xi_n \leq 1$, $n \in \mathbb{N}$, $x \in \mathbb{R}^N$. Here, we deal with $f \in C^2(\mathbb{R}^N)$, $N \geq 1$, with $f, f_\alpha \in L_1(\mathbb{R}^N)$, $|\alpha| = 2$, where $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, N$, and $|\alpha| = \sum_{i=1}^N \alpha_i$. Let μ_{ξ_n} be a Borel probability measure on $\mathbb{R}^N$. Suppose that for all $\alpha : |\alpha| = 2$, $j = 0, 1, \dots, r$, we have that both

$$I_{1j}^*(\alpha) := \int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{j \|s\|_2}{\xi_n} \right) d\mu_{\xi_n}(s), \quad (25)$$

and

$$I_2^* := \int_{\mathbb{R}^N} \frac{\|s\|_2}{\xi_n} d\mu_{\xi_n}(s), \quad (26)$$

are uniformly bounded in $\xi_n \in (0, 1]$.

Here, $\Delta_n(x)$ is as in (22).

Then, it holds

$$\|\Delta_n(x)\|_1 \leq \left\{ \sum_{j=0}^r |\alpha_j| \left[j^2 \sum_{|\alpha|=2} \left(\frac{2}{\prod_{i=1}^N \alpha_i!} \right) \omega_1(f_\alpha, \xi_n)_1 \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{j\|s\|_2}{\xi_n} \right) d\mu_{\xi_n}(s) \right) + \omega_1(f, \xi_n)_1 \left(\int_{\mathbb{R}^N} \left(1 + \frac{j\|s\|_2}{\xi_n} \right) d\mu_{\xi_n}(s) \right) \right] \right\}. \quad (27)$$

As $n \rightarrow \infty$ and $\xi_n \rightarrow 0$, by (27), we obtain that $\|\Delta_n\|_1 \rightarrow 0$ with rates. One also obtains by (27) that

$$\begin{aligned} \|\theta_n(f) - f\|_1 &\leq \left(\sum_{j=0}^r |\alpha_j| j \right) \sin(1) \left[\sum_{i=1}^N \left\| \frac{\partial f}{\partial x_i} \right\|_1 \left(\int_{\mathbb{R}^N} |s_i| d\mu_{\xi_n}(s) \right) \right] \\ &+ 2 \left(\sum_{j=0}^r |\alpha_j| j^2 \right) \sin^2\left(\frac{1}{2}\right) \left\{ \sum_{i=1}^N \left(\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s) \right) \left\| \frac{\partial^2 f}{\partial x_i^2} \right\|_1 + \right. \\ &\left. \sum_{\substack{i \neq j^*, \\ i, j^* \in \{1, \dots, N\}}} \left(\int_{\mathbb{R}^N} |s_i s_{j^*}| d\mu_{\xi_n}(s) \right) \left\| \frac{\partial^2 f}{\partial x_i \partial x_{j^*}} \right\|_1 \right\} + R.H.S. (27), \end{aligned} \quad (28)$$

given that $\|f_\alpha\|_1 < \infty$, $|\alpha| = \tilde{j}$, $\tilde{j} = 1, 2$. Assuming that $\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s)$, $\int_{\mathbb{R}^N} |s_i s_{j^*}| d\mu_{\xi_n}(s)$, $i, j^* \in \{1, \dots, N\}$, $i \neq j^*$, converge to zero as $\xi_n \rightarrow 0$, we derive $\|\theta_n(f) - f\|_1 \rightarrow 0$; that is $\theta_n \rightarrow I$ in L_1 norm, with rates.

3. Auxiliary Essential Results

We need

Theorem 5. Let $N \in \mathbb{N}$; $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, N$: $|\alpha| := \sum_{i=1}^N \alpha_i = 2$, $\xi_n \in (0, 1]$, $n \in \mathbb{N}$; $j = 0, 1, \dots, r \in \mathbb{N}$. Then

$$\begin{aligned} I_{1j}^*(\alpha) &:= \frac{1}{(\sqrt{\pi \xi_n})^N} \int_{\mathbb{R}^N} \left(1 + \frac{j\|s\|_1}{3\xi_n} \right) \left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 \dots ds_N \leq \\ &\sqrt{\xi_n} \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1 + jN) + (1 + j) \left(\frac{\lfloor 6e \rfloor}{e} \right)^N \right] \leq \\ &\left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1 + jN) + (1 + j) \left(\frac{\lfloor 6e \rfloor}{e} \right)^N \right] < +\infty, \end{aligned} \quad (29)$$

are uniformly bounded, and $I_{1j}^*(\alpha) \rightarrow 0$, as $\xi_n \rightarrow 0$.

Above, $\lfloor \cdot \rfloor$ denotes the integral part.

Proof. We have that

$$I_{1j}^*(\alpha) = \frac{1}{(\sqrt{\pi \xi_n})^N} \int_{\mathbb{R}^N} \left(1 + \frac{j\|s\|_1}{3\xi_n} \right) \left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 \dots ds_N =$$

$$\frac{2^N}{(\sqrt{\pi})^N (\sqrt{\xi_n})^N} \int_{\mathbb{R}_+^N} \left(1 + \frac{j \|s\|_1}{3\xi_n}\right) \left(\prod_{i=1}^N s_i^{\alpha_i}\right) e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 \dots ds_N = \quad (30)$$

$$\frac{2^N}{(\sqrt{\pi})^N (\sqrt{\xi_n})^N} \int_{\mathbb{R}_+^N} \left(1 + \frac{j \left(\sum_{i=1}^N s_i\right)}{3\xi_n}\right) \left(\prod_{i=1}^N s_i^{\alpha_i}\right) e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 \dots ds_N \leq$$

