

## Article

# Global Non-Existence of a Coupled Parabolic–Hyperbolic System of Thermoelastic Type with History

Jorge A. Esquivel-Avila

Departamento de Ciencias Básicas, Análisis Matemático y sus Aplicaciones, UAM-Azcapotzalco, Av. San Pablo 420, Col. Nueva el Rosario, Azcapotzalco 02128, Mexico; jaea@azc.uam.mx

**Abstract:** We consider two abstract systems of parabolic–hyperbolic type that model thermoelastic problems. We study the influence of the physical constants and the initial data on the nonexistence of global solutions that, in our framework, are produced by the blow-up in finite time of the norm of the solution in the phase space. We employ a differential inequality to find sufficient conditions that produce the blow-up. To that end, we construct a set that is positive-invariant for any positive value of the initial energy. As a result, we found that the coupling with the parabolic equation stabilizes the system, as well as the damping term in the hyperbolic equation. Moreover, for any pair of positive values  $(\zeta, \epsilon)$ , there exist initial data, such that the corresponding solution with initial energy  $\zeta$  blows up at a finite time less than  $\epsilon$ . Our purpose is to improve results previously published in the literature.

**Keywords:** parabolic–hyperbolic system; global non-existence; blow-up; thermoelasticity

**MSC:** 35B44; 35M30; 74F05



**Citation:** Esquivel-Avila, J.A. Global Non-Existence of a Coupled Parabolic–Hyperbolic System of Thermoelastic Type with History. *Mathematics* **2024**, *12*, 131. <https://doi.org/10.3390/math12010131>

Academic Editor: Dongfang Li

Received: 30 November 2023

Revised: 21 December 2023

Accepted: 28 December 2023

Published: 30 December 2023



**Copyright:** © 2023 by the author. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

## 1. Introduction

We analyze two abstract problems that cover some mathematical models of thermoelastic type reported in the literature. For a detailed explanation of the physics behind those models, the interested reader should consult the bibliography at the end of the article. The purpose here is to prove that under some conditions on the initial data, the corresponding solutions of the abstract systems exist only up to a finite time due to the blow-up of the solutions in the norm of the phase space. We will illustrate our main theorem with some concrete thermoelastic systems and show that the results published in previous articles are improved.

Firstly, we present two thermoelastic problems that have been analyzed by several authors in previous publications and give the corresponding abstract formulation within a functional framework where our analysis is carried out. In particular, we define what we mean by ‘solution’. Next, we present our main results and illustrate our analysis with some thermoelastic systems. We compare our conclusions with the results already published. Finally, we present some extensions.

Let us first introduce two parabolic–hyperbolic systems that will motivate our abstract formulations. We begin with a Cauchy one-dimensional thermoelastic problem with short memory.

$$\text{(The)}_1 \left\{ \begin{array}{l} \text{Given initial data } (u_0(x), u_1(x), \theta_0(x)) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \\ \text{find } u(x, t) \in \mathbb{R}, \theta(x, t) \in \mathbb{R}, \text{ such that} \\ \\ u_{tt} - au_{xx} - bu_x + \eta\theta_x + \delta u_t = f(t, u), & x \in (0, L), \quad t > 0, \\ c\theta_t - \kappa\theta_{xx} + \eta u_{xt} - h * \theta_{xx} - pu_x - q\theta_x = g(\theta), & x \in (0, L), \quad t > 0, \\ u = u_0, \quad u_t = u_1, \quad \theta = \theta_0, & x \in (0, L), \quad t = 0, \\ u = \partial_\nu u = 0, \quad \theta = 0, & x = 0, L, \quad t > 0, \end{array} \right.$$

where  $u(x, t)$  is the vertical or transversal displacement of a one-dimensional rod of length  $L$  at the point  $x \in [0, L]$  and time  $t \geq 0$ , and  $\theta(x, t)$  is the temperature,  $a, c, \eta, \kappa, \delta$  are positive numbers,  $b, p, q$  are nonnegative constants,  $(h * l)(t) = \int_0^t h(t-s)l(s) ds$ , is the convolution and  $h$  is a relaxation function, see [1–5] for details of this model and related, using either the Fourier's law of heat flux or the theory due to Gurtin–Pipkin. In those references,  $(\text{ThE})_1$  is analyzed without source term in the parabolic equation  $g(\theta) = 0$ . Here, we shall study this problem with  $q = p = b = 0, c = 1$  and an autonomous source term  $f$ . The assumptions on the initial data, constants, the relaxation function, and the source terms will be given later.

The second problem models the dynamics of an extensible plate equation with long thermal memory.

$$(\text{ThE})_2 \begin{cases} \text{Given initial data } (u_0, u_1, \theta_0(s)) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}, s \leq 0, \\ \text{find } u(x, t) \in \mathbb{R}^n, \theta(x, t) \in \mathbb{R}, \text{ such that} \\ u_{tt} + \Delta^2 u - \phi(\|\nabla u\|_2^2) \Delta u(t) + \eta \Delta \theta + \delta u_t = f(u), & x \in \Omega, t > 0, \\ \theta_t - \omega \Delta \theta - (1 - \omega) \int_0^\infty k(t-s) \Delta \theta(s) ds - \eta \Delta u_t = g(\theta), & x \in \Omega, t > 0, \\ u = u_0, u_t = u_1, & x \in \Omega, t = 0, \\ \theta = \theta_0, & x \in \Omega, t \leq 0, \\ u = \partial_\nu u = 0, \theta = 0, & x \in \partial\Omega, t > 0, \end{cases}$$

where  $\Omega \subset \mathbb{R}^2$  is a bounded domain, with a smooth boundary  $\partial\Omega$  and corresponding normal vector  $\nu$ ,  $k$  is the long memory relaxation kernel,  $\eta, \delta$  are as before, and

$$\phi(\|\nabla u\|_2^2) = 1 + \|\nabla u\|_2^{2p}, \quad p \geq 1.$$

See [6–8] for the physics of the model. These sources point out that this problem considers a heat flux theory due to the Coleman–Gurtin with parameter  $\omega \in (0, 1)$ . This theory has the following as limit cases: Fourier's law when  $\omega = 1$  and the theory due to Gurtin–Pipkin if  $\omega = 0$ . In [7,8], a rotational term is considered. As in the last problem, the assumptions on the initial data, constants, the relaxation kernel, and the source terms will be given later.

In the systems introduced above, we have two types of memory terms, a short one and a long or infinite memory. We also observe that the coupling terms in these systems are of two kinds. In  $(\text{ThE})_2$ , the corresponding operator has opposite signs in the equations of the system, and is symmetric and positive. In  $(\text{ThE})_1$ , the coupling operator is antisymmetric and the sign in both equations is the same. In order to handle these examples and some more, we shall work with two abstract systems. The first is a system with short memory and an antisymmetric coupling operator. The second is a system with long memory, a coupling positive coefficient, and a symmetric operator. Although more combinations of memory and coupling terms can be studied, their analysis can be performed in a similar way.

Let us now consider the following semilinear problems associated with abstract parabolic–hyperbolic systems with short and long memory terms, respectively. The first one models problems with short memory.

For every set of initial data  $u_0, u_1, v_0$ , the goal is to find functions  $t \mapsto (u(t), v(t))$ ,  $t \geq 0$ , such that the following system holds for every  $t > 0$ ,

$$(\text{P})_1 \begin{cases} Pu_{tt}(t) + A_1 u(t) - \eta Bv(t) + \delta Pu_t(t) = f(u(t)), \\ Pv_t(t) + A_2 v(t) + \int_0^t h(t-\tau) A_2 v(\tau) d\tau - \eta Bu_t(t) = g(v(t)), \\ u(0) = u_0, u_t(0) = u_1, v(0) = v_0. \end{cases}$$

In the second problem, we consider infinite memory as follows.

For every set of initial data  $u_0, u_1, v_0(\cdot)$ , the goal is to find functions  $t \mapsto (u(t), v(t))$ ,  $t \geq 0$ , such that the following system holds, for every  $t > 0$ ,

$$(\mathbf{P})_2 \begin{cases} Pu_{tt}(t) + A_1 u(t) - \eta Bv(t) + \delta Pu_t(t) = f(u(t)), \\ Pv_t(t) + \omega A_2 v(t) + (1 - \omega) \int_0^\infty k(s) A_2 v(t - s) ds + \eta Bu_t(t) = g(v(t)), \\ u(0) = u_0, u_t(0) = u_1, v(-t) = v_0(-t), t \geq 0. \end{cases}$$

Here,  $\eta > 0, \delta > 0$  and  $0 < \omega < 1$  are constants. Functions  $h(t)$ ,  $t \geq 0$  and  $k(t)$ ,  $t \in \mathbb{R}$  are short and long memory relaxation kernels, respectively. Functions  $f(u)$  and  $g(v)$  are nonlinear source terms. The following operators, defined on Banach spaces, are linear and continuous:

$$P : V_P \rightarrow V'_P, A_j : V_{A_j} \rightarrow V'_{A_j}, j = 1, 2, B : V_B \rightarrow V'_B.$$

We assume that

$$V_{A_1} \subset V_{A_2} \subset V_B \subset V_P \subset H,$$

are linear subspaces of a Hilbert space  $H$ , and the inner product  $(\cdot, \cdot)$ , norm  $\|\cdot\|$ , and  $H', V'_P, V'_{A_j}, j = 1, 2, V'_B$  are the corresponding dual spaces. We identify  $H = H'$ , then

$$H \subset V'_P \subset V'_B \subset V'_{A_2} \subset V'_{A_1}.$$

In terms of the corresponding duality pairs, we have the following bilinear forms:

$$\begin{aligned} \mathcal{P}(u, w) &\equiv (Pu, w)_{V'_P \times V_P}, u, w \in V_P, \mathcal{B}(u, w) \equiv (Bu, w)_{V'_B \times V_B}, u, w \in V_B, \\ \mathcal{A}_j(u, w) &\equiv (A_j u, w)_{V'_{A_j} \times V_{A_j}}, u, w \in V_{A_j}, j = 1, 2. \end{aligned}$$

We assume that  $P$  and  $A_j$ ,  $j = 1, 2$  are positive and symmetric; then we have the corresponding norms for  $V_P, V_{A_j}, j = 1, 2$ ,

$$\|u\|_{V_P}^2 \equiv \mathcal{P}(u, u), u \in V_P, \|u\|_{V_{A_j}}^2 \equiv \mathcal{A}_j(u, u), u \in V_{A_j}, j = 1, 2,$$

The following hypotheses are assumed to hold throughout the paper.