$$\frac{2^N (\sqrt{\xi_n})^2}{(\sqrt{\pi})^N} \int_{\mathbb{R}_+^N} \left(\frac{1}{\sqrt{\xi_n}} + j \frac{1}{\sqrt{\xi_n}} \left(\sum_{i=1}^N \frac{s_i}{\sqrt{\xi_n}}\right)\right)$$

$$\left(\prod_{i=1}^N \left(\frac{s_i}{\sqrt{\xi_n}}\right)^{\alpha_i}\right) e^{-\sum_{i=1}^N \left(\frac{s_i}{\sqrt{\xi_n}}\right)^2} \frac{ds_1}{\sqrt{\xi_n}} \dots \frac{ds_N}{\sqrt{\xi_n}} =$$

$$\frac{2^N \sqrt{\xi_n}}{(\sqrt{\pi})^N} \int_{\mathbb{R}_+^N} \left(1 + j \left(\sum_{i=1}^N z_i\right)\right) \left(\prod_{i=1}^N z_i^{\alpha_i}\right) e^{-\left(\sum_{i=1}^N z_i^2\right)} dz_1 \dots dz_N = \quad (31)$$

$$\frac{2^N \sqrt{\xi_n}}{(\sqrt{\pi})^N} \left[\int_{[0,1]^N} \left(1 + j \left(\sum_{i=1}^N z_i\right)\right) \left(\prod_{i=1}^N z_i^{\alpha_i}\right) e^{-\left(\sum_{i=1}^N z_i^2\right)} dz_1 \dots dz_N + \right.$$

$$\left. \int_{(\mathbb{R}_+ - [0,1])^N} \left(1 + j \left(\sum_{i=1}^N z_i\right)\right) \left(\prod_{i=1}^N z_i^{\alpha_i}\right) e^{-\left(\sum_{i=1}^N z_i^2\right)} dz_1 \dots dz_N \right] \leq$$

$$\frac{2^N \sqrt{\xi_n}}{(\sqrt{\pi})^N} \left[(1 + jN) + (1 + j) \int_{(\mathbb{R}_+ - [0,1])^N} \left(\sum_{i=1}^N z_i\right) \left(\prod_{i=1}^N z_i^{\alpha_i}\right) e^{-\sum_{i=1}^N z_i^2} dz_1 \dots dz_N \right] \leq$$

$$\frac{2^N \sqrt{\xi_n}}{(\sqrt{\pi})^N} \left[(1 + jN) + (1 + j) \int_{(\mathbb{R}_+ - [0,1])^N} \left(\prod_{i=1}^N z_i\right) \left(\prod_{i=1}^N z_i^{\alpha_i}\right) \prod_{i=1}^N e^{-z_i} \prod_{i=1}^N dz_i \right] =$$

$$\frac{2^N \sqrt{\xi_n}}{(\sqrt{\pi})^N} \left[(1 + jN) + (1 + j) \prod_{i=1}^N \int_1^\infty z_i^{(\alpha_i+2)-1} e^{-z_i} dz_i \right] = \quad (32)$$

(by [19], p. 348)

$$\frac{2^N \sqrt{\xi_n}}{(\sqrt{\pi})^N} \left[(1 + jN) + (1 + j) \prod_{i=1}^N \Gamma((\alpha_i + 2), 1) \right] =$$

(where $\Gamma(\cdot, \cdot)$ is the upper incomplete gamma function)

$$\frac{2^N \sqrt{\xi_n}}{(\sqrt{\pi})^N} \left[(1 + jN) + (1 + j) \prod_{i=1}^N \frac{[e(\alpha_i + 1)!]}{e} \right] \leq$$

$$\frac{2^N \sqrt{\xi_n}}{(\sqrt{\pi})^N} \left[(1 + jN) + (1 + j) \left(\frac{[6e]}{e}\right)^N \right] < +\infty.$$

That is,

$$I_{1j}^*(\alpha) \leq \sqrt{\xi_n} \left(\frac{2}{\sqrt{\pi}}\right)^N \left[(1 + jN) + (1 + j) \left(\frac{[6e]}{e}\right)^N \right] \leq \quad (33)$$

$$\left(\frac{2}{\sqrt{\pi}}\right)^N \left[(1 + jN) + (1 + j) \left(\frac{[6e]}{e}\right)^N \right] < +\infty,$$

are uniformly bounded; furthermore, $I_{1j}^*(\alpha) \rightarrow 0$, as $\xi_n \rightarrow 0$. $\square$

We make

Remark 3. By Theorem 5, $j = 0, 1, \dots, r \in \mathbb{N}; i, j^* \in \{1, \dots, N\}, i \neq j^*$, we have that

$$\begin{aligned} & \frac{1}{(\sqrt{\pi\zeta_n})^N} \int_{\mathbb{R}^N} |s_i| |s_{j^*}| e^{-\frac{\sum_{i=1}^N s_i^2}{\zeta_n}} ds_1 \dots ds_N, \\ & \frac{1}{(\sqrt{\pi\zeta_n})^N} \int_{\mathbb{R}^N} s_i^2 e^{-\frac{\sum_{i=1}^N s_i^2}{\zeta_n}} ds_1 \dots ds_N \leq I_{1j}^* < \infty, \end{aligned} \quad (34)$$

and

$$\begin{aligned} & \frac{1}{(\sqrt{\pi\zeta_n})^N} \int_{\mathbb{R}^N} |s_i| e^{-\frac{\sum_{i=1}^N s_i^2}{\zeta_n}} ds_1 \dots ds_N \leq \\ & \frac{1}{(\sqrt{\pi\zeta_n})^N} \int_{\mathbb{R}^N} s_i^2 e^{-\frac{\sum_{i=1}^N s_i^2}{\zeta_n}} ds_1 \dots ds_N < \infty, \end{aligned} \quad (35)$$

and all these integrals are uniformly bounded in $\zeta_n \in (0, 1]$. And all integrals converge to zero, as $\zeta_n \rightarrow 0$.

We continue with

Theorem 6. Let $N \geq 1, \zeta_n \in (0, 1], n \in \mathbb{N}$. Then

$$\begin{aligned} I_2^* &:= \frac{1}{(\sqrt{\pi\zeta_n})^N} \int_{\mathbb{R}^N} \frac{\|s\|_1}{\zeta_n} e^{-\frac{\sum_{i=1}^N s_i^2}{\zeta_n}} ds_1 \dots ds_N \leq \\ & \frac{1}{\sqrt{\zeta_n}} \left(\frac{2}{\sqrt{\pi}} \right)^N \left[N + \left(\frac{\lfloor e \rfloor}{e} \right)^N \right]. \end{aligned} \quad (36)$$

Proof. We observe that

$$\begin{aligned} I_2^* &= \frac{1}{(\sqrt{\pi\zeta_n})^N} \int_{\mathbb{R}^N} \frac{\|s\|_1}{\zeta_n} e^{-\frac{\sum_{i=1}^N s_i^2}{\zeta_n}} ds_1 \dots ds_N = \\ & \frac{2^N}{(\sqrt{\pi})^N (\sqrt{\zeta_n})^N} \int_{\mathbb{R}_+^N} \frac{\sum_{i=1}^N s_i}{\zeta_n} e^{-\frac{\sum_{i=1}^N s_i^2}{\zeta_n}} ds_1 \dots ds_N = \\ & \frac{2^N}{(\sqrt{\pi})^N} \frac{1}{\sqrt{\zeta_n}} \int_{\mathbb{R}_+^N} \left(\sum_{i=1}^N \left(\frac{s_i}{\sqrt{\zeta_n}} \right) \right) e^{-\sum_{i=1}^N \left(\frac{s_i}{\sqrt{\zeta_n}} \right)^2} d \frac{s_1}{\sqrt{\zeta_n}} \dots d \frac{s_N}{\sqrt{\zeta_n}} = \\ & \frac{2^N}{(\sqrt{\pi})^N} \frac{1}{\sqrt{\zeta_n}} \int_{\mathbb{R}_+^N} \left(\sum_{i=1}^N z_i \right) e^{-\sum_{i=1}^N z_i^2} dz_1 \dots dz_N = \\ & \left(\frac{2}{\sqrt{\pi}} \right)^N \frac{1}{\sqrt{\zeta_n}} \left[\int_{[0,1]^N} \left(\sum_{i=1}^N z_i \right) e^{-\sum_{i=1}^N z_i^2} dz_1 \dots dz_N + \right. \end{aligned} \quad (37)$$

$$\begin{aligned}
& \int_{(\mathbb{R}_+ - [0,1])^N} \left(\sum_{i=1}^N z_i \right) e^{-\sum_{i=1}^N z_i^2} dz_1 \dots dz_N \Bigg] \leq \\
& \left(\frac{2}{\sqrt{\pi}} \right)^N \frac{1}{\sqrt{\xi_n}} \left[N + \int_{(\mathbb{R}_+ - [0,1])^N} \left(\prod_{i=1}^N z_i \right) e^{-\sum_{i=1}^N z_i} dz_1 \dots dz_N \right] = \\
& \left(\frac{2}{\sqrt{\pi}} \right)^N \frac{1}{\sqrt{\xi_n}} \left[N + \prod_{i=1}^N \int_1^\infty z_i e^{-z_i} dz_i \right] = \\
& \left(\frac{2}{\sqrt{\pi}} \right)^N \frac{1}{\sqrt{\xi_n}} \left[N + \left(\int_1^\infty z^{2-1} e^{-z} dz \right)^N \right] = \tag{38}
\end{aligned}$$

(by [19], p. 348)

$$\begin{aligned}
& \left(\frac{2}{\sqrt{\pi}} \right)^N \frac{1}{\sqrt{\xi_n}} \left[N + \Gamma^N(2, 1) \right] = \\
& \left(\frac{2}{\sqrt{\pi}} \right)^N \frac{1}{\sqrt{\xi_n}} \left[N + \left(\frac{\lfloor e \rfloor}{e} \right)^N \right],
\end{aligned}$$

proving the claim. $\square$

We need the following.

Theorem 7 ([3], p. 403). Let $r, N, m \in \mathbb{N}$, with $m > r$; $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, N$: $|\alpha| := \sum_{i=1}^N \alpha_i = m$, $\xi_n \in (0, 1]$, $n \in \mathbb{N}$; $p > 1$. Then

$$\begin{aligned}
\tilde{A}_{\xi_n}(\alpha) &:= \frac{1}{(\sqrt{\pi \xi_n})^N} \int_{\mathbb{R}^N} \left(\left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right)^p e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 \dots ds_N \leq \tag{39} \\
& \sqrt{\xi_n}^{p(m-r)} \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^{rp} + 2^{rp} \Gamma^N((m+r)p+1, 1) \right] \leq \\
& \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1+N)^{rp} + 2^{rp} \Gamma^N((m+r)p+1, 1) \right] < +\infty,
\end{aligned}$$

are uniformly bounded, where $m > r$.

Also, $\tilde{A}_{\xi_n}(\alpha) \rightarrow 0$, as $\xi_n \rightarrow 0$, $n \rightarrow +\infty$.

Above, $\Gamma(\cdot, \cdot)$ is the upper incomplete gamma function.