(i) There are constants  $c > 0, \tilde{c} > 0$ , such that

$$(H0) \quad \|u\|_{V_{A_1}}^2 \geq c \|u\|_{V_P}^2, u \in V_{A_1}, |\mathcal{B}(u, v)| \leq \tilde{c} \|u\|_{V_{A_1}} \|v\|_{V_P}, u \in V_{A_1}, v \in V_B.$$

For the problem  $(\mathbf{P})_1$ , we assume that the operator  $B$  is antisymmetric

$$(H0)_1 \quad \mathcal{B}(u, v) = -\mathcal{B}(v, u), u, v \in V_B,$$

in particular

$$\mathcal{B}(v, v) = 0, v \in V_B.$$

For the problem  $(\mathbf{P})_2$ , we assume that the operator  $B$  is symmetric and positive, then we define a norm for  $B$

$$(H0)_2 \quad \|u\|_{V_B}^2 \equiv \mathcal{B}(v, v), v \in V_B.$$

(ii) The nonlinear source term  $f : V_{A_1} \rightarrow H$  is a potential operator with potential  $F : V_{A_1} \rightarrow \mathbb{R}$ ; that is,  $f(u) = D_u F(u)$ . We assume that  $f(0) = 0 = g(0)$ , and that there exists a constant  $r > 2$ , such that

$$(H1)_1 \quad (f(u), u) - rF(u) \geq 0, u \in V_{A_1},$$

and

$$(H1)_2 \quad \|u\|_{V_{A_2}}^2 - (g(u), u) \geq 0, \quad u \in V_{A_2}.$$

(iii) The relaxation kernel  $h \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ , satisfies the following conditions

$$(H2)_1 \quad h(0) > 0, \quad l \equiv 1 - \int_0^\infty h(t) \, dt > 0, \quad \dot{h}(t) \equiv \frac{d}{dt}h(t) \leq 0, \quad t \geq 0,$$

and  $h$  is a positive definite kernel; that is

$$(H2)_2 \quad \int_0^t (h * w)(s)w(s) \, ds \geq c_0 \int_0^t |(h * w)(s)|^2 \, ds,$$

for every  $w \in L_{loc}^1(\mathbb{R}^+, V_{A_2})$  and some constant  $c_0 > 0$ , where

$$(h * w)(s) \equiv \int_0^s h(s - \tau)w(\tau) \, d\tau \geq 0,$$

is the convolution of  $h$  and  $w$ .

(iv) The long memory relaxation kernel  $k \in C^1(\mathbb{R}, \mathbb{R}^+)$  satisfies the following hypotheses:

$$(H2)_3 \quad \zeta(t) \equiv -(1 - \omega)\dot{k}(t) \geq 0, \quad \dot{\zeta}(t) \leq 0,$$

$$(H2)_4 \quad \zeta(t) \rightarrow 0, \quad k(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

The phase space, where we study the dynamics of  $(\mathbf{P})_1$ , is

$$\mathcal{H}_1 \equiv V_{A_1} \times V_P \times V_P,$$

with the corresponding square norm

$$\|(u, w, v)\|_{\mathcal{H}_1}^2 \equiv \|u\|_{V_{A_1}}^2 + \|w\|_{V_P}^2 + \eta \|v\|_{V_P}^2.$$

For the problem  $(\mathbf{P})_2$ , we introduce the following new memory function

$$\rho(t, s) \equiv \int_0^s v(t - \tau) \, d\tau = \int_{t-s}^t v(y) \, dy,$$

for every  $t \geq 0$ ,  $s \geq 0$  and, hence,

$$\rho_t(t, s) + \rho_s(t, s) = v(t),$$

with

$$\rho(t, 0) = 0, \quad \rho(0, s) = \rho_0(s) \equiv \int_{-s}^0 v_0(y) \, dy.$$

With respect to this new function, we define the following space

$$\mathcal{M}_{V_{A_2}} \equiv L_\xi^2(\mathbb{R}^+, V_{A_2}) \equiv \left\{ w : \mathbb{R}^+ \rightarrow V_{A_2}, \left| \int_0^\infty \xi(s) \|w(s)\|_{V_{A_2}}^2 \, ds \right| < \infty \right\},$$

with norm

$$\|w\|_{\mathcal{M}_{V_{A_2}}}^2 \equiv \int_0^\infty \xi(s) \|w(s)\|_{V_{A_2}}^2 \, ds.$$

We notice that, by an integration of parts,

$$(1 - \omega) \int_0^\infty k(s) A_2 v(t - s) ds = -(1 - \omega) \int_0^\infty \dot{k}(s) A_2 \rho(t, s) ds.$$

Then, the problem  $(\mathbf{P})_2$  is equivalent to the following:

For every set of initial data  $u_0, u_1, v_0, \rho_0$ , the goal is to find functions  $t \mapsto (u(t), v(t), \rho(t))$ ,  $t \geq 0$ , such that the following system holds, for every  $t > 0$ ,

$$(\mathbf{P})_2^* \begin{cases} Pu_{tt}(t) + A_1 u(t) - \eta Bv(t) + \delta Pu_t(t) = f(u(t)), \\ Pv_t + \omega A_2 v(t) + \int_0^\infty \zeta(s) A_2 \rho(t, s) ds + \eta Bu_t(t) = g(v(t)), \\ \rho_t(t, s) + \rho_s(t, s) = v(t), \\ u(0) = u_0, u_t(0) = u_1, v(0) = v_0, \rho(0, \cdot) = \rho_0(\cdot) \quad t \geq 0, \\ \text{such that } \dot{\rho}_0(0) = v_0. \end{cases}$$

The phase space, where we study the dynamics of  $(\mathbf{P})_2^*$ , is

$$\mathcal{H}_2 \equiv V_{A_1} \times V_P \times V_P \times \mathcal{M}_{V_{A_2}},$$

with the corresponding square norm

$$\|(u, w, v, \rho)\|_{\mathcal{H}_2}^2 \equiv \|u\|_{V_{A_1}}^2 + \|w\|_{V_P}^2 + \eta \|v\|_{V_P}^2 + \eta \|\rho\|_{\mathcal{M}_{V_{A_2}}}^2.$$

The concavity argument, introduced by Professor Howard Levine [9,10], is one of the methods to study the nonexistence of global solutions of evolution equations due to blow-up and has been generalized by means of differential inequalities. See [11] for an account of several methods to study blow ups in equations of mathematical physics. The purpose of this work is to prove that under some conditions on the initial data, there exist non-global solutions to the abstract problems  $(\mathbf{P})_1$  and  $(\mathbf{P})_2^*$  by means of the analysis of a differential inequality recently studied in [12]. Our main result can be applied to some mathematical problems in thermoelasticity. Without attempting to provide the state of the art of thermoelastic models, we only mention some cases outside the scope of our analysis. For instance, in [13,14], chemical potentials are included. In [15], time fractional parabolic-hyperbolic and time fractional hyperbolic thermo-elasticity equations are studied. Other nonlinearities like p-Laplacian and fractional powers of operators are focused on in [16–20]. Equations with delay terms are studied in [21,22]. In the last section, we mention some extensions of this work; for instance, we comment on a thermoelastic system in n-dimensions with short memory, as studied in [23,24]. Finally, parabolic-hyperbolic systems similar to the ones presented in the introduction have been analyzed in [25,26].

The analysis of problem  $(\mathbf{P})_1$  will be conducted for weak solutions in the following sense,

**Definition 1.** For every set of initial data,

$$(u_0, u_1, v_0) \in \mathcal{H}_1,$$

the map, for  $t > 0$ ,

$$(u_0, u_1, v_0) \mapsto (u(t), \dot{u}(t), v(t)) \in \mathcal{D}_1 \equiv V_{A_1} \times V_B \times V_{A_2} \subset \mathcal{H}_1,$$

is a weak local solution of problem  $(\mathbf{P})_1$ , if there exists some  $T > 0$ , such that

$$(u, \dot{u}, v) \in C([0, T]; \mathcal{H}_1)$$

with

$$\dot{u} \in L^2([0, T]; V_P), \quad v \in L^2([0, T]; V_{A_2}),$$

$$u(0) = u_0, \quad \dot{u}(0) = u_1, \quad v(0) = v_0,$$

and

$$\frac{d}{dt} \mathcal{P}(\dot{u}(t), w) + \mathcal{A}_1(u(t), w) - \eta \mathcal{B}(v(t), w) + \delta \mathcal{P}(\dot{u}(t), w) = (f(u(t)), w),$$

$$\begin{aligned} \frac{d}{dt} \mathcal{P}(v(t), \tilde{w}) + \mathcal{A}_2(v(t), \tilde{w}) + \int_0^t h(t - \tau) \mathcal{A}_2(v(\tau), \tilde{w}) \, d\tau - \eta \mathcal{B}(\dot{u}(t), \tilde{w}) \\ = (g(v(t)), \tilde{w}), \end{aligned}$$

a. e.  $t \in (0, T)$ , for every  $w \in V_{A_1}$ ,  $\tilde{w} \in V_{A_2}$ .