Clearly, for $(m > r)$,

$$B_{\xi_n}(\alpha) := \frac{1}{(\sqrt{\pi \xi_n})^N} \int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 \dots ds_N < \infty, \tag{40}$$

uniformly bounded in $\xi_n \in (0, 1]$. And, clearly, $B_{\xi_n}(\alpha) \rightarrow 0$, as $\xi_n \rightarrow 0$, $n \rightarrow \infty$, by (39).

We need the following

Theorem 8. Let $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, N \in \mathbb{N} : |\alpha| := \sum_{i=1}^N \alpha_i = 2$, $\xi_n \in (0, 1]$, $n \in \mathbb{N}$, $p \geq 1$, and $j = 0, 1, \dots, r \in \mathbb{N}$. Then

$$\begin{aligned} \tilde{A}_{j\xi_n}(\alpha) &:= \frac{1}{(\sqrt{\pi\xi_n})^N} \int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{p\alpha_i} \right) \left(1 + \frac{j^p \|s\|_2^p}{\xi_n^p} \right) e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 \dots ds_N \leq \quad (41) \\ &\left(\sqrt{\xi_n} \right)^p \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1 + j^p N^p) + (1 + j^p) \Gamma^N(3p + 1, 1) \right] \leq \\ &\left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1 + j^p N^p) + (1 + j^p) \Gamma^N(3p + 1, 1) \right] < +\infty, \end{aligned}$$

are uniformly bounded; furthermore, $\tilde{A}_{j\xi_n}(\alpha) \rightarrow 0$, as $\xi_n \rightarrow 0$, $n \rightarrow \infty$.

Let $\tilde{A}_{j\xi_n}(\alpha)$ be denoted as $\tilde{\tilde{A}}_{j\xi_n}(\alpha)$ when $p = 1$.

Proof. We estimate ($p \geq 1$)

$$\begin{aligned} \tilde{A}_{j\xi_n}(\alpha) &= \frac{1}{(\sqrt{\pi\xi_n})^N} \int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{p\alpha_i} \right) \left(1 + \frac{j^p \|s\|_2^p}{\xi_n^p} \right) e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 \dots ds_N = \\ &\frac{2^N}{(\sqrt{\pi})^N (\sqrt{\xi_n})^N} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^N s_i^{p\alpha_i} \right) \left(1 + \frac{j^p \|s\|_2^p}{\xi_n^p} \right) e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 \dots ds_N \leq \quad (42) \end{aligned}$$

$$\begin{aligned} &\frac{2^N (\sqrt{\xi_n})^{2p}}{(\sqrt{\pi})^N} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^N \left(\frac{s_i}{\sqrt{\xi_n}} \right)^{p\alpha_i} \right) \\ &\left(\frac{1}{(\sqrt{\xi_n})^p} + \left(\frac{j}{\sqrt{\xi_n}} \right)^p \left(\sum_{i=1}^N \frac{s_i}{\sqrt{\xi_n}} \right)^p \right) e^{-\sum_{i=1}^N \left(\frac{s_i}{\sqrt{\xi_n}} \right)^2} d \frac{s_1}{\sqrt{\xi_n}} \dots d \frac{s_N}{\sqrt{\xi_n}} = \\ &\left(\frac{2}{\sqrt{\pi}} \right)^N (\sqrt{\xi_n})^p \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^N z_i^{p\alpha_i} \right) \left(1 + j^p \left(\sum_{i=1}^N z_i \right)^p \right) e^{-\sum_{i=1}^N z_i^2} dz_1 \dots dz_N = \\ &(\sqrt{\xi_n})^p \left(\frac{2}{\sqrt{\pi}} \right)^N \left[\int_{[0,1]^N} \left(\prod_{i=1}^N z_i^{p\alpha_i} \right) \left(1 + j^p \left(\sum_{i=1}^N z_i \right)^p \right) e^{-\sum_{i=1}^N z_i^2} dz_1 \dots dz_N + \quad (43) \right. \\ &\left. \int_{(\mathbb{R}_+ - [0,1])^N} \left(\prod_{i=1}^N z_i^{p\alpha_i} \right) \left(1 + j^p \left(\sum_{i=1}^N z_i \right)^p \right) e^{-\sum_{i=1}^N z_i^2} dz_1 \dots dz_N \right] \leq \end{aligned}$$

$$\begin{aligned} &(\sqrt{\xi_n})^p \left(\frac{2}{\sqrt{\pi}} \right)^N \\ &\left[(1 + j^p N^p) + (1 + j^p) \int_{(\mathbb{R}_+ - [0,1])^N} \left(\prod_{i=1}^N z_i^{p\alpha_i} \right) \left(\sum_{i=1}^N z_i \right)^p e^{-\sum_{i=1}^N z_i^2} dz_1 \dots dz_N \right] \leq \\ &(\sqrt{\xi_n})^p \left(\frac{2}{\sqrt{\pi}} \right)^N \\ &\left[(1 + j^p N^p) + (1 + j^p) \int_{(\mathbb{R}_+ - [0,1])^N} \left(\prod_{i=1}^N z_i^{p\alpha_i} \right) \left(\prod_{i=1}^N z_i^p \right) \left(\prod_{i=1}^N e^{-z_i} \right) \prod_{i=1}^N dz_i \right] = \end{aligned}$$

$$\begin{aligned}
& \left(\sqrt{\xi_n} \right)^p \left(\frac{2}{\sqrt{\pi}} \right)^N \\
& \left[(1 + j^p N^p) + (1 + j^p) \int_{(\mathbb{R}_+ - [0,1])^N} \left(\prod_{i=1}^N z_i^{p(\alpha_i+1)} \right) \left(\prod_{i=1}^N e^{-z_i} \right) \prod_{i=1}^N dz_i \right] = \\
& \left(\sqrt{\xi_n} \right)^p \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1 + j^p N^p) + (1 + j^p) \prod_{i=1}^N \left(\int_1^\infty z_i^{p(\alpha_i+1)} e^{-z_i} dz_i \right) \right] = \quad (44)
\end{aligned}$$

(by [19], p. 348)

$$\left(\sqrt{\xi_n} \right)^p \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1 + j^p N^p) + (1 + j^p) \prod_{i=1}^N \Gamma((p(\alpha_i + 1) + 1), 1) \right] \leq$$

(by [19], p. 909)

$$\left(\sqrt{\xi_n} \right)^p \left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1 + j^p N^p) + (1 + j^p) \Gamma^N(3p + 1, 1) \right] \leq \quad (45)$$

$$\left(\frac{2}{\sqrt{\pi}} \right)^N \left[(1 + j^p N^p) + (1 + j^p) \Gamma^N(3p + 1, 1) \right] < +\infty,$$

are uniformly bounded. $\square$

We need the following.

Theorem 9. Let $p \geq 1$, $\xi_n \in (0, 1]$, $n \in \mathbb{N}$, $N \geq 1$. Then

$$K_{p\xi_n} := \frac{1}{(\sqrt{\pi\xi_n})^N} \int_{\mathbb{R}^N} \left(\frac{\|s\|_2}{\xi_n} \right)^p e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 \dots ds_N \leq \quad (46)$$

$$\frac{1}{(\sqrt{\xi_n})^p} \left(\frac{2}{\sqrt{\pi}} \right)^N \left[N^p + \Gamma^N(p + 1, 1) \right].$$

We set $K_{p\xi_n}$ as $K_{\xi_n}^*$ when $p = 1$.

Proof. We have ($p \geq 1$)

$$\begin{aligned}
K_{p\xi_n} &= \frac{1}{(\sqrt{\pi\xi_n})^N} \int_{\mathbb{R}^N} \left(\frac{\|s\|_2}{\xi_n} \right)^p e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 \dots ds_N = \\
& \frac{2^N}{(\sqrt{\pi})^N (\sqrt{\xi_n})^N} \int_{\mathbb{R}_+^N} \left(\frac{\|s\|_2}{\xi_n} \right)^p e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 \dots ds_N \leq \\
& \frac{2^N}{(\sqrt{\pi})^N (\sqrt{\xi_n})^N} \int_{\mathbb{R}_+^N} \left(\frac{\sum_{i=1}^N s_i}{\xi_n} \right)^p e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 \dots ds_N = \quad (47) \\
& \frac{2^N}{(\sqrt{\pi})^N} \frac{1}{(\sqrt{\xi_n})^p} \int_{\mathbb{R}_+^N} \left(\sum_{i=1}^N \left(\frac{s_i}{\sqrt{\xi_n}} \right) \right)^p e^{-\sum_{i=1}^N \left(\frac{s_i}{\sqrt{\xi_n}} \right)^2} d \frac{s_1}{\sqrt{\xi_n}} \dots d \frac{s_N}{\sqrt{\xi_n}} =
\end{aligned}$$

$$\begin{aligned}
& \frac{2^N}{(\sqrt{\pi})^N} (\sqrt{\xi_n})^{-p} \int_{\mathbb{R}_+^N} \left(\sum_{i=1}^N z_i \right)^p e^{-\sum_{i=1}^N z_i^2} dz_1 \dots dz_N = \\
& \frac{2^N}{(\sqrt{\pi})^N} (\sqrt{\xi_n})^{-p} \left[\int_{[0,1]^N} \left(\sum_{i=1}^N z_i \right)^p e^{-\sum_{i=1}^N z_i^2} dz_1 \dots dz_N + \right. \\
& \quad \left. \int_{(\mathbb{R}_+ - [0,1])^N} \left(\sum_{i=1}^N z_i \right)^p e^{-\sum_{i=1}^N z_i^2} dz_1 \dots dz_N \right] \leq \\
& \frac{2^N}{(\sqrt{\pi})^N} (\sqrt{\xi_n})^{-p} \left[N^p + \int_{(\mathbb{R}_+ - [0,1])^N} \left(\prod_{i=1}^N z_i^p \right) e^{-\sum_{i=1}^N z_i} dz_1 \dots dz_N \right] = \\
& \frac{2^N}{(\sqrt{\pi})^N} (\sqrt{\xi_n})^{-p} \left[N^p + \prod_{i=1}^N \int_1^\infty z_i^p e^{-z_i} dz_i \right] = \\
& \frac{2^N}{(\sqrt{\pi})^N} (\sqrt{\xi_n})^{-p} \left[N^p + \left(\int_1^\infty z^p e^{-z} dz \right)^N \right] =
\end{aligned} \tag{48}$$

(by [19], p. 348)

$$\frac{2^N}{(\sqrt{\pi})^N} (\sqrt{\xi_n})^{-p} \left[N^p + \Gamma^N(p+1, 1) \right].$$

□

4. Main Results

The general smooth multivariate Gauss–Weierstrass singular integral operators are defined as:

$$\begin{aligned}
W_n(f; x_1, \dots, x_N) &:= W_{r,n}^{[m]}(f; x_1, \dots, x_N) := \\
& \frac{1}{(\sqrt{\pi \xi_n})^N} \sum_{j=0}^r \alpha_{j,r}^{[m]} \int_{\mathbb{R}^N} f(x_1 + s_1 j, x_2 + s_2 j, \dots, x_N + s_N j) e^{-\sum_{i=1}^N \frac{s_i^2}{\xi_n}} ds_1 \dots ds_N.
\end{aligned} \tag{49}$$

Observe that

$$\frac{1}{(\sqrt{\pi \xi_n})^N} \int_{\mathbb{R}^N} e^{-\sum_{i=1}^N \frac{s_i^2}{\xi_n}} ds_1 \dots ds_N = 1, \tag{50}$$

see [2], p. 15.