We shall consider that the solution in this sense is unique and satisfies the following energy equation for  $T > t \geq t_0 \geq 0$ ,

$$\begin{aligned} E(u(t), \dot{u}(t), v(t)) - E(u(t_0), \dot{u}(t_0), v(t_0)) = -\delta \int_{t_0}^t \|\dot{u}(s)\|_{V_P}^2 \, ds \\ - \eta \int_{t_0}^t \left( \|v(s)\|_{V_{A_2}}^2 - (g(v(s)), v(s)) \right) \, ds - \eta \int_{t_0}^t \mathcal{A}_2((h * v)(s), v(s)) \, ds, \end{aligned}$$

where

$$\begin{aligned} E(t) \equiv E(u(t), \dot{u}(t), v(t)) &\equiv \frac{1}{2} \|\dot{u}(t)\|_{V_P}^2 + J(u(t), v(t)), \\ J(u(t), v(t)) &\equiv \frac{1}{2} \left( \|u(t)\|_{V_{A_1}}^2 + \eta \|v(t)\|_{V_P}^2 \right) - F(u(t)). \end{aligned}$$

Due to  $(H1)_2$  and  $(H2)_2$ ,

$$\begin{aligned} E(u(t), \dot{u}(t), v(t)) - E(u(t_0), \dot{u}(t_0), v(t_0)) &\leq -\delta \int_{t_0}^t \|\dot{u}(s)\|_{V_P}^2 \, ds \\ &- \eta \int_{t_0}^t \left( \|v(s)\|_{V_{A_2}}^2 - (g(v(s)), v(s)) + c_0 \|(h * v)(s)\|_{A_2}^2 \right) \, ds \leq 0. \end{aligned}$$

Then,

$$E(t) = \frac{1}{2} \|(u(t), \dot{u}(t), v(t))\|_{\mathcal{H}_1}^2 - F(u(t)) \leq E_0$$

where

$$E_0 \equiv E(0) = \frac{1}{2} \|(u_0, u_1, v_0)\|_{\mathcal{H}_1}^2 - F(u_0).$$

Furthermore, if the maximal time of existence  $T_{MAX} < \infty$ , then

$$\lim_{t \rightarrow T_{MAX}} \|(u(t), \dot{u}(t), v(t))\|_{\mathcal{H}_1} = \infty,$$

consequently,

$$\lim_{t \rightarrow T_{MAX}} F(u(t)) = \infty.$$

The analysis of problem  $(\mathbf{P})_2^*$  will be conducted for weak solutions in the following sense,

**Definition 2.** For every set of initial data

$$(u_0, u_1, v_0, \rho_0) \in \mathcal{H}_2,$$

the map, for  $t > 0$ ,

$$(u_0, u_1, v_0, \rho) \mapsto (u(t), \dot{u}(t), v(t), \rho(t)) \in \mathcal{D}_2 \equiv V_{A_1} \times V_B \times V_{A_2} \times \mathcal{M}_{V_{A_2}} \subset \mathcal{H}_2,$$

where  $\rho(t) \equiv \rho(t, \cdot)$ , is a weak local solution of problem  $(\mathbf{P})_2^*$ , if there exists some  $T > 0$ , such that

$$(u, \dot{u}, v, \rho) \in C([0, T]; \mathcal{H}_2)$$

with

$$\begin{aligned} \dot{u} &\in L^2([0, T]; V_P), \quad v \in L^2([0, T]; V_{A_2}), \\ u(0) &= u_0, \quad \dot{u}(0) = u_1, \quad v(0) = v_0, \quad \rho_0(0) = v_0, \end{aligned}$$

and

$$\frac{d}{dt} \mathcal{P}(\dot{u}(t), w) + \mathcal{A}_1(u(t), w) - \eta \mathcal{B}(v(t), w) + \delta \mathcal{P}(\dot{u}(t), w) = (f(u(t)), w),$$

$$\begin{aligned} \frac{d}{dt} \mathcal{P}(v(t), \tilde{w}) + \omega \mathcal{A}_2(v(t), \tilde{w}) + \int_0^\infty \xi(s) \mathcal{A}_2(\rho(t, s), \tilde{w}) \, ds + \eta \mathcal{B}(\dot{u}(t), \tilde{w}) \\ = (g(v(t)), \tilde{w}), \end{aligned}$$

$$\mathcal{A}_2(\rho_t(t, s), \tilde{w}) + \mathcal{A}_2(\rho_s(t, s), \tilde{w}) = \mathcal{A}_2(v(t), \tilde{w}),$$

a. e.  $t \in (0, T)$ ,  $s \geq 0$ , for every  $w \in V_{A_1}$ ,  $\tilde{w} \in V_{A_2}$ .

We shall consider that the solution in this sense is unique and satisfies the following energy equation for  $T > t \geq t_0 \geq 0$ ,

$$\begin{aligned} E(u(t), \dot{u}(t), v(t), \rho(t)) - E(u(t_0), \dot{u}(t_0), v(t_0), \rho(t_0)) &= -\delta \int_{t_0}^t \|\dot{u}(\tau)\|_{V_P}^2 \, d\tau \\ &- \eta \int_{t_0}^t \left( \omega \|v(\tau)\|_{V_{A_2}}^2 - (g(v(\tau)), v(\tau)) \right) \, d\tau + \frac{\eta}{2} \int_{t_0}^t \int_0^\infty \xi(s) \|\rho(\tau, s)\|_{V_{A_2}}^2 \, ds \, d\tau, \end{aligned}$$

where

$$\begin{aligned} E(t) &\equiv E(u(t), \dot{u}(t), v(t), \rho(t)) \equiv \frac{1}{2} \|\dot{u}(t)\|_{V_P}^2 + J(u(t), v(t), \rho(t)), \\ J(u(t), v(t), \rho(t)) &\equiv \frac{1}{2} \left( \|u(t)\|_{V_{A_1}}^2 + \eta \|v(t)\|_{V_P}^2 + \eta \|\rho(t)\|_{\mathcal{M}_{V_{A_2}}}^2 \right) - F(u(t)). \end{aligned}$$

Indeed, from this last definition

$$\begin{aligned} \frac{d}{dt} E(t) &= -\delta \int_{t_0}^t \|\dot{u}(s)\|_{V_P}^2 \, ds - \eta \int_{t_0}^t \left( \omega \|v(s)\|_{V_{A_2}}^2 - (g(v(s)), v(s)) \right) \, ds \\ &- \eta \int_0^\infty \xi(s) \mathcal{A}_2(\rho(t, s), v(t)) \, ds + \frac{\eta}{2} \frac{d}{dt} \int_0^\infty \xi(s) \|\rho(t, s)\|_{V_{A_2}}^2 \, ds. \end{aligned}$$

But,

$$\begin{aligned} & - \int_0^\infty \xi(s) \mathcal{A}_2(\rho(t, s), v(t)) \, ds + \frac{1}{2} \frac{d}{dt} \int_0^\infty \xi(s) \|\rho(t, s)\|_{V_{A_2}}^2 \, ds \\ &= \int_0^\infty \xi(s) \mathcal{A}_2(\rho(t, s), \rho_t(t, s) - v(t)) \, ds = - \int_0^\infty \xi(s) \mathcal{A}_2(\rho(t, s), \rho_s(t, s)) \, ds \\ &= - \frac{1}{2} \int_0^\infty \xi(s) \frac{\partial}{\partial s} \|\rho(t, s)\|_{V_{A_2}}^2 \, ds = \frac{1}{2} \int_0^\infty \xi(s) \|\rho(t, s)\|_{V_{A_2}}^2 \, ds \end{aligned}$$

Due to  $(H1)_2$  and  $(H2)_3$ ,

$$E(t) - E(t_0) \leq 0.$$

Then,

$$E(t) = \frac{1}{2} \|(u(t), \dot{u}(t), v(t), \rho(t))\|_{\mathcal{H}_2}^2 - F(u(t)) \leq E_0$$

where

$$E_0 \equiv E(0) = \frac{1}{2} \|(u_0, u_1, v_0, \rho_0)\|_{\mathcal{H}_2}^2 - F(u_0).$$

Furthermore, if the maximal time of existence  $T_{MAX} < \infty$ , then

$$\lim_{t \rightarrow T_{MAX}} \|(u(t), \dot{u}(t), v(t), \rho(t))\|_{\mathcal{H}_2} = \infty,$$

consequently,

$$\lim_{t \rightarrow T_{MAX}} F(u(t)) = \infty.$$

## 2. Main Result

In this section, we shall analyze the nonexistence of global solutions for both problems presented in the introduction and any positive value of the initial energy. To this end, we define the following constants

$$\left. \begin{aligned} \alpha &\equiv \frac{1}{4}(r-2), & \gamma &\equiv 2rE_0, \\ \beta &\equiv c\left(r-2-\frac{1}{2}\tilde{c}\eta\right), \end{aligned} \right\} \quad (1)$$

We assume that

$$(H3) \quad r \geq \frac{1}{2}\tilde{c}, \quad r > 2 + \frac{1}{2}\tilde{c}\eta, \quad \text{then } \alpha > 0, \quad \beta > 0.$$

Along the solutions in the sense of Definitions 1 and 2, respectively, we define the following functions

$$\left. \begin{aligned} \psi(t) &\equiv \|u(t)\|_{V_P}^2, \\ \phi(t) &\equiv \left( \frac{\frac{d}{dt}\psi(t)}{\psi^{\frac{1}{2}}(t)} - \frac{\delta}{\alpha}\psi^{\frac{1}{2}}(t) \right)^2 + \frac{\beta}{\alpha}\psi(t), \\ \sigma_v(t) &\equiv \frac{1+2\alpha}{2} \left( \phi(t) - \frac{\beta v}{\alpha}\psi(t) \right), \\ \mu_\lambda(t) &\equiv \frac{1+2\alpha}{2} \left( \phi(t) - \frac{\beta}{\alpha(1+2\alpha)}\psi(t) \left( \frac{\lambda\beta\psi(t)}{\alpha\phi(t)} \right)^{2\alpha} \right), \end{aligned} \right\} \quad (2)$$

for  $t \geq 0$ ,  $\nu > 0$ ,  $\lambda \in (0, 1)$ , and

$$\psi_0 \equiv \psi(0), \quad \phi_0 \equiv \phi(0) = \left( \frac{\dot{\psi}_0}{\psi_0^{\frac{1}{2}}} - \frac{\delta}{\alpha} \psi_0^{\frac{1}{2}} \right)^2 + \frac{\beta}{\alpha} \psi_0, \quad \dot{\psi}_0 \equiv \frac{d}{dt} \psi(0). \quad (3)$$

**Theorem 1.** Consider any solution either from problem  $(\mathbf{P})_1$  or problem  $(\mathbf{P})_2^*$ , in the sense of Definitions 1 and 2, respectively. Assume that hypotheses (H0) – (H3) hold. If

$$\dot{\psi}_0 > \frac{\delta}{\alpha} \psi_0 > 0, \quad (4)$$

is satisfied, then there exists a nonempty interval

$$\mathcal{I} \equiv (\mathbf{a}, \mathbf{b}) \subset \left( 0, \frac{1+2\alpha}{2} \phi_0 \right),$$

with the following consequences:

(i) If  $\gamma = 2rE_0 \in \mathcal{I}$ , then  $\psi(t)$  blows up at a finite time  $t^* > 0$ ; that is,

$$\lim_{t \rightarrow t^*} \psi(t) = \infty.$$

Hence, the corresponding solutions of both problems are not global.

(ii)  $\mathbf{a} = \sigma_\nu(0)$  and  $\mathbf{b} = \mu_\lambda(0)$ , moreover,

$$\mathbf{a} = \frac{\beta \psi_0}{((1+2\alpha)\nu^*)^{\frac{1}{2\alpha}}} < \frac{\beta \psi_0}{(1+2\alpha)^{\frac{1}{\alpha}}},$$

$$\mathbf{b} = \frac{\alpha \phi_0}{\lambda^*} > \frac{1+2\alpha}{2} \phi_0 - \left( \frac{1+2\alpha}{2\alpha} - \chi(\lambda^*) \right) \beta \psi_0 > \frac{1+2\alpha}{2} \phi_0 - \frac{\beta}{2\alpha} \psi_0,$$

for some  $\frac{2\alpha}{1+2\alpha} < \lambda^* < 1$  and  $\nu^* > 1+2\alpha$ , where  $1 < \chi(\lambda^*) < \frac{1+2\alpha}{2\alpha}$  is a function of  $\lambda^*$ .

(iii) For fixed  $\psi_0, \dot{\psi}_0$ ,

$\delta \mapsto t^*$ , is strictly increasing, and

$\delta \mapsto |\mathcal{I}| = \mathbf{b} - \mathbf{a}$ , is strictly decreasing.

For fixed  $\psi_0, \delta$ ,

$\dot{\psi}_0 \mapsto t^*$ , is strictly decreasing, and

and

$\dot{\psi}_0 \mapsto |\mathcal{I}|$ , is strictly increasing.

We have the bounds

$$0 < \frac{1+2\alpha}{2} \phi_0 - |\mathcal{I}| < \left( \frac{1+2\alpha}{2\alpha} - \chi(\lambda^*) + \frac{1}{((1+2\alpha)\nu^*)^{\frac{1}{2\alpha}}} \right) \beta \psi_0,$$

$$t^* \geq \left( \alpha \frac{\dot{\psi}_0}{\psi_0} - \delta \right)^{-1}.$$

(iv) Furthermore, for  $\psi_0$  fixed, we have the limit values as  $\psi_0 \rightarrow \infty$ ,

$$\begin{aligned} \mathbf{a} &\rightarrow 0, & \left| \mathbf{b} - \frac{1+2\alpha}{2} \phi_0 \right| &\rightarrow 0, & t^* &\rightarrow 0, \\ \nu^* &\rightarrow \infty, & \lambda^* &\rightarrow \frac{2\alpha}{1+2\alpha}, & \chi(\lambda^*) &\rightarrow \frac{1+2\alpha}{2\alpha}. \end{aligned}$$

**Corollary 1.** Consider any solution either from problem  $(\mathbf{P})_1$  or problem  $(\mathbf{P})_2^*$ , in the sense of Definitions 1, 2, respectively. Assume that the hypotheses of Theorem 1 are met. Given any numbers  $\xi > 0, \epsilon > 0$ , we can choose initial data with  $\mathcal{P}(u_0, u_1)$  being large enough, so that the conclusions of Theorem 1 are satisfied for initial energy with  $E_0 = \xi$  at a blow-up time  $t^* < \epsilon$ .

For the proof, we will employ the following definitions.

First, we consider the orthogonal decomposition of the velocity

$$\begin{aligned} \dot{u} &= \frac{\mathcal{P}(\dot{u}, u)}{\|u\|_{V_P}^2} u + h, \quad \mathcal{P}(u, h) = 0, \\ \|\dot{u}\|_{V_P}^2 &= \|h\|_{V_P}^2 + \frac{|\mathcal{P}(\dot{u}, u)|^2}{\|u\|_{V_P}^2} \geq \frac{|\mathcal{P}(\dot{u}, u)|^2}{\|u\|_{V_P}^2} \equiv \mathcal{Q}(\dot{u}, u). \end{aligned} \quad (5)$$

Second, since the conditions on the initial data that produce the nonexistence of the global solution in both problems are only on  $u_0, u_1$ , we define the auxiliary space

$$\tilde{\mathcal{H}} \equiv V_{A_1} \times V_P,$$

then the phase spaces for the problems, in the sense of Definitions 1 and 2, become, respectively,

$$\mathcal{H}_1 = \tilde{\mathcal{H}} \times V_P, \quad \mathcal{H}_2 = \tilde{\mathcal{H}} \times V_P \times \mathcal{M}_{V_{A_2}}.$$

Third, we define the concept of a positive-invariant set with respect to any solution from problem  $(\mathbf{P})_1$  or problem  $(\mathbf{P})_2^*$ , in the sense of Definitions 1 and 2, respectively. Indeed,  $\mathcal{W}_1 \in \mathcal{H}_1$  is a positive-invariant set, along  $(u, \dot{u}, v)$ , respectively, and  $\mathcal{W}_2 \in \mathcal{H}_2$  is a positive-invariant set, along  $(u, \dot{u}, v, \rho)$ , if

$$\begin{aligned} (u_0, u_1, v_0) &\equiv (u(0), \dot{u}(0), v(0)) \in \mathcal{W}_1 \\ &\Rightarrow (u(t), \dot{u}(t), v(t)) \in \mathcal{W}_1, \text{ for any } t > 0, \end{aligned}$$

respectively

$$\begin{aligned} (u_0, u_1, v_0, \rho_0) &\equiv (u(0), \dot{u}(0), v(0), \rho(0)) \in \mathcal{W}_2 \\ &\Rightarrow (u(t), \dot{u}(t), v(t), \rho(t)) \in \mathcal{W}_2, \text{ for any } t > 0. \end{aligned}$$

Fourth, from (2) and (3), and by introducing the function,

$$\mathcal{G}(t) \equiv \psi^{-\alpha}(t) e^{\delta t},$$

the inequality in (4) has the equivalent forms

$$2\mathcal{P}(u, \dot{u}) > \frac{\delta}{\alpha} \|u\|_{V_P}^2 \Leftrightarrow \dot{\psi}_0 > \frac{\delta}{\alpha} \psi_0 \Leftrightarrow \dot{\mathcal{G}}_0 < 0.$$