That is, $W_{r,n}^{[m]}$ are the $\theta_{r,n}^{[m]}$ operators applied for the Borel probability measures on $\mathbb{R}^N$, $N \geq 1$,

$$d\mu_{\xi_n}(s) = \frac{1}{(\sqrt{\pi \xi_n})^N} e^{-\sum_{i=1}^N \frac{s_i^2}{\xi_n}} ds_1 \dots ds_N, \quad s \in \mathbb{R}^N, \tag{51}$$

where $0 < \xi_n \leq 1$, $n \in \mathbb{N}$.

We will apply to $W_{r,n}^{[m]}$ the Theorems 1–4, with the help of all of Section 3. This section presents an approximation of properties of W_n .

Here, first apply Theorem 1 to the W_n operators.

Theorem 10. We consider $f \in C^2(\mathbb{R}^N)$ and let all $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, N$, $N \geq 1$, $|\alpha| := \sum_{i=1}^N \alpha_i = 2$; $x \in \mathbb{R}^N$, and all the partials f_α of order 2, along with $f \in C_B(\mathbb{R}^N)$; or all f_α of order 2, $f \in C_U(\mathbb{R}^N)$; $0 < \xi_n \leq 1$, $n \in \mathbb{N}$.

Denote ($n \in \mathbb{N}$)

$$\begin{aligned} \Delta_n^*(x) &:= W_n(f, x) - f(x) - \left(\sum_{j=0}^r \alpha_j j \right) \sin(1) \\ &\quad \left[\sum_{i=1}^N \frac{\partial f(x)}{\partial x_i} \frac{1}{(\sqrt{\pi \xi_n})^N} \left(\int_{\mathbb{R}^N} s_i e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 \dots ds_N \right) \right] \\ &\quad - 2 \left(\sum_{j=0}^r \alpha_j j^2 \right) \sin^2 \left(\frac{1}{2} \right) \left\{ \sum_{i=1}^N \frac{1}{(\sqrt{\pi \xi_n})^N} \left(\int_{\mathbb{R}^N} s_i^2 e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 \dots ds_N \right) \frac{\partial^2 f(x)}{\partial x_i^2} \right. \\ &\quad \left. + \sum_{\substack{i \neq j^*, \\ i, j^* \in \{1, \dots, N\}}} \frac{1}{(\sqrt{\pi \xi_n})^N} \left(\int_{\mathbb{R}^N} s_i s_{j^*} e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 \dots ds_N \right) \frac{\partial^2 f(x)}{\partial x_i \partial x_{j^*}} \right\}. \end{aligned} \quad (52)$$

Then

(i)

$$\begin{aligned} |\Delta_n^*(x)| &\leq \|\Delta_n^*\|_\infty \leq \sum_{j=0}^r |\alpha_j| \\ &\quad \left[\left[j^2 \sum_{\substack{\alpha_i \in \mathbb{Z}^+, \\ \alpha: |\alpha|=2}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \omega_1(f_\alpha, \xi_n) I_{1j}^*(\alpha) \right] + \right. \\ &\quad \left. \frac{1}{2} \omega_1(f, \xi_n) \left[1 + \frac{j}{3} I_2^* \right] \right] =: \varphi_{\xi_n}^*, \end{aligned} \quad (53)$$

where $I_{1j}^*(\alpha)$ is as in (29), and I_2^* is as in (36).

In the case of all f_α of order 2 and $f \in C_U(\mathbb{R}^N)$ and $\xi_n \rightarrow 0$, as $n \rightarrow \infty$, then $\Delta_n^*(x)$, $\|\Delta_n^*\|_\infty \rightarrow 0$ with rates.

(ii) If $\frac{\partial f(x)}{\partial x_i} = 0$, $i = 1, \dots, N$, and $f_\alpha(x) = 0$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, N$, with $|\alpha| = 2$, then

$$|W_n(f, x) - f(x)| \leq \varphi_{\xi_n}^*. \quad (54)$$

And $W_n(f, x) \rightarrow f(x)$ in the uniformly continuous case.

(iii) Additionally, assume all partials of order ≤ 2 are bounded. Hence,

$$\begin{aligned} \|W_n(f) - f\|_\infty &\leq \left(\sum_{j=0}^r |\alpha_j| j \right) (0.8414) \\ &\quad \left[\sum_{i=1}^N \left\| \frac{\partial f}{\partial x_i} \right\|_\infty \frac{1}{(\sqrt{\pi \xi_n})^N} \left(\int_{\mathbb{R}^N} |s_i| e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 \dots ds_N \right) \right] + \\ &\quad \left(\sum_{j=0}^r |\alpha_j| j^2 \right) (0.4596) \left\{ \sum_{i=1}^N \frac{1}{(\sqrt{\pi \xi_n})^N} \left(\int_{\mathbb{R}^N} s_i^2 e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 \dots ds_N \right) \left\| \frac{\partial^2 f}{\partial x_i^2} \right\|_\infty + \right. \end{aligned}$$

$$\sum_{\substack{i \neq j^*, \\ i, j^* \in \{1, \dots, N\}}} \frac{1}{(\sqrt{\pi \xi_n})^N} \left(\int_{\mathbb{R}^N} |s_i| |s_{j^*}| e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 \dots ds_N \right) \left\| \frac{\partial^2 f}{\partial x_i \partial x_{j^*}} \right\|_{\infty} \Bigg\} + \varphi_{\xi_n}^*. \quad (55)$$

If all f_{α} of order 2, $f \in C_U(\mathbb{R}^N)$, then

$$\|W_n(f) - f\|_{\infty} \rightarrow 0 \text{ with rates, as } \xi_n \rightarrow 0, n \rightarrow +\infty.$$

Proof. By Theorems 1, 5, 6 and Remark 3. $\square$

Next, we apply Theorem 2 to W_n operators.

Theorem 11. Let $f \in C^m(\mathbb{R}^N)$, $m \in \mathbb{N}$, $m > r$, $N \geq 1$, with $f_{\alpha} \in L_p(\mathbb{R}^N)$, $|\alpha| = m$, $x \in \mathbb{R}^N$. Let $p, q > 1$: $\frac{1}{p} + \frac{1}{q} = 1$; $0 < \xi_n \leq 1$, $n \in \mathbb{N}$. For $\tilde{j} = 1, \dots, m$, and $\alpha := (\alpha_1, \dots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, N$, $|\alpha| := \sum_{i=1}^N \alpha_i = \tilde{j}$, call

$$c_{\alpha, n, \tilde{j}}^* := \frac{1}{(\sqrt{\pi \xi_n})^N} \int_{\mathbb{R}^N} \left(\prod_{i=1}^N s_i^{\alpha_i} \right) e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 \dots ds_N. \quad (56)$$

Then

$$\begin{aligned} \|E_{r, n}^{*[m]}\|_p &:= \left\| W_{r, n}(f; x) - f(x) - \sum_{\tilde{j}=1}^m \delta_{\tilde{j}, r}^{[m]} \left(\sum_{|\alpha|=\tilde{j}} \frac{c_{\alpha, n, \tilde{j}}^* f_{\alpha}(x)}{\left(\prod_{i=1}^N \alpha_i! \right)} \right) \right\|_{p, x} \leq \\ &\left[\left(\frac{N^m}{(m-1)!} \right)^{\frac{p}{q}} \left(\frac{m}{(q(m-1)+1)^{\frac{p}{q}}} \right) \left(\sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^N \alpha_i!} \tilde{A}_{\xi_n}(\alpha) \omega_r(f_{\alpha}, \xi_n)_p^p \right) \right]^{\frac{1}{p}}, \end{aligned} \quad (57)$$

where $\tilde{A}_{\xi_n}(\alpha)$ as in (39).

As $n \rightarrow \infty$ and $\xi_n \rightarrow 0$, by (57), we obtain that $\|E_{r, n}^{*[m]}\|_p \rightarrow 0$ with rates.

One also obtains by (57) that

$$\begin{aligned} \|W_n(f; x) - f(x)\|_{p, x} &\leq \\ &\sum_{\tilde{j}=1}^m \left| \delta_{\tilde{j}, r}^{[m]} \right| \left(\sum_{|\alpha|=\tilde{j}} \frac{|c_{\alpha, n, \tilde{j}}^*|}{\prod_{i=1}^N \alpha_i!} \|f_{\alpha}\|_p \right) + R.H.S. (57), \end{aligned} \quad (58)$$

given that $\|f_{\alpha}\|_p < \infty$, $|\alpha| = \tilde{j}$, $\tilde{j} = 1, \dots, m$.

Finally, we get $\|W_n(f) - f\|_p \rightarrow 0$ as $\xi_n \rightarrow 0$, and $n \rightarrow \infty$. That is, $W_n \rightarrow I$ the unit operator, in L_p norm, with rates.

Proof. By Theorems 2, 7 and (40). $\square$

Next, we apply Theorem 3 to W_n operators.

Theorem 12. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1, 0 < \xi_n \leq 1, n \in \mathbb{N}$. Here, we deal with $f \in C^2(\mathbb{R}^N)$, $N \geq 1$, with $f, f_\alpha \in L_p(\mathbb{R}^N)$, $|\alpha| = 2$, where $\alpha_i \in \mathbb{Z}^+, i = 1, \dots, N$, and $|\alpha| = \sum_{i=1}^N \alpha_i; x \in \mathbb{R}^N; j = 0, 1, \dots, r$.

Denote $(n \in \mathbb{N})$

$$\begin{aligned} \bar{\Delta}_n(x) &:= W_n(f, x) - f(x) - \left(\sum_{j=0}^r \alpha_j j \right) \sin(1) \\ &\quad \left[\sum_{i=1}^N \frac{\partial f(x)}{\partial x_i} \frac{1}{(\sqrt{\pi \xi_n})^N} \left(\int_{\mathbb{R}^N} s_i e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 \dots ds_N \right) \right] \\ &\quad - 2 \left(\sum_{j=0}^r \alpha_j j^2 \right) \sin^2 \left(\frac{1}{2} \right) \left\{ \sum_{i=1}^N \frac{1}{(\sqrt{\pi \xi_n})^N} \left(\int_{\mathbb{R}^N} s_i^2 e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 \dots ds_N \right) \frac{\partial^2 f(x)}{\partial x_i^2} + \right. \\ &\quad \left. \sum_{\substack{i \neq j^*, \\ i, j^* \in \{1, \dots, N\}}} \frac{1}{(\sqrt{\pi \xi_n})^N} \left(\int_{\mathbb{R}^N} s_i s_{j^*} e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 \dots ds_N \right) \frac{\partial^2 f(x)}{\partial x_i \partial x_{j^*}} \right\}. \end{aligned} \quad (59)$$

Then

$$\begin{aligned} \|\bar{\Delta}_n\|_p &\leq \left(\sum_{j=0}^r |\alpha_j|^q \right)^{\frac{1}{q}} \left\{ \frac{2}{(q+1)^{\frac{1}{q}}} \left(\frac{N(N+1)+2}{2} \right)^{\frac{1}{q}} \right\} \\ &\quad \left\{ \sum_{j=0}^r \left[j^{2p} \sum_{|\alpha|=2} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right)^p \omega_1(f_\alpha, \xi_n)_p \tilde{A}_{j\xi_n}(\alpha) + \omega_1(f, \xi_n)_p [1 + j^p K_{p\xi_n}] \right] \right\}^{\frac{1}{p}}, \end{aligned} \quad (60)$$

where $\tilde{A}_{j\xi_n}(\alpha)$ as in (41), and $K_{p\xi_n}$ as in (46).