Finally, we define the sets

$$\begin{aligned}\mathcal{V} &\equiv \left\{ (u, \dot{u}) \in \tilde{\mathcal{H}} : 2\mathcal{P}(u, \dot{u}) > \frac{\delta}{\alpha} \|u\|_{V_P}^2 \right\} \\ &= \left\{ (u, \dot{u}) \in \tilde{\mathcal{H}} : \dot{\psi} > \frac{\delta}{\alpha} \psi \right\} = \{ (u, \dot{u}) \in \tilde{\mathcal{H}} : \dot{\mathcal{G}} < 0 \}.\end{aligned}$$

$$\mathcal{V}_1 \equiv \{ (u, \dot{u}, v) \in \mathcal{H}_1 : (u, \dot{u}) \in \mathcal{V} \} \quad \mathcal{V}_2 \equiv \{ (u, \dot{u}, v, \rho) \in \mathcal{H}_2 : (u, \dot{u}) \in \mathcal{V} \}$$

**Lemma 1.** Consider any solution either from problem  $(\mathbf{P})_1$  or problem  $(\mathbf{P})_2^*$ , in the sense of Definitions 1 and 2, respectively. Assume that hypotheses (H0) – (H3) hold and (4) is satisfied. If there exists a constant  $\kappa_0^2 > 0$ , such that

$$\mathcal{J}(s) \geq \kappa_0^2 > 0, \quad s \geq 0, \quad (6)$$

where the function  $\mathcal{J}(s)$  is defined by

$$\mathcal{J}(s) \equiv \frac{2\alpha^2}{1+2\alpha} \gamma s^{\frac{1+2\alpha}{\alpha}} - \alpha\beta s^2 + \alpha^2 \psi_0^{-(1+2\alpha)} \left( \phi_0 - \frac{2\gamma}{1+2\alpha} \right), \quad s \geq 0.$$

then the corresponding set  $\mathcal{V}_j, j = 1, 2$  is positive-invariant. Furthermore,

$$\dot{\mathcal{G}}(t) \leq -\kappa_0 < 0, \quad \text{for any } t \geq 0.$$

**Proof of Lemma 1.** Consider a solution either from problem  $(\mathbf{P})_1$  or problem  $(\mathbf{P})_2^*$ , such that the initial data are in the corresponding  $\mathcal{V}_j, j = 1, 2$ . Then, in any case,  $(u_0, u_1) \equiv (u(0), \dot{u}(0)) \in \mathcal{V}$ . To show the invariance property, we proceed by contradiction. Assume that there exists some  $\hat{t} > 0$ , such that

$$(u(t), \dot{u}(t)) \in \mathcal{V}, \text{ for } t \in [0, \hat{t}), \quad \text{and} \quad (u(\hat{t}), \dot{u}(\hat{t})) \notin \mathcal{V},$$

that is

$$\dot{\mathcal{G}}(t) < 0, \quad t \in [0, \hat{t}), \quad \dot{\mathcal{G}}(\hat{t}) = 0.$$

We shall prove that the time  $\hat{t}$  is never reached. To this end, we first construct a differential inequality for the function

$$\psi(t) \equiv \|u(t)\|_{V_P}^2 \in \mathbb{R}^+, \quad t \geq 0.$$

We calculate the first and second derivatives of  $\psi(t)$ , and we use Definitions 1 and 2. First, we only use the hyperbolic equation, which is the same in both problems. Then, we conclude the following for  $t \geq 0$ ,

$$\left. \begin{aligned} \frac{d}{dt} \psi(t) &= 2\mathcal{P}(u(t), \dot{u}(t)) \\ \frac{d^2}{dt^2} \psi(t) &= 2(\|\dot{u}(t)\|_{V_P}^2 - \|u(t)\|_{V_{A_1}}^2 + \eta \mathcal{B}(v(t), u(t)) + (f(u(t)), u(t)) \\ &\quad - 2\delta \mathcal{P}(u(t), \dot{u}(t))). \end{aligned} \right\} \quad (7)$$

We shall estimate the terms on the right-hand side of the second derivative of  $\psi(t)$ . First, we consider the problem  $(\mathbf{P})_1$ . By the corresponding energy equation and hypothesis (H1), we obtain the following:

$$\begin{aligned} &2(\|\dot{u}(t)\|_{V_P}^2 - \|u(t)\|_{V_{A_1}}^2 + (f(u(t), u(t))) + 2r(E(t) - E(t)) \\ &\geq (r+2)\|\dot{u}(t)\|_{V_P}^2 + (r-2)\|u(t)\|_{V_{A_1}}^2 + r\eta\|v(t)\|_{V_P}^2 - 2rE_0. \end{aligned}$$

For the problem  $(\mathbf{P})_2^*$ , we estimate in a similar way the terms on the right-hand side of the second derivative of  $\psi(t)$

$$\begin{aligned} & 2(\|\dot{u}(t)\|_{V_P}^2 - \|u(t)\|_{V_{A_1}}^2 + (f(u(t), u(t))) + 2r(E(t) - E(t)) \\ & \geq (r+2)\|\dot{u}(t)\|_{V_P}^2 + (r-2)\|u(t)\|_{V_{A_1}}^2 + r\eta\|v(t)\|_{V_P}^2 + r\eta\|\rho(t)\|_{\mathcal{M}_{V_{A_2}}}^2 - 2rE_0 \\ & \geq (r+2)\|\dot{u}(t)\|_{V_P}^2 + (r-2)\|u(t)\|_{V_{A_1}}^2 + r\eta\|v(t)\|_{V_P}^2 - 2rE_0. \end{aligned}$$

Consequently, from (7), hypothesis  $(H0)$ ,  $(H0)_1$ , and  $(H0)_2$ , (5), and the last inequalities, we obtain for both problems and for  $t \geq 0$ ,

$$\begin{aligned} \frac{d^2}{dt^2}\psi(t) & \geq (r+2)\|\dot{u}(t)\|_{V_P}^2 - 2\delta\mathcal{P}(u(t), \dot{u}(t)) \\ & \quad + (r-2)\|u(t)\|_{V_{A_1}}^2 + r\eta\|v(t)\|_{V_P}^2 \\ & \quad - \eta\tilde{c}\frac{1}{2}\left(\|u(t)\|_{V_{A_1}}^2 + \|v(t)\|_{V_P}^2\right) - 2rE_0 \\ & \geq -\delta\frac{d}{dt}\psi(t) + (r+2)\frac{\left(\frac{d}{dt}\psi(t)\right)^2}{\psi(t)} \\ & \quad + \left(r-2-\frac{1}{2}\eta\tilde{c}\right)\|u(t)\|_{V_{A_1}}^2 + \eta\left(r-\frac{1}{2}\tilde{c}\right)\|v(t)\|_{V_P}^2 - 2rE_0 \\ & \geq -\delta\frac{d}{dt}\psi(t) + (r+2)\frac{\left(\frac{d}{dt}\psi(t)\right)^2}{\psi(t)} + c\left(r-2-\frac{1}{2}\eta\tilde{c}\right)\psi(t) - 2rE_0. \end{aligned}$$

That is, for  $t \geq 0$ , the following inequality is satisfied

$$\frac{d^2}{dt^2}\psi(t) + \delta\frac{d}{dt}\psi(t) - (r+2)\frac{\left(\frac{d}{dt}\psi(t)\right)^2}{\psi(t)} - c\left(r-2-\frac{1}{2}\eta\tilde{c}\right)\psi(t) + 2rE_0 \geq 0. \quad (8)$$

From  $(H3)$ , we can simplify the notation by substituting the constants defined in (1). After multiplying the differential inequality (8) by  $\psi(t)$ , we obtain

$$\psi(t)\frac{d^2}{dt^2}\psi(t) + \delta\psi(t)\frac{d}{dt}\psi(t) - (1+\alpha)\left(\frac{d}{dt}\psi(t)\right)^2 - \beta\psi^2(t) + \gamma\psi(t) \geq 0, \quad t \geq 0. \quad (9)$$

This is the differential inequality studied in [12]. Here, we will apply it to prove the nonexistence of global solutions of the abstract problems  $(\mathbf{P})_1$  and  $(\mathbf{P})_2^*$ .

We set  $\mathcal{G}(t) \equiv \psi^{-\alpha}(t)e^{\delta t}$  in (9), then this inequality becomes

$$\frac{d^2}{dt^2}\mathcal{G}(t) - \delta\frac{d}{dt}\mathcal{G}(t) + \alpha\beta\mathcal{G}(t) - \alpha\gamma\mathcal{G}^{\frac{1+\alpha}{\alpha}}(t)e^{-\frac{\delta t}{\alpha}} \leq 0, \quad t \geq 0.$$

From the definition of  $\hat{t} > 0$ ,

$$-\delta\frac{d}{dt}\mathcal{G}(t) > 0, \quad \text{for } t \in [0, \hat{t}),$$

and the last differential inequality, in terms of  $\mathcal{G}(t)$ , we obtain

$$\frac{d^2}{dt^2}\mathcal{G}(t) + \alpha\beta\mathcal{G}(t) < \alpha\gamma\mathcal{G}^{\frac{1+\alpha}{\alpha}}(t)e^{-\frac{\delta t}{\alpha}} \leq \alpha\gamma\mathcal{G}^{\frac{1+\alpha}{\alpha}}(t), \quad t \in [0, \hat{t}).$$

Consequently, we arrive to

$$\frac{d^2}{dt^2}\mathcal{G}(t) + \alpha\beta\mathcal{G}(t) - \alpha\gamma\mathcal{G}^{\frac{1+\alpha}{\alpha}}(t) < 0, \quad t \in [0, \hat{t}). \quad (10)$$

Multiplying (10) by  $\frac{d}{dt}\mathcal{G}(t) < 0$ , we conclude the following integral:

$$\left(\frac{d}{dt}\mathcal{G}(t)\right)^2 < \mathcal{J}(\mathcal{G}(t)), \quad t \in [0, \hat{t}). \quad (11)$$

By the hypotheses, there is a constant  $\kappa_0^2 > 0$ , such that

$$\mathcal{J}(s) \geq \kappa_0^2 > 0, \quad s \geq 0,$$

then, from (11)

$$\dot{\mathcal{G}}(t) = \frac{d}{dt}\mathcal{G}(t) < -\kappa_0 < 0, \quad t \in [0, \hat{t}).$$

By continuity, when  $t \rightarrow \hat{t}$ ,

$$\dot{\mathcal{G}}(\hat{t}) \leq -\kappa_0 < 0,$$

which contradicts the definition of  $\hat{t}$ . Hence, as long as the solution exists,

$$\dot{\mathcal{G}}(t) \leq -\kappa_0 < 0,$$

and the corresponding  $\mathcal{V}_j, j = 1, 2$  is positive-invariant.  $\square$

**Proof of Theorem 1.** If the solution is global, then

$$t \rightarrow \psi(t) \equiv \|u(t)\|_{V_p}^2 \in \mathbb{R}^+,$$

that is, it is well-defined for any  $t \geq 0$ . The conclusions of Theorem 1 are derived from the analysis of (9), as was made in [12]. However, for completeness, we shall sketch the proof. First, from Lemma 1,

$$\frac{d}{dt}\mathcal{G}(t) \leq -\kappa_0 < 0.$$

And consequently,

$$0 \leq \psi^{-\alpha}(t)e^{\delta t} = \mathcal{G}(t) \leq \psi_0^{-\alpha} - t\kappa_0.$$

Then  $t \rightarrow t^* \equiv (\kappa_0\psi_0^\alpha)^{-1}$  implies that  $\psi(t) \rightarrow \infty$ . That is,  $\psi(t)$  blows up at  $t^*$ .

The proof of (6) is as follows. First, we notice that  $\mathcal{J}(s)$  attains an absolute minimum at  $s_0 \equiv \left(\frac{\beta}{\gamma}\right)^\alpha$ ; that is

$$\mathcal{J}(s) \geq \mathcal{J}(s_0) = \alpha^2\psi_0^{-(1+2\alpha)}(\phi_0 - \mathcal{K}(\gamma)),$$

where

$$\mathcal{K}(\gamma) \equiv \frac{2\gamma}{1+2\alpha} + \frac{\beta}{\alpha(1+2\alpha)}\left(\frac{\beta}{\gamma}\right)^{2\alpha}\psi_0^{1+2\alpha}.$$

We define  $\kappa_0^2 \equiv \mathcal{J}(s_0)$ . Then, (6) holds if and only if

$$\mathcal{K}(\gamma) < \phi_0. \quad (12)$$

We notice that

$$\mathcal{K}(\gamma) \rightarrow \infty \quad \text{as either} \quad \gamma \rightarrow 0 \quad \text{or} \quad \gamma \rightarrow \infty.$$

Furthermore,

$$\mathcal{K}(\gamma) \geq \mathcal{K}(\gamma_0) = \frac{\beta}{\alpha}\psi_0, \quad \gamma > 0,$$

where  $\gamma_0 \equiv \beta\psi_0$ . Hence, there exist two different roots, denoted by **a** and **b**, of

$$\mathcal{K}(\gamma) = \phi_0.$$

That is, there exists a nonempty interval  $\mathcal{I} \equiv (\mathbf{a}, \mathbf{b})$ , such that

$$0 < \mathbf{a} < \gamma_0 < \mathbf{b} < \frac{1+2\alpha}{2}\phi_0,$$

and

$$\frac{\beta}{\alpha}\psi_0 < \mathcal{K}(\gamma) < \phi_0 \Leftrightarrow \gamma \in \mathcal{I} \equiv (\mathbf{a}, \mathbf{b}), \gamma \neq \gamma_0.$$

Then, (6) holds if and only if  $\gamma \in \mathcal{I}$ . The strict monotonicity of  $\mathcal{K}$  for  $\gamma < \gamma_0$  and  $\gamma > \gamma_0$ , implies that, for fixed  $\psi_0$ , the interval  $\mathcal{I}$  grows as  $\psi_0$  grows. That is,

$$\lim_{\psi_0 \rightarrow \infty} \left| \frac{1+2\alpha}{2}\phi_0 - \mathbf{b} \right| = 0 = \lim_{\psi_0 \rightarrow \infty} \mathbf{a}.$$

The rest of the conclusions follow as in [12].  $\square$

**Proof of Corollary 1.** Since  $\psi_0 \rightarrow \infty \Rightarrow \mathbf{a} \rightarrow 0, \mathbf{b} \rightarrow \infty$  and  $t^* \rightarrow 0$ , then, for every  $\xi > 0$  there exists  $\eta_1 > 0$ , such that  $\psi_0 > \eta_1 \Rightarrow \gamma = 2r\xi \in \mathcal{I} = (\mathbf{a}, \mathbf{b})$ . Also, for every  $\epsilon > 0$  there exists  $\eta_2 > 0$ , such that  $\psi_0 > \eta_2 \Rightarrow t^* < \epsilon$ . Hence, any solution with  $\gamma/2r = E_0 = \xi$  blows up at a finite time  $t^* < \epsilon$  if  $\psi_0 > \eta \equiv \max\{\eta_1, \eta_2\}$ .  $\square$

**Remark 1.** Notice that  $\frac{\beta}{\alpha}\psi_0 < \phi_0$ , the assumption (4) in Theorem 1, is the condition that allows the existence of  $\mathcal{I}$ , and  $\gamma \in \mathcal{I}$  characterizes the condition that implies the positive invariance of  $\mathcal{V}_j, j = 1, 2$  and, hence, the blow-up of  $\psi(t)$  in finite time and, consequently, the nonexistence of global solutions.

**Remark 2.** The blow-up of the norm of the solution comes from two different sources. (i) The physical properties of the model:  $\delta, r, \eta$ . (ii) The initial data:  $\psi_0, \dot{\psi}_0$ . The source term  $f$  destabilizes the system and it is the main cause of the blow-up, meanwhile, the source,  $g$ , is controlled by the hypothesis,  $(H1)_2$ . The blow-up property is reached for a larger set of values of  $r$  as long as  $\tilde{c}\eta$  decreases. That is, if the ‘decouple’ coefficient  $\eta$  decreases, then the source terms  $f$  that produce the blow-up are larger. If we decouple the system,  $\eta = 0$ , then the blow-up is reached as if the parabolic equation did not exist. The coupling with the parabolic equation stabilizes the system, as does the damping term in the hyperbolic equation. Indeed, the numbers **a** and **b** become closer to each other as the damping coefficient  $\delta$  or the coupling factor  $\tilde{c}\eta$  grows. Hence, the length of the blow-up interval  $\mathcal{I}$  decreases as  $\delta$  or  $\tilde{c}\eta$  increases. Therefore, as the damping coefficient or the coupling factor grows, the set of initial energies where global nonexistence can occur becomes smaller. On the other hand, a notable property that should be highlighted is that the blow-up time approaches zero and the length of the blow-up interval  $\mathcal{I}$  becomes infinite as  $\psi_0$  approaches infinity. That is, if the inner product between the displacement and velocity at the initial time increases in value, then the range of initial energy values where the blow-up is reached becomes larger, and the blow-up time is closer to zero.

### 3. Applications and Some Extensions

First, we shall apply the results proved in the last section to the following two problems, which are related to the ones introduced at the beginning of this work. Then, we will present two more problems where our analysis can be extended.

### 3.1. Cauchy Problem of a One-Dimensional Thermoelastic Model with a Short Memory

$$(\text{ThE})_1^* \left\{ \begin{array}{l} \text{Given initial data } (u_0(x), u_1(x), \theta_0(x)) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \\ \text{find } u(x, t) \in \mathbb{R}, \theta(x, t) \in \mathbb{R}, \text{ such that} \\ u_{tt} - au_{xx} + \eta\theta_x + \delta u_t = f(u), \quad x \in \Omega, \quad t > 0, \\ \theta_t - \kappa\theta_{xx} + \eta u_{xt} - \int_0^t h(t-s)\theta_{xx}(s) ds = g(\theta), \quad x \in \Omega, \quad t > 0, \\ u = u_0, u_t = u_1, \theta = \theta_0, \quad x \in \Omega, \quad t = 0, \\ u = 0, \theta = 0, \quad x \in \partial\Omega, \quad t > 0. \end{array} \right.$$

Here,  $\Omega = (0, L)$ , the short memory kernel  $h$  satisfies hypotheses  $(H2)_1, (H2)_2$ , and

$$\begin{aligned} \mathcal{P}(u, w) &= (u, w), \quad u, w \in V_p = H = L_2(\Omega), \\ \mathcal{A}_1(u, w) &= a(u_x, w_x), \quad u, w \in V_{A_1} = H_0^1(\Omega), \\ \mathcal{A}_2(\theta, w) &= \kappa(\theta_x, w_x), \quad u, w \in V_{A_2} = H_0^1(\Omega), \\ \mathcal{B}(u, \theta) &= -\eta(u_x, \theta)_{V_B' \times V_B} = \eta(\theta_x, u)_{V_B' \times V_B} = -\mathcal{B}(\theta, u), \quad u, \theta \in V_B = H^{1/2}(\Omega), \\ |\mathcal{B}(u, \theta)| &\leq \tilde{c}\|u\|_{H_0^1(\Omega)}\|\theta\|_{L_2(\Omega)}, \quad u \in V_{A_1}, \theta \in V_B. \end{aligned}$$

Then, hypotheses  $(H0)$  and  $(H0)_1$  are satisfied. Moreover, the nonlinearities satisfy hypotheses  $(H1)_1, (H1)_2$ . We assume that  $(H3)$  holds. Consider a solution in the sense of Definition 1, such that the initial data satisfy

$$(u_0, u_1) > \frac{2\delta}{r-2}\|u_0\|^2 > 0,$$

then there exists a nonempty blow-up interval  $\mathcal{I}$ , given by Theorem 1. If the initial energy is such that  $2rE_0 \in \mathcal{I}$ , then the corresponding solution is not global and blows up in finite time. Furthermore, for every positive value of the initial energy, there exist initial data, such that the corresponding solution blows up. Since we consider a larger set of initial energies where the solutions can blow-up in finite time, our conclusions improve the ones for blow-up showed in [1,4], under our hypotheses on the physical constants and memory kernel.