One also obtains from (60) that

$$\begin{aligned} \|W_n(f) - f\|_p &\leq \\ &\quad \left(\sum_{j=0}^r |\alpha_j| j \right) \sin(1) \left[\sum_{i=1}^N \left\| \frac{\partial f}{\partial x_i} \right\|_p \frac{1}{(\sqrt{\pi \xi_n})^N} \left(\int_{\mathbb{R}^N} |s_i| e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 \dots ds_N \right) \right] \\ &\quad + 2 \left(\sum_{j=0}^r |\alpha_j| j^2 \right) \sin^2 \left(\frac{1}{2} \right) \left\{ \sum_{i=1}^N \frac{1}{(\sqrt{\pi \xi_n})^N} \left(\int_{\mathbb{R}^N} s_i^2 e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 \dots ds_N \right) \left\| \frac{\partial^2 f}{\partial x_i^2} \right\|_p + \right. \\ &\quad \left. \sum_{\substack{i \neq j^*, \\ i, j^* \in \{1, \dots, N\}}} \frac{1}{(\sqrt{\pi \xi_n})^N} \left(\int_{\mathbb{R}^N} |s_i s_{j^*}| e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 \dots ds_N \right) \left\| \frac{\partial^2 f}{\partial x_i \partial x_{j^*}} \right\|_p \right\} + R.H.S. (60), \end{aligned} \quad (61)$$

given that $\|f_\alpha\|_p < \infty, |\alpha| = \tilde{j}, \tilde{j} = 1, 2$.

Additionally, assuming that $\omega_1(f, \xi_n)_p \leq \lambda \xi_n, \lambda > 0$, then as $n \rightarrow \infty$ and $\xi_n \rightarrow 0$, by (60), we obtain that $\|\bar{\Delta}_n\|_p \rightarrow 0$ with rates, and furthermore by (61), we derive that $\|W_n(f) - f\|_p \rightarrow 0$; that is, $W_n \rightarrow I$ the unit operator, in L_p norm, with rates.

Proof. By Theorems 3, 8, 9 and Remark 3. $\square$

We finish with an application of Theorem 4 to W_n operators.

Theorem 13. Let $0 < \xi_n \leq 1$, $n \in \mathbb{N}$, $x \in \mathbb{R}^N$. Here, we deal with $f \in C^2(\mathbb{R}^N)$, $N \geq 1$, with f , $f_\alpha \in L_1(\mathbb{R}^N)$, $|\alpha| = 2$, where $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, N$, and $|\alpha| = \sum_{i=1}^N \alpha_i$; $j = 0, 1, \dots, r$. Here $\bar{\Delta}_n$ as in (59). Then

$$\|\bar{\Delta}_n\|_1 \leq \left\{ \sum_{j=0}^r |\alpha_j| \left[j^2 \sum_{|\alpha|=2} \left(\frac{2}{\prod_{i=1}^N \alpha_i!} \right) \omega_1(f_\alpha, \xi_n)_1 \tilde{A}_{j\xi_n}(\alpha) + \omega_1(f, \xi_n)_1 [1 + jK_{\xi_n}^*] \right] \right\}, \quad (62)$$

where $\tilde{A}_{j\xi_n}(\alpha)$ as in Theorem 8, and $K_{\xi_n}^*$ as in Theorem 9.
We also derive that

$$\begin{aligned} \|W_n(f) - f\|_1 &\leq \left(\sum_{j=0}^r |\alpha_j| j \right) \sin(1) \left[\sum_{i=1}^N \left\| \frac{\partial f}{\partial x_i} \right\|_1 \frac{1}{(\sqrt{\pi\xi_n})^N} \left(\int_{\mathbb{R}^N} |s_i| e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 \dots ds_N \right) \right] \\ &+ 2 \left(\sum_{j=0}^r |\alpha_j| j^2 \right) \sin^2\left(\frac{1}{2}\right) \left\{ \sum_{i=1}^N \frac{1}{(\sqrt{\pi\xi_n})^N} \left(\int_{\mathbb{R}^N} s_i^2 e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 \dots ds_N \right) \left\| \frac{\partial^2 f}{\partial x_i^2} \right\|_1 + \right. \\ &\left. \sum_{\substack{i \neq j^*, \\ i, j^* \in \{1, \dots, N\}}} \frac{1}{(\sqrt{\pi\xi_n})^N} \left(\int_{\mathbb{R}^N} |s_i s_{j^*}| e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 \dots ds_N \right) \left\| \frac{\partial^2 f}{\partial x_i \partial x_{j^*}} \right\|_1 \right\} + R.H.S. (62), \end{aligned} \quad (63)$$

given that $\|f_\alpha\|_1 < \infty$, $|\alpha| = \tilde{j}$, $\tilde{j} = 1, 2$. Additionally, assuming that $\omega_1(f, \xi_n)_1 \leq \lambda^* \xi_n$, $\lambda^* > 0$, then as $n \rightarrow \infty$ and $\xi_n \rightarrow 0$, by (62), we obtain that $\|\bar{\Delta}_n\|_1 \rightarrow 0$ with rates. Furthermore, by (63), we derive that $\|W_n(f) - f\|_1 \rightarrow 0$; that is, $W_n \rightarrow I$ the unit operator, in L_1 norm, with rates.

Proof. By Theorems 4, 8, 9 and Remark 3. $\square$

5. Conclusions

A new type of approximation was introduced for singular integrals. This approximation is an interesting trigonometric-based one.

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