### 3.2. Plate Equation with a Long Thermal Memory

$$(\text{ThE})_2^* \left\{ \begin{array}{l} \text{Given initial data } (u_0, u_1, \theta_0(s)) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \quad s \leq 0, \\ \text{find } u(x, t) \in \mathbb{R}, \theta(x, t) \in \mathbb{R}, \text{ such that} \\ u_{tt} + \Delta^2 u - \Delta u(t) - \|\nabla u\|_2^{2p} \Delta u(t) + \eta\Delta\theta + \delta u_t = \hat{f}(u), \quad x \in \Omega, \quad t > 0, \\ \theta_t - \omega\Delta\theta - (1-\omega) \int_0^\infty k(t-s)\Delta\theta(s) ds - \eta\Delta u_t = g(\theta), \quad x \in \Omega, \quad t > 0, \\ u = u_0, u_t = u_1, \quad x \in \Omega, \quad t = 0, \\ \theta = \theta_0, \quad x \in \Omega, \quad t \leq 0, \\ u = \partial_\nu u = 0, \quad \theta = 0, \quad x \in \partial\Omega, \quad t > 0, \end{array} \right.$$

Here,  $\Omega \subset \mathbb{R}^2$  is a bounded domain, with a smooth boundary  $\partial\Omega$  and normal vector  $\nu$ , the long memory kernel  $k$  satisfies hypotheses  $(H2)_3, (H2)_4$  with  $\omega \in (0, 1)$ , and

$$\begin{aligned} \mathcal{P}(u, w) &= (u, w), \quad u, w \in V_p = H = L_2(\Omega), \\ \mathcal{A}_1(u, w) &= (\Delta u, \Delta w) + (\nabla\theta, \nabla w), \quad u, w \in V_{A_1} = H_0^2(\Omega), \\ \mathcal{A}_2(\theta, w) &= \mathcal{B}(\theta, w) = (\nabla\theta, \nabla w), \quad u, w \in V_{A_2} = V_B = H_0^1(\Omega), \\ |\mathcal{B}(u, \theta)| &\leq \tilde{c}\|u\|_{H_0^2(\Omega)}\|\theta\|_{L_2(\Omega)}, \quad u \in V_{A_1}, \theta \in V_B. \end{aligned}$$

Then, hypotheses  $(H0)$  and  $(H0)_2$  are satisfied. The nonlinearity  $g$  satisfies  $(H1)_2$ . The nonlinear term in the hyperbolic equation satisfies  $(H1)_1$ . Indeed, the nonlinear source  $\hat{f}$  is such that

$$(\hat{f}(u), u) - \rho \hat{F}(u) \geq 0, \quad \rho > 2,$$

where  $\hat{F}$  is the potential of  $\hat{f}$ . Then,

$$f(u) = \hat{f}(u) + \|\nabla u\|_{2p}^2 \Delta u,$$

has the potential

$$F(u) = \hat{F}(u) - \frac{1}{2(p+1)} \|\nabla u\|^{2(p+1)}.$$

Hence,  $(H1)_1$  is satisfied if

$$\rho \geq r \geq 2(p+1).$$

That is, the nonlinearity of the source term  $\hat{f}$  is stronger than the one of  $\phi$ .

We assume that  $(H3)$  holds. Consider a solution in the sense of Definition 2, such that the initial data satisfy

$$(u_0, u_1) > \frac{2\delta}{r-2} \|u_0\|^2 > 0,$$

then there exists a nonempty blow-up interval  $\mathcal{I}$  given by Theorem 1. If the initial energy is such that  $2rE_0 \in \mathcal{I}$ , then the corresponding solution is not global and, in fact, blows up in finite time. Furthermore, for every positive value of the initial energy, there exist initial data, such that the corresponding solution blows up. Problems similar to  $(\text{ThE})_2^*$  have studied long-time dynamics and decay estimates of the solution to zero, in [6,27,28]. However, to our knowledge, our analysis is the first to address the blow-up.

In  $(\text{ThE})_2^*$ , as well as in  $(\text{ThE})_1^*$

$$\mathcal{I} \subset \left( 0, \left\{ \frac{\sqrt{r}}{2\|u_0\|} \left[ (u_0, u_1) - \frac{2\delta}{r-2} \|u_0\|^2 \right] \right\}^2 \right),$$

and as  $(u_0, u_1) \rightarrow \infty$ ,  $\mathcal{I} \rightarrow (0, \infty)$ . Next, we present two problems where we can extend our analysis.

### 3.3. Thermoelastic System in n-Dimensions with a Short Memory

Consider the following n-dimensional system of thermoelasticity under Fourier's law of heat flux.

$$(\text{ThE})_3 \left\{ \begin{array}{ll} \text{Given initial data } (u_0(x), u_1(x), \theta_0(x)) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}, \\ \text{find } u(x, t) \in \mathbb{R}^n, \theta(x, t) \in \mathbb{R}, \text{ such that} \\ \begin{array}{ll} u_{tt} - \mu \Delta u - (\mu + \lambda) \nabla (\nabla \cdot u) + \eta \nabla \theta + \delta u_t = f(u), & x \in \Omega, \quad t > 0, \\ c \theta_t - \kappa \Delta \theta - \kappa \int_0^t \Delta h(t - \tau) \theta(\tau) d\tau + \eta \nabla \cdot u_t = g(\theta), & x \in \Omega, \quad t > 0, \\ u = u_0, \quad u_t = u_1, \quad \theta = \theta_0, & x \in \Omega, \quad t = 0, \\ u = 0, \quad \theta = 0, & x \in \partial\Omega, \quad t > 0, \end{array} \end{array} \right.$$

where  $u(x, t)$  is the displacement vector,  $\theta(x, t)$  is the difference temperature,  $\Omega \subset \mathbb{R}^n$  is a domain with a smooth boundary  $\partial\Omega$ . Here,  $\lambda, \mu$  denotes the Lamé moduli,  $\kappa$  is the Fourier heat conduction coefficient,  $\delta > 0$  is the damping coefficient, and  $c, \eta$  denote positive constants. Finally,  $f(u), g(v)$  are nonlinear source terms. This problem was studied in [23] with second sound, and in [24,29–32], it was investigated with viscoelastic dissipation acting on a part of the boundary, but without the source term in the parabolic equation  $g(\theta) = 0$ , without damping  $\delta = 0$ , and without the viscoelastic term in the parabolic equation. An abstract formulation was studied in [33] with Cattaneo's law and inertial terms. Notice that the hyperbolic component is a system of equations in contrast with the

parabolic one, which is a single equation; thus, our formulation cannot be applied directly and must be adapted. However, we can extend our results and obtain the same conclusions. That is, if the initial data satisfy

$$(u_0, u_1) > \frac{2\delta}{r-2} \|u_0\|^2 > 0,$$

then  $\mathcal{I} \neq \emptyset$  and the solution blows up if  $2rE_0 \in \mathcal{I}$ . As far as we know, this analysis about blow-up would be the first one for this problem **(ThE)**<sub>3</sub>.

### 3.4. Kirchhoff Equation with a Long Thermal Memory

$$(\text{ThE})_4 \left\{ \begin{array}{l} \text{Given initial data } (u_0, u_1, \theta_0(s)) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \ s \leq 0, \\ \text{find } u(x, t) \in \mathbb{R}, \ \theta(x, t) \in \mathbb{R}, \text{ such that} \\ u_{tt} - \Delta u(t) - \|\nabla u\|_2^{2p} \Delta u(t) + \eta \Delta \theta + \delta u_t = \hat{f}(u), \quad x \in \Omega, \ t > 0, \\ \theta_t - \omega \Delta \theta - (1 - \omega) \int_0^\infty k(t-s) \Delta \theta(s) \, ds - \eta \Delta u_t = g(\theta), \quad x \in \Omega, \ t > 0, \\ u = u_0, \ u_t = u_1, \quad x \in \Omega, \ t = 0, \\ \theta = \theta_0, \quad x \in \Omega, \ t \leq 0, \\ u = 0, \ \theta = 0, \quad x \in \partial\Omega, \ t > 0, \end{array} \right.$$

Here,  $\Omega \subset \mathbb{R}^n$  is a bounded domain, with a smooth boundary  $\partial\Omega$ , and the long memory kernel  $k$  satisfies hypotheses  $(H2)_3, (H2)_4$  with  $\omega \in (0, 1)$ . This is a quasilinear problem and our functional framework does not cover it. However, we can extend our analysis and conclude blow-up if the nonlinearities hold the same relations as in **(ThE)**<sub>2</sub><sup>\*</sup>. That is, the nonlinearity of the source term  $\hat{f}$  is stronger than the one of  $\phi$ . Then, if the initial data satisfy

$$(u_0, u_1) > \frac{2\delta}{r-2} \|u_0\|^2 > 0,$$

then  $\mathcal{I} \neq \emptyset$  and the solution blows up if  $2rE_0 \in \mathcal{I}$ . To our knowledge, this analysis would be the first one about blow-up for problem **(ThE)**<sub>4</sub>.

## 4. Conclusions

We introduce two parabolic–hyperbolic abstract problems, motivated by two corresponding thermoelastic models, to prove the nonexistence of global solutions due to blow-up in finite time. We exemplify our main theorem with some concrete models found in existing literature, showing that our sufficient conditions on the initial data, which lead to blow-up, improve those already published. We also provide some extensions of the main result.

**Funding:** This work was supported by CONACYT under grant 684340, and by the Universidad Autónoma Metropolitana.

**Data Availability Statement:** No new data were created or analyzed in this study. Data sharing is not applicable to this article.

**Acknowledgments:** We would like to thank the reviewers for their constructive comments that helped to improve the final form of this article.

**Conflicts of Interest:** The author declares that there are no conflicts of interest.

## References

1. Qin, Y.; Rivera, J.M. Blow-up of solutions to the Cauchy problem in nonlinear one-dimensional thermoelasticity. *J. Math. Anal. Appl.* **2004**, *292*, 160–193. [\[CrossRef\]](#)
2. Qin, Y.; Rivera, J.M. Global existence and exponential stability of solutions to thermoelastic equations of hyperbolic type. *J. Elast.* **2004**, *75*, 125–145. [\[CrossRef\]](#)
3. Rivera, J.M. Energy Decay Rates in Linear Thermoelasticity. *Funkc. Ekvacioj* **1992**, *35*, 19–30.

4. Kirane, M.; Tatar, N. A nonexistence result to a Cauchy problem in nonlinear one dimensional thermoelasticity. *J. Math. Anal. Appl.* **2001**, *254*, 71–86. [\[CrossRef\]](#)
5. Grasselli, M.; Rivera, J.M.; Pata, V. On the energy decay of the linear thermoelastic plate with memory. *J. Math. Anal. Appl.* **2005**, *309*, 1–14. [\[CrossRef\]](#)
6. Ngoc, L.; Khanh, P.; Nhan, N.; Long, N. On a nonlinear boundary problem for thermoelastic coupled beam equations with memory term. *Math. Meth. Appl. Sci.* **2022**, *46*, 3614–3648. [\[CrossRef\]](#)
7. Aguiar-Barbosa, A.R.; Ma, T.F. Long-time dynamics of an extensible plate equation with thermal memory. *J. Math. Anal. Appl.* **2014**, *416*, 143–165. [\[CrossRef\]](#)
8. Grasselli, M.; Squassina, M. Exponential stability and singular limit for a linear thermoelastic plate with memory effects. *Adv. Math. Sci. Appl.* **2006**, *16*, 15–31.
9. Levine, H.A. Some nonexistence and instability theorems for solutions of formally parabolic equations of the form  $Pu_t = -Au + \mathcal{F}(u)$ . *Arch. Rational Mech. Anal.* **1973**, *51*, 371–386. [\[CrossRef\]](#)
10. Levine, H.A. Instability and nonexistence of global solutions to nonlinear wave equations of the form  $Pu_{tt} = -Au + \mathcal{F}(u)$ . *Trans. Amer. Math. Soc.* **1974**, *192*, 1–21.
11. Korpusov, M.O.; Ovchinnikov, A.V.; Sveshnikov, A.G.; Yushkov, E.V. *Blow-Up in Nonlinear Equations of Mathematical Physics. Theory and Methods*; De Gruyter Series in Nonlinear Analysis and Applications 27; De Gruyter: Berlin, Germany, 2018.
12. Esquivel-Avila, J.A. A differential inequality and the blow-up of its solutions. *Appl. Math. E-Notes* **2022**, *22*, 178–183.
13. Aouadi, M.; Miranville, A. Smooth attractor for a nonlinear thermoelastic diffusion thin plate based on Gurtin-Pipkin's model. *Asymptot. Anal.* **2015**, *95*, 129–160. [\[CrossRef\]](#)
14. Aouadi, M.; Miranville, A. Quasi-stability and global attractor in nonlinear thermoelasticity diffusion plate with memory. *Evol. Equ. Control Theory* **2015**, *4*, 241–263. [\[CrossRef\]](#)
15. Ahmad, B.; Alsaedi, A.; Kirane, M. Nonexistence results for the Cauchy problem of time fractional nonlinear systems of thermo-elasticity. *Math. Meth. Appl. Sci.* **2017**, *40*, 4272–4279. [\[CrossRef\]](#)
16. Fatoria, L.; Silva, M.J.; Mab, T.F. Yang, Z. Long-time behavior of a class of thermoelastic plates with nonlinear strain. *J. Diff. Eq.* **2015**, *259*, 4831–4862. [\[CrossRef\]](#)
17. Raposo, C.A.; Ribeiro, J.O.; Cattai, A.P. Global solution for a thermoelastic system with p-Laplacian. *Appl. Math. Lett.* **2018**, *86*, 119–125. [\[CrossRef\]](#)
18. Ding, H.; Zhou, J. Global existence and blow-up for a thermoelastic system with p-Laplacian. *Appl. Anal.* **2022**, *101*, 6690–6708. [\[CrossRef\]](#)
19. Boutefnouchet, M.; Erjaee, H.; Kirane, M.; Qafsaoui, M. Nonexistence results for the Cauchy problem for some fractional nonlinear systems of thermo-elasticity type. *Z. Angew. Math. Mech.* **2016**, *96*, 1119–1128. [\[CrossRef\]](#)
20. Hao, J.; Liu, Z. Stability of an abstract system of coupled hyperbolic and parabolic equations. *Z. Angew. Math. Phys.* **2013**, *64*, 1145–1159. [\[CrossRef\]](#)
21. Liu, W.; Chen, D.; Chen, Z. Long-Time Behavior for a Thermoelastic Microbeam Problem with Time Delay and the Coleman-Gurtin Thermal Law. *Acta Math. Sci.* **2021**, *41*, 609–632. [\[CrossRef\]](#)
22. Moumen, A.; Ouchenane, D.; Choucha, A.; Zennir, K.; Zubair, S.A. Exponential stability of Timoshenko system in thermoelasticity of second sound with a memory and distributed delay term. *Open Math.* **2021**, *19*, 1636–1647. [\[CrossRef\]](#)
23. Messaoudi, S.A. Local existence and blow up in nonlinear thermoelasticity with second sound. *Commun. Partial. Differ. Eq.* **2002**, *27*, 1681–1693. [\[CrossRef\]](#)
24. Messaoudi, S.A.; Al-Shehri, A. General boundary stabilization of memory-type thermoelasticity. *J. Math. Phys.* **2010**, *51*, 16. [\[CrossRef\]](#)
25. Kalantarova, J.V.; Kalantarov, V.K. Blow-up of solutions of coupled parabolic systems and hyperbolic equations. *Mat. Zametki* **2022**, *112*, 391–397; translation in *Math. Notes* **2022**, *112*, 406–411. [\[CrossRef\]](#)
26. Kalantarova, J.V. Blow up of solutions of systems of nonlinear equations of thermoelasticity. *Math. Meth. Appl. Sci.* **2023**, *46*, 13797–13805. [\[CrossRef\]](#)
27. Chen, D.; Liu, W.; Chen, Z. General decay for a thermoelastic problem of a microbeam with Gurtin-Pipkin thermal law. *Acta Math. Appl. Sin. Engl. Ser.* **2022**, *38*, 426–440. [\[CrossRef\]](#)
28. Wang, D.; Zhang, J. Long-time dynamics of N-dimensional structure equations with thermal memory. *Bound. Value Probl.* **2017**, *2017*, 21. [\[CrossRef\]](#)
29. Boulanouar, F.; Drabla, S. General boundary stabilization result of memory-type thermoelasticity with second sound. *Electron. J. Differ. Equations* **2014**, *2014*, 18.
30. Messaoudi, S.; Al-Shehri, A. General boundary stabilization of memory-type thermoelasticity with second sound. *Z. Anal. Anwend.* **2012**, *31*, 441–461. [\[CrossRef\]](#)
31. Messaoudi, S.; Al-Shehri, A. General boundary stabilization of memory type in thermoelasticity of type III. *Z. Angew. Math. Phys.* **2011**, *62*, 469–481. [\[CrossRef\]](#)

32. Mustafa, M. Boundary stabilization of memory-type thermoelasticity with second sound. *Z. Angew. Math. Phys.* **2012**, *63*, 777–792. [[CrossRef](#)]
33. Han, Z.; Kuang, Z.; Zhang, Q. Stability analysis for abstract theomoelastic systems with Cattaneo’s law and inertial terms. *Math. Control Relat. Fields* **2023**, *13*, 1639–1673. [[CrossRef](#)]

**Disclaimer/Publisher’s Note:** The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.